BRUNO FRANCHI
GUOZHEN LU
RICHARD L. WHEEDEN

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REPRESENTATION FORMULAS AND WEIGHTED POINCARÉ INEQUALITIES FOR HÖRMANDER VECTOR FIELDS

by B. FRANCHI, G. LU & R.L. WHEEDEN (*)

1. Introduction.

In this paper, we derive the Poincaré inequality

(1.1)
$$\left(\frac{1}{|B|}\int_B |f(x)-f_B|^q dx\right)^{\frac{1}{q}} \le cr\left(\frac{1}{|B|}\int_B \left(\sum_j |\langle X_j, \nabla f(x)\rangle|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$$

in Euclidean space $\mathbb{R}^N$ for $1 \le p < \infty$ and certain values $q > p$, where $\{X_j\}$ is a collection of smooth vector fields which satisfy the Hörmander condition (see [H]). Here, B denotes any suitably restricted ball of radius r relative to a metric ρ which is naturally associated with $\{X_j\}$ as, *e.g.*, in [FP] (although similar results hold for more general regions), $f_B = |B|^{-1}\int_B f(x)dx$, and c is a constant independent of f and B.

Inequality (1.1) was derived in [J] for $q = p$ and $1 \le p < \infty$, and this result was improved in case $p > 1$ in [L2] by showing that the estimate holds for $1 < p < Q$ and $q = pQ/(Q-p)$, where Q ($\ge N$) denotes

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the homogeneous dimension of $\mathbb{R}^N$ associated with $\{X_j\}$ (see §2 for the definition). We will show that this result also holds in case $p = 1$.

In fact, we will show that (1.1) holds for $1 \le p < q < \infty$ if p and q are related by a natural balance condition which involves the local doubling order of Lebesgue measure (for metric balls). This condition will allow values of q which may be larger than those in [L2] and which may be different for different balls. We will also derive weighted versions of (1.1) for $1 \le p \le q < \infty$, and our estimates of this kind include those in [L1]. We note that it is shown in [BM1], [BM2] that, in very general settings, Poincaré's inequality with $p = q = p_0$, for a value $p_0 \ge 1$, together with the doubling property of the underlying measure implies some Sobolev-Poincaré results of a different type for $q \ge p \ge p_0$, with q related to the doubling order. Some results in the same spirit were proved in [S-Cos] for compactly supported functions. We also mention here that embedding theorems for Hörmander vector fields on Campanato-Morrey spaces, and from Morrey spaces to BMO and non-isotropic Lipschitz spaces have been obtained in [L3] and [L4], together with some applications to subelliptic problems.

As a corollary of our results for $p = 1$, we will derive *relative* isoperimetric inequalities for vector fields, including weighted versions. Such inequalities are more local than standard isoperimetric estimates. They remain valid for the classes of degenerate vector fields introduced in [FL] (see also [FS], [F], [FGuW]), which are not smooth but satisfy appropriate geometric conditions instead of the Hörmander condition. For $p = 1$ and vector fields of this second type, weighted Poincaré estimates are proved in [FGuW]. In this way, we obtain relative versions of the isoperimetric estimates in [FGaW1], [FGaW2], which are derived by using Sobolev's inequality (for $p = 1$), *i.e.*, the inequality like (1.1) in which the constant f_B is omitted but f is assumed to be supported in B.

Our results of Poincaré type are based on a new representation formula for a function in terms of the vector fields $\{X_j\}$, and this formula is one of our main results. One form of the representation states that if ρ denotes the metric corresponding to $\{X_j\}$, then

$$(1.2) \qquad |f(x) - f_B| \le C \int_{cB} |Xf(y)| \frac{\rho(x,y)}{|B(x,\rho(x,y))|} dy, \quad x \in B,$$

where B is any suitably small ρ-ball. Here, C and c are appropriate constants, $|Xf|^2 = \sum_j |\langle X_j, \nabla f\rangle|^2$, f_B is the Lebesgue average $|B|^{-1} \int_B f dy$, $B(x,r)$ is the metric ball with center x and radius r, and cB denotes

$B(x, cr)$ if $B = B(x, r)$. This estimate is more difficult to prove than the corresponding formula (without the constant f_B on the left) for functions f with compact support in B. In fact, that formula follows easily from the estimates in [NSW] and [SCal] for the fundamental solution of the operator $\sum_j X_j^* X_j$.

Inequality (1.2) was shown to be true on graded nilpotent Lie groups for the left invariant vector fields in [L1] (see Lemma (3.1) there). For general Hörmander vector fields, (1.2) improves an analogous fractional integral estimate in [L1] (Lemma (3.2) there) in several ways. For example, it only involves the original vector fields $\{X_j\}$ and metric ρ rather than their "lifted" versions $\{\tilde{X}_j\}$ and $\tilde{\rho}$ as defined in [RS] (see §2 below). Furthermore, the representation in [L1] also involves the Hardy-Littlewood maximal function of $|\tilde{X}f|+|f|$. Since the maximal function is not a bounded operator on L^1, its elimination is an important step in deriving Poincaré estimates for $p = 1$. Another important step involves eliminating the zero order term $|f|$. We will do this and also derive a sharper local version of (1.2) by modifying an argument in [SW] (see also [FGuW] and [FGaW1], [FGaW2]). The main modification we need in order to eliminate the zero order term is to use the known unweighted Poincaré inequality from L^1 to L^1 (see for example [J]). The precise argument is given in §2. The more local version of (1.2) is stated in Proposition 2.12 and will be especially important for our Poincaré estimates in case $p = 1$.

In order to state our results more precisely, we now introduce some additional notation (see §2 for more detail). Let Ω be an open, connected set in $\mathbb{R}^N$. Let $X_1, \dots, X_m$ be real C^∞ vector fields which satisfy Hörmander's condition, i.e., the rank of the Lie algebra generated by $X_1, \dots, X_m$ equals N at each point of a neighborhood Ω_0 of $\bar{\Omega}$. As is well-known, it is possible to naturally associate with $\{X_j\}$ a metric $\rho(x, y)$ for $x, y \in \Omega$. The geometry of the metric space (Ω, ρ) is described in [NSW], [FP] and [S-Cal]. In particular, the ρ-topology and the Euclidean topology are equivalent in Ω, each metric ball

$$B(x, r) = \{y \in \Omega : \rho(x, y) < r\}, \ x \in \Omega, \ r > 0,$$

contains some Euclidean ball with center x, and if K is a compact subset of Ω, there are positive constants c and r_0 such that

$$|B(x, 2r)| \le c|B(x, r)|, \quad x \in K, \quad 0 < r < r_0, \tag{1.3}$$

where $|E|$ denotes the Lebesgue measure of a measurable set E. This doubling property of Lebesgue measure is crucial for our results. If $B = B(x, r)$, we will use the notation $r(B)$ for the radius r of B.

By [NSW], given a ball $B = B(x,r)$, $x \in K$, $r < r_0$, there exist positive constants γ and c, depending on B, so that

$$|J| \leq c \left(\frac{r(J)}{r(I)} \right)^{N\gamma} |I| \tag{1.4}$$

for all balls I, J with $I \subset J \subset B$. We will call γ the (local) doubling order of Lebesgue measure for B. In fact, by [NSW], $N\gamma$ lies somewhere in the range $N \leq N\gamma \leq Q$, where Q is the homogeneous dimension. We can always choose $N\gamma = Q$, but smaller values may arise for particular vector fields, and these values may vary with $B(x,r)$. See §2 for some further comments about (1.4).

Given any real-valued function $f \in \operatorname{Lip}(\Omega)$, we denote

$$X_j f(x) = \langle X_j(x), \nabla f(x) \rangle, \; j = 1, \ldots, m,$$

and

$$|Xf(x)|^2 = \sum_{j=1}^{m} |X_j f(x)|^2,$$

where ∇f is the usual gradient of f and $\langle \; , \; \rangle$ is the usual inner product on $\mathbb{R}^N$.

The Poincaré estimate that we will prove in the unweighted case is as follows.

THEOREM 1. — *Let K be a compact subset of Ω. There exists r_0 depending on K, Ω and $\{X_j\}$ such that if $B = B(x,r)$ is a ball with $x \in K$ and $0 < r < r_0$, and if $1 \leq p < N\gamma$ and $1/q = 1/p - 1/(N\gamma)$, where γ is defined by (1.4) for B, then*

$$\left(\frac{1}{|B|} \int_B |f(x) - f_B|^q dx \right)^{\frac{1}{q}} \leq cr \left(\frac{1}{|B|} \int_B |Xf(x)|^p dx \right)^{\frac{1}{p}}$$

for any $f \in \operatorname{Lip}(\bar{B})$. The constant c depends on K, Ω, $\{X_j\}$, and the constants c and γ in (1.4). Also, f_B may be taken to be the Lebesgue average of f, $f_B = |B|^{-1} \int_B f(x)dx$.

As mentioned earlier, we may always choose $N\gamma = Q$, and then with $p > 1$ we obtain the principal result of [L2]. The theorem also improves the estimate in [J] for $p = 1$, where the L^1 norm appears on the left side of the conclusion.

