Replication of period-doubling route to chaos in impulsive systems

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Abstract. In this study, we investigate the dynamics of impulsive systems driven by a chaotic system. It is shown that the response impulsive system replicates the sensitivity and the period-doubling cascade of the drive. Illustrative examples that support the theoretical results are provided.

Keywords: impulsive systems, sensitivity, period-doubling cascade, unidirectional coupling.

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1 Introduction

One of the routes to chaos is the period-doubling cascade, which was first observed by Myrberg in quadratic maps [27–29]. This phenomenon is based on the successive emergence of periodic motions with twice period of the previous oscillation as some parameter is varied in a system [14, 26, 36]. The period-doubling onset of chaos exhibits a universal behavior [14]. Period-doubling route to chaos can be observed in various fields such as mechanics, electrical circuits, lasers, magnetism, photochemistry, neural processes, and predator-prey systems [7, 20, 23, 30–32, 39, 40].

It is known that if a function $I(t)$ with a certain property such as boundedness, periodicity, or almost periodicity is considered as an *input* for an evolution equation $u' = L[u] + I(t)$, where $L[u]$ is a linear operator with spectra placed in the left half of the complex plane, then the equation produces a solution, an *output*, with a similar property of boundedness, periodicity, or almost periodicity [11, 15]. Some applications of the input-output systems can be found in the studies [7, 8, 13, 38].

In this paper, we take into account the problem whether chaotic inputs generate chaotic outputs in systems of impulsive differential equations. Such differential equations describe the dynamics of real world processes in which abrupt changes occur, and they play an increasingly important role in mechanics, electronics, biology, neural networks, communication systems, chaos theory, and population dynamics [4, 21, 24, 34, 43–45]. In the present paper, we consider

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unidirectionally coupled systems such that the drive system is chaotic and the response system admits impulsive actions. We understand chaos in terms of sensitivity and the existence of infinitely many unstable periodic solutions in a bounded region. The sensitivity feature can be considered as the main ingredient of chaos [12,25,33,42]. It is theoretically proved that the impulsive response system replicates the sensitivity and the period-doubling cascade of the drive.

The usage of perturbations to generate chaos in systems of differential equations was initiated by Akhmet [1, 2], and replication of various types of chaos in generator–replicator systems was considered in the paper [5]. The existence of Li-Yorke chaos in impulsive systems was investigated in [3] by taking advantage of the chaotic behavior of the impulsive moments. On the other hand, perturbations were utilized in [6] to demonstrate the presence of Li–Yorke chaos in systems with impulses such that the proximality and frequent separation features were rigorously proved. Differently from the papers [3,6], in this study, we investigate the replication of sensitivity and period-doubling cascade in the dynamics of impulsive systems. Furthermore, the approach used in the present paper is different from synchronization of chaotic systems [16] since we do not consider the coupled systems from the asymptotic point of view.

The rest of the paper is organized as follows. In Section 2, we introduce the coupled systems of differential equations that will be investigated and provide sufficient conditions for the replication of period-doubling route to chaos. In Section 3, the replication of sensitivity by the impulsive response system is theoretically proved. Section 4 is devoted to the replication of period-doubling cascade, and finally, illustrative examples that support the theoretical results are given in Section 5.

2   The model

Let us consider the system

$$x' = F(t, x), \tag{2.1}$$

where the function $F : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous in all of its arguments and there exists a positive number T such that $F(t + T, x) = F(t, x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Our main assumption on system (2.1) is the existence of a nonempty set $\mathscr{A}$ of all solutions of (2.1) that are uniformly bounded on $\mathbb{R}$. In this case, there exists a compact set $\Lambda \subset \mathbb{R}^m$ such that the trajectories of all solutions that belong to $\mathscr{A}$ lie inside Λ.

Next, we take into account the impulsive system

$$\begin{aligned}
y' &= Ay + f(t, y) + g(x(t)), \qquad t \neq \theta_k, \\
\Delta y|_{t=\theta_k} &= By + W(y),
\end{aligned} \tag{2.2}$$

where $x(t)$ is a solution of (2.1), the functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^n$ and $W : \mathbb{R}^n \to \mathbb{R}^n$ are continuous in all their arguments, the function $f(t, y)$ satisfies the periodicity condition $f(t + T, y) = f(t, y)$ for all $t \in \mathbb{R}$, $y \in \mathbb{R}^n$, A and B are constant, $n \times n$ real matrices, the sequence $\{\theta_k\}$, $k \in \mathbb{Z}$, of impulsive moments is strictly increasing, $\Delta y|_{t=\theta_k} = y(\theta_k+) - y(\theta_k)$, and $y(\theta_k+) = \lim_{t \to \theta_k^+} y(t)$. We suppose that there exists a natural number p such that $\theta_{k+p} = \theta_k + T$ for all $k \in \mathbb{Z}$.

We will rigorously prove that if system (2.1) is chaotic, then the impulsive system (2.2) replicates the chaotic structure of (2.1). Our results are based on sensitivity and the existence

of infinitely many unstable periodic solutions in a bounded region. For the latter case we will take into account a period-doubling cascade of system (2.1). The descriptions of period-doubling cascade for system (2.1) and its replication by (2.2) are provided in Section 4.

The nonlinear terms $f(t, y)$ and $W(y)$ used in system (2.2) make the results of the present study more amenable to real world phenomena with impulsive actions since nonlinearity is essential in most cases [4,7,35]. On the other hand, the periodicity conditions on systems (2.1) and (2.2) are required for the existence of periodic solutions.

Throughout the paper, we make use of the usual Euclidean norm for vectors and the spectral norm for square matrices [19]. Moreover, we denote by $\log(I + B)$ the principal logarithm of the matrix $I + B$ assuming that $I + B$ has no eigenvalues on the closed negative real axis [18].

The following conditions are required.

(A1) The matrices A and B commute, and $\det(I + B) \neq 0$, where I is the $n \times n$ identity matrix;

(A2) The eigenvalues of the matrix $A + \frac{p}{T} \log(I + B)$ have negative real parts;

(A3) There exist positive numbers M_f and M_W such that $\sup_{t \in \mathbb{R}, y \in \mathbb{R}^n} \|f(t, y)\| \leq M_f$ and $\sup_{y \in \mathbb{R}^n} \|W(y)\| \leq M_W$.

(A4) There exist positive numbers L_F, L_f, L_1, L_2, and L_W such that

(i) $\|F(t, x_1) - F(t, x_2)\| \leq L_F \|x_1 - x_2\|$ for all $t \in \mathbb{R}$, $x_1, x_2 \in \Lambda$,

(ii) $\|f(t, y_1) - f(t, y_2)\| \leq L_f \|y_1 - y_2\|$ for all $t \in \mathbb{R}$, $y_1, y_2 \in \mathbb{R}^n$,

(iii) $L_1 \|x_1 - x_2\| \leq \|g(x_1) - g(x_2)\| \leq L_2 \|x_1 - x_2\|$ for all $x_1, x_2 \in \Lambda$,

(iv) $\|W(y_1) - W(y_2)\| \leq L_W \|y_1 - y_2\|$ for all $y_1, y_2 \in \mathbb{R}^n$.

