REPRESENTATION FORMULAE OF GENERAL SOLUTIONS IN THE THEORY OF HEMITROPIC ELASTICITY

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Summary

We consider the steady state oscillation equations of the theory of elasticity of hemitropic materials. We derive general representation formulae for the displacement and microrotation vectors by means of six scalar metaharmonic functions. These formulae are very convenient and useful in many particular problems for domains with concrete geometry. Here we consider two canonical transmission problems for piecewise homogeneous bodies with spherical interfaces and with the help of the representation formulae construct explicit solutions in the form of absolutely and uniformly convergent series. The representations can also be applied to multi-layered bodies with spherical and cylindrical interfaces.

1. Introduction

Technological and industrial developments, and also great success in biological and medical sciences, require us to use more refined models for elastic bodies. In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or Cosserat solids. They model composites with a complex inner structure whose material particles have six degrees of freedom (three displacement components and three microrotation components). Recall that the classical elasticity theory allows only three degrees of freedom (three displacement components).

Experiments have shown that micropolar materials possess quite different properties in comparison with classical elastic materials (**1** to **6**). For example, in non-centrosymmetric micropolar materials (which are also called hemitropic or chiral materials) left-handed and right-handed waves can propagate. Moreover, the twisting behaviour under an axial stress is a purely hemitropic (chiral) phenomenon and has no counterpart in classical elasticity.

Hemitropic solids are not isotropic with respect to inversion: they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

Materials may exhibit chirality on the atomic scale, as in quartz and biological molecules (DNA), as well as on a large scale, as in composites with helical or screw-shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films and twisted fibres. For more details see (**1**, **2**, **4**, **7** to **14**).

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Mathematical models describing the chiral properties of elastic hemitropic materials have been proposed by Aero and Kuvshinski (**1**, **2**). For historical notes see also (**4**, **11**, **12**).

In the mathematical theory of hemitropic elasticity there are introduced the asymmetric force stress tensor and moment stress tensor, which are kinematically related with the asymmetric strain tensor and torsion (curvature) tensor via the constitutive equations. All these quantities are expressed in terms of the components of the displacement and microrotation vectors. In turn these satisfy a coupled complex system of second-order partial differential equations of dynamics. When the mechanical characteristics (displacements, microrotations, body force and body couple vectors) do not depend on a time variable t we have the differential equations of statics. If the characteristics are time harmonic (that is, they are represented as the product of $e^{-i\sigma t}$ and a function of the spatial variable $x \in \mathbb{R}^3$) then we have the steady state oscillation equations. Here σ is a real frequency parameter. Note that if $\sigma = 0$, then we obtain the equations of statics. If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter, then we have the so-called pseudo-oscillation equations (which are related to the dynamical equations via the Laplace transform). All the above equations generate a 6×6 strongly elliptic, formally self-adjoint differential operator involving nine material constants.

The Dirichlet, Neumann and mixed type boundary-value problems (BVPs) corresponding to this model are well investigated for general domains of arbitrary shape and uniqueness and existence theorems are proved, and regularity results for solutions are established by potential methods as well as by variational methods; see (**12**, **15** to **18**).

Our main goal is to derive general representation formulae for the displacement and microrotation vectors by means of metaharmonic functions, solutions of the Helmholtz equations with different wave numbers. That is, we can represent solutions to the very complicated coupled system of simultaneous differential equations of ‘hemitropic elasticity’ with the help of solutions of a simpler canonical metaharmonic equation (similar formulae in classical elastostatics are well known as Papkovich–Neuber representation formulae). We prove that the six components of the field vectors (three displacement and three microrotation components) can be expressed linearly by six scalar metaharmonic functions. Moreover, we show that this correspondence is one-to-one. The representation formulae obtained have proved to be very useful in the study of many problems for domains with concrete geometry.

In particular, here we apply these representation formulae to construct explicit solutions of two canonical boundary-value and transmission problems with a spherical interface. In the first case both components are hemitropic with different material constants and on the interface we have transmission conditions relating limiting values of the displacement, microrotation, force stress, and couple stress vectors, twelve conditions (chiral–chiral coupling). In the second problem the interior ball is occupied by the usual isotropic elastic material described by the classical Lamé model, while in the exterior part we have again a hemitropic material. In this case, the interface conditions relate the corresponding displacement and force stress vectors and, in addition, on the interface there are given either components of the microrotation vector or the couple stress vector, in all nine conditions (chiral–achiral coupling). We represent the solutions of these problems in the form of Fourier–Laplace series and prove that these series along with their derivatives of the first order are absolutely and uniformly convergent in closed domains if the boundary data satisfy appropriate smoothness conditions.

The motivation for the choice of the transmission problems treated in the paper is that by the same approach one can construct explicit solutions to similar transmission problems for layered composites with finitely many spherical interfaces. Moreover, the representations obtained can be applied to some generalizations of the classical Eshelby type inclusion problems for hemitropic materials (**19**, **20**). Applications are anticipated in complex structures such as composite thin films, bones, DNA, nanotubes among others. For a wider overview of the subject concerning different areas of applications we refer to (**5**, **7**, **9**, **14**, **21** to **23**).

2. Field equations. Auxiliary material

Here we collect some auxiliary material from the theory of elasticity of hemitropic bodies.

2.1 *Constitutive equations*

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded, simply connected domain with a piecewise smooth connected Lipschitz boundary $S := \partial\Omega^+$ and $\overline{\Omega^+} = \Omega \cup S$. Then it follows that $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$ is also simply connected and $\partial\Omega^- = \partial\Omega^+$. Let $\overline{\Omega} \in \{\overline{\Omega^+},\ \overline{\Omega^-}\}$ be filled with an elastic material with hemitropic properties: each material particle has 6 degrees of freedom, corresponding to displacements and microrotations. Let $B(r)$ be a ball of radius r, centred at the origin, with spherical boundary Σ_r.

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the *displacement vector* and the *microrotation vector*, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the microrotation vector in the hemitropic elasticity theory is kinematically distinct from the *macrorotation vector* $\frac{1}{2}\,\mathrm{curl}\ u$.

The *force stress tensor* $\{\tau_{pq}\}$ and the *couple stress tensor* $\{\mu_{pq}\}$ in the linear theory are related to u and ω by the following constitutive equations (see (**1**, **11**)):

$$\begin{aligned}\tau_{pq} = \tau_{pq}(U) &:= (\mu+\alpha)\frac{\partial u_q}{\partial x_p} + (\mu-\alpha)\frac{\partial u_p}{\partial x_q} + \lambda\delta_{pq}\,\mathrm{div}\,u + \delta\,\delta_{pq}\,\mathrm{div}\,\omega \\ &\quad + (\varkappa+\nu)\frac{\partial \omega_q}{\partial x_p} + (\varkappa-\nu)\frac{\partial \omega_p}{\partial x_q} - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k, \qquad (2.1)\\ \mu_{pq} = \mu_{pq}(U) &:= \delta\,\delta_{pq}\,\mathrm{div}\,u + (\varkappa+\nu)\left[\frac{\partial u_q}{\partial x_p} - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\right] + \beta\,\delta_{pq}\,\mathrm{div}\,\omega \\ &\quad + (\varkappa-\nu)\left[\frac{\partial u_p}{\partial x_q} - \sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\right] + (\gamma+\varepsilon)\frac{\partial \omega_q}{\partial x_p} + (\gamma-\varepsilon)\frac{\partial \omega_p}{\partial x_q},\end{aligned}$$

where $U = (u, \omega)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi-Civitá) symbol, and α, β, γ, δ, λ, μ, ν, $\varkappa$ and ε are material constants. Concerning experimental determination of the hemitropic material parameters and comparison of centrosymmetric and hemitropic (acentric) models see (**14**).

The *strain* and *torsion* (*curvature*) tensors for hemitropic bodies are calculated by

$$u_{pq} = \partial_p u_q - \sum_{k=1}^{3}\varepsilon_{pqk}\,\omega_k, \qquad \omega_{pq} = \partial_p\,\omega_q, \quad p, q = 1, 2, 3. \qquad (2.2)$$

Clearly, all the above tensors are asymmetric.

The so called *energy bilinear form* reads as follows:

$$E(U,U') = E(U',U) = \sum_{p,q=1}^{3} \{(\mu+\alpha)u'_{pq}u_{pq} + (\mu-\alpha)u'_{pq}u_{qp} + (\varkappa+\nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) + (\varkappa-\nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) + (\gamma+\varepsilon)\omega'_{pq}\omega_{pq} + (\gamma-\varepsilon)\omega'_{pq}\omega_{qp} + \delta(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}) + \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\}.$$

For $U' = U$ we have the potential energy density $E(U,U)$ which due to physical considerations is assumed to be positive definite with respect to the variables (2.2). For the material constants this implies (see (**16**))

$$\mu > 0, \quad \alpha > 0, \quad 3\lambda + 2\mu > 0, \quad \mu\gamma - \varkappa^2 > 0, \quad \alpha\varepsilon - \nu^2 > 0,$$
$$(3\lambda + 2\mu)(3\beta + 2\gamma) - (3\delta + 2\varkappa)^2 > 0, \tag{2.3}$$

whence

$$\gamma > 0, \quad \varepsilon > 0, \quad \lambda + \mu > 0, \quad \beta + \gamma > 0,$$
$$d_1 := (\mu+\alpha)(\gamma+\varepsilon) - (\varkappa+\nu)^2 > 0, \quad d_2 := (\lambda+2\mu)(\beta+2\gamma) - (\delta+2\varkappa)^2 > 0. \tag{2.4}$$

The components of the force stress vector $\tau^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a normal vector $n = (n_1, n_2, n_3)$, read as

$$\tau_q^{(n)}(U) = \sum_{p=1}^{3} \tau_{pq}(U)\, n_p, \quad \mu_q^{(n)}(U) = \sum_{p=1}^{3} \mu_{pq}(U)\, n_p, \quad q = 1, 2, 3.$$

Further we introduce the generalized stress operator (6×6 matrix differential operator)

$$T(\partial, n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) \end{bmatrix}_{6\times 6}, \quad T^{(j)} = \left[T^{(j)}_{pq}\right]_{3\times 3}, \quad j = \overline{1,4}, \tag{2.5}$$

where $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_j = \partial/\partial x_j$, $\partial/\partial n$ denotes the directional derivative along the vector n (normal derivative),

$$T^{(1)}_{pq}(\partial,n) = (\mu+\alpha)\,\delta_{pq}\frac{\partial}{\partial n} + (\mu-\alpha)\,n_q\frac{\partial}{\partial x_p} + \lambda\, n_p\frac{\partial}{\partial x_q},$$

$$T^{(2)}_{pq}(\partial,n) = (\varkappa+\nu)\,\delta_{pq}\frac{\partial}{\partial n} + (\varkappa-\nu)\,n_q\frac{\partial}{\partial x_p} + \delta\, n_p\frac{\partial}{\partial x_q} - 2\,\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\,n_k,$$

$$T^{(3)}_{pq}(\partial,n) = (\varkappa+\nu)\,\delta_{pq}\frac{\partial}{\partial n} + (\varkappa-\nu)\,n_q\frac{\partial}{\partial x_p} + \delta\, n_p\frac{\partial}{\partial x_q},$$

$$T^{(4)}_{pq}(\partial,n) = (\gamma+\varepsilon)\,\delta_{pq}\frac{\partial}{\partial n} + (\gamma-\varepsilon)\,n_q\frac{\partial}{\partial x_p} + \beta\, n_p\frac{\partial}{\partial x_q} - 2\,\nu\sum_{k=1}^{3}\varepsilon_{pqk}\,n_k.$$

It can be easily checked that $T(\partial, n)\, U = \big(\tau^{(n)}(U),\ \mu^{(n)}(U)\big)^{\top}$. In what follows we refer to $T(\partial, n)$ as the *hemitropic stress operator*. The first three components of the vector $T(\partial, n)\, U$ correspond to the *force stress vector*,

$$\mathcal{T}(U) := T^{(1)}(\partial, n)\, u + T^{(2)}(\partial, n)\, \omega,$$

while the second three components describe the *couple stress vector*,

$$\mathcal{M}(U) := T^{(3)}(\partial, n)\, u + T^{(4)}(\partial, n)\, \omega.$$

Direct calculations show that

$$\begin{aligned} \mathcal{T}(U) = 2\mu\, \frac{\partial u}{\partial n} + \lambda\, n \operatorname{div} u + (\mu - \alpha)\,[n \times \operatorname{curl} u] + 2\varkappa\, \frac{\partial \omega}{\partial n} \\ + \delta\, n \operatorname{div} \omega + (\varkappa - \nu)\,[n \times \operatorname{curl} \omega] + 2\alpha\,[n \times \omega], \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathcal{M}(U) = 2\varkappa\, \frac{\partial u}{\partial n} + \delta\, n \operatorname{div} u + (\varkappa - \nu)\,[n \times \operatorname{curl} u] + 2\gamma\, \frac{\partial \omega}{\partial n} \\ + \beta\, n \operatorname{div} \omega + (\gamma - \varepsilon)\,[n \times \operatorname{curl} \omega] + 2\nu\,[n \times \omega], \end{aligned} \tag{2.7}$$

where the symbol $\times$ denotes the cross product in $\mathbb{R}^3$.