After the preparation of this paper, a result similar to Theorem 1 was proved in [MS-Cos] by using a different approach. In fact, in [MS-Cos], the authors do not derive a representation formula like (1.2), which is one of the main results of the present paper. Moreover, formula (1.2) enables us to prove two-weight Sobolev-Poincaré inequalities (see Theorem 2 below), and at present seems essential for deriving such inequalities. Some related arguments have been given on graphs and manifolds in [Cou].

As the proof of Theorem 1 will show, if the conclusion is weakened by replacing the integration over B on the right by integration over an appropriate larger ball cB, then (1.4) may be replaced by the condition

$$|B| \leq c\left(\frac{r(B)}{r(I)}\right)^{N\gamma} |I|$$

for all balls I with center in cB and $r(I) \leq r(B)$.

Some weighted versions of Poincaré's inequality for Hörmander vector fields are proved in [L1] when $p > 1$, and our methods allow us to improve these and also extend them to $p = 1$. A weight function $w(x)$ on Ω is a nonnegative function on Ω which is locally integrable with respect to Lebesgue measure. We say that a weight $w \in A_p(= A_p(\Omega, \rho, dx))$, $1 \leq p < \infty$, if

$$\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}dx\right)^{p-1} \leq C \quad \text{when } 1 < p < \infty$$

$$\frac{1}{|B|}\int_B w\,dx \leq C \operatorname*{ess\,inf}_B w \quad \text{when } p = 1$$

for all metric balls $B \subset \Omega$. The fact that Lebesgue measure satisfies the doubling condition (1.3) allows us to develop the usual theory of such weight classes as in [Ca], at least for balls $B = B(x,r)$ with $0 < r < r_0$ and x belonging to a compact subset of Ω. It follows easily from the definition and (1.3) that if $w \in A_p$ then

$$w(B(x, 2r)) \leq Cw(B(x,r))$$

if $0 < r < r_0$ and $x \in K \subset \Omega$, K compact, with $C = C(r_0, K)$, where we use the standard notation $w(E) = \int_E w dx$. We say that any such weight is doubling. All the weights we shall consider will be doubling weights.

Given two weight functions w_1, w_2 on Ω and $1 \leq p < q < \infty$, we will assume that the following local balance condition holds for w_1, w_2 and a ball B with center in K and $r(B) < r_0$:

$$\frac{r(I)}{r(J)}\left(\frac{w_2(I)}{w_2(J)}\right)^{\frac{1}{q}} \leq c\left(\frac{w_1(I)}{w_1(J)}\right)^{\frac{1}{p}} \tag{1.5}$$

for all metric balls I, J with $I \subset J \subset B$. Note that in the case of Lebesgue measure ($w_1 = w_2 = 1$), (1.5) reduces to (1.4) when $1/q = 1/p - 1/(N\gamma)$.

Our main result of Poincaré type for $p < q$ is then as follows.

THEOREM 2. — *Let K be a compact subset of Ω. Then there exists r_0 depending on K, Ω and $\{X_j\}$ such that if $B = B(x,r)$ is a ball with $x \in K$ and $0 < r < r_0$, and if $1 \le p < q < \infty$ and w_1, w_2 are weights satisfying the balance condition (1.5) for B, with $w_1 \in A_p(\Omega, \rho, dx)$ and w_2 doubling, then*

$$\left(\frac{1}{w_2(B)}\int_B |f(x) - f_B|^q w_2(x)dx\right)^{\frac{1}{q}} \le cr\left(\frac{1}{w_1(B)}\int_B |Xf(x)|^p w_1(x)dx\right)^{\frac{1}{p}}$$

for any $f \in \mathrm{Lip}(\bar{B})$, with $f_B = w_2(B)^{-1}\int_B f(x)w_2(x)dx$. The constant c depends only on K, Ω, $\{X_j\}$ and the constants in the conditions imposed on w_1 and w_2.

This result includes Theorem 1 and the weighted results in [L1].

We note here that the proof of Theorem 2 will show that if we replace the integration over B on the right side of the conclusion by integration over a suitably enlarged ball cB, then we may also choose f_B to be $|B|^{-1}\int_B f(x)dx$. Moreover, the inequality with the enlarged ball cB on the right also holds by assuming a slightly different balance condition : see (3.1). We also remark here that in the usual Poincaré inequality for Hörmander vector fields, one can replace the average f_B by $f(x_0)$ for any fixed distinguished interior point x_0 of B if f is a solution of a certain type of degenerate subelliptic differential equation (see [L5]).

Remark 1.6. — Theorem 2 has an analogue in case $q = p$ and $1 \le p < \infty$. In fact, the theorem remains true as stated if $1 < p < \infty$ and $q = p$ provided $w_1 \in A_p$ and there exists $s > 1$ such that w_2^s is a doubling weight and the balance condition (1.5) is replaced by the condition

$$\left(\frac{r(I)}{r(J)}\right)^p \frac{\mathcal{A}_s(I,w_2)}{w_2(J)} \le c\frac{w_1(I)}{w_1(J)} \tag{1.7}$$

for all balls I, J with $I \subset J \subset B$, where

$$\mathcal{A}_s(I,w_2) = |I|\left(\frac{1}{|I|}\int_I w_2^s dx\right)^{\frac{1}{s}}.$$

Note that $w_2(I) \le \mathcal{A}_s(I,w_2)$ for $s > 1$ by Hölder's inequality, and, as is well known, $w_2(I)$ and $\mathcal{A}_s(I,w_2)$ are equivalent if w_2 belongs to some A_{p_0} class and s is sufficiently close to 1. For some discussion concerning this remark, and for a result in case $p = q = 1$, see the end of §3.

We mention in passing that it is possible to use the Poincaré estimates above to derive analogous estimates for domains other than balls. In particular, this can be done for domains which satisfy the Boman chain condition; see the end of the proof of Theorem 2 in §3 for a result of this type. In fact, the technique used for Boman domains is also needed in order to prove Theorem 2. John metric domains (see [BKL]) and bounded (ϵ,∞) metric domains (see [LW]) have been shown to be Boman chain domains, and thus Poincaré inequalities hold on such domains by the argument in this paper (see also [FGuW]). For the problem of extending Poincaré inequalities to other domains than balls, see also [CDG].

We will use the Poincaré estimates for $p = 1$ to derive analogues of the relative isoperimetric inequality. The classical relative isoperimetric inequality for a bounded open set $E \subset \mathbb{R}^N$ with sufficiently regular boundary ∂E and a Euclidean ball B is

$$\min\{|B\cap E|, |B\backslash E|\}^{1-\frac{1}{N}} \leq cH_{N-1}(B\cap\partial E),$$

where H_{N-1} denotes $(N-1)$-dimensional Hausdorff measure. This estimate is more local than the standard isoperimetric inequality $|E|^{1-\frac{1}{N}} \leq cH_{N-1}(\partial E)$. Some analogues of the standard estimate which are related to either Hörmander vector fields or vector fields of the type [FL], including weighted versions, are derived in [FGaW1], [FGaW2]. By adapting the arguments there, we will prove the following corresponding result of relative type in §4.

THEOREM 3. — *Let $\{X_j\}$ be vector fields of Hörmander type on $\Omega_0 \supset \bar{\Omega}$, and let K be a compact subset of Ω. Let w_1, w_2 be weights with w_1 continuous and in $A_1(\Omega,\rho,dx)$, and w_2 doubling. Suppose also that (1.5) holds for $p=1$ and some $q>1$ uniformly in $B=B(x,r)$ with $x\in K$ and $0<r<r_0$. Let E be an open, bounded, connected subset of Ω whose boundary ∂E is an oriented C^1 manifold such that E lies locally on one side of ∂E. If r_0 is sufficiently small and $B=B(x,r)$ is any ball with $x\in K$ and $0<r<r_0$, then*

$$\min\{w_2(B\cap E), w_2(B\backslash E)\}^{1/q} \leq c\int_{\partial E\cap B}\Big(\sum_j \langle X_j,\nu\rangle^2\Big)^{1/2} w_1 dH_{N-1},$$

where ν is the unit outer normal to ∂E, and the constants c, r_0 are independent of E and B.

In particular, in the case of Lebesgue measure, *i.e.*, in case $w_1 = w_2 = 1$, the conclusion holds with $q = Q/(Q-1)$. In any case, the assumption that (1.5) holds uniformly in B may be deleted by allowing the constant c in the conclusion to depend on the constant in (1.5).

The analogous isoperimetric result in [FGaW1], [FGaW2] amounts to the special case when E lies in the middle half of B. Theorem 3 has an analogue for the degenerate vector fields of type [FL] : see the remarks at the end of §4.

Some of the results of the present paper were announced in [FLW], where applications to Harnack's inequality for degenerate elliptic equations are given.

