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REPRESENTATION AND DUALITY OF UNIMODULAR C^*-DISCRETE QUANTUM GROUPS

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Abstract. Suppose that $\mathcal{D}$ is a C^*-discrete quantum group and $\mathcal{D}_0$ a discrete quantum group associated with $\mathcal{D}$. If there exists a continuous action of $\mathcal{D}$ on an operator algebra $L(H)$ so that $L(H)$ becomes a $\mathcal{D}$-module algebra, and if the inner product on the Hilbert space H is $\mathcal{D}$-invariant, there is a unique C^*-representation θ of $\mathcal{D}$ associated with the action. The fixed-point subspace under the action of $\mathcal{D}$ is a Von Neumann algebra, and furthermore, it is the commutant of $\theta(\mathcal{D})$ in $L(H)$.

1. Introduction

Let G be a group with a unit e and D is the vector space of complex functions on G with finite support. If G is finite, under the pointwise operation D can be made into a Hopf algebra if we define comultiplication, counit and antipode respectively by

$$\begin{aligned}(\Delta(f))(s,t) &= f(st),\\ \varepsilon(f) &= f(e),\\ (Sf)(t) &= f(t^{-1}),\end{aligned}$$

where $f \in D$ and $s,t \in G$. We have $\Delta(D) \subseteq D \otimes D$ if we identify $D \otimes D$ with functions on $G \times G$. If G is infinite, the range of Δ is no longer in $D \otimes D$. Notice that for any $f, g \in D$, $\Delta(f)(g \otimes 1)$ and $\Delta(f)(1 \otimes g)$ are elements in $D \otimes D$, where 1 is the unit in the multiplier algebra $M(D)$ of D. This leads to the concept of multiplier Hopf algebras ([3]).

Definition 1.1. Let D be an algebra with a non-degenerate product. A comultiplication on D is a homomorphism Δ: $D \to M(D \otimes D)$ so that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ are in $D \otimes D$ and that Δ is coassociative in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c),$$

where $a, b, c \in D$, and 1 is the unit of $M(D)$.

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Now consider the linear mappings T_1 and T_2 defined on $D \otimes D$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b),$$

$$T_2(a \otimes b) = (a \otimes 1)\Delta(b).$$

If the mappings T_1 and T_2 are both bijections from $D \otimes D$ onto $D \otimes D$, (D, Δ) is called a multiplier Hopf algebra. If the algebra has also a $*$-structure so that the comultiplication Δ is a $*$-homomorphism, we then call D a multiplier Hopf $*$-algebra. It is easy to see that in a multiplier Hopf $*$-algebra D, the mappings

$$a \otimes b \to \Delta(a)(b \otimes 1), \quad a \otimes b \to (1 \otimes a)\Delta(b),$$

are both bijective.

A discrete quantum group is a multiplier Hopf $*$-algebra (D, Δ) where the algebra D is a direct sum of full matrix algebras over $\mathbb{C}$ with the natural involution. Discrete quantum groups were studied firstly as duals of compact matrix quantum groups [13]. Since the theory of compact quantum groups was initiated with the fundamental papers of S. L. Woronowicz (For detail, see [18] and [19]), this class of quantum groups then was carried out and the theory of discrete quantum groups on the level of C^*-algebras was developed as well ([13]). The algebraic counterparts and generalizations of this theory came in the form of E. G. Effros, Z. J. Ruan and Van Daele's theory on duality for multiplier Hopf algebras with invariant functionals ([5], [7]). It was also Van Daele who use multiplier Hopf algebras to give a description for discrete quantum groups without reference to their compact duals ([4]).

Let D be a discrete quantum group. There is a unique element z in D satisfying $z = z^* = z^2$, $\varepsilon(z) = 1$ and $\forall h \in D$,