In what follows, we will denote by $i(\Gamma)$ the number of the terms of the sequence $\{\theta_k\}$, $k \in \mathbb{Z}$, which belong to an interval Γ. One can confirm that $i((a, b)) \leq p + \frac{p}{T}(b - a)$, where a and b are numbers such that $b > a$.

Let us denote by $U(t, s)$ the transition matrix of the linear homogeneous impulsive system

$$u' = Au, \qquad t \neq \theta_k,$$
$$\Delta u|_{t=\theta_k} = Bu.$$

Under the condition $(A1)$ we have $U(t, s) = e^{A(t-s)}(I + B)^{i([s,t))}$ for $t > s$ and $U(s, s) = I$. Moreover, if condition $(A2)$ additionally holds, then there exist positive numbers N and ω such that

$$\|U(t, s)\| \leq Ne^{-\omega(t-s)} \tag{2.3}$$

for $t \geq s$ [4,35]. The inequality (2.3) can be verified, for example, using the equation (4.114) [35, p. 192] and the proof technique of Theorem 34 [35, p. 115]. Further details about transition matrices can be found in the book [4].

The following conditions are also needed.

(A5) $N\left(\frac{L_f}{\omega} + \frac{pL_W}{1 - e^{-\omega T}}\right) < 1$;

(A6) $-\omega + NL_f + \frac{p}{T} \ln(1 + NL_W) < 0$;

(A7) $L_W \|(I + B)^{-1}\| < 1$.

For a fixed solution $x(t)$ of (2.1), a left-continuous function $y(t) : \mathbb{R} \to \mathbb{R}^n$ is a solution of (2.2) if: (i) It has discontinuities only at the points θ_k, $k \in \mathbb{Z}$, and these discontinuities are jump discontinuities; (ii) The derivative $y'(t)$ exists at each point $t \in \mathbb{R} \setminus \{\theta_k\}$, and the left-sided derivative exists at the points θ_k, $k \in \mathbb{Z}$; (iii) The differential equation is satisfied by $y(t)$ on $\mathbb{R} \setminus \{\theta_k\}$, and it holds for the left derivative of $y(t)$ at every point θ_k, $k \in \mathbb{Z}$; (iv) The jump equation is satisfied by $y(t)$ for every $k \in \mathbb{Z}$.

According to the results of [4, 35], under the conditions $(A1)$, $(A2)$, $(A3)$, (ii), (iv), $(A4)$, (ii), (iv), and $(A5)$, for each $x(t) \in \mathscr{A}$ there exists a unique solution $\phi_{x(t)}(t)$ of system (2.2) that is bounded on the whole real axis, and it satisfies the relation

$$\phi_{x(t)}(t) = \int_{-\infty}^{t} U(t,s) \left[f(s, \phi_{x(t)}(s)) + g(x(s)) \right] ds + \sum_{-\infty < \theta_k < t} U(t, \theta_k+) W(\phi_{x(t)}(\theta_k)). \qquad (2.4)$$

Let us denote

$$M_g = \sup_{x \in \Lambda} \|g(x)\|.$$

Making use of the equation (2.4) and the inequality

$$\sum_{-\infty < \theta_k < t} e^{-\omega(t-\theta_k)} \leq \frac{p}{1 - e^{-\omega T}},$$

it can be verified that the inequality $\sup_{t \in \mathbb{R}} \left\| \phi_{x(t)}(t) \right\| \leq K_0$ is valid for each $x(t) \in \mathscr{A}$, where

$$K_0 = \frac{N(M_f + M_g)}{\omega} + \frac{pNM_W}{1 - e^{-\omega T}}.$$

Moreover, if condition $(A6)$ additionally holds, then for a fixed solution $x(t) \in \mathscr{A}$ of (2.1) the bounded solution $\phi_{x(t)}(t)$ attracts all other solutions of (2.2), i.e., $\left\| y(t) - \phi_{x(t)}(t) \right\| \to 0$ as $t \to \infty$ for each solution $y(t)$ of (2.2). In other words, for each fixed $x(t) \in \mathscr{A}$, the impulsive system (2.2) is dissipative [17].

For the theoretical investigation of replication of sensitivity, let us denote by $\mathscr{B}$ the set of all bounded solutions $\phi_{x(t)}(t)$ of the impulsive system (2.2), where $x(t) \in \mathscr{A}$.

It is worth noting that the results of the present paper are valid even if we replace the non-autonomous system (2.1) by an autonomous one of the form

$$x' = \overline{F}(x)$$

with the counterpart of condition $(A4)$, (i), where $\overline{F} : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function.

3   Sensitivity analysis

System (2.1) is called sensitive if there exist positive numbers ϵ_0 and Δ such that for an arbitrary positive number δ_0 and for each $x(t) \in \mathscr{A}$, there exist $\overline{x}(t) \in \mathscr{A}$, $t_0 \in \mathbb{R}$, and an interval $J \subset [t_0, \infty)$, with a length no less than Δ, such that $\|x(t_0) - \overline{x}(t_0)\| < \delta_0$ and $\|x(t) - \overline{x}(t)\| > \epsilon_0$ for all $t \in J$ [5].

We say that system (2.2) replicates the sensitivity of (2.1) if there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that for an arbitrary positive number δ_1 and for each bounded solution $\phi_{x(t)}(t) \in \mathscr{B}$, there exist a bounded solution $\phi_{\overline{x}(t)}(t) \in \mathscr{B}$, $t_0 \in \mathbb{R}$, and an interval $\widetilde{J} \subset [t_0, \infty)$, with a length no less than $\overline{\Delta}$, which contains at most one element of the sequence $\{\theta_k\}$ such that $\left\| \phi_{x(t)}(t_0) - \phi_{\overline{x}(t)}(t_0) \right\| < \delta_1$ and $\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \epsilon_1$ for all $t \in \widetilde{J}$.

For the proof of replication of sensitivity, the following analogue of the Gronwall's inequality for piecewise continuous functions is required.

Lemma 3.1 ([10]). *Suppose that for $\gamma_1 \leq t \leq \gamma_2$ the following inequality holds:*

$$u(t) \leq a(t) + \int_{\gamma}^{t} b(s)u(s)ds + \sum_{\gamma < t_k < t} \beta_k u(t_k),$$

where β_k, $k \in \mathbb{N}$, are constants and $u(t) : \mathbb{R} \to \mathbb{R}$, $a(t) : \mathbb{R} \to \mathbb{R}$, $b(t) : \mathbb{R} \to [0, \infty)$ are piecewise continuous functions that have jump discontinuities at the points t_k, $k = 1, 2, \ldots$, only and are left continuous at each t_k. Then, for $\gamma_1 \leq t \leq \gamma_2$,

$$u(t) \leq a(t) + \int_{\gamma_1}^{t} a(s)b(s) \prod_{s < t_k < t} (1 + \beta_k) e^{\int_s^t b(\tau)d\tau} ds + \sum_{\gamma_1 < t_k < t} a(t_k)\beta_k \prod_{t_k < t_j < t} (1 + \beta_j) e^{\int_{t_k}^t b(\tau)d\tau}.$$

Note that in the notation $\sum_{\gamma < t_k < t} \beta_k u(t_k)$ in Lemma 3.1, the numbers $\beta_k u(t_k)$ are being summed over all natural numbers k such that the inequality $\gamma < t_k < t$ is valid, and the remaining summation and product notations used in the lemma have similar meanings.