2. Proof of the representation formula.

We begin by briefly recalling some definitions and facts about Hörmander vector fields. For details, we refer to [NSW], [FP], [S-Cal], [RS] and [J]. Following [FP], we say that an absolutely continuous curve $\gamma : [0,T] \to \Omega$ is a sub-unit curve if

$$|\langle \gamma'(t), \xi \rangle|^2 \le \sum_j |\langle X_j(\gamma(t)), \xi \rangle|^2$$

for all $\xi \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. The metric $\rho(x,y)$ mentioned in the introduction is then defined for $x, y \in \Omega$ by

$$\rho(x,y) = \inf\{T : \exists \text{ a sub-unit curve } \gamma : [0,T] \to \Omega \text{ with } \gamma(0) = x, \gamma(T) = y\}.$$

By [NSW], Lebesgue measure satisfies the doubling condition (1.3) for ρ-balls. In fact, by the results of [NSW], we have

$$c_1 \left(\frac{r}{s}\right)^\alpha \le \frac{|B(x,r)|}{|B(x,s)|} \le c_2 \left(\frac{r}{s}\right)^\beta \tag{2.1}$$

for $x \in K$ and $0 \le s < r < r_0$ for suitable $\alpha = \alpha(x)$ and $\beta = \beta(x)$, with $N \le \alpha \le \beta$. To prove (2.1), first remember that, by the Hörmander condition, there exists a positive integer M such that, among $X_1, \ldots, X_m$ and their commutators of degree (length) less than or equal to M, we can find at least one N-tuple of vector fields which are linearly independent at x. Now define

$$\alpha = \alpha(x) = \min\{\deg Y_1 + \ldots + \deg Y_N\}$$
$$\beta = \beta(x) = \max\{\deg Y_1 + \ldots + \deg Y_N\},$$

where $\deg Y_i$ is the formal degree of Y_i (a fixed integer ≥ 1) and $\{Y_1, \ldots, Y_N\}$ ranges over all collections of N vectors chosen from $\{X_j\}$

and its commutators up to degree M such that $Y_1, \dots, Y_N$ are linearly independent at x. Clearly, $\alpha \geq N$ and $\beta \leq MN$. Actually, $\beta \leq Q$, where Q is the homogeneous dimension defined below.

By the results of [NSW], p. 110, if $x \in K$ and $0 < r < r_0$, then

$$|B(x,r)| \sim \Lambda(x,r) = \sum_I |\lambda_I(x)| r^{d(I)},$$

with constants of equivalence depending on K and r_0, where the sum is over all N-tuples $I = (i_1, \dots, i_N)$ of integers such that $Y_{i_1}, \dots, Y_{i_N}$ is a collection of N vectors chosen from $\{X_j\}$ and its commutators up to degree M,

$$\lambda_I(x) = \det(Y_{i_1}, \dots, Y_{i_N})(x),$$

and

$$d(I) = \deg Y_{i_1} + \ldots + \deg Y_{i_N}.$$

Since

$$\Lambda(x,r) = \sum_I |\lambda_I(x)| r^{d(I)} = \sum_I |\lambda_I(x)| s^{d(I)} \left(\frac{r}{s}\right)^{d(I)},$$

and since for $s < r$ and $\lambda_I(x) \neq 0$ we have

$$\left(\frac{r}{s}\right)^{\alpha} \leq \left(\frac{r}{s}\right)^{d(I)} \leq \left(\frac{r}{s}\right)^{\beta},$$

it follows that

$$\left(\frac{r}{s}\right)^{\alpha} \Lambda(x,s) \leq \Lambda(x,r) \leq \left(\frac{r}{s}\right)^{\beta} \Lambda(x,s), \ 0 < s < r,$$

which proves (2.1).

The second inequality on (2.1) leads easily to a natural choice for the local doubling order γ defined in (1.4). In fact, let $x \in K$ and $r < r_0$, and let I and J be balls satisfying $I \subset J \subset B(x,r)$. Then, assuming as we may by doubling that I and J are concentric, we have by (2.1) that

$$|J| \leq c \left(\frac{r(J)}{r(I)}\right)^{N\gamma} |I|$$

where γ is chosen so that $N\gamma = \max\{\beta(y) : y \in B(x,r)\}$.

We may adjoin new variables $(t_1, \dots, t_d) = t \in \mathbb{R}^d$ to $(x_1, \dots, x_N)$ as in [RS] and form new C^∞ vector fields $\{\tilde{X}_j\}$ in $\Omega \times \mathbb{R}^d$,

$$\langle \tilde{X}_j, \nabla_{x,t} \rangle = \langle X_j, \nabla_x \rangle + \sum_{l=1}^{d} a_{jl}(x,t) c \frac{\partial}{\partial t_l}, \ j = 1, \dots, m,$$

so that the new vector fields $\{\tilde{X}_j\}_{j=1}^m$ together with their commutators $\{\tilde{X}_\alpha\}_{|\alpha| \leq M}$ of length at most M span the tangent space in $\mathbb{R}^{N+d}$ at each point of $\Omega_0 \times \mathbb{R}^d$, and are also *free* of order M, *i.e.*, the commutators of length at most M satisfy no linear relationships other than antisymmetry and the Jacobi identity. The collection $\{\tilde{X}_j\}_{j=1}^m$ is referred to as the lifted or free vector fields. If m_i denotes the number of linearly independent commutators of length i (the length of each $\tilde{X}_j$ itself being 1), then the number

$$Q = \sum_{i=1}^{M} i m_i$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{X_j\}$. In what follows, we set $\tilde{\Omega} = \Omega \times U_0$ where U_0 is the unit ball in $\mathbb{R}^d$, and we denote by $\tilde{\rho}$ the metric on $\tilde{\Omega} \times \tilde{\Omega}$ associated with the lifted vector fields $\tilde{X}_1, \dots, \tilde{X}_m$. The corresponding metric balls will be denoted $\tilde{B} = \tilde{B}(\xi, r)$. Given a compact set $K \subset \Omega$ and $r_0 > 0$, we have

$$|\tilde{B}(\xi, r)| \approx r^Q \tag{2.2}$$

with constants of equivalence independent of $\xi \in K \times U_0$ and $0 < r < r_0$.

We will also use the following basic facts :

$$\tilde{\rho}((x,s),(y,t)) \geq \rho(x,y) \tag{2.3}$$

and

$$\int_{\mathbb{R}^d} \chi_{\tilde{B}((x,0),r)}(y,t)\,dt \leq c\,\frac{|\tilde{B}((x,0),r)|}{|B(x,r)|} \tag{2.4}$$

provided $x \in K$ and $0 < r < r_0$; see Lemmas 3.1 and 3.2 of [NSW] (see also Lemma 4.4 of [J] for a result about the inequality opposite to (2.4)).

As a first step in deriving the representation formula, we now prove the following pointwise estimate for the lifted vector fields $\{\tilde{X}_j\}$.

LEMMA 2.5. — *Let $\tilde{K}$ be a compact subset of $\tilde{\Omega}$ and $\tilde{B} = \tilde{B}(\xi_0, r)$ with $\xi_0 \in \tilde{K}$, $0 < r < r_0$. Then there are constants $c, c_{\tilde{B}}$ such that*

$$|f(\xi) - c_{\tilde{B}}| \leq c \int_{c\tilde{B}} \frac{(|\tilde{X}f| + |f|)(\eta)}{\tilde{\rho}(\xi,\eta)^{Q-1}}\,d\eta, \quad \xi \in \tilde{B},$$

for any $f \in \mathrm{Lip}(\overline{c\tilde{B}})$, where c is independent of f and $\tilde{B}$, and $|\tilde{X}f|^2 = \sum_j \langle \tilde{X}_j, \nabla f\rangle^2$.

This lemma improves Lemma 3.2 in [L1] by replacing the term $M[(|\tilde{X}f| + |f|)\chi_{c\tilde{B}}]$ in the fractional integral there by $|\tilde{X}f| + |f|$, where M is the Hardy-Littlewood maximal operator. The proof of Lemma 2.5

will be a modification of the one in [L1], and due to the complexity of notation, we will only point out the changes that are needed in the proof there.

Proof of Lemma 2.5. — For simplicity, we will delete the tildas from the notations $\tilde{X}_j$, $\tilde{p}$, $\tilde{B}$. It is shown in [L1], p. 384-388, that

$$|M_1h(\xi) - f(\xi)| \leq C \int_{cB} \frac{(|Xf| + |f|)(\eta)}{\rho(\xi,\eta)^{Q-1}} d\eta \tag{2.6}$$

for all $\xi \in B = B(\xi_0, r)$. We need to show that for some constant c_B, $|M_1h(\xi) - c_B|$ is also bounded by the quantity on the right of (2.6).