$$hz = zh = \varepsilon(h)z.$$

We call such an element a cointegral [6]. Using the cointegral, [9] presents a duality theory between a finite dimensional discrete quantum group ant its fixed-point subalgebra in an operator algebra $L(H)$ where $L(H)$ is a D-module algebra, and H is a Hilbert space. Such a duality theory has its inherent physical meaning. In detail, suppose that G is a finite group and $D(G)$ is its double algebra. Also suppose that $\mathcal{G}$ is the field algebra of G-spin model. There is a natural action of $D(G)$ on $\mathcal{G}$ so that $\mathcal{G}$ becomes a $D(G)$-module algebra. Then the observable algebra O, which is a $D(G)$-invariant subalgebra of $\mathcal{G}$, is obtained. When an irreducible representation, associated to a $D(G)$-invariant state, π of $\mathcal{G}$ is given, there emerges a realization of $D(G)$ so that $D(G)$ and $\pi(O)$ are commutants of each other ([16]). Besides this example, the Schur-Weyl duality between the symmetric group and the general linear group ([8], [17]), the Jimbo-Schur-Weyl duality between quantum group of type A and Hecke algebra ([10], [11]), and so on, fit into the scheme given in [9]. This paper extends the result of [9] and gets a duality result between a unimodular C^*-discrete quantum group (i.e., a C^*-discrete quantum group whose left and right integrals coincide) and its fixed-point subalgebra in $L(H)$ as well. As to the nonunimodular case, it is under consideration now.

All algebras in this paper will be $*$-algebras over the complex field $\mathbb{C}$. For general results on Hopf algebras one can refer to the books of Abe ([1]) and Sweedler ([15]). We shall use m, Δ, ε and S for the multiplication, the comultiplication, the counit and the antipode respectively. Also we shall adopt the summation convention, which is standard in Hopf algebra theory. In a multiplier Hopf algebra D, the formular $m\,(id \otimes S)\,\Delta\,(b) = \varepsilon\,(b)\,1$ can be understand as, for example,

$$\sum_{(b)} ab_{(1)} S\left(b_{(2)}\right) = \varepsilon\,(b)\,a,$$

where $a, b \in D$ and $(a \otimes 1)\,\Delta\,(b) = \sum_{(b)} ab_{(1)} \otimes b_{(2)}$. Here we call $b_{(1)}$ is covered by a. Also the formular $(a \otimes 1 \otimes 1)(\Delta \otimes \iota)\,(\Delta\,(b)\,(1 \otimes c))$ can be written as

$$\sum_{(b)} ab_{(1)} \otimes b_{(2)} \otimes b_{(3)}c,$$

where $a, b, c \in D$. One has to make sure that at least all but one factor $b_{(k)}$ is covered by an element in D.

2. Unimodular C^*-discrete quantum group and its fixed-point algebra in $L(H)$

Suppose that $(\mathcal{D}, \Delta_{\mathcal{D}})$ is a C^*-discrete quantum group. There exists a compact quantum group $(\mathcal{B}, \Delta_{\mathcal{B}})$ so that $(\mathcal{D}, \Delta_{\mathcal{D}})$ is the dual of $(\mathcal{B}, \Delta_{\mathcal{B}})$ in the sense of [13]. Let Λ be the set of equivalent classes of all irreducible unitary representations of $\mathcal{B}$, then

$$\mathcal{D} = \oplus_{\lambda \in \Lambda} M_{n_\lambda},$$

where n_λ is the dimension of the representation corresponding to λ ([14]). This means that $\mathcal{D}$ consists of infinite families $(m_\lambda)_{\lambda \in \Lambda}$ such that $m_\lambda \in M_{n_\lambda}$, and for any $\varepsilon > 0$ there exists a finite subset $F \subseteq \Lambda$ such that $\|m_\lambda\| < \varepsilon$ for all $\lambda \in \Lambda \setminus F$. In other words, $\mathcal{D}$ is the restricted direct sum of the family of matrix algebras $(M_{n_\lambda})_{\lambda \in \Lambda}$ ([12]). Let $\mathcal{D}_0$ be the Pederson ideal of $\mathcal{D}$, i.e., the minimal dense ideal of $\mathcal{D}$. It is easy to see that $\mathcal{D}_0$ is the algebraic direct sum of the same family of matrix algebras $(M_{n_\lambda})_{\lambda \in \Lambda}$. Suppose that Δ is the restriction of the map $\Delta_{\mathcal{D}}$ to $\mathcal{D}_0$. The pair $(\mathcal{D}_0, \Delta)$ is a discrete quantum group. We shall refer to $(\mathcal{D}_0, \Delta)$ as a discrete quantum group associated with $(\mathcal{D}, \Delta_{\mathcal{D}})$.