The next theorem is concerned with the replication of sensitivity of system (2.1).

Theorem 3.2. *Suppose that the conditions* (A1)–(A7) *hold. If system* (2.1) *is sensitive, then the impulsive system* (2.2) *replicates the sensitivity of* (2.1).

Proof. Fix an arbitrary number $\delta_1 > 0$ and a bounded solution $\phi_{x(t)}(t) \in \mathscr{B}$ of system (2.2). Let $\alpha = \omega - NL_f - \frac{p}{T} \ln(1 + NL_W)$. Note that the number α is positive by condition $(A6)$. Suppose that $\bar{\epsilon}$ is a sufficiently small positive number satisfying the inequality

$$\left[1 + \frac{NL_2}{\omega} \left(1 + \frac{NL_f}{\alpha}(1 + NL_W)^p + \frac{pNL_W}{1 - e^{-\alpha T}}(1 + NL_W)^p \right) \right] \bar{\epsilon} \leq \delta_1.$$

Take a number $R < 0$ sufficiently large in absolute value such that

$$\left(\frac{2N(M_f + M_g)}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} \right) (1 + NL_W)^p e^{\alpha R} \leq \bar{\epsilon},$$

and let $\delta_0 = \bar{\epsilon} e^{L_F R}$.

Since (2.1) is sensitive, there exist positive numbers ϵ_0 and Δ such that $\|x(t_0) - \bar{x}(t_0)\| < \delta_0$ and $\|x(t) - \bar{x}(t)\| > \epsilon_0$, $t \in J$, for some $\bar{x}(t) \in \mathscr{A}$, $t_0 \in \mathbb{R}$ and for some interval $J = (\rho_0, \rho_0 + \Delta_0)$, where ρ_0 and Δ_0 are numbers with $\rho_0 \geq t_0$ and $\Delta_0 \geq \Delta$. In the first part of the proof, we will show that $\|\phi_{x(t)}(t_0) - \phi_{\bar{x}(t)}(t_0)\| < \delta_1$.

The solutions $x(t)$ and $\bar{x}(t)$ of system (2.1) satisfy the equation

$$x(t) - \bar{x}(t) = x(t_0) - \bar{x}(t_0) + \int_{t_0}^{t} [F(s, x(s)) - F(s, \bar{x}(s))]ds.$$

Therefore, we have for $t \in [t_0 + R, t_0]$ that

$$\|x(t) - \bar{x}(t)\| \leq \|x(t_0) - \bar{x}(t_0)\| + \left| \int_{t_0}^{t} L_F \|x(s) - \bar{x}(s)\| ds \right|.$$

By means of the Gronwall–Bellman inequality, one can confirm that

$$\|x(t) - \bar{x}(t)\| \leq \|x(t_0) - \bar{x}(t_0)\| e^{L_F |t - t_0|}.$$

Hence, $\|x(t) - \bar{x}(t)\| < \bar{\epsilon}$ for $t \in [t_0 + R, t_0]$.

Since the relation

$$\phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) = \int_{-\infty}^{t_0+R} U(t,s) \left[f\left(s, \phi_{x(t)}(s)\right) + g(x(s)) - f\left(s, \phi_{\overline{x}(t)}(s)\right) - g(\overline{x}(s)) \right] ds$$
$$+ \int_{t_0+R}^{t} U(t,s) \left[f\left(s, \phi_{x(t)}(s)\right) - f\left(s, \phi_{\overline{x}(t)}(s)\right) \right] ds$$
$$+ \int_{t_0+R}^{t} U(t,s) \left[g(x(s)) - g(\overline{x}(s)) \right] ds$$
$$+ \sum_{-\infty < \theta_k \leq t_0+R} U(t, \theta_k+) \left[W\left(\phi_{x(t)}(\theta_k)\right) - W\left(\phi_{\overline{x}(t)}(\theta_k)\right) \right]$$
$$+ \sum_{t_0+R < \theta_k < t} U(t, \theta_k+) \left[W\left(\phi_{x(t)}(\theta_k)\right) - W\left(\phi_{\overline{x}(t)}(\theta_k)\right) \right]$$

holds, we have that

$$\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| \leq \left(\frac{2N(M_f + M_g)}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} \right) e^{-\omega(t-t_0-R)}$$
$$+ \frac{NL_2\overline{\epsilon}}{\omega} \left(1 - e^{-\omega(t-t_0-R)} \right)$$
$$+ \int_{t_0+R}^{t} NL_f e^{-\omega(t-s)} \left\| \phi_{x(t)}(s) - \phi_{\overline{x}(t)}(s) \right\| ds$$
$$+ \sum_{t_0+R < \theta_k < t} NL_W e^{-\omega(t-\theta_k)} \left\| \phi_{x(t)}(\theta_k) - \phi_{\overline{x}(t)}(\theta_k) \right\|. \qquad (3.1)$$

Let us define the functions $v(t) = e^{\omega t} \left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\|$ and $h(t) = c + \frac{NL_2\overline{\epsilon}}{\omega} e^{\omega t}$, where

$$c = \left(\frac{2N(M_f + M_g) - NL_2\overline{\epsilon}}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} \right) e^{\omega(t_0+R)}.$$

The inequality (3.1) implies that

$$v(t) \leq h(t) + \int_{t_0+R}^{t} NL_f v(s) ds + \sum_{t_0+R < \theta_k < t} NL_W v(\theta_k), \qquad t \in [t_0 + R, t_0].$$

By applying Lemma 3.1 one can verify that

$$v(t) \leq h(t) + \int_{t_0+R}^{t} NL_f (1 + NL_W)^{i((s,t))} e^{NL_f(t-s)} h(s) ds$$
$$+ \sum_{t_0+R < \theta_k < t} NL_W (1 + NL_W)^{i((\theta_k,t))} e^{NL_f(t-\theta_k)} h(\theta_k).$$

Using the equation

$$1 + \int_{t_0+R}^{t} NL_f (1 + NL_W)^{i((s,t))} e^{NL_f(t-s)} ds + \sum_{t_0+R < \theta_k < t} NL_W (1 + NL_W)^{i((\theta_k,t))} e^{NL_f(t-\theta_k)}$$
$$= (1 + NL_W)^{i((t_0+R,t))} e^{NL_f(t-t_0-R)}$$

together with the inequality

$$(1 + NL_W)^{i((a,b))} e^{NL_f(b-a)} \leq (1 + NL_W)^p e^{(\omega-\alpha)(b-a)}, \qquad b \geq a,$$

we obtain that

$$v(t) \leq c(1 + NL_W)^p e^{(\omega - \alpha)(t - t_0 - R)} + \frac{NL_2 \overline{\epsilon}}{\omega} e^{\omega t}$$

$$+ \int_{t_0 + R}^{t} \frac{N^2 L_f L_2 \overline{\epsilon}}{\omega} (1 + NL_W)^p e^{(\omega - \alpha)(t - s)} e^{\omega s} ds$$

$$+ \sum_{t_0 + R < \theta_k < t} \frac{N^2 L_2 L_W \overline{\epsilon}}{\omega} (1 + NL_W)^p e^{(\omega - \alpha)(t - \theta_k)} e^{\omega \theta_k}.$$