We also have

$$|X_iM_1h(\eta)| \leq Cr^{-Q} \int_{\rho(\zeta,\eta)\leq cr} (|X_ih| + |h|)(\zeta)d\zeta$$

and

$$|E_i^{\xi_0,r}h(\eta)| \leq Cr^{-Q} \int_{\rho(\zeta,\eta)\leq cr} |h(\zeta)|d\zeta$$

by pp. 387 and 388, respectively, of [L1]. Now, unlike what is done in [L1], we keep the two expressions on the right above rather than bounding them by maximal functions. As in [L1], we then obtain for $\xi \in B$,

$$\begin{aligned} &|M_1h(\xi) - C_1| \\ &\leq C \int_{\rho(\xi_0,\eta)<cr} \frac{r[r^{-Q}\int_{\rho(\zeta,\eta)\leq cr}(|Xh| + |h|)(\zeta)d\zeta]}{r^{-Q+1}\rho(\xi,\eta)^{Q-1}} r^{-Q} d\eta \\ &\leq C \int_{\rho(\xi_0,\zeta)\leq cr} r^{-Q} \left(\int_{\rho(\xi_0,\eta)\leq cr} \frac{d\eta}{\rho(\xi,\eta)^{Q-1}} \right) (|Xh| + |h|)(\zeta)d\zeta. \end{aligned}$$

A simple computation based on (2.2) gives

$$\int_{\rho(\xi_0,\eta)\leq cr} \frac{d\eta}{\rho(\xi,\eta)^{Q-1}} \leq Cr$$

uniformly for $\xi \in B$. Thus,

$$\begin{aligned} |M_1h(\xi) - C_1| &\leq Cr^{1-Q} \int_{cB} (|Xh| + |h|)d\zeta \\ &\leq C \int_{cB} \frac{(|Xh| + |h|)(\zeta)}{\rho(\xi,\zeta)^{Q-1}} d\zeta \end{aligned}$$

since $\rho(\xi,\zeta) \leq cr$. Therefore, since $|Xh| \leq C(|Xf| + |f|)$ and $|h| \leq C|f|$, and by (2.6),

$$|f(\xi) - C_1| \leq C \int_{cB} \frac{(|Xf| + |f|)(\eta)}{\rho(\xi,\eta)^{Q-1}} d\eta, \ \xi \in B,$$

which proves the lemma.

The next lemma concerns the original vector fields.

LEMMA 2.7. — *Let K be a compact subset of Ω and $B = B(x_0, r)$ be a ρ-ball with $x_0 \in K$ and $0 < r < r_0$. Then*

$$|f(x) - f_B| \le c \int_{cB} (|Xf| + |f|)(y) \frac{\rho(x,y)}{|B(x, \rho(x,y))|} dy, \ x \in B,$$

for any $f \in \mathrm{Lip}(\overline{cB})$, where c is independent of f and B, and $f_B = |B|^{-1} \int_B f(y) dy$.

Proof. — We first show that the conclusion holds with f_B replaced by some constant c_B. We will deduce this from Lemma 2.5 by using the same sort of argument as in [RS] and [NSW] (see, in particular, Theorem 5 of [NSW]).

Let $B = B(x_0, r)$ and $\tilde{B} = \tilde{B}(\xi_0, r)$ where $\xi_0 = (x_0, 0)$. Note that $\tilde{B} \subset B \times \mathbb{R}^d$ by (2.3). Extend f to the closure of $c\tilde{B}$ by making it constant in t, *i.e.*, if $\xi = (x,t)$ then $f(\xi) = f(x)$. Then by Lemma 2.5, since $\tilde{X} f(\eta) = Xf(y)$ if $\eta = (y,t)$,

$$\begin{aligned}
|f(x) - c_{\tilde{B}}| &\le c \int_{c\tilde{B}} \frac{(|Xf| + |f|)(y)}{\tilde{\rho}((x,0),(y,t))^{Q-1}} \, dydt, \ x \in B \\
&= c \int_{\mathbb{R}^N} (|Xf| + |f|)(y) \left\{ \int_{\mathbb{R}^d} \chi_{c\tilde{B}}(y,t) \frac{dt}{\tilde{\rho}((x,0),(y,t))^{Q-1}} \right\} dy \\
&\le c \int_{cB} (|Xf| + |f|)(y) \left\{ \int_{\mathbb{R}^d} \frac{dt}{\tilde{\rho}((x,0),(y,t))^{Q-1}} \right\} dy.
\end{aligned}$$

Momentarily fix y and let $\rho = \rho(x,y)$. Since $\tilde{\rho}((x,0),(y,t)) \ge \rho(x,y)$ by (2.3), we have the following estimate for the inner integral :

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{dt}{\tilde{\rho}((x,0),(y,t))^{Q-1}} \le \sum_{k=0}^{\infty} \frac{1}{(2^k \rho)^{Q-1}} \int_{2^k \rho \le \tilde{\rho}((x,0),(y,t)) \le 2^{k+1}\rho} dt \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k\rho)^{Q-1}} \int_{\mathbb{R}^d} \chi_{\tilde{B}((x,0),2^{k+1}\rho)}(y,t) dt \\
&\le c \sum_{k=0}^{\infty} \frac{1}{(2^k\rho)^{Q-1}} \frac{(2^{k+1}\rho)^Q}{|B(x, 2^{k+1}\rho)|} \quad \text{by (2.4) and (2.2)} \\
&\le c \sum_{k=0}^{\infty} \frac{2^k \rho}{|B(x, 2^{k+1}\rho)|} \le c \sum_{k=0}^{\infty} 2^{k(1-N)} \frac{\rho}{|B(x,\rho)|} \quad \text{by (2.1) with } \alpha = N. \\
&= c \frac{\rho}{|B(x,\rho)|} \quad \text{since } N > 1.
\end{aligned}$$

Hence with $c_B = c_{\tilde{B}}$, we have

$$|f(x) - c_B| \leq c \int_{cB} (|Xf| + |f|)(y) \frac{\rho(x,y)}{|B(x,\rho(x,y))|} dy, \ x \in B. \tag{2.8}$$

It remains to show that c_B can be taken to be $f_B = |B|^{-1} \int_B f(z)dz$. In fact,

$$\begin{aligned} |f_B - c_B| &\leq \frac{1}{|B|} \int_B |f(z) - c_B| dz \\ &\leq \frac{1}{|B|} \int_B \left\{ C \int_{cB} (|Xf| + |f|)(y) \frac{\rho(z,y)}{|B(z,\rho(z,y))|} dy \right\} dz \text{ by (2.8)} \\ &\leq C \int_{cB} (|Xf| + |f|)(y) \left\{ \frac{1}{|B|} \int_B \frac{\rho(z,y)}{|B(y,\rho(z,y))|} dz \right\} dy \end{aligned}$$

since $|B(y,\rho(z,y)|$ and $|B(z,\rho(z,y)|$ are equivalent by doubling. Hence, it is enough to show the following condition of A_1-type :

$$\frac{1}{|B|} \int_B \frac{\rho(z,y)}{|B(y,\rho(z,y))|} dz \leq c \frac{\rho(x,y)}{|B(y,\rho(x,y))|}, \ x, y \in cB. \tag{2.9}$$

To prove (2.9), fix $y \in cB$. Since $\rho(z,y) \leq cr(B)$ for $z \in B$, the expression on the left in (2.9) is at most

$$\begin{aligned} &\frac{1}{|B|} \sum_{k=0}^{\infty} \left(\int_{c2^{-k}r(B) \leq \rho(z,y) \leq c2^{1-k}r(B)} dz \right) \frac{c2^{1-k}r(B)}{|B(y, c2^{-k}r(B))|} \\ \leq &\frac{1}{|B|} \sum_{k=0}^{\infty} \frac{|B(y, c2^{1-k}r(B))|}{|B(y, c2^{-k}r(B))|} c2^{1-k}r(B) = c\frac{r(B)}{|B|} \end{aligned}$$

by doubling. However, if $x \in cB$ (and $y \in cB$) then $\rho(x,y)$ is at most $cr(B)$, and by (2.1) with $\alpha = N$,

$$\begin{aligned} \frac{|B(x,\rho(x,y))|}{\rho(x,y)} &\leq c \left(\frac{\rho(x,y)}{r(B)} \right)^{N-1} \frac{|B(x,r(B))|}{r(B)} \\ &\leq c \frac{|B(x,r(B))|}{r(B)} \quad \text{since } N > 1 \\ &\leq c \frac{|B|}{r(B)} \quad \text{by doubling.} \end{aligned} \tag{2.10}$$

We obtain (2.9) by combining estimates, and the proof of Lemma 2.7 is complete. □

In the following result, we use Lemma 2.7 to obtain the basic estimate (1.2) as well as a more local estimate. A similar argument given in [SW] (see also [FGuW]) needs modification due to the presence of the zero order term $|f|$ in the integral in the conclusion of Lemma 2.7. We will use the notation

$$Tf(x) = \int_\Omega f(y)\frac{\rho(x,y)}{|B(x,\rho(x,y))|}dy, \ x \in \Omega, \tag{2.11}$$

for the fractional integral transform on Ω.