It is well known that every C^*-algebra has an approximate unit. Now suppose that $\mathcal{D} = \oplus_{\lambda \in \Lambda} M_{n_\lambda}$ is a C^*-discrete quantum group, and that $\mathcal{D}_0$ is the discrete quantum group associated with $\mathcal{D}$. Suppose that $F \subseteq \Lambda$ is a finite set, under the inclusion relation $\mathcal{F} := \{F | F \subseteq \Lambda\}$ is a directed set. Set $e_F = \sum_{\lambda \in F} e_\lambda$, where e_λ is the unit of matrix algebra M_{n_λ}.

Proposition 2.1. *The net $(e_F)_{F \in \mathcal{F}}$ is an approximate unit in $\mathcal{D}$.*

Proof. It is clear that e_F is a central idempotent element in $\mathcal{D}_0$, and that for any finite sets $F_1 \subseteq F_2$, $e_{F_1} \leq e_{F_2}$. Now for $x \in \mathcal{D}$, notice that $\mathcal{D}_0$ is dense in $\mathcal{D}$, $\forall \varepsilon > 0$, there exists an element $x_0 \in \mathcal{D}_0$ so that $\|x - x_0\| < \frac{\varepsilon}{2}$. One can suppose that $x_0 = \sum_{\lambda \in F_0} a_\lambda$, where $F_0 \subseteq \Lambda$ is a finite set and $a_\lambda \in M_{n_\lambda}$. Also for an arbitrary finite set F so that $\Lambda \supseteq F \supseteq F_0$,

$$x_0 e_F = e_F x_0 = x_0.$$

Therefore for $F \supseteq F_0$,

$$\begin{aligned} \|xe_F - x\| &= \|(x - x_0) e_F + x_0 e_F - x_0 + x_0 - x\| \\ &\leq \|(x - x_0) e_F\| + \|x_0 e_F - x_0\| + \|x_0 - x\| \\ &= \|(x - x_0) e_F\| + \|x_0 - x\| \\ &\leq \|(x - x_0)\| \, \|e_F\| + \|x_0 - x\| \\ &< \varepsilon. \end{aligned}$$

Similarly for $F \supseteq F_0$ one have $\|e_F x - x\| < \varepsilon$. This completes the proof. □

Definition 2.2. A linear functional φ on $\mathcal{D}$ is called a left integral if

$$(id \otimes \varphi) \Delta (a) = \varphi (a) 1$$

for all $a \in \mathcal{D}$. Similarly a linear functional ψ on $\mathcal{D}$ is called right integral if $(\psi \otimes id) \Delta (a) = \psi (a) 1$ for all $a \in \mathcal{D}$. We call a (C^*-) discrete quantum group unimodular if its left and right integrals coincide.

We have the following characterizations of unimodularity for C^*-discrete quantum group [13].

Lemma 2.3. *Let $(\mathcal{D}, \Delta_{\mathcal{D}})$ be a C^*-discrete quantum group and $(\mathcal{D}_0, \Delta)$ the discrete quantum group associated with $(\mathcal{D}, \Delta_{\mathcal{D}})$. The following three statements are equivalent:*
1) *$(\mathcal{D}, \Delta)$ is unimodular;*
2) *the antipode S of $(\mathcal{D}_0, \Delta)$ is bounded;*
3) *the antipode S of $(\mathcal{D}_0, \Delta)$ is involutive, namely, $S^2 = id$.*

From now on, suppose that $\mathcal{D} = \oplus_{\lambda \in \Lambda} M_{n_\lambda}$ is a unimodular C^*-discrete quantum group and that $\mathcal{D}_0$ is the discrete quantum group associated with $\mathcal{D}$. Then the antipode S on $\mathcal{D}_0$ is a $*$-anti-homomorphism and satisfies the relation $S^2 = id$.

Lemma 2.4. *If $S^2 = id$, for all $a, b \in \mathcal{D}_0$ we have*

$$\sum_{(b)} aS(b_{(2)})b_{(1)} = \varepsilon(b)a,$$

$$\sum_{(b)} b_{(2)}S(b_{(1)})a = \varepsilon(b)a,$$

where $\Delta (b) (1 \otimes a) = \sum_{(b)} b_{(1)} \otimes b_{(2)}a$.