The last inequality implies for $t \in [t_0 + R, t_0]$ that

$$\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| \leq \left(\frac{2N(M_f + M_g) - NL_2 \overline{\epsilon}}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} \right) (1 + NL_W)^p e^{-\alpha(t - t_0 - R)}$$

$$+ \frac{NL_2 \overline{\epsilon}}{\omega} + \frac{N^2 L_f L_2 \overline{\epsilon}}{\alpha \omega} (1 + NL_W)^p \left(1 - e^{-\alpha(t - t_0 - R)} \right)$$

$$+ \frac{pN^2 L_2 L_W \overline{\epsilon}}{(1 - e^{-\alpha T}) \omega} (1 + NL_W)^p \left(1 - e^{-\alpha(t - t_0 - R + T)} \right).$$

Hence,

$$\left\| \phi_{x(t)}(t_0) - \phi_{\overline{x}(t)}(t_0) \right\| < \left(\frac{2N(M_f + M_g)}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} \right) (1 + NL_W)^p e^{\alpha R}$$

$$+ \frac{NL_2 \overline{\epsilon}}{\omega} \left(1 + \frac{NL_f}{\alpha} (1 + NL_W)^p + \frac{pNL_W}{1 - e^{-\alpha T}} (1 + NL_W)^p \right)$$

$$\leq \left[1 + \frac{NL_2}{\omega} \left(1 + \frac{NL_f}{\alpha} (1 + NL_W)^p + \frac{pNL_W}{1 - e^{-\alpha T}} (1 + NL_W)^p \right) \right] \overline{\epsilon}$$

$$\leq \delta_1.$$

Next, we will show the existence of positive numbers ϵ_1 and $\overline{\Delta}$ such that

$$\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \epsilon_1$$

for all $t \in \widetilde{J}$, where $\widetilde{J} \subset [t_0, \infty)$ is an interval which has length $\overline{\Delta}$ and contains at most one element of the sequence $\{\theta_k\}$, $k \in \mathbb{Z}$, of impulsive moments.

Let us denote $M_F = \sup_{t \in \mathbb{R}, x \in \Lambda} \|F(t, x)\|$. Since for each $x(t) \in \mathscr{A}$ the inequality $\sup_{t \in \mathbb{R}} \|x'(t)\| \leq M_F$ holds, one can conclude that the set $\mathscr{A}$ is an equicontinuous family on $\mathbb{R}$. Suppose that $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$, where each g_j, $1 \leq j \leq n$, is a real valued function. Because the function $\overline{g} : \Lambda \times \Lambda \to \mathbb{R}^n$ defined as $\overline{g}(x_1, x_2) = g(x_1) - g(x_2)$ is uniformly continuous on $\Lambda \times \Lambda$, the set consisting of the elements of the form $g_i(x(t)) - g_i(\overline{x}(t))$, $i = 1, 2, \ldots, n$, where $x(t), \overline{x}(t) \in \mathscr{A}$, is an equicontinuous family on $\mathbb{R}$. Therefore, there exists a positive number $\tau < \Delta$, which does not depend on the functions $x(t)$ and $\overline{x}(t)$, such that for each $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \tau$, the inequality

$$|(g_i(x(t_1)) - g_i(\overline{x}(t_1))) - (g_i(x(t_2)) - g_i(\overline{x}(t_2)))| < \frac{L_1 \epsilon_0}{2\sqrt{n}} \tag{3.2}$$

is valid for all $i = 1, 2, \ldots, n$.

Now, let η be the midpoint of the interval J, i.e., $\eta = \rho_0 + \Delta_0 / 2$, and set $\zeta = \eta - \tau / 2$. There exists an integer j, $1 \leq j \leq n$, such that

$$\left| g_j(x(\eta)) - g_j(\overline{x}(\eta)) \right| \geq \frac{1}{\sqrt{n}} \|g(x(\eta)) - g(\overline{x}(\eta))\|,$$

and therefore, condition $(A4)$, (iii), implies that

$$\left|g_j(x(\eta)) - g_j(\overline{x}(\eta))\right| \geq \frac{L_1}{\sqrt{n}} \left\|x(\eta) - \overline{x}(\eta)\right\| > \frac{L_1\epsilon_0}{\sqrt{n}}.$$

According to (3.2), we have for all $t \in [\zeta, \zeta + \tau]$ that

$$\left|g_j(x(t)) - g_j(\overline{x}(t))\right| > \left|g_j(x(\eta)) - g_j(\overline{x}(\eta))\right| - \frac{L_1\epsilon_0}{2\sqrt{n}} > \frac{L_1\epsilon_0}{2\sqrt{n}}.$$

One can confirm by using the last inequality that

$$\left\|\int_\zeta^{\zeta+\tau} [g(x(s)) - g(\overline{x}(s))]ds\right\| > \frac{\tau L_1\epsilon_0}{2\sqrt{n}}.$$

For $t \in [\zeta, \zeta + \tau]$, the functions $\phi_{x(t)}(t)$ and $\phi_{\overline{x}(t)}(t)$ satisfy the relations

$$\phi_{x(t)}(t) = \phi_{x(t)}(\zeta) + \int_\zeta^t \left[A\phi_{x(t)}(s) + f\left(s, \phi_{x(t)}(s)\right) + g(x(s))\right] ds$$
$$+ \sum_{\zeta \leq \theta_k < t} \left[B\phi_{x(t)}(\theta_k) + W\left(\phi_{x(t)}(\theta_k)\right)\right]$$

and

$$\phi_{\overline{x}(t)}(t) = \phi_{\overline{x}(t)}(\zeta) + \int_\zeta^t \left[A\phi_{\overline{x}(t)}(s) + f\left(s, \phi_{\overline{x}(t)}(s)\right) + g(\overline{x}(s))\right] ds$$
$$+ \sum_{\zeta \leq \theta_k < t} \left[B\phi_{\overline{x}(t)}(\theta_k) + W\left(\phi_{\overline{x}(t)}(\theta_k)\right)\right],$$

respectively. Thus, we have that

$$\left\|\phi_{x(t)}(\zeta + \tau) - \phi_{\overline{x}(t)}(\zeta + \tau)\right\|$$
$$\geq \left\|\int_\zeta^{\zeta+\tau} [g(x(s)) - g(\overline{x}(s))]ds\right\| - \left\|\phi_{x(t)}(\zeta) - \phi_{\overline{x}(t)}(\zeta)\right\|$$
$$- \int_\zeta^{\zeta+\tau} (\|A\| + L_f) \left\|\phi_{x(t)}(s) - \phi_{\overline{x}(t)}(s)\right\| ds - \sum_{\zeta \leq \theta_k < \zeta+\tau} (\|B\| + L_W) \left\|\phi_{x(t)}(\theta_k) - \phi_{\overline{x}(t)}(\theta_k)\right\|$$
$$> \frac{\tau L_1\epsilon_0}{2n} - \left[1 + \tau(\|A\| + L_f) + p\left(1 + \frac{\tau}{T}\right)(\|B\| + L_W)\right] \sup_{t \in [\zeta, \zeta+\tau]} \left\|\phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t)\right\|.$$

The last inequality implies that $\sup_{t \in [\zeta, \zeta+\tau]} \left\|\phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t)\right\| > \overline{M}$, where

$$\overline{M} = \frac{\tau L_1\epsilon_0}{2\sqrt{n}\left[2 + \tau(\|A\| + L_f) + p\left(1 + \frac{\tau}{T}\right)(\|B\| + L_W)\right]}.$$

Set $\underline{\theta} = \min_{1 \leq k \leq p}(\theta_{k+1} - \theta_k)$, and define the numbers

$$\epsilon_1 = \frac{\overline{M}}{2} \min\left\{1, \frac{1 - L_W \left\|(I + B)^{-1}\right\|}{\left\|(I + B)^{-1}\right\|}, \frac{1}{\|I + B\| + L_W}\right\}$$

and

$$\overline{\Delta} = \min\left\{ \underline{\theta}, \frac{\overline{M}}{4[(\|A\| + L_f)K_0 + M_g](1 + \|I + B\| + L_W)}, \right.$$
$$\left. \frac{\overline{M}\left(1 - L_W \left\|(I + B)^{-1}\right\|\right)}{4[(\|A\| + L_f)K_0 + M_g][1 + (1 - L_W)\left\|(I + B)^{-1}\right\|]} \right\}.$$

It is worth noting that the numbers ϵ_1 and $\overline{\Delta}$ are positive according to condition $(A7)$.