PROPOSITION 2.12. — *Let K be a compact subset of Ω, and $B = B(x_0, r)$, $x_0 \in K$, $0 < r < r_0$. There are positive constants C, c, and r_0 depending only on K, Ω and $\{X_j\}$ such that if $k = 0, \pm 1, \pm 2, \dots$, and S_k, S_k^* are defined by*

$$S_k = \{x \in B : 2^k < |f(x) - f_B| \le 2^{k+1}\},$$
$$S_k^* = \{x \in cB : 2^k < |f(x) - f_B| \le 2^{k+1}\},$$

then

$$|f(x) - f_B| \le CT(|Xf|\chi_{S_{k-1}^*})(x) + C\frac{r}{|B|}\int_B |Xf|dy$$

for all $x \in S_k$ and all $f \in \mathrm{Lip}(\overline{cB})$. Moreover,

$$|f(x) - f_B| \le CT(|Xf|\chi_{cB})(x), \ x \in B.$$

Proof. — For $x \in \Omega$, define

$$f_k(x) = \begin{cases} 2^{k-1} & \text{if } \ |f(x) - f_B| \le 2^{k-1} \\ |f(x) - f_B| & \text{if } \ 2^{k-1} < |f(x) - f_B| < 2^k \\ 2^k & \text{if } \ |f(x) - f_B| \ge 2^k. \end{cases}$$

Then $2^{k-1} \le f_k(x) \le 2^{k-1} + |f(x) - f_B|$. Thus if $x \in S_k$,

$$\begin{aligned} 2^k = f_k(x) &\le |f_k(x) - (f_k)_B| + (f_k)_B \\ &\le CT([|Xf_k| + f_k]\chi_{cB})(x) + 2^{k-1} + \frac{1}{|B|}\int_B |f - f_B|dz \end{aligned}$$

by Lemma 2.7

$$\le CT(|Xf|\chi_{S_{k-1}^*})(x) + CT(f_k\chi_{cB})(x) + 2^{k-1} + \frac{1}{|B|}\int_B |f - f_B|dz$$

since $|Xf_k|\chi_{cB} \le |Xf|\chi_{S_{k-1}^*}$.

Since $2^{k-1} \le f_k \le 2^k$,

$$T(f_k\chi_{cB})(x) \le 2^k \int_{cB} \frac{\rho(x,y)}{|B(x,\rho(x,y))|}dy \le Cr2^k$$

where $r = r(B)$ (*cf.* (2.9)). By applying the known Poincaré inequality for Lebesgue measure and $p = q = 1$ (see for example [J]), we obtain

$$\frac{1}{|B|}\int_B |f(z) - f_B|dz \le C\frac{r}{|B|}\int_B |Xf(y)|dy.$$

Combining estimates, we have for $x \in S_k$ that

$$2^k \leq CT(|Xf|\chi_{S^*_{k-1}})(x) + Cr2^k + 2^{k-1} + C\frac{r}{|B|}\int_B |Xf|dy.$$

Since $2^{k-1} = \frac{1}{2}2^k$ and $Cr2^k < \frac{1}{3}2^k$ when r is small (independent of k), we obtain by subtracting that

$$2^k \leq CT(|Xf|\chi_{S^*_{k-1}})(x) + C\frac{r}{|B|}\int_B |Xf|dy, \ x \in S_k,$$

for small r. The first statement in Proposition 2.12 follows from this since $|f(x) - f_B| \leq 2^{k+1}$ for $x \in S_k$. The second statement in the proposition (which is (1.2)) follows from the first one by simply noting that $T(|Xf|\chi_{S^*_{k-1}}) \leq T(|Xf|\chi_{cB})$ and (by (2.10))

$$\begin{aligned}\frac{r}{|B|}\int_B |Xf|dy &\leq C\int_B |Xf(y)|\frac{\rho(x,y)}{|B(x,\rho(x,y))|}dy, \ x \in B,\\ &\leq CT(|Xf|\chi_{cB})(x), \ x \in B.\end{aligned}$$

This completes the proof of Proposition 2.12. □

3. Proof of the Poincaré estimates.

As noted in the introduction, Theorem 1 is a special case of Theorem 2. To prove Theorem 2, we first derive a weaker version in which the domain of integration on the right side of the Poincaré inequality is an enlarged ball cB, $c > 1$, rather than B. In order to prove this weaker version for a given B, we will use the following slightly different form of the balance condition (1.5) :

$$\frac{r(I)}{r(B)}\left(\frac{w_2(I)}{w_2(B)}\right)^{\frac{1}{q}} \leq c\left(\frac{w_1(I)}{w_1(B)}\right)^{\frac{1}{p}} \tag{3.1}$$

for all balls I with center in cB and $r(I) \leq r(B)$. The restriction $r(I) \leq r(B)$ may be replaced by $r(I) \leq cr(B)$ by doubling. Theorem 2 itself will follow from its weaker version for the same values of p and q by the results in §5 of [FGuW]. Some further comments about how to do this, including an indication of how (3.1) is used in conjunction with (1.5), are given at the end of the proof of Theorem 2. In fact, by a similar method, it is possible to prove a version of Theorem 2 for domains other than balls, as mentioned in the introduction.

Using Proposition 2.12, we will be able to derive the weaker version of Theorem 2 by the sort of argument used in [SW], including the adaptation of this argument to the case $p = 1$ given in [FGuW]. We need the following estimate for Tf which is essentially a special case of Theorem 4.1 (and Remark 4.3) of [FGuW] (see also [GGK] and [SW]).

LEMMA 3.2. — *Let $1 \le p < q < \infty$, K be a compact subset of Ω, $B = B(x_0, r)$, $x_0 \in K$, and*

$$T_B f(x) = \int_B f(y) \frac{\rho(x,y)}{|B(x,\rho(x,y))|} dy$$

for $x \in B$ and $f \ge 0$. There are constants r_0 and C depending only on K, Ω and $\{X_j\}$ such that if $r < r_0$ and w_1, w_2 are nonnegative weights then

$$\int_{B \cap \{T_B f > t\}} w_2 dx \le (CL\|f\|_{L^p_{w_1}(B)}/t)^q, \ t > 0,$$

where

$$L = \begin{cases} \sup \ w_2(B(x,s))^{1/q} \left(\int_B k_s(x,y)^{p'} w_1(y)^{-p'/p} dy \right)^{1/p'} & \text{if } p > 1 \\ \sup \ w_2(B(x,s))^{1/q} \left(\operatorname*{ess\ sup}_{y \in B} [k_s(x,y)/w_1(y)] \right) & \text{if } p = 1. \end{cases}$$

Here,

$$k_s(x,y) = \min\left\{ \frac{s}{|B(x,s)|}, \frac{\rho(x,y)}{|B(x,\rho(x,y))|} \right\},$$

$p' = p/(p-1)$, and the sup *is taken over all x and s with $x \in B$ and $B(x,s) \subset 5B$.*

We now prove the version of Theorem 2 with an enlarged ball cB on the right. Let B be a ball of radius r for which the conclusion of Proposition 2.12 is valid. Then

(3.3)

$$\int_B |f(x) - f_B|^q w_2(x) dx = \int_{B \cap \{|f - f_B| \le 2^{M+1}\}} \cdots + \int_{B \cap \{|f - f_B| > 2^{M+1}\}} \cdots,$$

where M is selected so that

$$2^{M-1} < C \frac{r}{|B|} \int_B |Xf| dx \le 2^M,$$

C being the same constant which appears in the second term on the right of the conclusion of Proposition 2.12. With S_k as defined there, the right side of (3.3) is bounded by

$$2^{(M+1)q} w_2(B) + \sum_{k \ge M+1} \int_{S_k} \cdots \le 2^{(M+1)q} w_2(B) + \sum_{k \ge M+1} 2^{(k+1)q} w_2(S_k).$$

For $k \geq M+1$, it follows from the choice of M and Proposition 2.12 (with T and S_k^* as defined there) that

$$S_k \subset \{x \in B : T(|Xf|\chi_{S_{k-1}^*})(x) > 2^{k-1}/C\}.$$

Set $A = \sup\ tw_2(B \cap \{Tf > t\})^{1/q}$ where the sup is taken over all $t > 0$ and all $f \geq 0$ with $\operatorname{supp}(f) \subset cB$ and $\|f\|_{L^p_{w_1}} \leq 1$. Then

$$w_2(S_k) \leq A^q \left\{ \frac{2^{k-1}/C}{\||Xf|\chi_{S_{k-1}^*}\|_{L^p_{w_1}}} \right\}^{-q}.$$

Therefore, (3.3) is bounded by

$$\begin{aligned}
&2^{(M+1)q} w_2(B) + \sum_{k \geq M+1} 2^{(k+1)q} A^q \left\{ C\||Xf|\chi_{S_{k-1}^*}\|_{L^p_{w_1}}/2^{k-1} \right\}^q \\
&\leq 2^{(M+1)q} w_2(B) + (4CA)^q \left(\sum_{k \geq M+1} \int_{S_{k-1}^*} |Xf|^p w_1 dx \right)^{q/p} \qquad \text{since } q \geq p \\
&\leq (4C)^q w_2(B) \left(\frac{r}{|B|} \int_B |Xf| dx \right)^q + (4CA)^q \left(\int_{cB} |Xf|^p w_1 dx \right)^{q/p}
\end{aligned}$$

by definition of M and since the S_{k-1}^* are disjoint. Dividing by $w_2(B)$ and taking the q^{th} root, we obtain

$$\begin{aligned}
&\left(\frac{1}{w_2(B)} \int_B |f(x) - f_B|^q w_2 dx \right)^{1/q} \\
&\leq 8C \frac{r}{|B|} \int_B |Xf| dx + \frac{8CA}{w_2(B)^{1/q}} \left(\int_{cB} |Xf|^p w_1 dx \right)^{1/p}.
\end{aligned} \tag{3.4}$$