Proof. Notice that $\mathcal{D}_0$ is a discrete quantum group, for $a, b \in \mathcal{D}_0$

$$\sum_{(b)} S\left(b_{(1)}\right) b_{(2)} a = \varepsilon(b)\, a.$$

Applying the antipode S on both sides, one have

$$\begin{aligned} \varepsilon(b)\, S(a) &= \sum\nolimits_{(b)} S(a)\, S\left(b_{(2)}\right) S^2\left(b_{(1)}\right), \\ &= \sum\nolimits_{(b)} S(a)\, S\left(b_{(2)}\right) b_{(1)}. \end{aligned}$$

Replacing $S(a)$ by a, then

$$\sum_{(b)} aS\left(b_{(2)}\right) b_{(1)} = \varepsilon(b)\, a.$$

Similarly we can prove the second equation and we omit it here. □

Definition 2.5. Let A be a $*$-closed operator algebra in $L(H)$ with a unit I. If there is a continuous bilinear mapping $\langle\cdot,\cdot\rangle : \mathcal{D} \times A \to A$ so that for $a, b \in \mathcal{D}$ and $P, T \in A$,

$$\begin{aligned} \langle a, I\rangle &= \varepsilon(a)\, I, \\ \langle ab, P\rangle &= \langle a, \langle b, P\rangle\rangle, \\ \langle a, P^*\rangle &= \langle S(a^*), P\rangle^*, \\ \langle a, P \cdot \langle b, T\rangle\rangle &= \sum\nolimits_{(a)} \left\langle a_{(1)}, P\right\rangle \left\langle a_{(2)} b, T\right\rangle, \end{aligned}$$

where $\Delta(a)(1 \otimes b) = \sum_{(a)} a_{(1)} \otimes a_{(2)} b$, A is called a $\mathcal{D}$-module algebra.

We will use $a(T)$ for the element $\langle a, T\rangle$. If A is a $\mathcal{D}$-module algebra, it is easy to see

$$a(b(T)\, P) = \sum_{(a)} a_{(1)} b(T)\, a_{(2)}(P).$$

Also if $\mathcal{D}$ is a finite dimensional C^*-discrete quantum group, it has a unit and $\mathcal{D}_0 = \mathcal{D}$. By z we denote its cointegral element, the mapping $z(\cdot)$ is a conditional expectation. That is to say, it is a positive mapping with bimodular property, and therefore is automatically continuous [9].

Proposition 2.6. *Suppose that $\mathcal{D}$ is a C^*-discrete quantum group and $\mathcal{D}_0$ the discrete quantum group associated with $\mathcal{D}$. If $L(H)$ is a $\mathcal{D}$-module algebra and z is the cointegral in $\mathcal{D}_0$, set*

$$O = \{T \in L(H) : z(T) = T\}.$$

Then

$$O = \{T \in L(H) : h(T) = \varepsilon(h) T, \quad \forall h \in \mathcal{D}_0\},$$

and it is a nonzero C^*-algebra.

Proof. Since $\varepsilon(z) = 1$, one have $z(I) = I$ and O is a nonzero subspace in $L(H)$. It is clear that

$$\{T \in L(H) : h(T) = \varepsilon(h) T, \quad \forall h \in \mathcal{D}_0\} \subseteq O.$$

On the other hand, suppose that $T \in L(H)$ so that $z(T) = T$, i.e. $T \in O$. Then for $h \in \mathcal{D}_0$,

$$h(T) = h(z(T)) = hz(T) = \varepsilon(h) z(T) = \varepsilon(h) T.$$

Thus $O = \{T \in L(H) : h(T) = \varepsilon(h)T, \quad \forall h \in \mathcal{D}_0\}$.

For $T \in O$, since $S(z^*) = z^*$,

$$z(T^*) = (S(z^*)(T))^* = (z(T))^* = T^*,$$

one have $T^* \in O$, and O is closed under the *-operation.

Now we prove that O is an algebra. Indeed, $\forall F, G \in O$,

$$\begin{aligned} z(FG) &= z(F(z(G))) \\ &= \textstyle\sum_{(z)} z_{(1)}(F) z_{(2)} z(G) \\ &= z(F) z(G) \\ &= FG. \end{aligned}$$

Here we use the relation $\Delta(z)(1 \otimes z) = \sum_{(z)} z_{(1)} \otimes z_{(2)} z = z \otimes z$ [6]. Therefore O is a nonzero $*$-algebra.

