Representation of a finite graph by a set of intervals on the real line

by

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1. Introduction. Let there be given a finite family of sets $A_1, A_2, \ldots, A_n$. The sets may be thought of as subsets of a given set. For each pair of indices i, j $(i \neq j)$ the sets A_i, A_j may overlap or may not overlap. We wish to establish necessary and sufficient conditions in order that the family $\{A_i\}$ be representable by a family of intervals $\alpha_1, \ldots, \alpha_n$ on the real line, in such a way that

$$\alpha_i \cap \alpha_j \neq \emptyset \quad \text{if and only if} \quad A_i \cap A_j \neq \emptyset,$$

$\emptyset$ denoting the empty set. It is immaterial whether we take the intervals α_i to be open or closed.

An equivalent but more transparent formulation of the problem is obtained, if we take what is known in algebraic topology as the one-dimensional skeleton of the nerf of the family $\{A_i\}$. This is a graph G consisting of n points $a_1, \ldots, a_n$, such that, for each two indices i, j $(i \neq j;\ i, j = 1, 2, \ldots, n)$, the two points a_i, a_j are joined if and only if the corresponding sets A_i, A_j meet. Two points a_i, a_j which are joined will also be called neighbouring points, or neighbours, and we shall write $a_i \nu a_j$. Clearly, the relation ν is *symmetric*. Our problem then takes the following form:

PROBLEM. *To decide for which graphs $G = \{a_1, a_2, \ldots, a_n\}$ it is possible to assign to each point a_i an interval α_i on the real line, in such a way that*

$$(*) \qquad \alpha_i \cap \alpha_j \neq \emptyset \quad \textit{if and only if} \quad a_i \nu a_j \quad (i \neq j;\ i, j = 1, \ldots, n).$$

Any graph possessing the above property will be called *representable (by intervals)*.

We note that it is convenient for our purposes to define the relation ν only for (certain) pairs of distinct points. More generally, one could write $a_i \nu a_j$ if the corresponding sets A_i, A_j meet, including the case that the indices are equal. Then, clearly, the relation ν is reflexive. Now, we can conceive a graph G abstractly as a set on which an arbitrary binary relation ν is defined. From this standpoint, we are dealing here exclusively with the case that the relation ν is symmetric and nonreflexive.

In this paper we shall prove two theorems each of which gives an answer to the problem stated above (theorems 3 and 4). They are of the following type. Let the concepts of subgraph, path, irreducible cycle, neighbour of a path be defined as in section 2; we emphasize that we use the concept of a subgraph in a rather restricted sense. Then we have

I. A finite graph G is representable by intervals if and only if it fulfills the following two conditions

(α) G does not contain an irreducible cycle with more than three points,

(β) if a_1, a_2, a_3 are three points of G, which are mutually distinct and no two of which are neighbouring points, then at least one a_i is a neighbour of every path connecting the two other points.

II. A finite graph G is representable by intervals if and only if it does not contain a subgraph which is one of the graphs I, II, III_n, IV_n, V_n listed in fig. 5.

In our considerations an important rôle will be played by the notion of a *simplicial point* of a graph (see definition 1). Such a point can be seen as an end-point of the graph. It turned out that graphs which are subjected to the single condition (α) always contain simplicial points (see theorems 1, 2 and lemma 6). In other words, there always exist such points in G if G is admitted to contain triangles but not irreducible cycles of "length" greater than 3.

In the last section of this paper practical methods will be sketched by which we can decide whether a given graph is representable. These methods will be based on proposition I formulated above. We shall derive upper bounds for the number of operations needed for the verification of (α) and (β). A remarkable fact is that in the case of the condition (α) the larger number of operations is required. In general, this number is of the order $O(n^4)$, whereas, if only (α) is known to hold true, the verification of (β) does not need more than $O(n^3)$ operations.

The problem formulated at the beginning of this introduction was put by the American biologist S. Benzer. He was concerned with the fine-structure of genes. The problem is whether the sub-elements of genes are linked together in a linear order. He could deal with this problem succesfully for a certain microorganism. Of these microorganisms, there are a standard form and mutants, the latter arising if a certain connected portion of the genetic structure is blemished. By recombination tests, it is possible to decide whether the blemished parts of two given mutants overlap or not. Thus, for a large number of portions of the genetic structure, the experiments lead to data as to whether any two of these portions

overlap or not. The problem is to decide whether these data are compatible with a linear structure of the gene.