In case $p = 1$, the fact that $w_1 \in A_1$ implies that the first term on the right of (3.4) is bounded by a multiple of $(r/w_1(B)) \int_B |Xf| w_1 dx$. On the other hand, if $p > 1$, by Hölder's inequality and the fact that $w_1 \in A_p$, we have

$$\begin{aligned}
\frac{r}{|B|} \int_B |Xf| dx &\leq \frac{r}{|B|} \left(\int_B w_1^{-1(p-1)} dx \right)^{(p-1)/p} \left(\int_B |Xf|^p w_1 dx \right)^{1/p} \\
&\leq cr \left(\frac{1}{w_1(B)} \int_B |Xf|^p w_1 d\xi \right)^{1/p}.
\end{aligned}$$

Thus, in any case,

$$\begin{aligned}
&\left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2 dx \right)^{1/q} \\
&\leq cr \left[1 + \frac{A w_1(B)^{1/p}}{r w_2(B)^{1/q}} \right] \left(\frac{1}{w_1(B)} \int_{cB} |Xf|^p w_1 dx \right)^{1/p}.
\end{aligned} \tag{3.5}$$

We now show by using Lemma 3.2 that the term $Aw_1(B)^{1/p}/rw_2(B)^{1/q}$ which appears on the right in (3.5) is bounded. Consider first the case $p = 1$. By the definition of A, Lemma 3.2 applied to the ball $B_0 = cB$, and doubling, it is enough to show that if $x \in B_0$ and $s \leq cr(B_0)$, then

$$w_2(B(x,s))^{1/q} \operatorname*{ess\,sup}_{y \in B_0} \left[k_s(x,y) \frac{1}{w_1(y)} \right] \tag{3.6}$$

is bounded by a multiple of $r(B_0)w_2(B_0)^{1/q}/w_1(B_0)$. Indeed, the assertion then follows by doubling since $B_0 = cB$. Let $x, y \in B_0$ and suppose that $y \in B(x, 2^{k+1}s) \setminus B(x, 2^k s)$ for some $k = 0, 1, \ldots$. (The argument for the remaining case when $y \in B(x,s)$ will be similar but simpler.) We may assume that $2^k s \leq cr(B_0)$ since all points lie in B_0. Then

$$k_s(x,y)\frac{1}{w_1(y)} \leq c\frac{2^k s}{|B(x,2^k s)|}\frac{1}{w_1(y)}$$

by definition of $k_s(x,y)$ and (2.1) (with $\alpha = N > 1$)

$$\leq c\frac{2^k s}{|B(x,2^k s)|}\frac{|B(x,2^{k+1}s)|}{w_1(B(x,2^{k+1}s))}$$

for a.e. $y \in B(x, 2^{k+1}s)$ by the A_1 estimate on w_1

$$\leq c\frac{2^{ks}}{w_1(B(x,2^k s))} \quad \text{by doubling.}$$

This estimate is also valid with $k = 0$ in case $y \in B(x,s)$. Multiplying both sides by $w_2(B(x,s))^{1/q}$ and using the balance condition (3.1) in the form

$$\frac{2^k s}{r(B_0)}\left(\frac{w(B_2(x,2^k s))}{w_2(B_0)}\right)^{1/q} \leq c\frac{w_1(B(x,2^k s))}{w_1(B_0)}$$

(recall that $p = 1$ and $2^k s \leq cr(B_0)$) and the doubling property of the weights, we see that (3.6) is at most a multiple of

$$\frac{r(B_0)w_2(B_0)^{1/q}}{w_1(B_0)}\left\{\sup_{k \geq 0}\frac{w_2(B(x,s))}{w_2(B(x,2^k s))}\right\}^{1/q} = \frac{r(B_0)w_2(B_0)^{1/q}}{w_1(B_0)},$$

as desired.

In case $p > 1$, with the same notation as above, if $x \in B_0$ and $s \leq cr(B_0)$,

$$\int_{B_0} k_s(x,y)^{p'} w_1(y)^{-p'/p} dy$$

$$\leq \sum_{\substack{k \geq 0 \\ B(x,2^k s) \subset cB_0}} \left(\frac{2^k s}{|B(x,2^k s)|}\right)^{p'} \int_{B(x,2^{k+1}s)} w_1^{-p'/p} dy$$

$$\leq C \sum_{\substack{k \geq 0 \\ 2^k s \leq cr(B_0)}} \left[\frac{2^k s}{w_1(B(x,2^k s))^{1/p}}\right]^{p'}$$

since $w_1 \in A_p$ and by doubling

$$\leq C\left[\frac{r(B_0)w_2(B_0)^{1/q}}{w_1(B_0)^{1/p}}\right] \sum_{\substack{k\geq 0 \\ 2^k s \leq cr(B_0)}} w_1(B(x,2^k s))^{-p'/q}$$

by (3.1)

$$\leq C\left[\frac{r(B_0)w_2(B_0)^{1/q}}{w_1(B_0)^{1/p}}\right] \frac{1}{w_2(B(x,s))^{p'/q}}$$

since the reverse doubling condition $w_2(B(x,2s)) \geq \gamma w_2(B(x,s))$ for some $\gamma > 1$ implies that the last sum is at most $\sum_{k\geq 0} \{\gamma^k w_2(B(x,s))\}^{-p'/q} = cw_2(B(x,s))^{-p'/q}$. This form of the reverse doubling condition follows easily from the doubling condition in any metric space in which annuli are not empty (see [W], (3.21); in fact, the value of α there can be chosen to be 2 in the case of a metric space, as the argument shows). If we combine the estimate above with Lemma 3.2 and the definition of A, we obtain the desired estimate for A.

Thus it follows by (3.5) that

$$\left(\frac{1}{w_2(B)}\int_B |f-f_B|^q w_2 dx\right)^{1/q} \leq Cr\left(\frac{1}{w_1(B)}\int_{cB} |Xf|^p w_1 dx\right)^{1/p},$$

which is the weaker version of Poincaré's inequality. Note that the constant f_B in this weaker version can be taken to be the Lebesgue average of f over B.

As mentioned earlier, this weaker version leads to Theorem 2 itself by using the results in section 5 of [FGuW], and we now briefly outline those results, which hold in a more general context. If (S,d) is a metric space, we say that an open set $D \subset S$ satisfies the Boman chain condition $\mathcal{F}(\tau, M)$, $\tau \geq 1$, $M \geq 1$, if there exists a covering W of D consisting of balls B such that

(i) $\sum_{B\in W} \chi_{\tau B}(x) \leq M\chi_D(x)$ for all $x \in S$.

(ii) There is a "central" ball $B_1 \in W$ which can be connected to every ball $B \in W$ by a finite chain of balls $B_1, \ldots, B_{\ell(B)} = B$ of W so that $B \subset MB_j$ for $j = 1, \ldots, \ell(B)$. Moreover, $B_j \cap B_{j-1}$ contains a ball R_j such that $B_j \cup B_{j-1} \subset MR_j$ for $j = 2, \ldots, \ell(B)$.

We then have the following result.

THEOREM 3.7. — *Let τ, $M \geq 1$, $1 \leq p \leq q < \infty$ and D satisfy the Boman chain condition $\mathcal{F}(\tau, M)$ in a metric space (S,d). Also, let μ and ν*

be Borel measures and μ be doubling. Suppose that f and g are measurable functions on D and for each ball B with $\tau B \subset D$ there exists a constant f_B such that

$$\|f - f_B\|_{L^q_{d\mu}(B)} \leq A\|g\|_{L^p_{d\nu}(\tau B)}$$

with A independent of B. Then there is a constant f_D such that

$$\|f - f_D\|_{L^q_{d\mu}(D)} \leq cA\|g\|_{L^p_{d\nu}(D)}$$

where c depends only on τ, M, q and μ. Moreover, we may choose $f_D = f_{B_1}$ where B_1 is a central ball for D.

The proof of Theorem 3.7 consists simply of adapting the argument given in [Ch] in case $S = \mathbb{R}^n$ and $d(x, y) = |x - y|$. The result also holds in d is merely quasimetric. See the remarks following Theorem 5.2 of [FGuW], and see [Bo] and [IN] for earlier basic results.

For the next result, we impose the following "segment" (geodesic) property for a ball B_0 in the metric space (S, d) :

$$\text{(3.8)}\qquad \begin{aligned} &\text{If } B \text{ is a ball contained in } B_0 \text{ with center } x_B,\\ &\text{then for each } x \in B \text{ there is a continuous one-to-one curve}\\ &\gamma = \gamma_{x_B,x}(t),\ 0 \leq t \leq 1, \text{ in } B\\ &\text{with } \gamma(0) = x_B,\ \gamma(1) = x \text{ and } d(x_B, z) = d(x_B, y) + d(y, z)\\ &\text{for all } y, z \in \gamma \text{ with } y = \gamma(s),\ z = \gamma(t) \text{ and } 0 \leq s \leq t \leq 1. \end{aligned}$$

We then have

THEOREM 3.9. — *Let (S, d) be a locally compact metric space, B_0 be an open ball in S which satisfies condition (3.8), and μ be a doubling measure on B_0. Then B_0 satisfies the Boman chain condition $\mathcal{F}(\tau, M)$ for any given τ with M depending only on τ and the doubling constant of μ.*

For a proof, see the proof of Theorem 5.4 of [FGuW].

