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Replica derivation of Sompolinsky free energy functional for mean field spin glasses

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Résumé. — Nous établissons une forme de la fonctionnelle d'énergie libre pour les verres de spin dans la limite du champ moyen que Sompolinsky a obtenue récemment *via* une approche dynamique. Nous utilisons ici la méthode des répliques avec une division en blocs analogue à celle employée pour établir la solution de Sommers et avec une procédure d'iteration proche de celle de Parisi mais portant sur les blocs diagonaux et hors diagonaux.

Abstract. — We derive a form of the free energy functional for mean field spin glasses that has been recently obtained by Sompolinsky *via* a dynamic approach. Here we use replicas with a block division along the lines used to derive Sommers solution and with an iterative procedure close to that of Parisi applied to both off diagonal and diagonal blocks.

1. **Introduction.** — The infinite ranged spin glass model of Sherrington and Kirkpatrick [1] (SK) that describes a mean field approximation to real spin glasses, has given rise to abundant work but has resisted so far, a full understanding. Even though the early description of the gelation transition was associated with anomalous behaviour of the large time limit of spin correlation functions [2], most of the work has been done with a static approach. In particular, but, by large, not exclusively, attention has been focused on trying to solve this model by using a replica trick [2, 3] that allows to directly take « quenched » averages at the price of having, in the end, to take the unphysical limit of the number n of replicas going to zero.

The so called SK solution describes the transition with a single, replica independent, Edwards Anderson (EA) order parameter

$$q_0 = \overline{\langle \sigma_i^\alpha \sigma_i^\beta \rangle}, \quad \alpha \neq \beta . \tag{1}$$

Here the thermodynamic average $\langle \ \rangle$ is taken with a weight $\exp - \beta H$

$$H = - \sum_{(i,j)} J_{ij} \sigma_i \sigma_j \tag{2}$$

the sum being over all pairs of sites (i, j), σ_i an Ising spin. The bar stands for average over the gaussian distribution of bonds

$$P(J_{jl}) \simeq \exp - N J_{jl}^2 / 2 J^2 \tag{3}$$

and α, β are replica indices ($\alpha = 1, 2, ..., n$).

The SK solution unfortunately leads to unphysical results [1] and is indeed unstable [4] with respect to fluctuations. A distinct solution has been exhibited by Sommers [5] that involves, besides the EA order parameter q_0, an anomaly a_0 to the linear response function. In terms of the replica approach, this solution is identified [6, 7] with a limiting case of an extension to the Blandin *et al.* [8] symmetry breaking scheme. One introduces an $n \times n$ order parameter matrix

$$q_{\alpha\beta} = \left(\begin{array}{c|c|c} q_0 & r_0 & r_0 \\ \hline r_0 & q_0 & r_0 \\ \hline r_0 & r_0 & q_0 \end{array} \right). \tag{4}$$

Here there are $(n/p_0)^2$ constant blocks, each one of size $p_0 \times p_0$, with value q_0 for a diagonal, r_0 for an off diagonal block. The limit $p_0 \to \infty$ is taken after $n \to 0$. In that limit

$$q_0 = r_0 \tag{5}$$

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is the EA order parameter, and the anomaly, for consistency with reference [12], is

$$- \Delta_0' \underset{p_0 \to \infty}{=} p_0(q_0 - r_0), \quad \Delta_0' < 0 . \tag{6}$$

Sommers solution, even though it does not carry distasteful features of the SK solution, remains unstable [6].

In a bold generalization, and guided by requirements of positive definiteness on the free energy, Parisi [9] has introduced a symmetry breaking scheme that leads to remarkable physical properties [9*b*] and verifies some stability criteria [10]. Parisi defines a self similar iterative procedure to build $q_{\alpha\beta}$:

The starting (zero) step being the constant, $n \times n$, q_0 matrix, step one is obtained by the transform T

$$(q_0)_{\overrightarrow{T}} \begin{pmatrix} q_1 & q_0 & q_0 \\ q_0 & q_1 & q_0 \\ q_0 & q_0 & q_1 \end{pmatrix} \tag{7}$$

characterized by (t_i) a division into $(n/m_1)^2$ blocks, each one $m_1 \times m_1$ in size, (t_{ii}) a shift ($q_0 \to q_1$) in the value of the diagonal blocks only. The transform is iterated (t_i) each q_1 block is divided into $(m_1/m_2)^2$ blocks of size $m_2 \times m_2$ and (t_{ii}) the new diagonal blocks are shifted ($q_1 \to q_2$). And so on, with the restriction

$$1 \leqslant m_K \leqslant \cdots \leqslant m_2 \leqslant m_1 \leqslant n . \tag{8}$$

When $n \to 0$, the continuous limit is obtained by

$$m_j = j/(K + 1), \quad K \to \infty \tag{9}$$

and

$$q_j = q(x), \quad m_j < x < m_{j+1} . \tag{10}$$

Questions left open include (i) are Parisi iteration and solution unique, (ii) what is the physical meaning of the parameter x [11].

In a far reaching recent paper, Sompolinsky [12] has reattacked the problem from a dynamical point of view. He has come out with a description that involves a *double* continuum of order parameters $q(x)$ and $\Delta'(x)$, the x index labelling now the physical continuum of infinite relaxation times that characterize the system. He ends up with an explicit functional of $q(x)$ and $\Delta'(x)$ that contains the main features of Parisi solution.

In this note we show that one recovers Sompolinsky functional by taking Sommers matrix (4) as a zero step, and applying, *both* on the diagonal (q_0) and off diagonal (r_0) blocks a self similar iterative procedure described below.

2. **Step zero : Sommers free energy functional** [6, 7]. — The free energy functional for the SK model writes

$$- \beta f = \frac{\beta^2 J^2}{4} + \frac{\partial}{\partial n}\bigg|_{n=0} \text{Max}\left[- \frac{\beta^2 J^2}{4} \sum_{\alpha \neq \beta} q_{\alpha\beta}^2 + \ln \text{Tr}_\sigma \exp \frac{\beta^2 J^2}{2} \sum_{\alpha \neq \beta} q_{\alpha\beta}\, \sigma_\alpha\, \sigma_\beta \right]. \tag{11}$$

It is readily evaluated when $q_{\alpha\beta}$ has the structure (4) *via*

$$\sum_{\alpha \neq \beta} q_{\alpha\beta}^2 = r_0^2\, n(n-1) + (q_0^2 - r_0^2)\, n(p_0 - 1) \tag{12}$$

$$\sum_{\alpha \neq \beta} q_{\alpha\beta}\, \sigma_\alpha\, \sigma_\beta = r_0 \left(\sum_{j_0, \alpha} \sigma_{j_0,\alpha} \right)^2 + (q_0 - r_0) \sum_{j_0=1}^{n/p_0} \left(\sum_{\alpha=1}^{p_0} \sigma_{j_0,\alpha} \right)^2 - n q_0 . \tag{13}$$

Here each spin is indexed by the block number $j_0 = 1, 2, \ldots, n/p_0$ and inside each block by $\alpha = 1, 2, \ldots, p_0$. Using z_0 and y_{j_0} to linearize (13), after taking traces over spins, we obtain for the log term of (11)

$$\ln \int \frac{\mathrm{d}z_0}{(2\pi)^{1/2}} \mathrm{e}^{-z_0^2/2} \prod_{j_0=1}^{n/p_0} \left(\int \frac{\mathrm{d}y_{j_0}}{(2\pi)^{1/2}} \exp \left\{ - \frac{y_{j_0}^2}{2} + p_0 \ln 2 \cosh \left\{ \beta J z_0\, r_0^{1/2} + \beta J y_{j_0} (q_0 - r_0)^{1/2} \right\} \right\} \right).$$

If we use the anomaly $- \Delta_0'$ as given by (6), we can, in the limit $p_0 \to \infty$ (taken after $n \to 0$ [6, 7]) write the y integrals ($y_{j_0} \to p_0^{1/2}\, y_{j_0}$) as saddle point contributions with

$$y_{j_0}^c \equiv \beta J m_0(z_0)\, (- \Delta_0')^{1/2} . \tag{14}$$

The free energy functional follows

$$- \beta f = \frac{\beta^2 J^2}{4} [(q_0 - 1)^2 + 2\, q_0\, \Delta_0'] + \int \frac{\mathrm{d}z_0}{(2\pi J^2 q_0)^{1/2}} \mathrm{e}^{-z_0^2/2J^2 q_0} \times$$

$$\times \left\{ + \frac{\beta^2 J^2}{2} m_0^2\, \Delta_0' + \ln 2 \cosh \left\{ \beta z_0 - \beta^2 J^2\, m_0\, \Delta_0' \right\} \right\}. \tag{15}$$