Professor de Groot drew our attention to Benzer's problem. He found the forbidden graphs with the exception of V_n ($n > 1$) and his work was continued by the authors of this paper. The second author found and proved theorems 3 and 4. His proofs were simplified by the first author, who introduced in this context the notion of a simplicial point. Sections 4 and 7 are entirely due to the first author.

2. Notations and definitions.

In the following G will always be a *finite* graph.

If a, b are two (distinct) neighbouring points of the graph G, then we write

$$a \nu b .$$

The relation ν is symmetric.

A *subgraph* of G is a graph H such that each point of H belongs to G and that, for two distinct points $a, b \in H$, the relation $a \nu b$ is true whenever it is true in G. In other words, if G is conceived abstractly as a finite set of elements, together with a certain set of non-ordered pairs (a, b), then H is obtained from G by removing certain elements and those pairs for which at least one constituent does not belong to H.

By the *union* of two subgraphs H_1 and H_2 of G that subgraph H of G is meant which consists of the points belonging to at least one of H_1, H_2. This union depends on G: if a, b are two points in H which do not belong to the same graph H_i, then the relation $a \nu b$ may or may not hold in H, and this cannot be decided from the structure of H_1 and H_2 alone. We therefore write $H = [H_1 \cup H_2]_G$. Only if no confusion can arise, we shall simply write $H_1 \cup H_2$.

The *complement* of a subgraph H of G is denoted by $G \backslash H$; it is the subgraph of G consisting of the points in G which do not belong to H (1). We have $[H \cup (G \backslash H)]_G = G$.

The graph consisting of a single point a is denoted by $\{a\}$.

A point a will be called a *neighbour* of a subgraph H of G and we shall write

$$a \nu H ,$$

if $a \notin H$ and $a \nu b$ for some point $b \in H$.

We further use the following terms:

path: $W = a_1 a_2 \ldots a_k$: any subgraph of G, such that $a_i \nu a_{i+1}$ $(i = 1, \ldots, k-1)$; it is not required that $a_j \neq a_i$ for all $j \neq i$;

irreducible path: a path $a_1 a_2 \ldots a_k$ such that $a_i \neq a_j$ if $i \neq j$ and $a_i \nu a_j$ only if $j = i \pm 1$;

cycle: a path of the form $a_1 a_2 \ldots a_k a_1$;

(1) Confer the previous definition of a subgraph.

irreducible cycle: a cycle $a_1a_2 \dots a_ka_1$ such that $a_i \neq a_j$ if $i \neq j$ and $a_i \, \nu \, a_j$ only if $j = i \pm 1$ or $i \pm (k-1)$;

star $S(a) = [S(a)]_G$ *of a point* $a \in G$: the subgraph of G consisting of a and all neighbours of a;

star $S(H) = [S(H)]_G$ *of a subgraph* H *of* G: the subgraph of G whose points are given by the points of H and the neighbours of H [2];

simplex: a graph G, such that $a \, \nu \, b$ for every two distinct points a, b of G.

We can now define the concepts which play a central rôle in our investigations.

Definition 1. Let $a \in G$. Then a is called a *simplicial point* of G, if $S(a)$ is a simplex.

Definition 2. A graph G is called *acyclic*, if it does not contain an irreducible cycle with more than three points.

Definition 3. A graph G is called *asteroidal* [3] if it contains three distinct points a_1, a_2, a_3 and three paths W_1, W_2, W_3 such that, for $i = 1, 2, 3$,

(i) W_i connects the two points a_j $(j \neq i)$;

(ii) a_i is not a neighbour of W_i [4].

Such a triple of points a_1, a_2, a_3 is called an asteroidal triple.

Definition 4. Let G be a graph. Suppose that there exists a set Γ of open intervals on the real line such that the following properties hold:

(i) there is a one-to-one correspondence between the points $a, b, \dots$ of G and the intervals $\alpha, \beta, \dots$ of Γ;

(ii) two intervals α, β intersect if and only if the corresponding points a, b satisfy $a \, \nu \, b$.

Then G is called *representable* and Γ is called a *model* of G. If, in particular, the union of the intervals of Γ is an interval, then Γ is called *connected*.

Finally, we wish to introduce the concept of duplication of a graph. Let H be a subgraph of G. Then we form a new graph K by taking two copies of G and by identifying corresponding points of H. In particular, if a, b are two points of $K \setminus H$, then the relation $a \, \nu \, b$ subsists if and only if a, b belong to the same copy of G and $a \, \nu \, b$ is true in G. We say that K is obtained by *duplication of* G *with respect to* H.