Let us now indicate how to use Theorems 3.7 and 3.9 to complete the proof of Theorem 2. Fix a ball $B = B(x, r)$ with $x \in K$ and $0 < r < r_0$. Condition (1.5) for B (together with the doubling property of the weights) clearly implies that

$$\frac{r(I)}{r(J)} \left(\frac{w_2(I)}{w_2(J)}\right)^{\frac{1}{q}} \leq C \left(\frac{w_1(I)}{w_1(J)}\right)^{\frac{1}{p}}$$

for all I with center in cJ and $r(I) \leq r(J)$ provided J is any ball with the property that $\tau J \subset B$ for a suitably large constant τ depending on c.

Since this is just condition (3.1) for J, we may apply the weaker version of Theorem 2 to any such J, obtaining

$$\left(\frac{1}{w_2(J)}\int_J |f - f_J|^q w_2 dx\right)^{\frac{1}{q}} \le Cr(J)\left(\frac{1}{w_1(J)}\int_{cJ} |Xf|^p w_1 dx\right)^{\frac{1}{p}}.$$

We also have

$$\frac{w_2(J)^{1/q} r(J)}{w_1(J)^{1/p}} \le C\frac{w_2(B)^{1/q} r(B)}{w_1(B)^{1/p}}$$

for all such J by (1.5). Moreover, condition (3.8) holds for B. In fact, (3.8) holds for balls in complete metric spaces of homogeneous type for which the metric is the infimum of the lengths of the curves between two points (a length space in the sense of [G]); indeed, this follows from [Bu] since any complete metric space of homogeneous type is locally compact. The conclusion of Theorem 2 now follows from Theorems 3.9 and 3.7, applied to B. Note that since the constant f_B in the weaker version of Theorem 2 can be chosen to be the Lebesgue average of f over B, it follows from Theorem 3.7 that the constant f_B in Theorem 2 can be chosen to be the Lebesgue average of f over a central sub-ball in B. By a standard argument, f_B can also be taken to be a $w_2(B)^{-1}\int_B f w_2 dx$.

In passing, we note that the argument used above in order to obtain Theorem 2 from its weaker version can be adapted to derive an analogue of Theorem 2 for suitable Boman domains. In fact, let D be a Boman domain of type $\mathcal{F}(\tau, M)$, and let B_1 be a central ball for D. By definition, each ball J in the covering W of D satisfies $\tau J \subset D$. Define

$$A = \sup_{\substack{I,J:\\ I\subset J, \tau J\subset D}} \left\{\frac{r(I)}{r(J)}\left(\frac{w_2(I)}{w_2(J)}\right)^{\frac{1}{q}}\left(\frac{w_1(I)}{w_1(J)}\right)^{-\frac{1}{p}}\right\}.$$

If $\bar{D}$ is a compact subset of Ω with small diameter, and if τ is sufficiently large, it follows from the argument used above that

$$\|f - f_D\|_{L^q_{w_2}(D)} \le cA\|Xf\|_{L^p_{w_1}(D)},$$

with f_D equal to the Lebesgue average of f over B_1.

The verification of the result for $p = q$, $1 < p < \infty$, which is mentioned in Remark 1.6 is analogous to that of Theorem II in [FGuW] and can be derived directly from the strong type estimates for T given in Theorem 3(a) of [SW], using only the representation (1.2) rather than the more local version. Actually, all the cases of Theorem 2 except the case $p = 1$ can also be derived in this way. We omit the details.

Finally, in case $p = q = 1$, if we assume that w_2 is doubling and w_1, w_2 satisfy the condition

$$\frac{1}{w_2(I)}\int_I \frac{\rho(x,y)}{|B(y,\rho(x,y))|}w_2(x)dx \le c\frac{r(I)}{w_1(I)}w_1(y) \text{ a.e. in } I \tag{3.10}$$

for all balls $I \subset B$, then the conclusion of Theorem 2 holds with $p = q = 1$. In fact, by Proposition 2.12 and Fubini's theorem, if J is a ball with $cJ \subset B$, $c > 1$, then

$$\begin{aligned}\frac{1}{w_2(J)}\int_J |f(x) - f_J| w_2(x)dx &\\ &\le c\frac{1}{w_2(J)}\int_{cJ}|Xf(y)|\left(\int_J \frac{\rho(x,y)}{|B(x,\rho(x,y))|}w_2(x)dx\right)dy\\ &\le c\frac{r(J)}{w_1(J)}\int_{cJ}|Xf(y)|w_1(y)dy\end{aligned}$$

by (3.10) with $I = cJ$ (note that $|B(x,\rho(x,y))| \sim |B(y,\rho(x,y))|$). Theorems 3.7 and 3.9 then show as before that

$$\frac{1}{w_2(B)}\int_B |f - c_B|w_2 dx \le c\frac{r(B)}{w_1(B)}\int_B |Xf|w_1 dx$$

for some constant c_B, and as usual c_B can be taken to be $w_2(B)^{-1}\int_B f w_2 dx$.

We would like to point out a misstatement in Lemma (6.1) of [L1, page 395]. It should instead be stated there that a condition like (3.1) above with (p, q) leads to the two-weighted Poincaré inequality in Lemma (6.1) with (p, q_0) for some $q_0 < q$, instead of with $q_0 = q$ as stated. This can easily be proved by using Lemma (6.9) in [L1]. The difficulty with the proof as given in [L1] comes in the passage from the balance condition (3.1) for the original vector fields to the one for the lifted vector fields. However, such a loss in q in Lemma (6.1) does no harm in deriving Theorem A in [L1]. On the other hand, by using the new representation formula in Proposition 2.12 above for the original vector fields, we have avoided such an argument altogether and proved the weighted Poincaré inequality for the same value of q that appears in the balance condition.

4. Isoperimetric results.

In this section, we prove Theorem 3 and briefly discuss some of its variants, including an analogue for the degenerate vector fields of type [F], [FGuW].

Let E be an open, bounded, connected set in Ω whose boundary ∂E is an oriented C^1 manifold such that E lies locally on one side of its boundary, *i.e.*, assume that for any $x \in \partial E$ there is a neighborhood O of x in $\mathbb{R}^N$ and a C^1 diffeomorphism $\varphi : O \to \varphi(O) \subset \mathbb{R}^n$ such that

$$\begin{aligned} \varphi(O \cap \partial E) &= \{y \in \varphi(O) : y_N = 0\} \\ \varphi(O \cap E) &= \{y \in \varphi(O) : y_N < 0\} \\ \varphi(O \backslash \bar{E}) &= \{y \in \varphi(O) : y_N > 0\}. \end{aligned} \tag{4.1}$$

Let $\{(O_j, \varphi_j) : j = 1, \ldots, l\}$ be a finite collection of such local coordinate systems such that $\bigcup_{j=1}^{l} O_j$ covers a neighborhood of ∂E, and let $\{\psi_j\}_{j=1}^{l}$ be a corresponding smooth partition of unity with supp $\psi_j \subset O_j$ and $\psi_j \geq 0$.

If $\varphi_j = (\varphi_{j1}, \ldots, \varphi_{jN})$, it is easy to see that the function

$$\sigma(x) = \sum_{j=1}^{l} \psi_j(x) \varphi_{jN}(x)$$

is a C^1 function in a neighborhood W_0 of ∂E such that

$$\begin{aligned} &\sigma(x) > 0 \text{ in } W_0 \backslash \bar{E},\ \sigma(x) < 0 \text{ in } W_0 \cap E, \\ &\text{and } \sigma(x) = 0 \text{ if } x \in \partial E. \end{aligned} \tag{4.2}$$

Moreover, $\nabla\sigma(x) \neq 0$ on ∂E, and hence we may assume that $\nabla\sigma(x) \neq 0$ in W_0, and without loss of generality we may also assume that $|\nabla\sigma(x)| \leq 1$ in W_0. In fact to show that $\nabla\sigma(x) \neq 0$ on ∂E, note that

$$\nabla\sigma(x) = \sum_j \psi_j(x) \nabla\varphi_{jN}(x) \text{ if } x \in \partial E,$$

since, by (4.1), $\varphi_{jN}(x) \equiv 0$ on ∂E, and so also $\nabla\psi_j(x)\varphi_{jN}(x) \equiv 0$ on ∂E. Now let $\nu(x)$ be the outer (relative to E) normal to ∂E at x. Since $\nabla\varphi_{jN}(x)$ is also normal to ∂E at x, and in fact by (4.1) is an outer normal, $\langle \nabla\varphi_{jN}(x), \nu(x) \rangle > 0$ for $x \in \partial E \cap O_j$, $j = 1, \ldots, l$. Thus, by the formula for $\nabla\sigma$ on ∂E, we have $\langle \nabla\sigma(x), \nu(x) \rangle > 0$ if $x \in \partial E$, and therefore $\nabla\sigma \neq 0$ on ∂E.