Here we have used (5). The order parameters q_0, Δ_0' and the local magnetization $m_0(z_0)$, are obtained by stationarity on (15),

$$m_0(z_0) = \tanh\left[\beta z_0 - \beta^2 J^2 m_0(z_0) \Delta_0'\right] \tag{16}$$

$$q_0 = \int \frac{\mathrm{d}z_0}{(2\pi J^2 q_0)^{1/2}} \mathrm{e}^{-z_0^2/2J^2 q_0} m_0^2(z_0) \tag{17}$$

$$\beta(1 - q_0 - \Delta_0') = \int \frac{\mathrm{d}z_0}{(2\pi J^2 q_0)^{1/2}} \mathrm{e}^{-z_0^2/2J^2 q_0} \frac{\partial}{\partial z_0} m_0(z_0)\,. \tag{18}$$

Introducing an external magnetic field h, adds a βh term inside the ln cosh, thus identifying $-\Delta_0'$ as the anomaly.

Note that the above results are also obtained for the reverse order of limits i.e. $p_0 \to \infty$ and then, $n = 0$.

3. **Iteration procedure : step one.** — We apply iteration (7) *both* on off diagonal blocks r_0 (with shift r_1 on the diagonal subblocks) and on diagonal blocks q_0 (shift q_1). Each spin is now indexed by block number $j_0 = 1, 2, \ldots, n/p_0$ and subblock number $j_1 = 1, 2, \ldots, p_0/p_1$ (for a given j_0), and inside each subblock by $\alpha = 1, 2, \ldots, p_1$. For step zero we had $p_0 \to \infty$. Here $p_0 \gg p_1$ and *both go to infinity* in succession. This procedure differs from Parisi's in two respects (i) it applies to both diagonal and off diagonal blocks, (ii) the division procedure ($p_i \to \infty$) leaves no variational parameters as is the case for Parisi m_i's.

With the above instructions equation (13) becomes

$$\sum_{\alpha \neq \beta} q_{\alpha\beta}\, \sigma_\alpha\, \sigma_\beta = r_0 \left(\sum_{j_0 j_1 \alpha} \sigma_{j_0 j_1 \alpha} \right)^2 + (q_0 - r_0) \sum_{j_0} \left(\sum_{j_1 \alpha} \sigma_{j_0 j_1 \alpha} \right)^2 - n q_1 + \\ + (r_1 - r_0) \sum_{j_1} \left(\sum_{j_0 \alpha} \sigma_{j_0 j_1 \alpha} \right)^2 + \left[(q_1 - q_0) - (r_1 - r_0)\right] \sum_{j_0 j_1} \left(\sum_{\alpha} \sigma_{j_0 j_1 \alpha} \right)^2 . \tag{19}$$

We introduce variables z_0 and z_{j_1} to unfold terms in r_0 and $(r_1 - r_0)$, variables y_{j_0} and $y_{j_0 j_1}$ for terms in $(q_0 - r_0)$ and $[(q_1 - q_0) - (r_1 - r_0)]$. Using (6) and

$$-\Delta_1' = p_1\left[(q_1 - q_0) - (r_1 - r_0)\right], \tag{20}$$

we get for the log term of (11)

$$\ln \int \frac{\mathrm{d}z_0}{(2\pi)^{1/2}} \mathrm{e}^{-z_0^2/2} \int \prod_{j_0} \left(\frac{\mathrm{d}y_{j_0}}{(2\pi)^{1/2}} \mathrm{e}^{-p_0 y_{j_0}^2/2} \right) \prod_{j_1=1}^{p_0/p_1} \left\{ \int \frac{\mathrm{d}z_{j_1}}{(2\pi)^{1/2}} \mathrm{e}^{-z_{j_1}^2/2} \times \right. \\ \times \int \prod_{j_0=1}^{n/p_0} \left(\frac{\mathrm{d}y_{j_1 j_0}}{(2\pi)^{1/2}} \exp\{ -p_1 y_{j_0 j_1}^2/2 + p_1 \ln 2\cosh[\beta J z_0 r_0^{1/2} + \beta J z_{j_1} (r_1 - r_0)^{1/2} \\ \left. + \beta J [y_{j_0}(-\Delta_0')^{1/2} + y_{j_0 j_1}(-\Delta_1')^{1/2}]] \} \right) \Bigg\} . \tag{21}$$