2.2 *The basic equations*

The equations of dynamics of the hemitropic theory of elasticity have the form

$$\sum_{p=1}^{3} \partial_p\, \tau_{pq}(x,t) + \varrho\, F_q(x,t) = \varrho\, \frac{\partial^2 u_q(x,t)}{\partial t^2},$$

$$\sum_{p=1}^{3} \partial_p\, \mu_{pq}(x,t) + \sum_{l,r=1}^{3} \varepsilon_{qlr}\, \tau_{lr}(x,t) + \varrho\, G_q(x,t) = \mathcal{I}\, \frac{\partial^2 \omega_q(x,t)}{\partial t^2}, \quad q = 1, 2, 3,$$

where t is the time variable, $F = (F_1, F_2, F_3)^{\top}$ and $G = (G_1, G_2, G_3)^{\top}$ are the body force and body couple vectors per unit mass, ϱ is the mass density of the elastic material, and $\mathcal{I}$ is a constant characterizing the so-called spin torque corresponding to the interior microrotations (that is, the moment of inertia per unit volume).

Using the constitutive equations (2.1) we can rewrite the above dynamical equations in terms of the displacement and microrotation vectors. If all the quantities involved in these equations have harmonic time dependence, that is, $u(x,t) = u(x)\, e^{-it\sigma}$, $\omega(x,t) = \omega(x)\, e^{-it\sigma}$, $F(x,t) = F(x)\, e^{-it\sigma}$ and $G(x,t) = G(x)\, e^{-it\sigma}$, with $\sigma \in \mathbb{R}$ and $i = \sqrt{-1}$, we obtain the *steady state oscillation equations* of the hemitropic theory of elasticity:

$$\begin{aligned} (\mu + \alpha)\Delta u(x) + (\lambda + \mu - \alpha) \operatorname{grad} \operatorname{div} u(x) + (\varkappa + \nu)\Delta \omega(x) \\ + (\delta + \varkappa - \nu) \operatorname{grad} \operatorname{div} \omega(x) + 2\alpha \operatorname{curl} \omega(x) + \varrho\, \sigma^2 u(x) = -\varrho\, F(x), \end{aligned} \tag{2.8}$$

$$\begin{aligned} (\varkappa + \nu)\Delta u(x) + (\delta + \varkappa - \nu) \operatorname{grad} \operatorname{div} u(x) + 2\alpha \operatorname{curl} u(x) + (\gamma + \varepsilon)\Delta \omega(x) \\ + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \omega(x) + 4\nu \operatorname{curl} \omega(x) + (\mathcal{I}\sigma^2 - 4\alpha)\, \omega(x) = -\varrho\, G(x), \end{aligned} \tag{2.9}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator and u, ω, F and G are complex-valued vector functions; σ is a *frequency parameter*.

If $\sigma = \sigma_1 + i\,\sigma_2$ is complex with $\sigma_2 \neq 0$, then the above equations are called the *pseudo-oscillation equations*, while for $\sigma = 0$ they represent the *equilibrium equations of statics*.

In this paper we deal with the steady state oscillation equations and assume that

$$\sigma > 0, \quad \mathcal{I}\sigma^2 - 4\alpha > 0. \tag{2.10}$$

Clearly, the first inequality is a natural condition while the second one is a restriction on σ. Due to this condition and the inequalities $\mathcal{I} > 0$ and $\alpha > 0$, we cannot pass to the limit as $\sigma \to 0$ in the arguments below. Therefore, the static case ($\sigma = 0$) needs special consideration.

Let us introduce the matrix differential operator corresponding to (2.8) and (2.9):

$$L(\partial, \sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma), & L^{(2)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma), & L^{(4)}(\partial,\sigma) \end{bmatrix}_{6\times 6}, \tag{2.11}$$

where

$$L^{(1)}(\partial,\sigma) := [(\mu+\alpha)\,\Delta + \varrho\,\sigma^2]\,I_3 + (\lambda+\mu-\alpha)\,Q(\partial),$$
$$L^{(2)}(\partial,\sigma) = L^{(3)}(\partial,\sigma) := (\varkappa+\nu)\,\Delta\,I_3 + (\delta+\varkappa-\nu)\,Q(\partial) + 2\,\alpha\,R(\partial),$$
$$L^{(4)}(\partial,\sigma) := [(\gamma+\varepsilon)\,\Delta + (\mathcal{I}\sigma^2 - 4\,\alpha)]\,I_3 + (\beta+\gamma-\varepsilon)\,Q(\partial) + 4\,\nu\,R(\partial).$$

Here and in the sequel I_k stands for the $k \times k$ unit matrix and

$$R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}_{3\times 3}, \qquad Q(\partial) := [\,\partial_k\partial_j\,]_{3\times 3}. \tag{2.12}$$

It is easy to see that $R(\partial)u = \operatorname{curl} u$, and $Q(\partial)\,u = \operatorname{grad}\operatorname{div} u$. Equations (2.8) and (2.9) can be rewritten in matrix form as

$$L(\partial,\sigma)U(x) = \Phi(x), \quad U = (u,\omega)^\top, \quad \Phi = (\Phi^{(1)}, \Phi^{(2)})^\top := (-\varrho\,F(x), -\varrho\,G(x))^\top.$$

2.3 *Orthogonal system of spherical vectors*

Denote by r, ϑ, φ ($0 \leqslant r < +\infty$, $0 \leqslant \vartheta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$) the spherical coordinates of a point $x \in \mathbb{R}^3$. Further, let Σ_1 be a unit sphere in $\mathbb{R}^3$. In $[L_2(\Sigma_1)]^3$ we introduce the following complete, orthonormal system of vector spherical harmonics (**24**, **25**):

$$\begin{aligned} \mathbf{X}_{mk}(\vartheta,\varphi) &= \mathbf{e}_r\, Y_k^{(m)}(\vartheta,\varphi), \quad k \geqslant 0, \\ \mathbf{Y}_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}} \left(\mathbf{e}_\vartheta \frac{\partial}{\partial\vartheta} + \frac{\mathbf{e}_\varphi}{\sin\vartheta} \frac{\partial}{\partial\varphi} \right) Y_k^{(m)}(\vartheta,\varphi), \quad k \geqslant 1, \\ \mathbf{Z}_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}} \left(\frac{\mathbf{e}_\vartheta}{\sin\vartheta} \frac{\partial}{\partial\varphi} - \mathbf{e}_\varphi \frac{\partial}{\partial\vartheta} \right) Y_k^{(m)}(\vartheta,\varphi), \quad k \geqslant 1, \end{aligned} \tag{2.13}$$

where $|m| \leqslant k$, $\mathbf{e}_r = (\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)^\top$, $\mathbf{e}_\vartheta = (\cos\varphi\cos\vartheta,\ \sin\varphi\cos\vartheta,\ -\sin\vartheta)^\top$ and $\mathbf{e}_\varphi = (-\sin\varphi,\ \cos\varphi,\ 0)^\top$ are unit $\mathbb{R}^3$-orthogonal vectors,

$$Y_k^{(m)}(\vartheta,\varphi) = \sqrt{\frac{2k+1}{4\pi}\ \frac{(k-m)!}{(k+m)!}}\ P_k^m(\cos\vartheta)\ e^{im\varphi} \tag{2.14}$$

and P_k^m is the associated Legendre function of the first kind of degree k and order m.

Let a vector function $f(\vartheta,\varphi)$ be representable in the form

$$f(\vartheta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{\alpha_{mk}\,\mathbf{X}_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)}\,[\beta_{mk}\,\mathbf{Y}_{mk}(\vartheta,\varphi) + \gamma_{mk}\,\mathbf{Z}_{mk}(\vartheta,\varphi)]\Big\},$$

with

$$\begin{aligned}
\alpha_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f(\vartheta,\varphi)\cdot\mathbf{X}_{mk}(\vartheta,\varphi)\ \sin\vartheta\ d\vartheta, \quad k \geqslant 0,\\
\beta_{mk} &= \frac{1}{\sqrt{k(k+1)}}\int_0^{2\pi} d\varphi \int_0^{\pi} f(\vartheta,\varphi)\cdot\mathbf{Y}_{mk}(\vartheta,\varphi)\ \sin\vartheta\ d\vartheta, \quad k \geqslant 1,\\
\gamma_{mk} &= \frac{1}{\sqrt{k(k+1)}}\int_0^{2\pi} d\varphi \int_0^{\pi} f(\vartheta,\varphi)\cdot\mathbf{Z}_{mk}(\vartheta,\varphi)\ \sin\vartheta\ d\vartheta, \quad k \geqslant 1.
\end{aligned} \tag{2.15}$$

Here and in what follows $a\cdot b$ denotes the usual scalar product of two complex vectors $a, b \in \mathbb{C}^m$: $a\cdot b = \sum_{j=1}^m a_j\,\overline{b_j}$, where the overbar denotes complex conjugation.

We remark that throughout the paper the summation index k for the summands involving the vectors $\mathbf{Y}_{mk}(\vartheta,\varphi)$ and $\mathbf{Z}_{mk}(\vartheta,\varphi)$ varies from 1 to ∞.

Now we formulate several technical lemmas.

LEMMA 2.1 *For the vectors $\mathbf{X}_{mk}(\vartheta,\varphi)$, $\mathbf{Y}_{mk}(\vartheta,\varphi)$ and $\mathbf{Z}_{mk}(\vartheta,\varphi)$ given by* (2.13) *the following inequalities hold:*

$$|\mathbf{X}_{mk}(\vartheta,\varphi)| \leqslant \sqrt{\frac{2k+1}{4\pi}}, \quad |\mathbf{Y}_{mk}(\vartheta,\varphi)| \leqslant \sqrt{\frac{k(k+1)}{2k+1}}, \quad |\mathbf{Z}_{mk}(\vartheta,\varphi)| \leqslant \sqrt{\frac{k(k+1)}{2k+1}}, \tag{2.16}$$

where the first inequality holds for $k \geqslant 0$, the second and third for $k \geqslant 1$.

LEMMA 2.2 *Let $f \in [C^\ell(\Sigma_1)]^3$ with $\ell \geqslant 1$. Then the coefficients α_{mk}, β_{mk} and γ_{mk} given by* (2.15) *have the properties*

$$\alpha_{mk} = \mathcal{O}(k^{-\ell}), \quad \beta_{mk} = \mathcal{O}(k^{-\ell-1}), \quad \gamma_{mk} = \mathcal{O}(k^{-\ell-1}) \ \text{ as } k \to \infty.$$

LEMMA 2.3 *A vector $v = (v_1, v_2, v_3)^\top$ solves the system*

$$\operatorname{curl} v(x) \mp \sigma\, v(x) = 0, \quad \operatorname{div} v(x) = 0, \quad \sigma > 0,$$

in some domain $\Omega \subset \mathbb{R}^3$ if and only if v can be represented as

$$v(x) = \operatorname{curl}\operatorname{curl}(x\,\Phi(x)) \pm \sigma\ \operatorname{curl}(x\,\Phi(x)),$$

where Φ is a scalar function satisfying the Helmholtz equation $(\Delta + \sigma^2)\,\Phi = 0$ in Ω.