The above construction will only be carried out in the case that H is a simplex.

[2] We could also write $S(H) = [\bigcup_{a \in H} S(a)]_G$.

[3] This term was suggested by the simplest examples of graphs of the type considered. Compare e.g. the diagrams I, IV_2, V_1 (see fig. 5).

[4] In particular, none of the relations $a_i \nu a_j$ $(i \neq j)$ holds.

3. Some lemmas.

Lemma 1. *If G is an acyclic graph and $a_1a_2 \dots a_ka_1$ is a cycle in G, with $k \geqslant 4$, then we necessarily have*

(i) $a_1 = a_3$ *or* $a_1 \, \nu \, a_3$ *or*

(ii) $a_2 = a_i$ *or* $a_2 \, \nu \, a_i$ *for some* i *with* $4 \leqslant i \leqslant k$.

Proof. For $k = 4$ the lemma is true, as the cycle is not irreducible, by definition. Now let $k > 4$ and suppose that for no $i \geqslant 4$ we have $a_2 = a_i$ or $a_2 \, \nu \, a_i$. Then, since the cycle is reducible, there must be two indices i', i'' with

$$a_{i'} = a_{i''} \quad \text{or} \quad a_{i'} \, \nu \, a_{i''}, \qquad i' < i''-1, \ i' \neq 2 .$$

Then we can make a shorter cycle, in which a_1, a_2, a_3 all occur and in which a'_i, a''_i occur as one point or as successive points. Assuming the lemma to be true for this cycle, we must have $a_1 = a_3$ or $a_1 \, \nu \, a_3$. Hence, the lemma follows by induction on k.

Lemma 2. *Each path $a_1a_2 \dots a_k$, with $a_k \neq a_1$, contains an irreducible path with the same endpoints.*

Proof. Put $i_0 = 1$. Take the maximal index i_1 with $a_{i_0} \nu \, a_{i_1}$, thereafter the maximal index i_2 with $a_{i_1} \nu \, a_{i_2}$, etc. Then the path $a_{i_0}a_{i_1}a_{i_2}\dots$ is irreducible.

Lemma 3. *Let G be an acyclic graph. If $ca_1a_2 \dots a_kc$ $(c \neq a_i$ for $i = 1, 2, \dots, k)$ is a cycle in G and $a_1a_2 \dots a_k$ is an irreducible path, then we have $c \, \nu \, a_i$ for $i = 1, 2, \dots, k$.*

Proof. By lemma 1, with c instead of a_2, we have $c \, \nu \, a_i$ for some i with $1 < i < k$. Then $ca_1 \dots a_ic$ and $ca_i \dots a_kc$ are cycles of the same type as the given one. The lemma now follows by induction.

Lemma 4. *Let G be an acyclic graph, and let $S \subset G$ be a simplex. Further, let K be the graph obtained by duplicating G with respect to S. Then K is acyclic. Furthermore, each point $a \in S$ which has a neighbour in $G \setminus S$, is not a simplicial point of K.*

Proof. A cycle C in K which has a point in each of the two copies of G is clearly not irreducible. From this the first assertion follows. The second assertion is also clear.

4. Existence of simplicial points.

We begin by proving the following fundamental

Theorem 1. *Each (finite) acyclic graph contains a simplicial point* [5].

Proof. We use induction on the number of points. Let G be an acyclic graph with $n > 1$ points, and suppose that the theorem is true for graphs with less than n points. Then we shall prove that G has a simplicial point.

[5] The theorem is no longer true for infinite graphs, as is seen from the example $G = \{a_k\}_{-\infty}^{\infty}$ with neighbour relations $a_k \nu a_{k+1}$ $(k = 0, \pm 1, \pm 2, \dots)$.

Let b be an arbitrary point of G and let a be a simplicial point of $G\backslash\{b\}$. Put $G_1 = G\backslash\{b\}$ and $S_1(a) = [S(a)]_{G_1}$.