At last, suppose that $T \in \overline{O}$, the closure of O under the uniform topology in $L(H)$, there exists a sequence $\{T_n\}$ in O so that $\lim_{n\to\infty} T_n = T$. Since the mapping $z(\cdot)$ is continuous,

$$z(T) = z\left(\lim_{n\to\infty} T_n\right) = \lim_{n\to\infty} z(T_n) = T.$$

This implies that $T \in O$ and O is closed under the C^*-norm. Therefore O is a C^*-algebra. □

Remark. The algebra O is called an observable algebra in $L(H)$. We will see in Theorem 3.3 that O is a Von Neumann algebra. Also since the discrete quantum group $\mathcal{D}_0$ is dense in $\mathcal{D}$ and $L(H)$ is a $\mathcal{D}$-module algebra,

$$O = \{T \in L(H) : h(T) = \varepsilon(h)T, \quad \forall h \in \mathcal{D}\}.$$

3. Representation and duality of $\mathcal{D}$

In this section we still suppose that $L(H)$ is a $\mathcal{D}$-module algebra and will build a duality theory between $\mathcal{D}$ and O, where H is a separate Hilbert space.

Definition 3.1. We call the inner product $(\,,\,)$ on a Hilbert space H is $\mathcal{D}$-invariant, if there is a vacuum vector $\Omega \in H$ of norm one so that $\forall a \in \mathcal{D}$, $T \in L(H)$,

$$(a(T)\Omega, \Omega) = \varepsilon(a)(T\Omega, \Omega).$$

Theorem 3.2. *If the inner product $(\,,\,)$ on H is $\mathcal{D}$-invariant, there exists a unique C^*-homomorphism $\theta : \mathcal{D} \to L(H)$ with the following two properties: $\forall a, b \in \mathcal{D}_0$, $T \in L(H)$,*

$$\theta(a)(\Omega) = \varepsilon(a)\Omega, \tag{3.1}$$

$$\theta(b)a(T) = \sum_{(a)} \theta\left(ba_{(1)}\right) T\theta\left(S\left(a_{(2)}\right)\right), \tag{3.2}$$

where $ba = \sum_{(a)} ba_{(1)}\varepsilon\left(a_{(2)}\right)$.

Proof. We divide the proof into four parts.

1) Firstly for $a \in \mathcal{D}_0$ we construct a mapping $\theta_0(a)$ on a dense subspace of H. To do this, let e be an arbitrary central idempotent element in $\mathcal{D}_0$. Given $a \in \mathcal{D}_0$ and $P, T \in L(H)$,

$$\begin{aligned}
(ae\,(T)\,\Omega, P\Omega) &= (ea\,(T)\,\Omega, P\Omega) \\
&= (P^*\,(ea\,(T)\,\Omega)\,,\Omega) \\
&= \textstyle\sum_{(a)} \left(\left(P^* \cdot ea_{(1)}\varepsilon\left(a_{(2)}\right)(T)\right)\Omega, \Omega\right) \\
&= \textstyle\sum_{(a)} \varepsilon\left(a_{(2)}\right)\left(\left(P^* \cdot ea_{(1)}\,(T)\right)\Omega, \Omega\right) \\
&= \textstyle\sum_{(a)} \varepsilon \cdot S\left(a_{(2)}\right)\left(\left(P^* \cdot ea_{(1)}\,(T)\right)\Omega, \Omega\right) \qquad (\varepsilon \cdot S = \varepsilon) \\
&= \textstyle\sum_{(a)} \left(S\left(a_{(2)}\right)\left(P^* \cdot ea_{(1)}\,(T)\right)\Omega, \Omega\right).
\end{aligned}$$