Suppose that there exists a number $\sigma \in [\zeta, \zeta + \tau]$ such that

$$\sup_{t \in [\zeta, \zeta + \tau]} \left\|\phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t)\right\| = \left\|\phi_{x(t)}(\sigma) - \phi_{\overline{x}(t)}(\sigma)\right\|.$$

Let

$$\kappa = \begin{cases} \sigma, & \text{if } \sigma \leq \zeta + \tau/2, \\ \sigma - \overline{\Delta}, & \text{if } \sigma > \zeta + \tau/2. \end{cases}$$

Since $\overline{\Delta} \leq \underline{\theta}$, there exists at most one impulsive moment on the interval $(\kappa, \kappa + \overline{\Delta})$.

First of all, we will consider the case $\sigma > \zeta + \tau/2$. Assume that there exists an impulsive moment $\theta_{k_0} \in (\kappa, \kappa + \overline{\Delta})$. For $t \in (\theta_{k_0}, \kappa + \overline{\Delta})$, we have that

$$\left\|\phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t)\right\| \geq \left\|\phi_{x(t)}(\kappa + \overline{\Delta}) - \phi_{\overline{x}(t)}(\kappa + \overline{\Delta})\right\| - \left\|\int_{\kappa + \overline{\Delta}}^{t} A\left(\phi_{x(t)}(s) - \phi_{\overline{x}(t)}(s)\right) ds\right\|$$
$$- \left\|\int_{\kappa + \overline{\Delta}}^{t} \left[f\left(s, \phi_{x(t)}(s)\right) - f\left(s, \phi_{\overline{x}(t)}(s)\right)\right] ds\right\|$$
$$- \left\|\int_{\kappa + \overline{\Delta}}^{t} [g(x(s)) - g(\overline{x}(s))] ds\right\|$$
$$> \overline{M} - 2\overline{\Delta}[K_0(\|A\| + L_f) + M_g]$$
$$> \frac{\overline{M}}{2}$$
$$\geq \epsilon_1.$$

Making use of the equations

$$\phi_{x(t)}(\theta_{k_0}+) = (I + B)\phi_{x(t)}(\theta_{k_0}) + W(\phi_{x(t)}(\theta_{k_0}))$$

and

$$\phi_{\overline{x}(t)}(\theta_{k_0}+) = (I + B)\phi_{\overline{x}(t)}(\theta_{k_0}) + W(\phi_{\overline{x}(t)}(\theta_{k_0}))$$

we obtain that

$$\left\|\phi_{x(t)}(\theta_{k_0}) - \phi_{\overline{x}(t)}(\theta_{k_0})\right\| > \frac{\overline{M} - 2\overline{\Delta}[K_0\left(\|A\| + L_f\right) + M_g]}{\|I + B\| + L_W}.$$

By means of the last inequality, one can verify for $t \in (\kappa, \theta_{k_0}]$ that

$$
\begin{aligned}
\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| &\geq \left\| \phi_{x(t)}(\theta_{k_0}) - \phi_{\overline{x}(t)}(\theta_{k_0}) \right\| - \left\| \int_{\theta_{k_0}}^{t} A \left(\phi_{x(t)}(s) - \phi_{\overline{x}(t)}(s) \right) ds \right\| \\
&\quad - \left\| \int_{\theta_{k_0}}^{t} \left[f \left(s, \phi_{x(t)}(s) \right) - f \left(s, \phi_{\overline{x}(t)}(s) \right) \right] ds \right\| \\
&\quad - \left\| \int_{\theta_{k_0}}^{t} [g(x(s)) - g(\overline{x}(s))] ds \right\| \\
&> \frac{\overline{M} - 2\overline{\Delta}[K_0(\|A\| + L_f) + M_g](1 + \|I + B\| + L_W)}{\|I + B\| + L_W} \\
&\geq \frac{\overline{M}}{2(\|I + B\| + L_W)} \\
&\geq \epsilon_1.
\end{aligned}
$$

Therefore, we have for $t \in (\kappa, \kappa + \overline{\Delta})$ that $\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \epsilon_1$.

On the other hand, if the interval $(\kappa, \kappa + \overline{\Delta})$ does not contain any impulsive moment, then one can confirm that $\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \overline{M}/2$ for all $t \in (\kappa, \kappa + \overline{\Delta})$. Hence, the inequality $\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \epsilon_1$ holds for all $t \in (\kappa, \kappa + \overline{\Delta})$ regardless of the existence of an impulsive moment inside the interval.

Next, let us take into account the case $\sigma \leq \zeta + \tau/2$. In the case that the interval $(\kappa, \kappa + \overline{\Delta})$ contains an impulsive moment θ_{k_0}, the inequality

$$
\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \overline{M} - 2\overline{\Delta}[K_0(\|A\| + L_f) + M_g] > \epsilon_1
$$

is valid for $t \in (\kappa, \theta_{k_0}]$. Therefore, we have that

$$
\begin{aligned}
\left\| \phi_{x(t)}(\theta_{k_0}+) - \phi_{\overline{x}(t)}(\theta_{k_0}+) \right\| &\geq \left(\frac{1 - L_W \left\| (I + B)^{-1} \right\|}{\|(I + B)^{-1}\|} \right) \left\| \phi_{x(t)}(\theta_{k_0}) - \phi_{\overline{x}(t)}(\theta_{k_0}) \right\| \\
&> \left(\frac{1 - L_W \left\| (I + B)^{-1} \right\|}{\|(I + B)^{-1}\|} \right) \left[\overline{M} - 2\overline{\Delta} \left(K_0(\|A\| + L_f) + M_g \right) \right].
\end{aligned}
$$

The last inequality implies for $t \in (\theta_{k_0}, \kappa + \overline{\Delta})$ that

$$
\begin{aligned}
\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| &> \left(\frac{1 - L_W \left\| (I + B)^{-1} \right\|}{\|(I + B)^{-1}\|} \right) \overline{M} \\
&\quad - 2\overline{\Delta} \left(1 + \frac{1 - L_W \left\| (I + B)^{-1} \right\|}{\|(I + B)^{-1}\|} \right) [K_0(\|A\| + L_f) + M_g] \\
&\geq \left(\frac{1 - L_W \left\| (I + B)^{-1} \right\|}{\|(I + B)^{-1}\|} \right) \frac{\overline{M}}{2} \\
&\geq \epsilon_1.
\end{aligned}
$$

If no impulsive moments take place inside the interval $(\kappa, \kappa + \overline{\Delta})$, then it can be deduced that

$$
\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \frac{\overline{M}}{2}, \qquad t \in (\kappa, \kappa + \overline{\Delta}).
$$

Thus, the inequality $\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \epsilon_1$, $t \in (\kappa, \kappa + \overline{\Delta})$, is valid for the case $\sigma \leq \zeta + \tau/2$ too.