First, we dispose of some trivial cases. If we do not have $a\,\nu\,b$, then a is also a simplicial point of G. More generally, if some point $c \in S_1(a)$ has no neighbour in $G\backslash S_1(a)$, then $S(c) = S_1(a)$, and so c is a simplicial point of G. If, on the other hand, $b\,\nu\,c$ for each point $c \in S_1(a)$, then $S(a) = S_1(a) \cup \{b\}$ is a simplex, so that a is a simplicial point of G. Henceforth, we may restrict ourselves to the case that the following three properties hold:

(i) $a\,\nu\,b$;

(ii) each point $c \neq a$ of $S_1(a)$ has at least one neighbour in $G\backslash S_1(a)$;

(iii) there is a point $c_0 \neq a$ in $S_1(a)$, such that not $b\,\nu\,c_0$.

We now consider the graph $G\backslash S_1(a)$. It need not be connected. We denote by C_1 the component of $G\backslash S_1(a)$ which contains the point b, and put $C_2 = G\backslash(S_1(a) \cup C_1)$. We shall prove that $c \in S_1(a)$, $c\,\nu\,C_1$ implies that $c\,\nu\,b$.

Let c be a point in $S_1(a)$, with $c\,\nu\,C_1$, and let d_1 be a neighbour of c in C_1. If $c = a$ or $d_1 = b$, then there is nothing to prove. Hence we may suppose that $c \neq a$ and $d_1 \neq b$. Then, since $d_1, b \in C_1$ and C_1 is connected, there is a path $d_1 \ldots d_k b$ in C_1, with $b \neq d_i$ for $i = 1, \ldots, k$ $(k \geqslant 1)$. Now $bacd_1 \ldots d_k b$ is a cycle. Further, we do not have $a = d_i$, nor $a\,\nu\,d_i$, for any $i = 1, 2, \ldots, k$, as the only neighbours of a are given by b and the points $\neq a$ of $S_1(a)$. Then lemma 1, with $\{a_1, a_2, a_3\} = \{b, a, c\}$, learns that $c\,\nu\,b$ (see fig. 1).

It follows now from (iii) that there is a point $c_0 \in S_1(a)$, which is not a neighbour of C_1. Then, by (ii), this point has a neighbour in C_2. This implies that C_2 is not empty. Now define

$$
\begin{aligned}
S_1 &= \text{graph of the points } c_1 \in S_1(a) \quad \text{with} \quad c_1\,\nu\,C_1, \text{ but not } c_1\,\nu\,C_2, \\
S_2 &= \text{graph of the points } c_2 \in S_1(a) \quad \text{with} \quad c_2\,\nu\,C_1,\ c_2\,\nu\,C_2, \\
S_3 &= \text{graph of the points } c_3 \in S_1(a) \quad \text{with} \quad c_3\,\nu\,C_2, \text{ but not } c_3\,\nu\,C_1.
\end{aligned}
$$

Then, by (ii), $S_1 \cup S_2 \cup S_3 = S_1(a)$. Further, the subsimplices S_1 and S_3 are not empty, as $a \in S_1$ and $c_0 \in S_3$. The subsimplex S_2 may be empty.

Next, we wish to duplicate a suitable part of the graph G (see fig. 2, where each S_i is represented by a single point). The subgraphs C_1 and $C_2 \cup S_3$ are disjoint and are separated by the simplex $S_1 \cup S_2$. More precisely, each point of $S_1 \cup S_2$ has a neighbour in C_1, by the definition of S_1 and S_2, and also a neighbour in $C_2 \cup S_3$, because $S_1 \cup S_2 \cup S_3 = S_1(a)$ is a simplex. Similarly, the subgraphs $C_1 \cup S_1$ and C_2 are disjoint, whereas each point of $S_2 \cup S_3$ has a neighbour in both of them.

Note that, by what we have proved above, in both cases the two subgraphs and the separating simplex are non-empty.

In at least one case the two subgraphs considered do not have the same number of points. Call them H_1 and H_2 and let H_1 have the smaller number of points. Denote by S the separating simplex. Now we duplicate $H_1 \cup S$ with respect to S. This gives a new graph K.

By lemma 4, K is acyclic, because $H_1 \cup S \subset G$ is acyclic. Further, K has less than n points. Hence, by our induction hypothesis, K has a simplicial point s. By the last clause of lemma 4, since each point of S has a neighbour in H_1, the point s does not belong to S. Then $[S(s)]_K = [S(s)]_{H_1 \cup S} = [S(s)]_G$. Consequently, s is a simplicial point of G.

The case $n = 1$ is trivial. So the theorem has been proved.