Replacing $P\Omega$ by $b(P)\Omega$, where $b \in \mathcal{D}_0$,

$$\begin{aligned}
&(ae\,(T)\,\Omega, b\,(P)\,\Omega) \\
= {} & \textstyle\sum_{(a)} \left(S\left(a_{(2)}\right)\left(b\,(P)^* \cdot ea_{(1)}\,(T)\right)\Omega, \Omega\right) \\
= {} & \textstyle\sum_{(a)} \left(S\left(a_{(2)}\right)\left(S\,(b^*)\,(P^*) \cdot ea_{(1)}\,(T)\right)\Omega, \Omega\right) \\
= {} & \textstyle\sum_{(a)} \left(S\left(a_{(3)}\right)S\,(b^*)\,(P^*) \cdot S\left(a_{(2)}\right)ea_{(1)}\,(T)\,\Omega, \Omega\right) \\
= {} & \textstyle\sum_{(a)} \left(S\left(a_{(3)}\right)S\,(b^*)\,(P^*) \cdot S\left(a_{(2)}\right)a_{(1)}e\,(T)\,\Omega, \Omega\right) \\
= {} & \textstyle\sum_{(a)} \varepsilon\left(a_{(1)}\right)\left(S\left(a_{(2)}\right)S\,(b^*)\,(P^*) \cdot e\,(T)\,\Omega, \Omega\right) \qquad (S^2 = id) \\
= {} & \textstyle\sum_{(a)} \left(S\left(b^*\varepsilon\left(a_{(1)}\right)a_{(2)}\right)(P^*) \cdot e\,(T)\,\Omega, \Omega\right) \\
= {} & \left(S\,(b^*a)\,(P^*) \cdot e\,(T)\,\Omega, \Omega\right) \\
= {} & \left((a^*b\,(P))^* \cdot e\,(T)\,\Omega, \Omega\right) \\
= {} & (e\,(T)\,\Omega, a^*b\,(P)\,\Omega).
\end{aligned}$$

Thus we have

$$(ae\,(T)\,\Omega, b\,(P)\,\Omega) = (ea\,(T)\,\Omega, b\,(P)\,\Omega) = (e\,(T)\,\Omega, a^*b\,(P)\,\Omega).$$

Considering the fact that the set $\{b\,(P)\,\Omega | b \in \mathcal{D}_0, P \in L(H)\}$ is dense in H, the relation $e\,(T)\,\Omega = 0$ implies that $ae\,(T)\,\Omega = 0$. Thus for $a \in \mathcal{D}_0$, set

$$\theta_0\,(a) : e\,(T)\,\Omega \to ae\,(T)\,\Omega,$$

where e is an arbitrary central idempotent element, $\theta_0(a)$ is well defined on the dense subspace $\{e\,(T)\,\Omega : T \in L(H)\}$ of H and satisfies the relation $\theta_0\,(a)^* = \theta_0\,(a^*)$.

2) Secondly we construct a C^*-homomorphism $\theta : \mathcal{D} \to L(H)$. For $T\Omega \in H$ and $b \in \mathcal{D}_0$, since the net $\{e_F | F \in \mathcal{F}\}$ is an approximate unit in $\mathcal{D}$, $\lim_{F \in \mathcal{F}} e_F\,(T)\,\Omega = T\Omega$. Thus

$$\begin{aligned}
\lim_{F \in \mathcal{F}} (\theta_0\,(a)\,(e_F\,(T)\,\Omega)\,, b\,(P)\,\Omega) &= \lim_{F \in \mathcal{F}} (ae_F\,(T)\,\Omega, b\,(P)\,\Omega) \\
&= \lim_{F \in \mathcal{F}} (e_F\,(T)\,\Omega, a^*b\,(P)\,\Omega) \\
&= (T\Omega, a^*b\,(P)\,\Omega) \\
&= (a\,(T)\,\Omega, b\,(P)\,\Omega).
\end{aligned}$$

Using the principle of uniform boundedness,

$$\lim_{F\in\mathcal{F}} \theta_0(a)(e_F(T)\Omega) = a(T)\Omega.$$

Therefore for $a \in \mathcal{D}_0$, the mapping

$$\theta(a) : T\Omega \to a(T)\Omega \quad (T \in L(H))$$

is well defined on H and exactly an extension of $\theta_0(a)$.

Now for $a \in \mathcal{D}_0$ we prove that $\theta(a) \in L(H)$. Indeed, for $x \in H$ with $\|x\| = 1$, set $T_x = x \otimes \Omega$, namely, $T_x(y) = (y, \Omega)x$, then $T_x\Omega = x$ and $\|T_x\| = 1$. Thus,

$$\begin{aligned} \|\theta(a)(x)\| &= \|\theta(a)(T_x\Omega)\| \\ &= \|a(T_x)\Omega\| \\ &\le \|a(\cdot)\|\,\|x\|, \end{aligned}$$

where the mapping $a(\cdot)$ is regarded as a linear bounded operator from $L(H)$ to $L(H)$. This means that $\theta(a) \in L(H)$. Also $\mathcal{D}_0$ is dense in $\mathcal{D}$, $\theta(\cdot)$ can be extended by continuity to a C^*-homomorphism from $\mathcal{D}$ into $L(H)$.