LEMMA 2.4 *A vector* $v = (v_1, v_2, v_3)^\top$ *solves the system*

$$(\Delta + \sigma^2)\, v(x) = 0, \quad \operatorname{curl} v(x) = 0, \ \ \sigma > 0,$$

in some domain $\Omega \subset \mathbb{R}^3$ *if and only if* v *can be represented as* $v(x) = \operatorname{grad}\Phi(x)$, *where* Φ *is a scalar function satisfying the Helmholtz equation* $(\Delta + \sigma^2)\,\Phi = 0$ *in* Ω.

Proofs of these lemmas can be found, for example, in (**26** to **28**).

3. Representation formulae of a general solution

Here we derive the basic representation formulae for a general solution to the system (2.8) and (2.9). As we shall see, the representation formulae have different forms for $\delta + 2\varkappa \neq 0$ and $\delta + 2\varkappa = 0$. From the basic inequalities (2.3) and (2.4) it follows that both cases are possible. Therefore we have to deal with these cases separately.

THEOREM 3.1 *Let* $\delta + 2\varkappa \neq 0$. *A vector* $U = (u, \omega)^\top$ *solves* (2.8) *and* (2.9) *if and only if*

$$u(x) = \sum_{j=1}^{6} v^{(j)}(x), \quad \omega(x) = \sum_{j=1}^{6} \beta_j\, v^{(j)}(x), \tag{3.1}$$

where $v^{(j)} = (v_1^{(j)}, v_2^{(j)}, v_3^{(j)})^\top$, $j = 1, 2, \dots, 6$, *satisfy the relations*

$$\begin{aligned}
&(\Delta + k_j^2)\, v^{(j)}(x) = 0, \quad \operatorname{curl} v^{(j)}(x) = 0, \ \ j = 1, 2,\\
&\operatorname{curl} v^{(j)}(x) - k_j\, v^{(j)}(x) = 0, \quad \operatorname{div} v^{(j)}(x) = 0, \ \ j = 3, 4,\\
&\operatorname{curl} v^{(j)}(x) + k_j\, v^{(j)}(x) = 0, \quad \operatorname{div} v^{(j)}(x) = 0, \ \ j = 5, 6;
\end{aligned} \tag{3.2}$$

$$\beta_j = [\rho\sigma^2 - (\lambda + 2\mu)k_j^2][(\delta + 2\varkappa)k_j^2]^{-1}, \quad j = 1, 2, \tag{3.3}$$

$$\beta_j = [\rho\sigma^2 - (\mu + \alpha)k_j^2][(\varkappa + \nu)k_j^2 - 2\alpha k_j]^{-1}, \quad j = 3, 4, \tag{3.4}$$

$$\beta_j = [\rho\sigma^2 - (\mu + \alpha)k_j^2][(\varkappa + \nu)k_j^2 + 2\alpha k_j]^{-1}, \quad j = 5, 6. \tag{3.5}$$

Here k_j *are positive constants;* k_1 *and* k_2 *are defined by the equations*

$$k_1^2 + k_2^2 = d_2^{-1}[(\beta + 2\gamma)\rho\sigma^2 + (\lambda + 2\mu)(\mathcal{I}\sigma^2 - 4\alpha)], \quad k_1^2 k_2^2 = d_2^{-1}\rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha), \tag{3.6}$$

while $k_3, k_4, -k_5, -k_6$ *are the roots of the equation*

$$d_1 t^4 + d_3 t^3 + d_4 t^2 + d_5 t + d_6 = 0 \tag{3.7}$$

with

$$\begin{aligned}
&d_1 = (\mu + \alpha)(\gamma + \varepsilon) - (\varkappa + \nu)^2 > 0, \quad d_2 = (\lambda + 2\mu)(\beta + 2\gamma) - (\delta + 2\varkappa)^2 > 0,\\
&d_3 = 4(\alpha\varkappa - \mu\nu), \quad d_4 = -[(\gamma + \varepsilon)\rho\sigma^2 + (\mu + \alpha)(\mathcal{I}\sigma^2 - 4\alpha) + 4\alpha^2],\\
&d_5 = 4\nu\rho\sigma^2, \quad d_6 = \rho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha).
\end{aligned} \tag{3.8}$$

Proof. We recall that $\sigma > 0$ and $\mathcal{I}\sigma^2 - 4\alpha > 0$ (see (2.10)). Let $U = (u, \omega)^\top$ solve (2.8) and (2.9). Applying the divergence operator to these equations gives

$$[(\lambda + 2\mu)\,\Delta + \rho\,\sigma^2]\,\mathrm{div}\,u + (\delta + 2\varkappa)\,\Delta\;\mathrm{div}\,\omega = 0,$$

$$(\delta + 2\varkappa)\,\Delta\,\mathrm{div}\,u + [(\beta + 2\gamma)\,\Delta + \mathcal{I}\,\sigma^2 - 4\alpha]\;\mathrm{div}\,\omega = 0,$$

whence

$$(\Delta + k_1^2)(\Delta + k_2^2)(\,\mathrm{div}\,u,\ \mathrm{div}\,\omega)^\top = 0, \tag{3.9}$$

where k_1^2 and k_2^2 are defined by (3.6).

Similarly, applying the curl operator to (2.8) and (2.9) we obtain

$$[(\mu + \alpha)\,\mathrm{curl}\,\mathrm{curl} - \rho\sigma^2 I_3]\,\mathrm{curl}\,u + [(\varkappa + \nu)\,\mathrm{curl}\,\mathrm{curl} - 2\alpha\,\mathrm{curl}]\,\mathrm{curl}\,\omega = 0,$$

$$[(\varkappa + \nu)\,\mathrm{curl}\,\mathrm{curl} - 2\alpha\,\mathrm{curl}]\,\mathrm{curl}\,u + [(\gamma + \varepsilon)\,\mathrm{curl}\,\mathrm{curl} - 4\nu\,\mathrm{curl} - (\mathcal{I}\sigma^2 - 4\alpha) I_3]\,\mathrm{curl}\,\omega = 0.$$

From these relations it follows that

$$[d_1\,\widetilde{R}^4(\partial) + d_3\,\widetilde{R}^3(\partial) + d_4\,\widetilde{R}^2(\partial) + d_5\,\widetilde{R}(\partial) + d_6\,I_6](\mathrm{curl}\,u,\ \mathrm{curl}\,\omega)^\top = 0, \tag{3.10}$$

where I_6 is the unit 6×6 matrix and

$$\widetilde{R}(\partial) = \begin{bmatrix} R(\partial) & 0 \\ 0 & R(\partial) \end{bmatrix}_{6\times 6},$$

$R(\partial)$ and the constants d_j are given by (2.12) and (3.8), respectively.

In (**18**) it is shown that for $\sigma > 0$ and $\mathcal{I}\sigma^2 - 4\alpha > 0$ all the roots of (3.7) are real. Since $d_4 < 0$ and $d_6 > 0$, it follows that (3.7) has two positive and two negative roots. Denote the positive roots by k_3 and k_4, and the negative roots by $-k_5$ and $-k_6$. Throughout the paper we assume that $k_j \neq k_p$ for $j \neq p,\ j, p = 1, 2, \ldots, 6$.

We can decompose then equation (3.10) as

$$[\widetilde{R}(\partial) - k_3\,I_6][\widetilde{R}(\partial) - k_4\,I_6][\widetilde{R}(\partial) + k_5\,I_6][\widetilde{R}(\partial) + k_6\,I_6](\mathrm{curl}\,u,\ \mathrm{curl}\,\omega)^\top = 0. \tag{3.11}$$

From (2.8) and (2.9) we have

$$u(x) = u'(x) + u''(x), \quad \omega(x) = \omega'(x) + \omega''(x), \tag{3.12}$$

where

$$u'(x) = -\frac{1}{\rho\,\sigma^2}[(\lambda + 2\mu)\ \mathrm{grad}\ \mathrm{div}\,u + (\delta + 2\varkappa)\ \mathrm{grad}\ \mathrm{div}\,\omega], \tag{3.13}$$

$$\omega'(x) = -\frac{1}{\mathcal{I}\,\sigma^2 - 4\alpha}\,[(\delta + 2\varkappa)\ \mathrm{grad}\ \mathrm{div}\,u + (\beta + 2\gamma)\ \mathrm{grad}\ \mathrm{div}\,\omega],$$

and

$$u''(x) = \frac{1}{\rho\sigma^2}\,[(\mu + \alpha)\,\mathrm{curl}\ \mathrm{curl}\,u + (\varkappa + \nu)\,\mathrm{curl}\ \mathrm{curl}\,\omega - 2\alpha\,\mathrm{curl}\,\omega], \tag{3.14}$$

$$\omega''(x) = \frac{1}{\mathcal{I}\sigma^2 - 4\alpha}\,[(\varkappa + \nu)\,\mathrm{curl}\,\mathrm{curl}\,u + (\gamma + \varepsilon)\,\mathrm{curl}\,\mathrm{curl}\,\omega - 2\alpha\,\mathrm{curl}\,u - 4\nu\,\mathrm{curl}\,\omega].$$

In view of (3.9) from (3.13) we get

$$(\Delta + k_1^2)\,(\Delta + k_2^2)\,(u', \omega')^\top = 0, \quad \mathrm{curl}\,u' = 0, \quad \mathrm{curl}\,\omega' = 0.$$

We can represent the vectors u' and ω' as

$$u'(x) = \sum_{j=1}^{2} v^{(j)}(x), \quad \omega'(x) = \sum_{j=1}^{2} \omega^{(j)}(x), \tag{3.15}$$

where

$$v^{(1)}(x) = [k_2^2 - k_1^2]^{-1}[\Delta + k_2^2]\, u'(x), \quad v^{(2)}(x) = [k_1^2 - k_2^2]^{-1}[\Delta + k_1^2]\, u'(x), \tag{3.16}$$

$$\omega^{(1)}(x) = [k_2^2 - k_1^2]^{-1}[\Delta + k_2^2]\, \omega'(x), \quad \omega^{(2)}(x) = [k_1^2 - k_2^2]^{-1}[\Delta + k_1^2]\, \omega'(x). \tag{3.17}$$

We see that, for $j = 1, 2$,

$$[\Delta + k_j^2]\, v^{(j)}(x) = 0, \quad \operatorname{curl} v^{(j)}(x) = 0, \quad [\Delta + k_j^2]\, \omega^{(j)}(x) = 0, \quad \operatorname{curl} \omega^{(j)}(x) = 0.$$

Since $\operatorname{div} u = \operatorname{div} u'$ and $\operatorname{div} \omega = \operatorname{div} \omega'$, with the help of (3.16), (3.17) and the identity grad div $v = \Delta v + \operatorname{curl}\operatorname{curl} v$ for any $v = (v_1, v_2, v_3)^\top$, from (3.13) we get

$$[(\lambda + 2\mu)k_j^2 - \rho\sigma^2]\, v^{(j)}(x) + (\delta + 2\varkappa)\, k_j^2\, \omega^{(j)}(x) = 0,$$

$$(\delta + 2\varkappa)\, k_j^2\, v^{(j)}(x) + [(\beta + 2\gamma)\, k_j^2 - (\mathcal{I}\sigma^2 - 4\alpha)]\, \omega^{(j)}(x) = 0, \quad j = 1, 2.$$

Hence, for $\delta + 2\varkappa \neq 0$, $\omega^{(j)}(x) = \beta_j v^{(j)}(x)$, $j = 1, 2$, where β_1 and β_2 are given by (3.3). Substituting these expressions into (3.15) gives

$$u'(x) = \sum_{j=1}^{2} v^{(j)}(x), \quad \omega'(x) = \sum_{j=1}^{2} \beta_j\, v^{(j)}(x), \quad \operatorname{curl} v^{(j)}(x) = 0, \quad j = 1, 2. \tag{3.18}$$