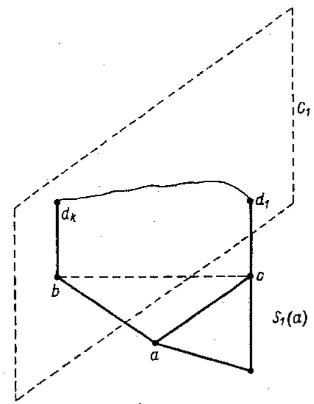

Fig. 1

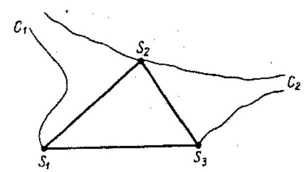

Fig. 2

We shall now investigate whether there are more simplicial points in a given acyclic graph G.

A simple consequence of definition 1 is the following

LEMMA 5. *Let G be a graph. If S is a connected subgraph of G and each point of S is a simplicial point of G, then S is a simplex. All points of S have the same star.*

Proof. Let a, b, c be three different points of S with $a\,\nu\,b$, $b\,\nu\,c$. Then, since b is simplicial, we necessarily have $a\,\nu\,c$, and so $\{a, b, c\}$ is a simplex. It is clear that, by a repetition of the argument, we find that S is a simplex.

Next, let a, b be two distinct points of S. If $c\,\nu\,a$ and $c \neq b$, then $c\,\nu\,b$, as a is simplicial. It follows that any two points of S have the same star.

Using the principle of duplication one easily deduces from theorem 1 that an acyclic graph with more than one point must have at least two simplicial points. More generally, this principle leads to a proof of the following

LEMMA 6. *Let G be an acyclic graph, which is not a simplex. Then G contains two non-neighbouring simplicial points.*

Proof. We may, clearly, suppose that G is connected. Let a be a simplicial point of G, and let S' denote the subsimplex of points $c \in S(a)$, which have no neighbour in $G\backslash S(a)$. These points c are just the points of $S(a)$ which are simplicial points of G.

We have $S' \neq \emptyset$, as $a \in S'$, and $S' \neq S(a)$, as otherwise we should have $G = S(a)$. Now we duplicate G with respect to S'. In virtue of lemma 4 and by the choice of S', we then get a graph K which is acyclic and which has no simplicial point in S'. Then K has a simplicial point outside S'. This point corresponds with a simplicial point $b \notin S'$ of G. Then, by the choice of S', $b \notin S(a)$. This proves the lemma.

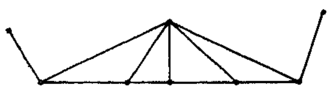

Fig. 3

There are various examples of acyclic graphs with exactly two simplicial points. For example the graphs which can be represented by a diagram of one of the following types:

1. a broken line (= an irreducible path);
2. a polygon with all diagonals through one given vertex;
3. a polygon with all diagonals except one;
4. figures like fig. 3.

Such examples, as well as lemma 6 and its proof, suggest that the simplicial points are to be sought at the "extremities" of the graph (cf. also lemma 7 in section 5). We can give a more precise meaning to this statement by proving the following theorem 2 (which is easily seen to be a generalization of lemma 6).

THEOREM 2. *Let G be an acyclic graph. Let H be a connected subgraph of G and suppose that $G\backslash S(H)$ is not empty. Then $G\backslash S(H)$ contains a point s which is a simplicial point of G.*

Proof. Let m be the number of points of $G\backslash S(H)$. We shall prove the theorem by induction on m.

First, let $m = 1$. Then $G\backslash S(H)$ consists of one point, a say. Let b_1, b_2 be two distinct neighbours of a. Then these points belong to $S(H)$, but not to H, as $a \notin S(H)$. Hence, there exists a path $W = c_1 \ldots c_k$ $(k \geqslant 1)$ in H, such that

$$b_1 \nu c_1 , \quad b_2 \nu c_k .$$

Now consider the cycle $ab_1c_1 \ldots c_kb_2a$. Since a is not a neighbour of W, it follows from lemma 1 that we must have $b_1 \nu b_2$. It follows that $S(a)$ is a simplex, i.e. that a is a simplicial point of G.

Now, let $m > 1$, and suppose that the theorem holds true if $G\backslash S(H)$ has less than m points.