Here we see that it is essential now that the limits be taken in the order (i) $p_0 \to \infty$, (ii) $p_1 \to \infty$, (iii) *only then* $n \to 0$. In this order, the saddle point values (14) and $\beta J m_1(z_0, z_1)(-\Delta_1')^{1/2}$ for $y_{j_0 j_1}^c$ are given by

$$m_1 = \tanh\{ \beta J [z_0 q_0^{1/2} + z_1 (q_1 - q_0)^{1/2}] - \beta^2 J^2 [m_0 \Delta_0' + m_1 \Delta_1'] \} \tag{22}$$

in which we have used (5) and $r_1 = q_1$, together with

$$m_0 = \int \frac{\mathrm{d}z_1}{(2\pi)^{1/2}} \mathrm{e}^{-z_1^2/2} m_1 \,. \tag{23}$$

In deriving (23) we have used the fact that the weight

$$\exp p_1 \sum_{j_0} \left[+ m_{j_0 1}^2 \Delta_1'/2 + \ln 2\cosh\left[\beta J [z_0 q_0^{1/2} + z_1 (q_1 - q_0)^{1/2}] - \beta^2 J^2 [m_{j_0} \Delta_0' + m_{j_0 1} \Delta_1']\right]\right]$$

disappears with the factor $\sum_{j_0} \equiv n/p_0$ when taken at the saddle point value with all $m_{j_0 1} \equiv m_1$.

The free energy functional replacing (15) is now

$$- \beta f = \frac{\beta^2 J^2}{4} \left[(q_1 - 1)^2 + 2 \sum_{j=0}^{1} q_j \Delta_j' \right] + \int \prod_{j_0=0}^{1} \left(\frac{dz_j}{[2 \pi J^2 (q_j - q_{j-1})]^{1/2}} \times \exp - z_j^2 / 2 J^2 (q_j - q_{j-1}) \right) \cdot$$

$$\cdot \left[+ \frac{\beta^2 J^2}{2} \sum_{j=0}^{1} m_j^2 \Delta_j' + \ln 2 \cosh \left\{ \beta h + \sum_{j=0}^{1} (\beta z_j - \beta^2 J^2 m_j \Delta_j') \right\} \right] . \quad (24)$$

We have here written the result in general form with $q_{-1} \equiv 0$. Equations (23, 22) derive from stationarity (saddle point) in $m_0(z_0)$, $m_1(z_0, z_1)$, stationarity in q_0, Δ_0' and $q_1 - q_0$, Δ_1' yield the analog of equations (18) and (17) for Δ_1' and q_1.

4. **General form and comments.** — The above procedure is trivially repeated (with sequences $q_1, q_2 \ldots, q_k, r_1, r_2, \ldots, r_k$). The kth iterated result is obtained by replacing index 1 by k in (24) and expressing stationarity in m_j, $q_j - q_{j-1}$, Δ_j'. This is then identical with the discretized form of Sompolinsky [12] free energy functional, definition of the continuous limit being given in that reference.

Several points may be stressed : (i) The order of limits $p_0 \gg p_1 \gg \cdots \gg p_k \to \infty$ and then $n \to 0$ is surprising, the other way around ($n \to 0$ first) seeming more « physical ». A reformulation of the replica trick in terms of $P \times P$, $P \to \infty$ matrices (instead of $n \times n$, $n \to 0$) that may help to render palatable the above order, is discussed elsewhere [13]. (ii) Contrary to Parisi approach, there is no way here in which results could be vibrational in the parameters $p_1, p_2, \ldots, p_k$. (iii) The present derivation may suggest a purely static interpretation of the order parameters that is currently under investigation.

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