Now, suppose that there exists an impulsive moment $\theta_{\widetilde{k}} \in [\zeta, \zeta + \tau]$ such that

$$\sup_{t \in [\zeta, \zeta + \tau]} \left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| = \left\| \phi_{x(t)}(\theta_{\widetilde{k}}+) - \phi_{\overline{x}(t)}(\theta_{\widetilde{k}}+) \right\|.$$

Let us denote

$$\kappa = \begin{cases} \theta_{\widetilde{k}}, & \text{if } \theta_{\widetilde{k}} \leq \zeta + \tau/2, \\ \theta_{\widetilde{k}} - \overline{\Delta}, & \text{if } \theta_{\widetilde{k}} > \zeta + \tau/2. \end{cases}$$

At first, we will consider the case $\theta_{\widetilde{k}} > \zeta + \tau/2$. Since the inequality

$$\left\| \phi_{x(t)}(\theta_{\widetilde{k}}) - \phi_{\overline{x}(t)}(\theta_{\widetilde{k}}) \right\| \geq \frac{\left\| \phi_{x(t)}(\theta_{\widetilde{k}}+) - \phi_{\overline{x}(t)}(\theta_{\widetilde{k}}+) \right\|}{\|I + B\| + L_W}$$

is valid, one can attain for $t \in (\kappa, \kappa + \overline{\Delta})$ that

$$\begin{aligned} \left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| &> \frac{\overline{M}}{\|I + B\| + L_W} - 2\overline{\Delta}[K_0(\|A\| + L_f) + M_g] \\ &> \frac{\overline{M}}{2\left(\|I + B\| + L_W\right)} \\ &\geq \epsilon_1. \end{aligned}$$

On the other hand, if $\theta_{\widetilde{k}} \leq \zeta + \tau/2$, then it can be shown for $t \in (\kappa, \kappa + \overline{\Delta})$ that

$$\left\| \phi_{x(t)}(t) - \phi_{\overline{x}(t)}(t) \right\| > \overline{M} - 2\overline{\Delta}[K_0(\|A\| + L_f) + M_g] > \frac{\overline{M}}{2} \geq \epsilon_1.$$

Consequently, system (2.2) replicates the sensitivity of (2.1). $\qquad\square$

Remark 3.3. Even though the impulsive system (2.2) replicates the sensitivity of (2.1) under the conditions (A1)–(A7), system (2.2) is not chaotic for fixed $x(t) \in \mathscr{A}$ since it is dissipative.

4   Period-doubling cascade

In this part of the paper, we suppose that there exists a function $G : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ satisfying the periodicity condition $G(t + T, x, \mu) = G(t, x, \mu)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $\mu \in \mathbb{R}$, where μ is a parameter, such that for some finite value μ_∞ of the parameter the function $F(t, x)$ on the right-hand side of system (2.1) is equal to $G(t, x, \mu_\infty)$.

System (2.1) is said to admit a period-doubling cascade [9, 14, 22, 36] if there exists a sequence $\{\mu_j\}$, $j \in \mathbb{N}$, of period-doubling bifurcation values with $\mu_j \to \mu_\infty$ as $j \to \infty$ such that as the parameter μ increases or decreases through μ_j the system

$$x' = G(t, x, \mu) \tag{4.1}$$

undergoes a period-doubling bifurcation, i.e., there exists a natural number λ such that for each $j \in \mathbb{N}$ a new periodic solution with period $\lambda 2^j T$ appears in the dynamics of system (4.1), and consequently, system (4.1) possesses infinitely many unstable periodic solutions all lying in a bounded region for $\mu = \mu_\infty$.

We say that the impulsive system (2.2) replicates the period-doubling cascade of system (2.1) if for each periodic solution $x(t) \in \mathscr{A}$ of (2.1) system (2.2) admits a periodic solution with the same period.

Under the conditions (A1)–(A5), one can verify using the results of [4, 35] that if $x(t) \in \mathscr{A}$ is a $\lambda_0 T$-periodic solution of system (2.1) for some natural number λ_0, then the corresponding bounded solution $\phi_{x(t)}(t)$ of (2.2) is also $\lambda_0 T$-periodic. Moreover, the instability of all periodic solutions of (2.2) is ensured by Theorem 3.2. Therefore, we have the following theorem.

Theorem 4.1. *Assume that the conditions* (A1)–(A7) *are valid. If system* (2.1) *admits a period-doubling cascade, then the impulsive system* (2.2) *replicates the period-doubling cascade of* (2.1).

A corollary of Theorem (4.1) is as follows.

Corollary 4.2. *Assume that the conditions* (A1)–(A7) *are valid. If system* (2.1) *admits a period-doubling cascade, then the same is true for the coupled system* (2.1)–(2.2).

Remark 4.3. One can confirm that the sequence $\{\mu_j\}$ of period-doubling bifurcation parameter values is exactly the same for both system (2.1) and the coupled system (2.1)–(2.2). Therefore, if system (2.1) obeys the Feigenbaum universality [14], then the same is true also for the coupled system (2.1)–(2.2). More precisely, when $\lim_{j \to \infty} \frac{\mu_j - \mu_{j+1}}{\mu_{j+1} - \mu_{j+2}}$ is evaluated, the universal constant $4.6692016\ldots$ known as the Feigenbaum number is achieved, and this number is the same for both system (2.1) and the coupled system (2.1)–(2.2).

The next section is devoted to illustrative examples that support the theoretical results.

5 Examples

In this part of the paper, two examples will be presented. In the first example the replication of sensitivity in an impulsive system driven by a chaotic Lorenz system will be demonstrated numerically, whereas in the second one replication of period-doubling cascade in an impulsive system driven by a Duffing equation will be discussed.

5.1 Example 1

Let us consider the Lorenz system [25]

$$
\begin{aligned}
x_1' &= -10x_1 + 10x_2, \\
x_2' &= -x_1 x_3 + 28x_1 - x_2, \\
x_3' &= x_1 x_2 - \frac{8}{3}x_3.
\end{aligned}
\tag{5.1}
$$

It was demonstrated in [25, 41] that system (5.1) is sensitive and it possesses a chaotic attractor.