For small $\epsilon > 0$, let

$$\begin{aligned} \sigma^+(x) &= \max\{0, \sigma(x)\} \text{ if } x \in W_0, \text{ and } \sigma^+(x) = 0 \\ &\quad \text{if } x \in E \backslash W_0, \\ f_\epsilon(x) &= \max\{0, 1 - \sigma^+(x)/\epsilon\}. \end{aligned}$$

Then both σ^+ and f_ϵ are Lipschitz continuous functions defined in a neighborhood of $\bar{E}$, and the Lipschitz constant of σ^+ is at most 1. Moreover,

$\sigma^+(x) = 0$ if and only if $x \in \bar{E}$, and $|\partial E| = 0$, so that f_ϵ converges a.e. to the characteristic function of E, χ_E, as $\epsilon \to 0$. Keeping in mind that $\sigma^+(x) = \sigma(x)$ if $\sigma(x) > 0$, we get from Theorem 2 (with $p = 1$) that

$$\begin{aligned}\Big(\int_B |f_\epsilon(x) - \frac{1}{w_2(B)}\int_B f_\epsilon(y) w_2 dy|^q w_2(x)dx\Big)^{1/q} \\ &\le Cr\frac{w_2(B)^{1/q}}{w_1(B)}\int_B |Xf_\epsilon(x)|w_1(x)dx \\ &\le Cr\frac{w_2(B)^{1/q}}{w_1(B)}\frac{1}{\epsilon}\int_{B\cap\{0<\sigma(x)\le\epsilon\}} |X\sigma(x)|w_1(x)dx\end{aligned}$$

if $B = B(\bar{x}, r)$, $\bar{x} \in K$, $0 < r < r_0$. We can cover K by a finite family of balls $\{B_j = B(x_j, r_0),\ j = 1, \ldots, J\}$. By (1.5) for $p = 1$,

$$r\frac{w_2(B)^{1/q}}{w_1(B)} \le cr_0\frac{w_2(B(\bar{x}, r_0))^{1/q}}{w_1(B(\bar{x}, r_0))}.$$

On the other hand, $\bar{x} \in B_j$ for some $j = 1, \ldots, J$, and hence $w_i(B(\bar{x}, r_0))$ is comparable to $w_i(B_j)$, $i = 1, 2$, for this j by doubling. Hence,

$$\begin{aligned}\frac{rw_2(B)^{1/q}}{w_1(B)} &\le c\max\Big\{r_0\frac{w_2(B_j)^{1/q}}{w_1(B_j)},\ j = 1, \ldots, J\Big\} \\ &= C_K.\end{aligned}$$

Hence,

(4.3)
$$\begin{aligned}&\Big(\int_B |f_\epsilon(x) - \frac{1}{w_2(B)}\int_B f_\epsilon(y)w_2(y)dy|^q w_2(x)dx\Big)^{1/q} \\ &\le C_K\frac{1}{\epsilon}\int_{B\cap\{0<\sigma(x)\le\epsilon\}}\Big(\sum_j |X_j\sigma(x)|^2\Big)^{1/2} w_1(x)dx \\ &= C_K\frac{1}{\epsilon}\int_0^\epsilon dt\int_{\{\sigma(x)=t\}}\Big(\sum_j \Big\langle X_j, \frac{\nabla\sigma}{|\nabla\sigma|}\Big\rangle^2\Big)^{1/2}\chi_B(x)w_1(x)dH_{N-1}(x)\end{aligned}$$

by the co-area formula ([Fe], Theorem 3.2.12) and since w_1 is continuous by hypothesis. Now let $\epsilon \to 0$. The term on the left in (4.3) tends to

$$\begin{aligned}&\Big(\int_B \Big|\chi_E(x) - \frac{w_2(B\cap E)}{w_2(B)}\Big|^q w_2(x)dx\Big)^{\frac{1}{q}} \\ &= \Big[\Big(\frac{w_2(B\setminus E)}{w_2(B)}\Big)^q w_2(B\cap E) + \Big(\frac{w_2(B\cap E)}{w_2(B)}\Big)^q w_2(B\setminus E)\Big]^{1/q} \\ &\ge \Big[\Big(\frac{1}{2}\Big)^q \min\{w_2(B\cap E), w_2(B\setminus E)\}\Big]^{1/q} \\ &= \frac{1}{2}\min\{w_2(B\cap E), w_2(B\setminus E)\}^{1/q}.\end{aligned}$$

To study the behavior of the expression on the right side of (4.3), we first consider the inner integral there with $\chi_B(x)$ replaced by a larger continuous analogue $\chi_\eta(x)$, $\eta > 0$, $0 \le \chi_\eta \le 1$, with $\chi_\eta = 1$ on B and supported in an η-neighborhood of B (in the usual Euclidean sense) : consider then

$$\int_{\{\sigma(x)=t\}} \left(\sum_j \left\langle X_j, \frac{\nabla\sigma}{|\nabla\sigma|} \right\rangle^2 \right)^{1/2} \chi_\eta(x) w_1(x) dH_{N-1}(x). \tag{4.4}$$

Consider the set $G = \{(x,t) \in W_0 \times (-\delta,\delta)\}$, for small $\delta > 0$, and the function $F(x,t) = \sigma(x) - t$ in G. Since $\nabla\sigma(x) \neq 0$ in $W_0 \supset \partial E$, we may assume by the implicit function theorem that $W_0 = \cup Q_i$ is the union of a finite number of Euclidean cubes Q_i with centers on ∂E such that

$$\{(x,t) \in Q_i \times (-\delta,\delta) : F(x,t) = 0\} = \{(x', g_i(x',t), t) : x' \in Q_i', t \in (-\delta,\delta)\},$$

for instance, where Q_i' is the projection of Q_i along one of the x-coordinates (we choose the x_N-coordinate above for simplicity; the coordinate may vary), and g_i is a C^1 function. Thus, for $t \in (-\delta,\delta)$,

$$\{x \in Q_i : \sigma(x) = t\} = \{(x', g_i(x',t)) : x' \in Q_i'\},$$

so that we can parametrize $\{\sigma(x) = t\} \cap Q_i$ for small t by means of $g_i(x',t)$, $x' \in Q_i'$. If $\{\tilde\psi_i\}$ is a corresponding smooth partition of unity, then (4.4) equals

$$\begin{aligned}\sum_i \int_{Q_i \cap \{\sigma(x)=t\}} &\left(\sum_j \langle X_j, \frac{\nabla\sigma}{|\nabla\sigma|} \rangle^2 \right)^{1/2} \chi_\eta \tilde\psi_i w_1 dH_{N-1} \\ &= \sum_i \int_{Q_i} (\dots)^{1/2}(x', g_i(x',t)) \chi_\eta(x', g_i(x',t)) \tilde\psi_i(x', g_i(x',t)) \cdot \\ &\qquad w_1(x', g_i(x',t))(1 + |\nabla g_i(x',t)|^2)^{1/2} dx'.\end{aligned}$$

By the continuity of all functions involved (recall that $\nabla\sigma$ is continuous and $|\nabla\sigma| \neq 0$ in W_0, so that $\nabla\sigma/|\nabla\sigma|$ is also continuous), this sum is continuous at $t = 0$. Since $\{x : \sigma(x) = 0\}$ equals ∂E by (4.2), and $\nabla\sigma/|\nabla\sigma| = \nu$ on ∂E, it follows that the limit as $\epsilon \to 0$ of the expression on the right of (4.3) is at most

$$c_K \int_{\partial E} \left(\sum_j \langle X_j, \nu \rangle^2 \right)^{1/2} \chi_\eta w_1 dH_{N-1}.$$

Theorem 3 now follows by letting $\eta \to 0$.

If E is not a regular domain, the following relative isoperimetric inequality can be proved by repeating the arguments used in the proof of

Theorem 3.2 of [FGaW2], except of course that we use Poincaré's inequality instead of Sobolev's inequality.

THEOREM 4.5. — *With the same assumptions as in Theorem 3, except that now w_1 need not be continuous and E may be any measurable set in Ω,*

$$\min\{w_2(B\cap\bar{E}), w_2(B\backslash\bar{E})\}^{1/q} \leq c \liminf_{\epsilon\to 0} \frac{1}{\epsilon} w_1(\{x \in B : \rho(x, \partial E) \leq \epsilon\}),$$

with c independent of B and E.

As in [FGaW1], [FGaW2], we can call the expression on the right above the $(N-1)$-dimensional lower Minkowski ρ-content of ∂E in B with respect to the measure w_1.

Analogues of Theorems 3 and 4.5 can be derived for vector fields of type [FL]. The required Poincaré estimates are discussed in Theorem 1 of [FGuW] and in the comments at the end of the introduction there. These analogues include the isoperimetric results in cases II and III of Theorem 3.2 of [FGaW2]. We omit their precise statements, and mention only that their proofs use the sort of representation formula in [FGuW].

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Bruno FRANCHI,
Dipartemento di Matematica
Università di Bologna
Piazza di Porta S. Donato, 5
I 40127 Bologna (Italy).

Guozhen LU,
Department of Mathematics
Wright State University
Dayton, Ohio 45435 (USA).

Richard L. WHEEDEN,
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903 (USA).