3) The homomorphism θ has properties (3.1) and (3.2) given in Theorem 3.2. Indeed the property (3.1) is obvious. Now for $a, b, c \in \mathcal{D}_0$ and $P \in L(H)$,

$$\begin{aligned} &\sum_{(a)} \theta\left(ba_{(1)}\right) T\theta\left(S\left(a_{(2)}\right)\right)(c(P)\Omega) \\ &= \sum_{(a)} \theta\left(ba_{(1)}\right) \cdot T\left(\theta\left(S\left(a_{(2)}\right)\right)(c(P)\Omega)\right) \\ &= \sum_{(a)} \theta\left(ba_{(1)}\right) \cdot T\left(S\left(a_{(2)}\right)c(P)\Omega\right) \\ &= \sum_{(a)} ba_{(1)}\left(T \cdot S\left(a_{(2)}\right)c(P)\right)\Omega \\ &= \sum_{(a)} ba_{(1)}(T) \cdot a_{(2)}S\left(a_{(3)}\right)c(P)\Omega \\ &= \sum_{(a)} ba_{(1)}(T) \cdot \varepsilon\left(a_{(2)}\right)c(P)\Omega \\ &= b\left(a(T)c(P)\right)\Omega \\ &= \theta(b)\left(a(T)c(P)\Omega\right), \end{aligned}$$

we have

$$\theta(b)a(T) = \sum_{(a)} \theta\left(ba_{(1)}\right) T\theta\left(S\left(a_{(2)}\right)\right).$$

4) Uniqueness. If there exists another C^*-homomorphism $\theta' : \mathcal{D} \to L(H)$ with properties (3.1) and (3.2). Namely for $a, b \in \mathcal{D}_0$ and $T \in L(H)$,

$$\begin{aligned} \theta'(a)(\Omega) &= \varepsilon(a)\Omega, \\ \theta'(b)a(T) &= \textstyle\sum_{(a)} \theta'\left(ba_{(1)}\right) T\theta'\left(S\left(a_{(2)}\right)\right). \end{aligned}$$

Then for $T \in L(H)$, notice that $\theta(b)(T\Omega) = b(T)\Omega$,

$$\begin{aligned}\theta'(a)\theta(b)(T\Omega) &= \theta'(a)\,b(T)\,\Omega \\ &= \textstyle\sum_{(b)} \theta'\left(ab_{(1)}\right) T\theta'\left(S\left(b_{(2)}\right)\right)\Omega \\ &= \textstyle\sum_{(b)} \theta'\left(ab_{(1)}\right) T\varepsilon\left(b_{(2)}\right)\Omega \\ &= \theta'(ab)(T\Omega) \\ &= \theta'(a)\,\theta'(b)(T\Omega).\end{aligned}$$

Thus for $a, b \in \mathcal{D}_0$, $\theta'(a)\,\theta(b) = \theta'(a)\,\theta'(b)$. Since θ is a C^*-homomorphism, the net $\{\theta(e_F) : F \in \mathcal{F}\}$ converges in the strong operator topology to the identity operator on H [2]. Replace $a \in \mathcal{D}_0$ by $e_F \in \mathcal{D}_0$, one can see for $b \in \mathcal{D}_0$, $\theta(b) = \theta'(b)$ and furthermore $\theta = \theta'$. This completes the proof. □

The following theorem is the main result of this paper, which gives a duality theory between $\theta(\mathcal{D})$ and O in $L(H)$.

Theorem 3.3. *Assumptions and notations as in Theorem* 3.2. *Then*

$$\theta(\mathcal{D})' = \theta(\mathcal{D}_0)' = O,$$

where the prime denotes the commutant in $L(H)$.