Further, from (3.11) with the help of (3.14) we get

$$[\widetilde{R}(\partial) - k_3 I_6][\widetilde{R}(\partial) - k_4 I_6][\widetilde{R}(\partial) + k_5 I_6][\widetilde{R}(\partial) + k_6 I_6](u'', \omega'')^\top = 0. \tag{3.19}$$

We can represent u'' and ω'' in the form

$$u''(x) = \sum_{j=3}^{6} v^{(j)}(x), \quad \omega''(x) = \sum_{j=3}^{6} \omega^{(j)}(x), \tag{3.20}$$

where

$$v^{(j)}(x) = M^{(j)}(\partial)\, u''(x), \quad \omega^{(j)}(x) = M^{(j)}(\partial)\, \omega''(x), \quad j = 3, 4, 5, 6, \tag{3.21}$$

and $M^{(j)}(\partial)$, $j = 3, 4, 5, 6$, are the following differential operators:

$$M^{(3)}(\partial) = \frac{1}{(k_3 - k_4)(k_3 + k_5)(k_3 + k_6)}\, [R(\partial) - k_4 I_3]\, [R(\partial) + k_5 I_3]\, [R(\partial) + k_6 I_3],$$

$$M^{(4)}(\partial) = \frac{1}{(k_4 - k_3)(k_4 + k_5)(k_4 + k_6)}\, [R(\partial) - k_3 I_3]\, [R(\partial) + k_5 I_3]\, [R(\partial) + k_6 I_3],$$

$$M^{(5)}(\partial) = \frac{1}{(k_6 - k_5)(k_5 + k_3)(k_5 + k_4)}\, [R(\partial) - k_3 I_3]\, [R(\partial) - k_4 I_3]\, [R(\partial) + k_6 I_3],$$

$$M^{(6)}(\partial) = \frac{1}{(k_5 - k_6)(k_6 + k_3)(k_6 + k_4)}\, [R(\partial) - k_3 I_3]\, [R(\partial) - k_4 I_3]\, [R(\partial) + k_5 I_3].$$

Due to the equalities $\operatorname{div} u'' = 0$ and $\operatorname{div} \omega'' = 0$ from (3.21) in view of (3.19) we have

$$\begin{aligned} &[R(\partial) - k_j\, I_3]\, v^{(j)}(x) = 0, \quad [R(\partial) - k_j\, I_3]\, \omega^{(j)}(x) = 0, \quad j = 3, 4, \\ &[R(\partial) + k_j\, I_3]\, v^{(j)}(x) = 0, \quad [R(\partial) + k_j\, I_3]\, \omega^{(j)}(x) = 0, \quad j = 5, 6, \\ &\operatorname{div} v^{(j)}(x) = 0, \quad \operatorname{div} \omega^{(j)}(x), \quad j = 3, 4, 5, 6. \end{aligned} \tag{3.22}$$

On the other side, since $\operatorname{curl} u = \operatorname{curl} u''$ and $\operatorname{curl} \omega = \operatorname{curl} \omega''$ from (3.14) with the help of (3.21) and (3.22) we conclude

$$[(\mu + \alpha)\, k_j^2 - \varrho\, \sigma^2]\, v^{(j)}(x) + [(\varkappa + \nu)\, k_j^2 - 2\alpha\, \eta_j]\, \omega^{(j)}(x) = 0,$$

$$[(\varkappa + \nu)\, k_j^2 - 2\alpha\, \eta_j]\, v^{(j)}(x) + [(\gamma + \varepsilon)\, k_j^2 - 4\nu\, \eta_j - (\mathcal{I}\sigma^2 - 4\alpha)]\, \omega^{(j)}(x) = 0,$$

where

$$\eta_3 = k_3, \quad \eta_4 = k_4, \quad \eta_5 = -k_5 \quad \text{and} \quad \eta_6 = -k_6. \tag{3.23}$$

From these relations it follows that

$$\omega^{(j)}(x) = \beta_j\, v^{(j)}(x), \quad j = 3, 4, 5, 6, \tag{3.24}$$

with β_j given by (3.4) and (3.5). Substituting $\omega^{(j)}(x)$ given by (3.24) into (3.20) gives

$$u''(x) = \sum_{j=3}^{6} v^{(j)}(x), \quad \omega''(x) = \sum_{j=3}^{6} \beta_j\, v^{(j)}(x), \tag{3.25}$$

where $\operatorname{curl} v^{(j)}(x) - \eta_j v^{(j)}(x) = 0$ and $\operatorname{div} v^{(j)}(x) = 0$ for $j = 3, 4, 5, 6$. Finally, inserting u', ω', u'', and ω'' given by (3.18) and (3.25) into (3.12) we get the representation (3.1).

Note that if $\delta + 2\varkappa = 0$, then an arbitrary solution $(u, \omega)^\top$ of (2.8) and (2.9) can be represented in the form

$$u(x) = v^{(1)}(x) + \sum_{j=3}^{6} v^{(j)}(x), \quad \omega(x) = v^{(2)}(x) + \sum_{j=3}^{6} \beta_j\, v^{(j)}(x),$$

where the constants β_j, $j = 3, 4, 5, 6$, are defined by (3.4) and (3.5),

$$[\Delta + k_j^2]\, v^{(j)}(x) = 0, \quad \operatorname{curl} v^{(j)}(x) = 0, \quad j = 1, 2,$$

$$\operatorname{curl} v^{(j)}(x) - \eta_j\, v^{(j)}(x) = 0, \quad \operatorname{div} v^{(j)}(x) = 0, \quad j = 3, 4, 5, 6,$$

with η_j given by (3.23), $k_1^2 = \varrho\sigma^2/(\lambda + 2\mu)$ and $k_2^2 = (\mathcal{I}\sigma^2 - 4\alpha)/(\beta + 2\gamma)$.

For definiteness, in what follows we assume that $\delta + 2\varkappa \neq 0$.

THEOREM 3.2 *A vector $U = (u, \omega)^\top$ is a solution of* (2.8) *and* (2.9) *if and only if it is representable in the form*

$$u(x) = \operatorname{grad} \sum_{j=1}^{2} \Phi_j(x) + \sum_{j=3}^{6} [\operatorname{curl}\operatorname{curl}(x\Phi_j(x)) + \eta_j \operatorname{curl}(x\Phi_j(x))], \tag{3.26}$$

$$\omega(x) = \operatorname{grad} \sum_{j=1}^{2} \beta_j \Phi_j(x) + \sum_{j=3}^{6} \beta_j [\operatorname{curl}\operatorname{curl}(x\Phi_j(x)) + \eta_j \operatorname{curl}(x\Phi_j(x))], \tag{3.27}$$

where η_j *are defined by* (3.23), β_j *are defined by* (3.3) *to* (3.5), *and the functions* Φ_j *solve the Helmholtz equations*, $[\Delta + k_j^2]\Phi_j(x) = 0$, *for* $j = 1, 2, \ldots, 6$, *with* k_j *as in Theorem* 3.1.

Proof. By direct calculation it can easily be shown that the pair of vectors given by (3.26) and (3.27) solves (2.8) and (2.9).

Now, let $U = (u, \omega)^\top$ be an arbitrary solution of (2.8) and (2.9). By Theorem 3.1, the representation formulae (3.1) with $v^{(j)}$, $j = 1, 2, \ldots, 6$, satisfying (3.2) are true. By Lemmas 2.3 and 2.4 we have $v^{(j)} = \operatorname{grad}\Phi_j$ for $j = 1, 2$ and $v^{(j)}(x) = \operatorname{curl}\operatorname{curl}(x\Phi_j(x)) + \eta_j \operatorname{curl}(x\Phi_j(x))$ for $j = 3, 4, 5, 6$, where η_j are given by (3.23) and $[\Delta + k_j^2]\Phi_j = 0$ for $j = 1, 2, \ldots, 6$. Substitution of these expressions for $v^{(j)}$ into (3.1) completes the proof.

COROLLARY 3.3 *Let* $U = (u, \omega)^\top$ *and* $\Phi_j(x)$, $j = 1, 2, \ldots, 6$, *be as in Theorem* 3.2. *The correspondence between* U *and* Φ_j, $j = 1, 2, \ldots, 6$, *in the ball* $B(R)$ *is one-to-one if*

$$\int_{\Sigma_r} \Phi_j(x)\, d\Sigma_r = 0, \quad j = 3, 4, 5, 6, \quad r = |x| \leqslant R, \tag{3.28}$$

that is, the totality of metaharmonic functions $\{\Phi_1, \Phi_2, \ldots, \Phi_6\}$ *is uniquely defined by the components of the vector* $U = (u, \omega)^\top$ *if* (3.28) *hold.*

Proof. From formulae (3.26) and (3.27) we get

$$\Phi_j(x) = \frac{(-1)^{\ell_j}(\delta + 2\varkappa)}{\varrho\,\sigma^2(k_2^2 - k_1^2)} k_{\ell_j}^2 [\beta_{\ell_j} \operatorname{div} u(x) - \operatorname{div}\omega(x)],$$

$$r^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k_j^2\right)\Phi_j(x) = \frac{\Psi_j(x, u)}{k_j(k_j - k_{\ell_j})(k_j + k_5)(k_j + k_6)},$$

$$r^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k_j^2\right)\Phi_j(x) = \frac{\Psi_j(x, \omega)}{k_j(k_j - k_{\ell_j})(k_j + k_3)(k_j + k_4)},$$

with $\ell_1 = 2$, $\ell_2 = 1$, $\ell_3 = 4$, $\ell_4 = 3$, $\ell_5 = 6$, $\ell_6 = 5$,

$$\Psi_j(x, u) = [R(\partial) - k_{\ell_j} I_3][R(\partial) + k_5 I_3][R(\partial) + k_6 I_3] \operatorname{curl} u \cdot x, \quad j = 3, 4,$$

$$\Psi_j(x, \omega) = [R(\partial) + k_{\ell_j} I_3][R(\partial) - k_3 I_3][R(\partial) - k_4 I_3] \operatorname{curl}\omega \cdot x, \quad j = 5, 6.$$

If $u = 0$ and $\omega = 0$ in $B(R)$ we have $\Phi_1 = \Phi_2 = 0$ and

$$r^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k_j^2\right)\Phi_j(x) = 0, \quad j = 3, 4, 5, 6. \tag{3.29}$$

It remains to show that $\Phi_j(x) = 0$, $j = 3, 4, 5, 6$. Applying the well-known series representation of metaharmonic functions (**29**) we can write, for $x \in B(R)$,

$$\Phi_j(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_j r)\, A_{mk}^{(j)}\, Y_k^{(m)}(\vartheta, \varphi), \quad j = 3, 4, 5, 6.$$

Here $Y_k^{(m)}$ are given by (2.14), $A_{mk}^{(j)}$ are constants, and $g_k(k_j r) = r^{-1/2} J_{k+1/2}(k_j r)$, where J_ν are the Bessel functions. With the help of the equality

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k_j^2\right) g_k(k_j r) = \frac{k(k+1)}{r^2} g_k(k_j r),$$

we get from (3.29)

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k} k(k+1)\, g_k(k_j r)\, A_{mk}^{(j)}\, Y_k^{(m)}(\vartheta,\varphi) = 0,$$

whence the equations $A_{mk}^{(j)} = 0$ follow for $k \geqslant 1$ and $j = 3, 4, 5, 6$. Therefore,

$$\Phi_j(x) = \frac{1}{2\sqrt{\pi}}\, g_0(k_j r)\, A_{00}^{(j)}, \quad j = 3, 4, 5, 6.$$

Further, from (3.28), we deduce that $A_{00}^{(j)} = 0$ for $j = 3, 4, 5, 6$.

By the same arguments we can show that Corollary 3.3 holds also for the domain $\mathbb{R}^3 \backslash \overline{B(R)}$ if the relations (3.28) hold for $r = |x| \geqslant R$.