Next, we take into account the impulsive system

$$y_1' = -3y_1 + 0.05\sin(\pi t) + 2.4x_1(t),$$
$$y_2' = -2y_2 + 0.15\cos y_2 - 2x_2(t),$$
$$y_3' = -4y_3 + 0.2\tanh y_1 + 0.6x_3(t), \ t \neq \theta_k,$$
$$\Delta y_1|_{t=\theta_k} = -\frac{2}{3}y_1, \tag{5.2}$$
$$\Delta y_2|_{t=\theta_k} = -\frac{2}{3}y_2 + 0.1\arctan y_3,$$
$$\Delta y_3|_{t=\theta_k} = -\frac{2}{3}y_3,$$

where $(x_1(t), x_2(t), x_3(t))$ is a solution of system (5.1) and $\theta_k = 2k, k \in \mathbb{Z}$. System (5.2) is in the form of (2.2) with $A = \text{diag}(-3, -2, -4)$, $B = \text{diag}\left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$,

$$f(t, y_1, y_2, y_3) = (0.05\sin(\pi t), 0.15\cos y_2, 0.2\tanh y_1),$$
$$g(x_1, x_2, x_3) = (2.4x_1, -2x_2, 0.6x_3),$$

and

$$W(y_1, y_2, y_3) = (0, 0.1\arctan y_3, 0).$$

One can verify that the conditions (A1)–(A7) are satisfied for system (5.2) with $N = 1$, $\omega = 2$, $T = 2$, $p = 1$, $M_f = 0.255$, $M_W = 0.05\pi$, $L_f = 0.2$, $L_1 = 0.6$, $L_2 = 2.4$, and $L_W = 0.1$. According to Theorem 3.2, the impulsive system (5.2) replicates the sensitivity of the Lorenz system (5.1). Figure 5.1 shows the 3-dimensional projections of two initially nearby solutions of the unidirectionally coupled systems (5.1)–(5.2) on the $y_1 - y_2 - y_3$ space. The trajectory in red corresponds to the initial data $x_1(0.5) = -7.61$, $x_2(0.5) = -2.35$, $x_3(0.5) = 33.04$, $y_1(0.5) = -0.53$, $y_2(0.5) = -5.15$, $y_3(0.5) = 5.19$, whereas the trajectory in blue corresponds to the initial data $x_1(0.5) = -7.65$, $x_2(0.5) = -2.42$, $x_3(0.5) = 33.02$, $y_1(0.5) = -0.51$, $y_2(0.5) = -5.16$, $y_3(0.5) = 5.18$. The time interval $[0.5, 3.65]$ is used in the simulation, and both trajectories make a jump at $t = 2$. Figure 5.1 supports the result of Theorem 3.2 such that even if the trajectories are initially nearby, later they diverge.

To show the irregular behavior of system (5.2), we depict in Figure 5.2 the 3-dimensional projection of the trajectory of the coupled system (5.1)–(5.2) with $x_1(0.5) = -10.74$, $x_2(0.5) = -13.35$, $x_3(0.5) = 26.51$, $y_1(0.5) = -5.94$, $y_2(0.5) = 7.67$, $y_3(0.5) = 3.52$ on the $y_1 - y_2 - y_3$ space. The irregular behavior seen in Figure 5.2 supports the existence of chaos in the dynamics of the coupled system (5.1)–(5.2). According to the impulse conditions in (5.2), the irregular trajectory represented in Figure 5.2 has discontinuities at the moments $t = \theta_k$.

5.2 Example 2

It was demonstrated in paper [37] that the Duffing equation

$$x'' + 0.3x' + x^3 = \mu\cos t, \tag{5.3}$$

where μ is a parameter, displays period-doubling bifurcations and leads to chaos at $\mu = \mu_\infty \equiv 40$.

Using the variables $x_1 = x$ and $x_2 = x'$, equation (5.3) can be rewritten as a system in the form

$$x_1' = x_2,$$
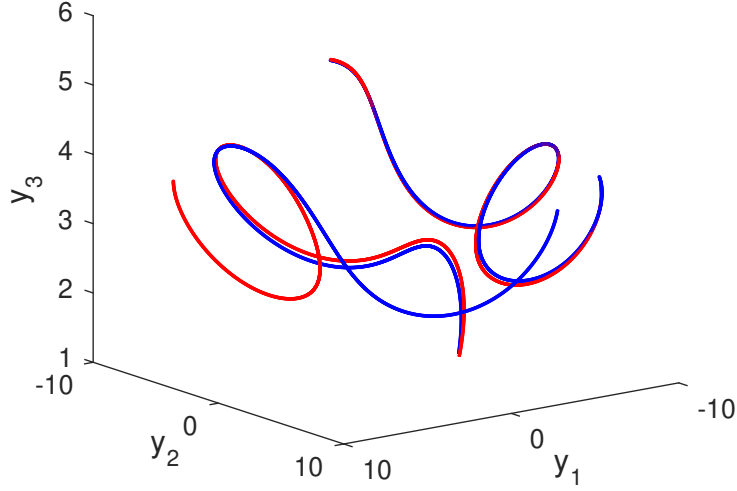$$x_2' = -0.3x_2 - x_1^3 + \mu\cos t. \tag{5.4}$$

Figure 5.1: Replication of sensitivity in system (5.2). The figure manifests the divergence of two initially nearby trajectories shown in red and blue, i.e., the impulsive system (5.2) replicates the sensitivity of the Lorenz system (5.1). The time interval $[0.5, 3.65]$ is used, and both trajectories make a jump at $t = 2$.

One can confirm that the chaotic attractor of system (5.4) takes place inside the compact region

$$\Lambda = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \ |x_1| \leq 5.5, \ |x_2| \leq 14 \right\}.$$

Next, we consider the impulsive system

$$
\begin{aligned}
y_1' &= -y_1 - 4y_2 + 0.12 \arctan y_2 + 2.1 x_1(t) - 0.3 \sin(x_1(t)), \\
y_2' &= y_1 - 3y_2 + 0.7 \cos t - 1.6 x_2(t) + 0.01 x_2^2(t), \ t \neq \theta_k, \\
\Delta y_1|_{t=\theta_k} &= -0.5 y_1 + 0.08 \sin y_2, \\
\Delta y_2|_{t=\theta_k} &= -0.5 y_2,
\end{aligned}
\tag{5.5}
$$

where $(x_1(t), x_2(t))$ is a solution of (5.4) and $\theta_k = \pi k, k \in \mathbb{Z}$.

The impulsive system (5.5) is in the form of (2.2) with

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -3 \end{pmatrix}, \qquad B = \mathrm{diag}\,(-0.5, -0.5),$$

$$f(t, y_1, y_2) = (0.5 \arctan y_2, 0.7 \cos t),$$
$$g(x_1, x_2) = (2.1 x_1 - 0.3 \sin x_1, -1.6 x_2 + 0.01 x_2^2),$$

and

$$W(y_1, y_2) = (0.08 \sin y_2, 0).$$

Let us denote by $U(t, s)$ the transition matrix of the linear homogeneous system

$$
\begin{aligned}
u_1' &= -u_1 - 4u_2, \\
u_2' &= u_1 - 3u_2, \ t \neq \theta_k, \\
\Delta u_1|_{t=\theta_k} &= -0.5 u_1, \\
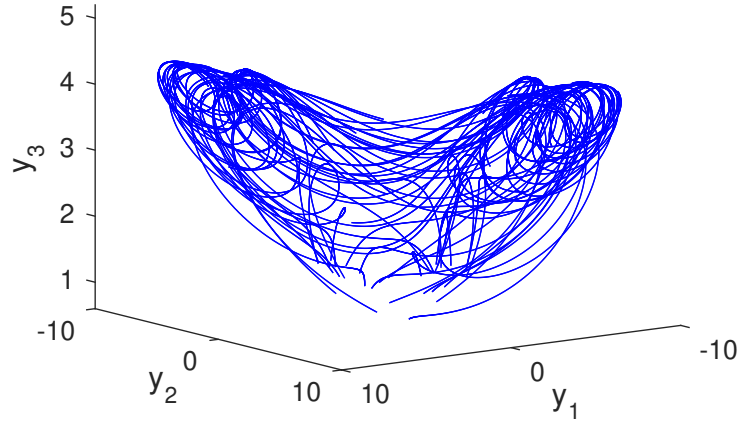\Delta u_2|_{t=\theta_k} &= -0.5 u_2.
\end{aligned}
\tag{5.6}
$$

Figure 5.2: Irregular behavior of system (5.2). The discontinuous irregular trajectory of the impulsive system (5.2) supports the result of Theorem 3.2 one more time.