Proof. We firstly prove that $\theta(a)P = P\theta(a)$, where $a \in \mathcal{D}_0$ and $P = z(P) \in O$. Indeed for $a, b \in \mathcal{D}_0$ and $T \in L(H)$, since

$$\begin{aligned}\theta(a)\,P\,(b(T)\,\Omega) &= a(Pb(T))\,\Omega \\ &= \textstyle\sum_{(a)} a_{(1)}(P)\,a_{(2)}b(T)\,\Omega \\ &= \textstyle\sum_{(a)} a_{(1)}z(P)\,a_{(2)}b(T)\,\Omega \\ &= \textstyle\sum_{(a)} P\varepsilon\left(a_{(1)}\right) a_{(2)}b(T)\,\Omega \\ &= Pab(T)\,\Omega \\ &= P\theta(a)(b(T)\,\Omega),\end{aligned}$$

we have $\theta(a)\,P = P\theta(a)$. Notice that $\mathcal{D}_0$ is dense in $\mathcal{D}$ and θ is a C^*-homomorphism, $\theta(x)\,P = P\theta(x)$ where $x \in \mathcal{D}$ and $P \in O$. Thus the inclusion relations

$$\theta(\mathcal{D}_0)' \supseteq \theta(\mathcal{D})' \supseteq O$$

hold, where the prime denotes the commutant in $L(H)$. On the other hand, suppose that $P \in \theta(\mathcal{D}_0)'$, that is to say, for $a \in \mathcal{D}_0$, $\theta(a)\,P = P\theta(a)$. Then for $x \in \mathcal{D}_0$, there exists a finite set $F \in \mathcal{F}$ so that $xe_F = e_F x = x$. Using Theorem 3.2,

$$\begin{aligned}\theta(e_F)\,x(P) &= \textstyle\sum_{(x)} \theta\left(e_F x_{(1)}\right) P\theta\left(S\left(x_{(2)}\right)\right) \\ &= P\textstyle\sum_{(x)} \theta\left(e_F x_{(1)}\right)\theta\left(S\left(x_{(2)}\right)\right) \\ &= \varepsilon(x)\,P\theta(e_F).\end{aligned}$$

Therefore $x(P) = \varepsilon(x)\,P$. This means that $P \in O$ and $\theta(\mathcal{D}_0)' \subseteq O$. Thus $\theta(\mathcal{D})' = \theta(\mathcal{D}_0)' = O$. The proof is completed. □

Remark. Under the conditions given above, O is a von Neumann algebra since $\theta(\mathcal{D})$ is closed under the $*$-operation, and O is the commutant of $\theta(\mathcal{D})$. In general when $\mathcal{D}$ is infinite dimensional, $\theta(\mathcal{D})$ and $\theta(\mathcal{D}_0)$ are not von Neumann algebras. Also, from Theorem 3.3 one can conclude that the irreducible representations of O are in one-to-one correspondence with those of $\theta(\mathcal{D})$. However even in the case of finite dimensional discrete quantum group, one in general could not get all irreducible representations of $\mathcal{D}$ in $L(H)$.

Furthermore assume that a C^*-algebra $\mathcal{A}$ is a $\mathcal{D}$-module algebra, and set

$$O = \{A \in \mathcal{A} | z(A) = A\},$$

where z is the cointegral in $\mathcal{D}_0$. Suppose that there is an irreducible representation π of $\mathcal{A}$ on a Hilbert space $H = \overline{(\pi(A)\Omega)}$ with a vacuum vector Ω, which gives rise to a $\mathcal{D}$-invariant state. Similar to the proofs of Theorem 3.2 and 3.3, one have the following result.

Corollary 3.4. *There exists a unique C^*-homomorphism $\theta : \mathcal{D} \to L(H)$ satisfying the relations: $\forall a, b \in \mathcal{D}_0$, $A \in \mathcal{A}$,*

$$\begin{aligned} \theta(a)(\Omega) &= \varepsilon(a)\Omega, \\ \theta(b)\pi(a(A)) &= \textstyle\sum_{(a)} \theta\left(ba_{(1)}\right) \pi(A) \theta\left(S\left(a_{(2)}\right)\right), \end{aligned}$$

and furthermore,

$$\theta(\mathcal{D})' = \overline{\pi(O)}, \quad \overline{\theta(\mathcal{D})} = \pi(O)',$$

where the prime denotes the commutant, and the bar denotes the weak closure in $L(H)$.

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