4. Applications of the representation formulae

Illustrating efficiency of the general representations obtained above, here we consider two canonical transmission problems, whose explicit solutions are obtained in the form of absolutely and uniformly convergent series. The motivation for the choice of problems is that by the same approach one can construct explicit solutions to similar problems for layered composites with finitely many spherical interfaces.

4.1 *Equations of classical elasticity and Sommerfeld–Kupradze radiation conditions*

Steady state oscillation equations in classical elasticity theory read as follows (**30**):

$$A(\partial,\sigma)\, u(x) = \mu\, \Delta\, u(x) + (\lambda+\mu)\ \mathrm{grad\ div}\, u(x) + \varrho\, \sigma^2\, u(x) = 0, \tag{4.1}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector, ϱ is the mass density, σ is the frequency parameter, and λ and μ are the Lamé constants satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$.

We denote the classical stress operator by $P(\partial, n) = [P_{kj}(\partial, n)]_{3\times 3}$ and the corresponding stress vector acting on a surface element with the unit normal $n = (n_1, n_2, n_3)$ by $P(\partial, n)u$:

$$P(\partial, n)u = 2\mu\, \partial u/\partial n + \lambda\, n\ \mathrm{div}\, u + \mu\, [n \times \mathrm{curl}\, u]. \tag{4.2}$$

We say that a vector $u = (u_1, u_2, u_3)^\top$ satisfies the Sommerfeld–Kupradze radiation conditions in $\mathbf{\Omega}^-$ if $u(x) = u^{(1)}(x) + u^{(2)}(x)$ in $\mathbf{\Omega}^-$, where the vectors $u^{(l)} = (u_1^{(l)}, u_2^{(l)}, u_3^{(l)})^\top$ satisfy the Helmholtz equations

$$[\Delta + (k_l^*)^2]\, u^{(l)}(x) = 0, \quad l = 1, 2,$$

and

$$\frac{\partial}{\partial |x|}\, u_p^{(l)}(x) - i\, k_l^*\, u_p^{(l)}(x) = o\,(|x|^{-1}) \quad \text{as } |x| \to \infty, \quad p = 1, 2, 3.$$

Here

$$(k_1^*)^2 = \varrho\sigma^2/\mu, \quad (k_2^*)^2 = \varrho\sigma^2/(\lambda + 2\mu). \tag{4.3}$$

For details see (**30**).

Note that for a general solution of the classical steady oscillation equations (4.1) we have the following representation formula (**25**):

$$u(x) = \operatorname{grad} \Phi_1(x) + \operatorname{curl} \operatorname{curl} (x\, \Phi_2(x)) + \operatorname{curl} (x\, \Phi_3(x)), \tag{4.4}$$

where

$$[\Delta + (k_1^*)^2]\, \Phi_1(x) = 0, \quad [\Delta + (k_2^*)^2]\, \Phi_j(x) = 0, \quad j = 2, 3. \tag{4.5}$$

We also have an analogue of Corollary 3.3: the triad of metaharmonic functions $\{\Phi_1, \Phi_2, \Phi_3\}$ is uniquely defined by the components of u if the conditions (3.28) hold for $j = 2, 3$.

4.2 *Sommerfeld–Kupradze type radiation conditions in hemitropic elasticity*

We say that a vector $U = (u, \omega)^\top$ satisfies the Sommerfeld–Kupradze type radiation conditions in Ω^- if

$$U(x) = \sum_{l=1}^{6} U^{(l)}(x) \quad \text{in } \Omega^-, \tag{4.6}$$

where the vectors $U^{(l)} = (U_1^{(l)}, \dots, U_6^{(l)})^\top$ satisfy the Helmholtz equations

$$[\Delta + k_l^2]\, U^{(l)}(x) = 0, \quad l = 1, 2, \dots, 6, \tag{4.7}$$

and for each l,

$$\frac{\partial}{\partial |x|}\, U_p^{(l)}(x) - i\, k_l\, U_p^{(l)}(x) = o\,(|x|^{-1}) \quad \text{as } |x| \to \infty, \quad p = 1, 2, \dots, 6.$$

Here k_l are as in Theorem 3.1. Such solutions will be referred to as *radiating* (for details see (**18**)). As is well known, for sufficiently large $|x|$ (as $|x| \to \infty$) there hold the asymptotic relations (**29**)

$$U^{(l)}(x) = \frac{\exp\{i\, k_l\, |x|\}}{|x|}\, U_\infty^{(l)}(\hat{x}) + \mathcal{O}\,(|x|^{-2}), \qquad \frac{\partial}{\partial |x|}\, U_p^{(l)}(x) - i\, k_l\, U_p^{(l)}(x) = \mathcal{O}\,(|x|^{-2}),$$

$$\frac{\partial}{\partial x_q}\, U_p^{(l)}(x) - i\, k_l\, \hat{x}_q\, U_p^{(l)}(x) = \mathcal{O}\,(|x|^{-2}), \quad \hat{x}_q = \frac{x_q}{|x|}, \quad q = 1, 2, 3,$$

where $U_\infty^{(l)}(\hat{x})$ is the so called *far-field pattern*,

$$U_\infty^{(l)}(\hat{x}) := -\frac{1}{4\pi} \int_{\partial\Omega^-} e^{-i\, k_l\, \hat{x}\cdot y} \{[\partial_{n(y)} U^{(l)}(y)]^- + i\, k_l\, (\hat{x} \cdot n(y))\, [U^{(l)}(y)]^-\} dS.$$

Here and in what follows the symbols $[\,\cdot\,]^\pm$ denote limits on $\partial\Omega^\pm$ from $\Omega^\pm$.

Recall the celebrated *Rellich–Vekua lemma*: if $U_p^{(l)}$ solves (4.7) in Ω^- with $k_l > 0$ and

$$\lim_{R\to\infty} \int_{\Sigma_R} |U_p^{(l)}(x)|^2\, d\Sigma_R = 0,$$

then $U_p^{(l)} = 0$ in Ω^-. As a consequence we get that $U_\infty^{(l)} = 0$ implies $U^{(l)} = 0$ in Ω^-.

It is evident that a vector $U = (u, \omega)^\top$, where u and ω are represented by formulae (3.1), is radiating if the vectors $v^{(l)}$ satisfy the Sommerfeld radiation conditions at infinity

$$\frac{\partial}{\partial |x|}\, v_p^{(l)}(x) - i\, k_l\, v_p^{(l)}(x) = o\,(|x|^{-1}) \text{ as } |x| \to \infty, \quad p = 1, 2, 3, \quad l = 1, 2, \dots, 6.$$

It is also obvious that, in this case, in (4.6) we can take $U^{(l)} = (v^{(l)},\ \beta_l v^{(l)})^\top$, $l = 1, 2, \dots, 6$.

4.3 *Formulation of transmission problems*

Let $\Omega^+ = \Omega_1 = B(R)$, a ball centred at the origin and radius R; $\Omega^- = \Omega_2 = \mathbb{R}^3 \setminus \overline{\Omega}_1$. We assume that the domains Ω_1 and Ω_2 are occupied by hemitropic elastic solids with different material constants involved in (2.8) and (2.9), and satisfying (2.3), (2.4) and (2.10). We endow these constants with a subscript l: $\lambda_l, \dots, \varkappa_l$, ϱ_l, $\mathcal{I}_l$, $l = 1, 2$. The corresponding operators will be denoted by $L^{[l]}(\partial, \sigma)$, $T^{[l]}(\partial, n)$, $\mathcal{T}^{[l]}$ and $\mathcal{M}^{[l]}$; see (2.5) and (2.11).

A vector function $V = (V_1, \dots, V_m)$ is said to be *regular* in Ω_p if $V \in [C^1(\overline{\Omega}_p)]^m$.

PROBLEM (H.H.) Find regular vector functions $U^{(l)} = (u^{(l)}, \omega^{(l)})^\top$ in Ω_l, $l = 1, 2$, satisfying the homogeneous differential equations $L^{[l]}(\partial, \sigma)\, U^{(l)}(x) = 0$, for $x \in \Omega_l$, $l = 1, 2$, with $U^{(2)}$ radiating in Ω_2, and the transmission conditions on the interface $\partial\Omega_1 = \partial\Omega_2 = \Sigma_R$,

$$[U^{(1)}(z)]^+ - [U^{(2)}(z)]^- = f(z), \tag{4.8}$$

$$[T^{[1]}(\partial, n)\, U^{(1)}(z)]^+ - [T^{[2]}(\partial, n)\, U^{(2)}(z)]^- = F(z), \tag{4.9}$$

where $n(z) = R^{-1} z$ is the unit exterior normal vector at $z \in \Sigma_R$, $f = (f^{(1)}, f^{(2)})^\top$ and $F = (f^{(3)}, f^{(4)})^\top$ with $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})^\top$ are given continuous vector functions on Σ_R.

We can rewrite the transmission conditions as follows:

$$[u^{(1)}(z)]^+ - [u^{(2)}(z)]^- = f^{(1)}(z), \quad [\omega^{(1)}(z)]^+ - [\omega^{(2)}(z)]^- = f^{(2)}(z), \tag{4.10}$$

$$[\mathcal{T}^{[1]}(U^{(1)})(z)]^+ - [\mathcal{T}^{[2]}(U^{(2)})(z)]^- = f^{(3)}(z), \tag{4.11}$$

$$[\mathcal{M}^{[1]}(U^{(1)})(z)]^+ - [\mathcal{M}^{[2]}(U^{(2)})(z)]^- = f^{(4)}(z), \tag{4.12}$$

where $\mathcal{T}^{[l]}(U^{(l)})$ and $\mathcal{M}^{[l]}(U^{(l)})$ are the force and couple stress vectors (cf. (2.6) and (2.7)),

$$\begin{aligned} \mathcal{T}^{[l]}(U^{(l)}) &= 2\,\mu_l \frac{\partial u^{(l)}}{\partial n} + \lambda_l\, n \operatorname{div} u^{(l)} + (\mu_l - \alpha_l)\,[n \times \operatorname{curl} u^{(l)}] + 2\,\varkappa_l \frac{\partial \omega^{(l)}}{\partial n} \\ &\quad + \delta_l\, n \operatorname{div} \omega^{(l)} + (\varkappa_l - \nu_l)\,[n \times \operatorname{curl} \omega^{(l)}] + 2\,\alpha_l\,[n \times \omega^{(l)}], \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathcal{M}^{[l]}(U^{(l)}) &= 2\varkappa_l \frac{\partial u^{(l)}}{\partial n} + \delta_l\, n \operatorname{div} u^{(l)} + (\varkappa_l - \nu_l)\,[n \times \operatorname{curl} u^{(l)}] + 2\gamma_l \frac{\partial \omega^{(l)}}{\partial n} \\ &\quad + \beta_l\, n \operatorname{div} \omega^{(l)} + (\gamma_l - \varepsilon_l)\,[n \times \operatorname{curl} \omega^{(l)}] + 2\,\nu_l\,[n \times \omega_l], \quad l = 1, 2. \end{aligned} \tag{4.14}$$

Equation (4.10) describes the jump of the displacement and microrotation vectors, while (4.11) and (4.12) describe the jumps of the force stress and couple stress vectors.

Further we formulate a transmission problem for the composed body where we consider the model of classical elasticity in Ω_1 and the model of hemitropic elasticity in Ω_2.

PROBLEM (C.H.) Find regular vector functions $u^{(1)}$ in Ω_1 and $U^{(2)} = (u^{(2)}, \omega^{(2)})^\top$ in Ω_2 satisfying $A^{[1]}(\partial, \sigma)\, u^{(1)}(x) = 0$ for $x \in \Omega_1$, $L^{[2]}(\partial, \sigma)\, U^{(2)}(x) = 0$ for $x \in \Omega_2$, with $U^{(2)}$ radiating in Ω_2, and the boundary transmission conditions on $\partial\Omega_1 = \partial\Omega_2 = \Sigma_R$,

$$[u^{(1)}(z)]^+ - [u^{(2)}(z)]^- = f^{(1)}(z), \quad [\omega^{(2)}(z)]^- = f^{(3)}(z), \tag{4.15}$$

$$[P^{[1]}(\partial, n) u^{(1)}(z)]^+ - [\mathcal{T}^{[2]}(U^{(2)})(z)]^- = f^{(2)}(z), \tag{4.16}$$

where $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})^\top$, $j = 1, 2, 3$, are given continuous vector functions on Σ_R, $A^{[1]}(\partial, \sigma)\, u^{(1)}$ and $P^{[1]}(\partial, n)u^{(1)}$ are defined by (4.1) and (4.2) with λ_1, μ_1, ϱ_1, and $u^{(1)}$ for λ, μ, ϱ, and u, and $\mathcal{T}^{[2]}(U^{(2)})$ is given by (4.13).