It can be verified that

$$U(t,s) = e^{-2(t-s)} \left(\frac{1}{2}\right)^{i([s,t))} P \left(\begin{array}{cc} \cos(\sqrt{3}(t-s)) & -\sin(\sqrt{3}(t-s)) \\ \sin(\sqrt{3}(t-s)) & \cos(\sqrt{3}(t-s)) \end{array} \right) P^{-1}, \qquad t > s,$$

where $P = \left(\begin{array}{cc} \sqrt{3} & 1 \\ 0 & 1 \end{array} \right)$. The conditions (A1)–(A7) are satisfied for system (5.5) with $N = 2.48421$, $\omega = 2$, $T = 2\pi$, $p = 2$, $M_f = 0.72494$, $M_W = 0.08$, $L_f = 0.12$, $L_1 = 1.32$, $L_2 = 2.4$, and $L_W = 0.08$. Moreover, the eigenvalues of the matrix

$$A + \frac{p}{T} \log(I + B) = \left(\begin{array}{cc} -1 - \dfrac{\ln 2}{\pi} & -4 \\ 1 & -3 - \dfrac{\ln 2}{\pi} \end{array} \right)$$

are $-2 - \frac{\ln 2}{\pi} \pm i\sqrt{3}$.

According to Theorem 4.1, the impulsive system (5.5) replicates the period-doubling cascade of the Duffing equation (5.3). In order to demonstrate this numerically, we depict in Figure 5.3 the periodic orbits as well as an irregular trajectory. Figure 5.3, (a), (b), and (c) respectively show the period 1, period 2, and period 4 orbits of (5.5). The parameter values $\mu = 31.7$, $\mu = 34.3$, and $\mu = 36.1$ are utilized in Figure 5.3, (a), (b), and (c), respectively. On the other hand, taking $\mu = 40$ in the coupled system (5.4)–(5.5), we represent in Figure 5.3 (d) the 2-dimensional projection of the trajectory of (5.4)–(5.5) corresponding to the initial data $x_1(0.2) = 3.16$, $x_2(0.2) = 1.86$, $y_1(0.2) = 0.71$, $y_2(0.2) = 0.18$ on the y_1–y_2 plane. Figure 5.3 supports the result of Theorem 4.1 such that the period-doubling cascade of (5.3) is replicated. Moreover, we represent in Figure 5.4, the time-series of the y_2-coordinate of the trajectory shown in Figure 5.3, (d). The irregular behavior of the time-series supports the presence of chaos in the coupled system (5.4)–(5.5) with $\mu = 40$.
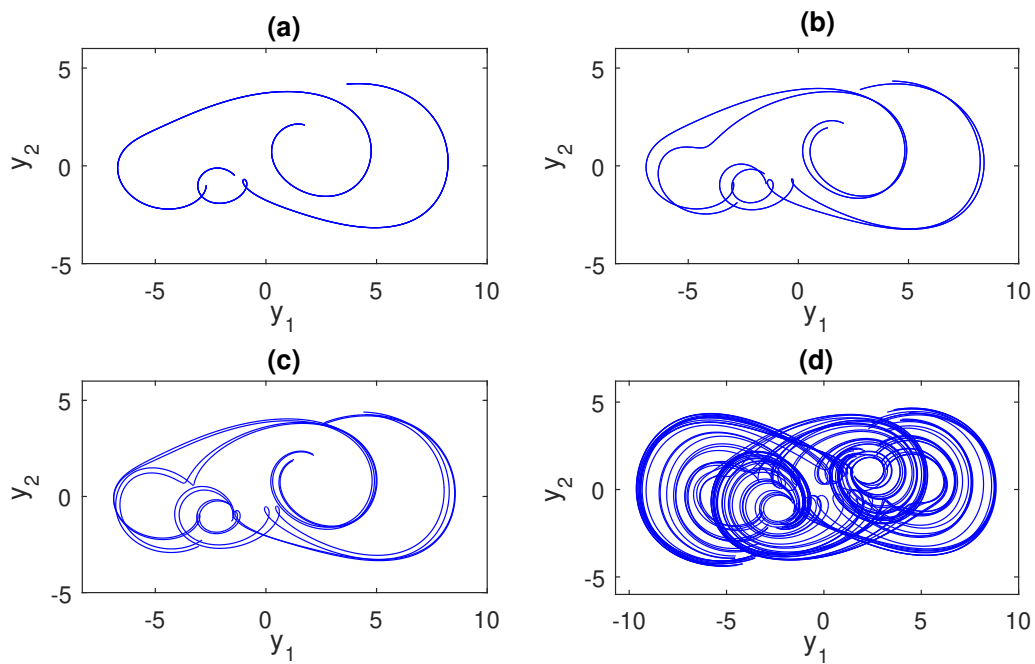
Figure 5.3: Periodic and irregular orbits of the impulsive system (5.5). (a) Period 1 orbit. (b) Period 2 orbit. (c) Period 4 orbit. (d) Irregular orbit. The parameter values $\mu = 31.7$, $\mu = 34.3$, $\mu = 36.1$, and $\mu = 40$ are utilized in (a), (b), (c), and (d), respectively. The figure reveals that the impulsive system (5.5) replicates the period-doubling cascade of the Duffing equation (5.3).
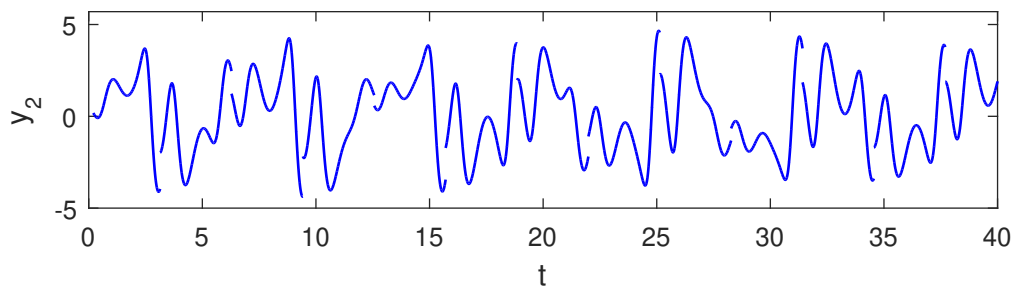

Figure 5.4: Time-series of the y_2-coordinate of system (5.5) with $\mu = 40$. The initial data $x_1(0.2) = 3.16$, $x_2(0.2) = 1.86$, $y_1(0.2) = 0.71$, and $y_2(0.2) = 0.18$ are utilized.

Acknowledgements

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