Note that, in the above problems, usually it is natural to require rigid bonding conditions on the interfaces between the adjacent bodies, that is, continuity of the displacement and microrotation fields, and force and couple stress vectors, which converts the transmission and boundary conditions (4.8), (4.9), (4.15) and (4.16) into homogeneous ones. However, non-homogeneous transmission conditions may occur for several reasons. For example, if we have non-zero body forces and body couples, then the differential equations in the corresponding domains become non-homogeneous. By a standard approach we can easily reduce them to homogeneous equations by introducing new unknown vector functions, $\tilde{U}^{(j)} = U^{(j)} - U_0^{(j)}$, $j = 1, 2$, where $U_0^{(j)}$ are some particular solutions to the non-homogeneous equations in the corresponding domains. For example, such solutions can be explicitly written by means of the Newtonian potentials since the matrices of fundamental solutions are known (**16**, **18**). Clearly, $\tilde{U}^{(j)}$ solves then the homogeneous differential equations, but now the transmission and boundary conditions become non-homogeneous since $U_0^{(1)}$ and $U_0^{(2)}$ do not satisfy homogeneous transmission and boundary conditions.

Non-homogeneous transmission conditions arise also in Eshelby-type inclusion problems. In this case, a region (‘inclusion’) in an infinite elastic medium undergoes a change of shape and size which, but for the constraint imposed by its surroundings (the ‘matrix’), would be an arbitrary homogeneous strain. It is required to determine the elastic state of inclusion and matrix. Due to Mura’s nomenclature (**31**), when the material properties of the inclusion and the matrix are the same, the problem is referred to as Eshelby’s first problem, while when the elastic properties are different the problem is referred to as Eshelby’s second problem; for details see (**19**, **20**) for elastic bodies, and (**14**) for hemitropic bodies.

THEOREM 4.1 *The homogeneous transmission problems* (H.H.) *and* (C.H.) *have only the trivial solution.*

Proof. Let $R_1 > R$ and $\Omega_{R_1} = \Omega_2 \cap B(R_1)$. We have the following Green’s formulae (**16**):

$$\int_{\Omega_1} [U^{(1)} \cdot L^{[1]}(\partial, \sigma)U^{(1)} - L^{[1]}(\partial, \sigma)U^{(1)} \cdot U^{(1)}]dx = \int_{\Sigma_R} [U^{(1)} \cdot T^{[1]}(\partial, n)U^{(1)} - T^{[1]}(\partial, n)U^{(1)} \cdot U^{(1)}]^+ dS,$$

$$\int_{\Omega_1} [u^{(1)} \cdot A^{[1]}(\partial, \sigma)u^{(1)} - A^{[1]}(\partial, \sigma)u^{(1)} \cdot u^{(1)}]dx = \int_{\Sigma_R} [u^{(1)} \cdot P^{[1]}(\partial, n)u^{(1)} - P^{[1]}(\partial, n)u^{(1)} \cdot u^{(1)}]^+ dS,$$

$$\int_{\Omega_{R_1}} [U^{(2)} \cdot L^{[2]}(\partial, \sigma)U^{(2)} - L^{[2]}(\partial, \sigma)U^{(2)} \cdot U^{(2)}]dx = -\int_{\Sigma_R} [U^{(2)} \cdot T^{[2]}(\partial, n)U^{(2)} - T^{[2]}(\partial, n)U^{(2)} \cdot U^{(2)}]^- dS + \int_{\Sigma_{R_1}} [U^{(2)} \cdot T^{[2]}(\partial, n)U^{(2)} - T^{[2]}(\partial, n)U^{(2)} \cdot U^{(2)}]dS.$$

From the homogeneous transmission conditions it follows that

$$\mathrm{Im}\int_{\Sigma_{R_1}} U^{(2)}\cdot T^{[2]}(\partial,n)U^{(2)}\,dS=0\quad\text{for arbitrary } R_1>R. \tag{4.17}$$

In (**18**) it is shown that for radiating vectors the condition (4.17) implies $U^{(2)}(x)=0$ in Ω_2. Therefore due to the homogeneous transmission conditions we get

$$[U^{(1)}(z)]^+=0,\quad [T^{[1]}(\partial,n)U^{(1)}(z)]^+=0,\quad z\in\Sigma_R, \tag{4.18}$$

in the case of Problem (H.H.) and

$$[u^{(1)}(z)]^+=0,\quad [P^{[1]}(\partial,n)u^{(1)}(z)]^+=0,\quad z\in\Sigma_R, \tag{4.19}$$

in the case of Problem (C.H.). Now, we recall the standard integral representation formulae for regular solutions $U^{(1)}$ and $u^{(1)}$ in Ω_1 (see, for example, (**16**, **30**))

$$U^{(1)}(x)=\int_{\Sigma_R}\{[T^{[1]}(\partial,n)\Gamma(x-y,\sigma)]^\top[U^{(1)}(z)]^+-\Gamma(x-y,\sigma)[T^{[1]}(\partial,n)U^{(1)}(z)]^+\}\,dS,$$

$$u^{(1)}(x)=\int_{\Sigma_R}\{[P^{[1]}(\partial,n)\widetilde{\Gamma}(x-y,\sigma)]^\top[u^{(1)}(z)]^+-\widetilde{\Gamma}(x-y,\sigma)[P^{[1]}(\partial,n)u^{(1)}(z)]^+\}\,dS,$$

where $\Gamma(x-y,\sigma)$ and $\widetilde{\Gamma}(x-y,\sigma)$ are the fundamental matrices of $L^{[1]}(\partial,\sigma)$ and $A^{[1]}(\partial,\sigma)$, respectively. These relations along with (4.18) and (4.19) complete the proof.

4.4 *Solution of Problem* (H.H.)

We look for a solution pair $U^{(1)}=(u^{(1)},\omega^{(1)})^\top$ and $U^{(2)}=(u^{(2)},\omega^{(2)})^\top$ of the transmission problem (H.H.) in the form (see (3.26) and (3.27))

$$u^{(l)}(x)=\mathrm{grad}\sum_{j=1}^{2}\Phi_j^{(l)}(x)+\sum_{j=3}^{6}[\mathrm{curl\ curl}\,(x\,\Phi_j^{(l)})+\eta_j^{(l)}\ \mathrm{curl}\,(x\,\Phi_j^{(l)}(x))], \tag{4.20}$$

$$\omega^{(l)}(x)=\mathrm{grad}\sum_{j=1}^{2}\beta_j^{(l)}\Phi_j^{(l)}(x)+\sum_{j=3}^{6}\beta_j^{(l)}[\mathrm{curl\,curl}(x\Phi_j^{(l)})+\eta_j^{(l)}\ \mathrm{curl}(x\Phi_j^{(l)}(x))], \tag{4.21}$$

where $l=1,2$,

$$\beta_j^{(l)}=[\rho_l\sigma^2-(\lambda_l+2\mu_l)k_{jl}^2][(\delta_l+2\varkappa_l)k_{jl}^2]^{-1},\quad j=1,2,$$

$$\beta_j^{(l)}=[\rho_l\sigma^2-(\mu_l+\alpha_l)k_{jl}^2][(\varkappa_l+\nu_l)k_{jl}^2-2\alpha_l\eta_j^{(l)}]^{-1},\quad j=3,4,5,6,$$

$$k_{1l}^2+k_{2l}^2=[d_{2l}]^{-1}[(\beta_l+2\gamma_l)\rho_l\sigma^2+(\lambda_l+2\mu_l)(\mathcal{I}_l\sigma^2-4\alpha_l)], \tag{4.22}$$

$$k_{1l}^2k_{2l}^2=[d_{2l}]^{-1}\rho_l\sigma^2(\mathcal{I}_l\sigma^2-4\alpha_l),\quad \eta_3^{(l)}=k_{3l},\ \eta_4^{(l)}=k_{4l},\ \eta_5^{(l)}=-k_{5l},\ \eta_6^{(l)}=-k_{6l}, \tag{4.23}$$

the constants $k_{3l}, k_{4l}, -k_{5l}, -k_{6l}$ ($k_{jl}>0,\ j=3,4,5,6$), are roots of the equation

$$d_{1l}t^4+d_{3l}t^3+d_{4l}t^2+d_{5l}t+d_{6l}=0$$

with

$$d_{1l} = (\mu_l + \alpha_l)(\gamma_l + \varepsilon_l) - (\varkappa_l + \nu_l)^2, \quad d_{2l} = (\lambda_l + 2\mu_l)(\beta_l + 2\gamma_l) - (\delta_l + 2\varkappa_l)^2,$$
$$d_{3l} = 4(\alpha_l\,\varkappa_l - \mu_l\nu_l), \quad d_{4l} = -[(\gamma_l + \varepsilon_l)\rho_l\sigma^2 + (\mu_l + \alpha_l)(\mathcal{I}_l\sigma^2 - 4\alpha_l) + 4\alpha_l^2],$$
$$d_{5l} = 4\nu_l\rho_l\sigma^2, \quad d_{6l} = \rho_l\sigma^2(\mathcal{I}_l\sigma^2 - 4\alpha_l), \quad l = 1, 2;$$

$\Phi_j^{(l)}$ are metaharmonic scalar functions,

$$[\Delta + k_{jl}^2]\Phi_j^{(l)}(x) = 0, \quad j = 1, 2, \ldots, 6, \ \ l = 1, 2. \tag{4.24}$$

Note that $U^{(2)}$ has to be radiating. Therefore $\Phi_j^{(l)}$ can be represented in $\mathbf{\Omega}_l$ as

$$\Phi_j^{(l)}(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} A_{mk}^{(j,l)}\, g_k^{(l)}(k_{jl}\, r)\, Y_k^{(m)}(\vartheta, \varphi), \tag{4.25}$$

where $A_{mk}^{(j,l)}$ are unknown constants, $Y_k^{(m)}$ are given by (2.14), and

$$g_k^{(1)}(k_{j1}\, r) = r^{-1/2} J_{k+1/2}(k_{j1}\, r), \quad g_k^{(2)}(k_{j2}\, r) = r^{-1/2} H_{k+1/2}^{(1)}(k_{j2}\, r), \quad j = 1, 2, \ldots, 6;$$

here J_ν are Bessel functions and $H_\nu^{(1)}$ are Hankel functions of the first kind. We assume that $\Phi_j^{(l)}(x)$ satisfy conditions similar to (3.28) for $j = 3, 4, 5, 6$ and $l = 1, 2$. It is evident that these restrictions are equivalent to the equalities

$$A_{00}^{(j,l)} = 0, \quad j = 3, 4, 5, 6, \quad l = 1, 2. \tag{4.26}$$

In what follows we assume that in the representation (4.25) these conditions are fulfilled.

We remark that for radiating metaharmonic functions $\Phi_j^{(2)}$ the series (4.25) converge absolutely and uniformly on compact subsets of $\mathbf{\Omega}_2$. Conversely, if the series (4.25) converge in the mean square sense on the sphere $|x| = R$ then they also converge absolutely and uniformly on compact subsets of $|x| > R$ and represent radiating solutions to the Helmholtz equation (4.24) for $|x| > R$; for details, see (**29**, Theorem 2.14).

Let us substitute (4.25) into (4.20) and (4.21), and apply the following identities:

$$\operatorname{grad}[a(r)\, Y_k^{(m)}(\vartheta, \varphi)] = \frac{da(r)}{dr}\, \mathbf{X}_{mk}(\vartheta, \varphi) + \frac{\sqrt{k(k+1)}}{r}\, a(r)\, \mathbf{Y}_{mk}(\vartheta, \varphi),$$
$$\operatorname{curl}(x\, a(r)\, Y_k^{(m)}(\vartheta, \varphi)) = \sqrt{k(k+1)}\, a(r)\, \mathbf{Z}_{mk}(\vartheta, \varphi),$$
$$\operatorname{curl}\operatorname{curl}(x\, a(r)\, Y_k^{(m)}) = \frac{k(k+1)}{r}\, a(r)\, \mathbf{X}_{mk} + \sqrt{k(k+1)}\left(\frac{d}{dr} + \frac{1}{r}\right) a(r)\, \mathbf{Y}_{mk},$$

where $\mathbf{X}_{mk}(\vartheta, \varphi)$, $\mathbf{Y}_{mk}(\vartheta, \varphi)$, and $\mathbf{Z}_{mk}(\vartheta, \varphi)$ are defined in (2.13) and $a(r)$ is an arbitrary differentiable scalar function of r. We arrive at the equalities

$$u^{(l)}(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\{u_{mk}^{(1,l)}(r)\, \mathbf{X}_{mk} + \sqrt{k(k+1)}\, [v_{mk}^{(1,l)}(r)\, \mathbf{Y}_{mk} + w_{mk}^{(1,l)}(r)\, \mathbf{Z}_{mk}]\}, \tag{4.27}$$

$$\omega^{(l)}(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\{u_{mk}^{(2,l)}(r)\, \mathbf{X}_{mk} + \sqrt{k(k+1)}[v_{mk}^{(2,l)}(r)\, \mathbf{Y}_{mk} + w_{mk}^{(2,l)}(r)\, \mathbf{Z}_{mk}]\}, \tag{4.28}$$

where

$$u_{mk}^{(p,l)}(r) = \sum_{j=1}^{2} \gamma_{jl}^{(p)} A_{mk}^{(j,l)} \frac{d}{dr} g_k^{(l)}(k_{jl}\, r) + \sum_{j=3}^{6} \frac{k(k+1)}{r} \gamma_{jl}^{(p)} A_{mk}^{(j,l)} g_k^{(l)}(k_{jl}\, r), \quad k \geqslant 0, \tag{4.29}$$

$$v_{mk}^{(p,l)}(r) = \sum_{j=1}^{2} \frac{1}{r} \gamma_{jl}^{(p)} A_{mk}^{(j,l)} g_k^{(l)}(k_{jl}\, r) + \sum_{j=3}^{6} \gamma_{jl}^{(p)} A_{mk}^{(j,l)} \left(\frac{d}{dr} + \frac{1}{r}\right) g_k^{(l)}(k_{jl}\, r), \quad k \geqslant 1, \tag{4.30}$$

$$w_{mk}^{(p,l)}(r) = \sum_{j=3}^{6} \eta_j^{(l)} \gamma_{jl}^{(p)} A_{mk}^{(j,l)} g_k^{(l)}(k_{jl}\, r), \quad k \geqslant 1, \quad p = 1, 2, \quad l = 1, 2, \tag{4.31}$$

$\gamma_{jl}^{(1)} = 1$, $\gamma_{jl}^{(2)} = \beta_j^{(l)}$ and $\eta_j^{(l)}$ are defined in (4.23). Further, let us substitute the above expressions of $u^{(l)}(x)$ and $w^{(l)}(x)$ into (4.13) and (4.14), and apply the identities $\mathbf{e}_r \times \mathbf{X}_{mk} = 0$, $\mathbf{e}_r \times \mathbf{Y}_{mk} = -\mathbf{Z}_{mk}$, $\mathbf{e}_r \times \mathbf{Z}_{mk} = \mathbf{Y}_{mk}$, $\operatorname{div}[a(r)\,\mathbf{Z}_{mk}] = 0$,

$$\operatorname{div}[a(r)\,\mathbf{X}_{mk}] = \left(\frac{d}{dr} + \frac{2}{r}\right) a(r)\, Y_k^{(m)}, \qquad \operatorname{div}[a(r)\,\mathbf{Y}_{mk}] = -\frac{\sqrt{k(k+1)}}{r}\, a(r)\, Y_k^{(m)},$$

$$\operatorname{curl}[a(r)\,\mathbf{X}_{mk}] = \frac{\sqrt{k(k+1)}}{r}\, a(r)\, \mathbf{Z}_{mk}, \quad \operatorname{curl}[a(r)\,\mathbf{Y}_{mk}] = -\left(\frac{d}{dr} + \frac{1}{r}\right) a(r)\, \mathbf{Z}_{mk},$$

$$\operatorname{curl}[a(r)\,\mathbf{Z}_{mk}(\vartheta,\varphi)] = \frac{\sqrt{k(k+1)}}{r}\, a(r)\, \mathbf{X}_{mk}(\vartheta,\varphi) + \left(\frac{d}{dr} + \frac{1}{r}\right) a(r)\, \mathbf{Y}_{mk}(\vartheta,\varphi).$$

Finally, we obtain the representation of the force stress and couple stress vectors in the form of series

$$\mathcal{T}^{[l]}(U^{(l)})(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \{a_{mk}^{(1,l)}(r)\, \mathbf{X}_{mk} + \sqrt{k(k+1)}[b_{mk}^{(1,l)}(r)\, \mathbf{Y}_{mk} + c_{mk}^{(1,l)} \mathbf{Z}_{mk}]\}, \tag{4.32}$$

$$\mathcal{M}^{[l]}(U^{(l)})(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \{a_{mk}^{(2,l)}(r)\, \mathbf{X}_{mk} + \sqrt{k(k+1)}[b_{mk}^{(2,l)}(r)\, \mathbf{Y}_{mk} + c_{mk}^{(2,l)} \mathbf{Z}_{mk}]\}, \tag{4.33}$$

where

$$\begin{aligned} a_{mk}^{(1,l)}(r) = &-\sum_{j=1}^{2} A_{mk}^{(j,l)} \left[\frac{4\xi_{jl}}{r}\frac{d}{dr} + \rho_l\, \sigma^2 - \frac{2k(k+1)}{r^2}\xi_{jl}\right] g_k^{(l)}(k_{jl}\, r) \\ &+ \sum_{j=3}^{6} \frac{2k(k+1)}{r} \xi_{jl} A_{mk}^{(j,l)} \left(\frac{d}{dr} - \frac{1}{r}\right) g_k^{(l)}(k_{jl}\, r), \quad k \geqslant 0, \end{aligned} \tag{4.34}$$

$$\begin{aligned} b_{mk}^{(1,l)}(r) = &\sum_{j=1}^{2} \frac{2\xi_{jl}}{r} A_{mk}^{(j,l)} \left(\frac{d}{dr} - \frac{1}{r}\right) g_k^{(l)}(k_{jl}\, r) \\ &- \sum_{j=3}^{6} A_{mk}^{(j,l)} \left[\frac{2\xi_{jl}}{r}\left(\frac{d}{dr} - \frac{k(k+1)-1}{r}\right) + \rho_l\, \sigma^2\right] g_k^{(l)}(k_{jl}\, r), \quad k \geqslant 1, \end{aligned} \tag{4.35}$$

$$c_{mk}^{(1,l)}(r) = -\sum_{j=1}^{2} \frac{2\alpha_l}{r}\,\beta_j^{(l)}\,A_{mk}^{(j,l)}\,g_k^{(l)}(k_{jl}\,r)$$

$$+\sum_{j=3}^{6} A_{mk}^{(j,l)}\left[\frac{\rho_l\,\sigma^2}{\eta_j^{(l)}}\left(\frac{d}{dr}+\frac{1}{r}\right)-\frac{2}{r}\,\xi_{jl}\,\eta_j^{(l)}\right] g_k^{(l)}(k_{jl}\,r), \quad k \geqslant 1, \tag{4.36}$$

$$a_{mk}^{(2,l)}(r) = -\sum_{j=1}^{2} A_{mk}^{(j,l)}\left[\frac{2\zeta_{jl}}{r}\left(2\frac{d}{dr}-\frac{k(k+1)}{r}\right)-\left(\delta_l+2\varkappa_l+(\beta_l+2\gamma_l)\beta_j^{(l)}\right)k_{jl}^2\right] g_k^{(l)}(k_{jl}\,r)$$

$$+\sum_{j=3}^{6}\frac{2k(k+1)}{r}\,A_{mk}^{(j,l)}\,\zeta_{jl}\left(\frac{d}{dr}-\frac{1}{r}\right) g_k^{(l)}(k_{jl}\,r), \quad k \geqslant 0, \tag{4.37}$$

$$b_{mk}^{(2,l)}(r) = \sum_{j=1}^{2}\frac{2\zeta_{jl}}{r}\,A_{mk}^{(j,l)}\left(\frac{d}{dr}-\frac{1}{r}\right) g_k^{(l)}(k_{jl}r) + \sum_{j=3}^{6} A_{mk}^{(j,l)}\left[-\frac{2\zeta_{jl}}{r}\left(\frac{d}{dr}-\frac{k(k+1)-1}{r}\right)\right.$$

$$\left.+(\varkappa_l+\nu_l)\,k_{jl}^2+(\gamma_l+\varepsilon_l)\,\beta_j^{(l)}\,k_{jl}^2+2\,\nu_l\,\beta_j^{(l)}\,\eta_j^{(l)}\right] g_k^{(l)}(k_{jl}\,r), \quad k \geqslant 1, \tag{4.38}$$

$$c_{mk}^{(2,l)}(r) = -\sum_{j=1}^{2}\frac{2\nu_l}{r}\beta_j^{(l)}A_{mk}^{(j,l)}g_k^{(l)}(k_{jl}r) + \sum_{j=3}^{6} A_{mk}^{(j,l)}\left\{\left[(\varkappa_l+\nu_l)\eta_j^{(l)}\right.\right.$$

$$\left.\left.+(\gamma_l+\varepsilon_l)\eta_j^{(l)}\beta_j^{(l)}-2\nu_l\beta_j^{(l)}\right]\left(\frac{d}{dr}+\frac{1}{r}\right)-\frac{2\zeta_{jl}}{r}\eta_j^{(l)}\right\} g_k^{(l)}(k_{jl}r), \quad k \geqslant 1. \tag{4.39}$$

Here $\xi_{jl} = \mu_l + \varkappa_l\beta_j^{(l)}$, $\zeta_{jl} = \varkappa_l + \gamma_l\beta_j^{(l)}$, $l = 1, 2$ and $j = 1, 2, \ldots, 6$.

Let the given vector functions $f^{(j)}(z)$, $j = 1, 2, 3, 4$, involved in the transmission conditions (4.10) to (4.12) be representable as the Fourier–Laplace series

$$f^{(j)}(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\{\alpha_{mk}^{(j)}\mathbf{X}_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}[\beta_{mk}^{(j)}\mathbf{Y}_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}\mathbf{Z}_{mk}(\vartheta,\varphi)]\}, \tag{4.40}$$

where the coefficients $\alpha_{mk}^{(j)}, \beta_{mk}^{(j)}, \gamma_{mk}^{(j)}$ are calculated by (2.15) with $f^{(j)}$ for f.

With the help of (4.27) to (4.39), and the expansions (4.40), from the transmission conditions (4.10) to (4.12) we get the following two groups of linear systems of algebraic equations for the unknowns $A_{mk}^{(j,l)}$:

1. for $k = 0$, $m = 0$ (four equations with four unknowns, $A_{00}^{(j,l)}$, $j = 1, 2$, $l = 1, 2$)

$$\left.\begin{aligned} u_{00}^{(1,1)}(R)-u_{00}^{(1,2)}(R)&=\alpha_{00}^{(1)}, \quad a_{00}^{(1,1)}(R)-a_{00}^{(1,2)}(R)=\alpha_{00}^{(3)},\\ u_{00}^{(2,1)}(R)-u_{00}^{(2,2)}(R)&=\alpha_{00}^{(2)}, \quad a_{00}^{(2,1)}(R)-a_{00}^{(2,2)}(R)=\alpha_{00}^{(4)}; \end{aligned}\right\}$$

2. for $k \geqslant 1$, $-k \leqslant m \leqslant k$ (12 equations with 12 unknowns for every fixed k and m, $A^{(j,l)}_{km}$, $j = 1, 2, \ldots, 6$, $l = 1, 2$)

$$\left.\begin{aligned}
u^{(1,1)}_{mk}(R) - u^{(1,2)}_{mk}(R) &= \alpha^{(1)}_{mk}, & a^{(1,1)}_{mk}(R) - a^{(1,2)}_{mk}(R) &= \alpha^{(3)}_{mk},\\
u^{(2,1)}_{mk}(R) - u^{(2,2)}_{mk}(R) &= \alpha^{(2)}_{mk}, & a^{(2,1)}_{mk}(R) - a^{(2,2)}_{mk}(R) &= \alpha^{(4)}_{mk},\\
v^{(1,1)}_{mk}(R) - v^{(1,2)}_{mk}(R) &= \beta^{(1)}_{mk}, & b^{(1,1)}_{mk}(R) - b^{(1,2)}_{mk}(R) &= \beta^{(3)}_{mk},\\
v^{(2,1)}_{mk}(R) - v^{(2,2)}_{mk}(R) &= \beta^{(2)}_{mk}, & b^{(2,1)}_{mk}(R) - b^{(2,2)}_{mk}(R) &= \beta^{(4)}_{mk},\\
w^{(1,1)}_{mk}(R) - w^{(1,2)}_{mk}(R) &= \gamma^{(1)}_{mk}, & c^{(1,1)}_{mk}(R) - c^{(1,2)}_{mk}(R) &= \gamma^{(3)}_{mk},\\
w^{(2,1)}_{mk}(R) - w^{(2,2)}_{mk}(R) &= \gamma^{(2)}_{mk}, & c^{(2,1)}_{mk}(R) - c^{(2,2)}_{mk}(R) &= \gamma^{(4)}_{mk}.
\end{aligned}\right\}$$

Due to Theorem 4.1 and Corollary 3.3 these systems are uniquely solvable with respect to the unknowns $A^{(j,l)}_{mk}$. Thus we can construct explicitly the formal solution of the transmission problem in the form of series. Further we have to investigate the convergence of these series and their derivatives. To this end we apply the asymptotic formulae for Bessel and Hankel functions as $k \to \infty$ (**32**) to obtain

$$g^{(1)}_k(k_{j1}r) \approx \frac{\sqrt{k_{j1}}\, 2^{k+1/2}\, k!\, (k_{j1}r)^k}{\sqrt{\pi}\,(2k+1)!}, \quad [g^{(1)}_k(k_{j1}r)]' \approx \frac{k}{k_{j1}r}\, g^{(1)}_k(k_{j1}r), \quad r < R,$$

$$g^{(2)}_k(k_{j2}r) \approx \frac{-i\sqrt{k_{j2}}\,(2k)!}{\sqrt{\pi}\, 2^{k-1/2} k!\, (k_{j2}r)^{k+1}}, \quad [g^{(2)}_k(k_{j2}r)]' \approx -\frac{k}{k_{j2}r} g^{(2)}_k(k_{j2}r), \quad R < r < R_1.$$

Here R_1 is an arbitrary number greater than R.

From these relations it follows that the series (4.27), (4.32) and (4.33) converge absolutely and uniformly on compact subsets of Ω_1 and Ω_2.

These series converge absolutely and uniformly on $\partial\Omega_1 = \partial\Omega_2 = \Sigma_R$ if the following dominating series (obtained with the help of (2.16)) converges

$$\sum_{k=k_0}^{\infty}\sum_{j=1}^{4} k^{\frac{3}{2}}[k(\delta_{1j}+\delta_{2j}) + \delta_{3j} + \delta_{4j}][|\alpha^{(j)}_{mk}| + k|\beta^{(j)}_{mk}| + k|\gamma^{(j)}_{mk}|], \tag{4.41}$$

where $\alpha^{(j)}_{mk}$, $\beta^{(j)}_{mk}$, and $\gamma^{(j)}_{mk}$ are the Fourier–Laplace coefficients of $f^{(j)}$ (see (4.40)).

Now, with the help of the above asymptotic formulae we conclude that the following asymptotic relations

$$\alpha^{(j)}_{mk} = \mathcal{O}(k^{-\tau-1}), \quad \beta^{(j)}_{mk} = \mathcal{O}(k^{-\tau-2}), \quad \gamma^{(j)}_{mk} = \mathcal{O}(k^{-\tau-2}), \quad j = 1, 2, \tag{4.42}$$

$$\alpha^{(j)}_{mk} = \mathcal{O}(k^{-\tau}), \quad \beta^{(j)}_{mk} = \mathcal{O}(k^{-\tau-1}), \quad \gamma^{(j)}_{mk} = \mathcal{O}(k^{-\tau-1}), \quad j = 3, 4, \tag{4.43}$$

with $\tau > 5/2$ are sufficient for convergence of the dominating series (4.41). In turn, from Lemmas 2.1 and 2.2, it follows that the inclusions

$$f^{(j)}(z) \in [C^4(\Sigma_R)]^3, \quad j = 1, 2, \qquad f^{(j)}(z) \in [C^3(\Sigma_R)]^3, \quad j = 3, 4, \tag{4.44}$$

imply (4.42) and (4.43). Thus, if the sufficient conditions (4.44) hold, then the series (4.27), (4.32) and (4.33) and their first-order derivatives converge uniformly and absolutely in $\overline{\Omega}_1$ and $\overline{\Omega}_2$ respectively, and define a regular pair of solutions $U^{(1)}$ and $U^{(2)}$ to Problem (H.H.).

4.5 *Solution of Problem* (C.H.)

We look for a regular vector $U^{(2)}$ in Ω_2 again in the form (4.20) and (4.21), while we seek a regular vector $u^{(1)}$ in Ω_1 in the form (4.4) and (4.5) with the wave numbers k_1^* and k_2^* defined by (4.3) with λ_1 and μ_1 for λ and μ. As before, we represent the functions Φ_j as

$$\Phi_j(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} A_{mk}^{(j)} \, g_k^{(1)}(k_j^* r) \, Y_k^{(m)}(\vartheta, \varphi), \quad j = 1, 2, 3, \tag{4.45}$$

where $k_3^* = k_2^*$, $A_{mk}^{(j)}$ are unknown constants, and $g_k^{(1)}(k_j^* r) = r^{-1/2} J_{k+1/2}(k_j^* r)$. We again assume that conditions (4.26) hold for $l = 2$ and also $A_{00}^{(j)} = 0$, $j = 2, 3$, which are equivalent to conditions similar to (3.28).

From (4.4) and (4.45) we derive the following expansion for the displacement vector:

$$u^{(1)}(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \{u_{mk}(r) \, \mathbf{X}_{mk} + \sqrt{k(k+1)}[v_{mk}(r) \, \mathbf{Y}_{mk} + w_{mk}(r) \, \mathbf{Z}_{mk}]\}, \tag{4.46}$$

where

$$u_{mk}(r) = A_{mk}^{(1)} \frac{d}{dr} g_k(k_1^* r) + A_{mk}^{(2)} \frac{k(k+1)}{R} g_k(k_2^* r), \quad k \geqslant 0,$$

$$v_{mk}(r) = A_{mk}^{(1)} \frac{g_k(k_1^* r)}{r} + A_{mk}^{(2)} \left(\frac{d}{dr} + \frac{1}{r} \right) g_k(k_2^* r), \quad w_{mk}(r) = A_{mk}^{(3)} g_k(k_2^* r), \quad k \geqslant 1.$$

With the help of formulae (4.2) and (4.46) we get a similar expansion for the stress vector:

$$P^{[1]}(\partial, n) u^{(1)}(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \{a_{mk}(r) \, \mathbf{X}_{mk} + \sqrt{k(k+1)}[b_{mk}(r) \, \mathbf{Y}_{mk} + c_{mk}(r) \, \mathbf{Z}_{mk}]\}, \tag{4.47}$$

where

$$a_{mk}(r) = -\mu_1 A_{mk}^{(1)} \left[\frac{4}{r} \frac{d}{dr} - \frac{2k(k+1)}{r^2} + k_2^{*2} \right] g_k(k_1^* r) + \frac{2\mu_1 k(k+1)}{r} A_{mk}^{(2)} \left(\frac{d}{dr} - \frac{1}{r} \right) g_k(k_2^* r), \quad k \geqslant 0,$$

$$b_{mk}(r) = \frac{2\mu_1}{r} A_{mk}^{(1)} \left(\frac{d}{dr} - \frac{1}{r} \right) g_k(k_1^* r) - \mu_1 A_{mk}^{(2)} \left(\frac{2}{r} \frac{d}{dr} - 2 \frac{k(k+1) - 1}{r^2} + k_2^{*2} \right) g_k(k_2^* r), \quad k \geqslant 1,$$

$$c_{mk}(r) = \mu_1 A_{mk}^{(3)} \left(\frac{d}{dr} - \frac{1}{r} \right) g_k(k_2^* r), \quad k \geqslant 1.$$

The representations (4.27) to (4.39), (4.46) and (4.47), and the transmission and boundary conditions (4.15) and (4.16), lead to the following linear systems of algebraic equations with respect to the unknown constants $A_{mk}^{(j)}$ and $A_{mk}^{(j,2)}$:

1. for $k = 0$, $m = 0$ (three equations with the three unknowns, $A_{00}^{(1)}$, $A_{00}^{(j,2)}$, $j = 1, 2$)

$$u_{00}(R) - u_{00}^{(1,2)}(R) = \alpha_{00}^{(1)}, \quad a_{00}(R) - a_{00}^{(1,2)}(R) = \alpha_{00}^{(2)}, \quad u_{00}^{(2,2)}(R) = \alpha_{00}^{(3)}; \tag{4.48}$$

2. for $k \geqslant 1$, $-k \leqslant m \leqslant k$ (nine equations with the nine unknowns for every fixed k and m, $A_{km}^{(p)}$, $p = 1, 2, 3$, $A_{km}^{(j,2)}$, $j = 1, 2, 3, 4, 5, 6$)

$$\left.\begin{aligned} &u_{mk}(R) - u_{mk}^{(1,2)}(R) = \alpha_{mk}^{(1)}, \;\; a_{mk}(R) - a_{mk}^{(1,2)}(R) = \alpha_{mk}^{(2)}, u_{mk}^{(2,2)}(R) = \alpha_{mk}^{(3)}, \\ &v_{mk}(R) - v_{mk}^{(1,2)}(R) = \beta_{mk}^{(1)}, \;\; b_{mk}(R) - b_{mk}^{(1,2)}(R) = \beta_{mk}^{(2)}, v_{mk}^{(2,2)}(R) = \beta_{mk}^{(3)}, \\ &w_{mk}(R) - w_{mk}^{(1,2)}(R) = \gamma_{mk}^{(1)}, c_{mk}(R) - c_{mk}^{(1,2)}(R) = \gamma_{mk}^{(2)}, w_{mk}^{(2,2)}(R) = \gamma_{mk}^{(3)}, \end{aligned}\right\} \tag{4.49}$$

where $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$ are the coefficients given by (2.15) with $f^{(j)}$ for f.

In accordance with Theorem 4.1 and Corollary 3.3 (see also section 4.1) the systems (4.48) and (4.49) are uniquely solvable and we can construct explicitly the vectors $u^{(1)}$ in $\mathbf{\Omega}_1$ and $U^{(2)}$ in $\mathbf{\Omega}_2$. Quite similarly, as in the previous case, we can show that if

$$f^{(1)}, \quad f^{(3)} \in [C^4(\Sigma_R)]^3, \quad f^{(2)} \in [C^3(\Sigma_R)]^3,$$

then the series obtained and their first-order derivatives converge absolutely and uniformly in $\overline{\mathbf{\Omega}}_1$ and $\overline{\mathbf{\Omega}}_2$. Therefore, $u^{(1)}$ and $U^{(2)}$ are regular in the corresponding domains.

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