Let b be an arbitrary point of $G\backslash S(H)$ and put $G_1 = G\backslash\{b\}$, $S_1(a) = [S(a)]_{G_1}$. Let a be a simplicial point of G_1 in $G_1\backslash S(H)$. If not $a \nu b$, then a is a simplicial point of G. Hence we may suppose

$$a \nu b .$$

We consider the components of $G\backslash S_1(a)$. Let C_1 be a component containing some point of H. Now H is connected and $H \cap S_1(a)$ is empty, as $a \notin S(H)$. Hence C_1 contains the whole graph H. Then it also contains $S(H)\backslash S_1(a)$. We now distinguish two cases.

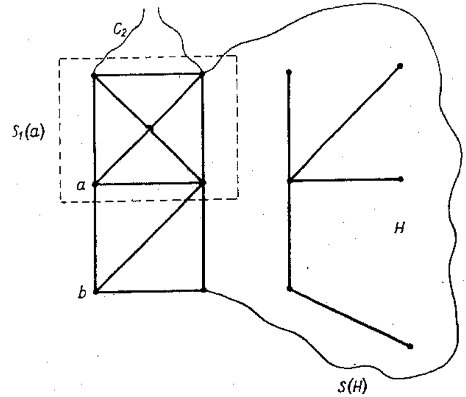

Fig. 4

Case 1. $G\backslash S_1(a)$ is connected, i.e. $G\backslash S_1(a) = C_1$. In particular $b \in C_1$. Then, as in the proof of theorem 1, it is true that $c \in S_1(a)$, $c \nu C_1$ implies $c \nu b$. There are two possibilities.

Case 1a. $c \nu C_1$ for each point $c \in S_1(a)$. Then a is a simplicial point of G.

Case 1b. There is a point $c_0 \neq a$ in $S_1(a)$, such that not $c_0 \nu C_1$. Then, *a fortiori*, c_0 is not a neighbour of H, and so $c_0 \notin S(H)$. Further, $S(c_0) = S_1(a)$. Hence c_0 is a simplicial point of G.

Case 2. $G\backslash S_1(a)$ is not connected. Let C_2 be a second component of $G\backslash S_1(a)$ and let $D = C_2 \cup S_1(a)$.

The subgraph D is either a simplex not contained in $S_1(a)$ or else, by lemma 6, it has two non-neighbouring simplicial points, which cannot both belong to $S_1(a)$. Hence, there is a point $s \in C_2$ which is a simplicial point of D. It does not belong to $S(H)$, because $S(H)$ is contained in $C_1 \cup S_1(a)$. Further, $[S(s)]_D = [S(s)]_G$, and so s is a simplicial point of G.

So, in all cases there is a simplicial point of G in $G \backslash S(H)$. This proves the theorem.

5. Representability of graphs. In this section we wish to derive a criterion for representability. Here the notion of asteroidal graph will come in. We shall further have to consider two types of simplicial points. Therefore, we define

DEFINITION 5. A simplicial point a of a graph G is called *strongly simplicial* if $G \backslash S(a)$ is connected, and *weakly simplicial* if $G \backslash S(a)$ is not connected. Further, an acyclic graph G is called extremal if it is connected and if all its simplicial points are strongly simplicial.

If a graph G formed by points $a, b, \ldots$ is representable, then we shall denote the corresponding intervals in a model Γ by corresponding Greek letters $\alpha, \beta, \ldots$. The left-hand and right-hand end points of an interval α will be denoted by $l(\alpha), r(\alpha)$ respectively. Then, by an *end-interval* of a model Γ we shall mean an interval $\alpha \in \Gamma$ such that either

(i) $r(\beta) > l(\alpha)$ for each interval $\beta \in \Gamma$, or

(ii) $l(\beta) < r(\alpha)$ for each interval $\beta \in \Gamma$.

In these cases, α is a left-hand or a right-hand endinterval respectively.

LEMMA 7. *If G is representable and a is a strongly simplicial point of G, then, in each model Γ of G, α is an endinterval.*

Proof. First observe that any model of a connected graph is connected. Now consider the simplex $S(a)$. If $G = S(a)$, then the assertion is trivial. If not, then take in Γ the submodel of $G \backslash S(a)$. It is connected, and no interval meets α. From this and the definition of $S(a)$ it follows that α is an enditerval.

We now come to the main result of this section:

THEOREM 3. *A graph G is representable if and only if it is acyclic and not asteroidal.*

Proof. The proof of the "only if" part is easy. Indeed, let there be a model Γ of G. First, suppose that G contains an irreducible cycle $a_1 a_2 \ldots a_k a_1$, with $k \geqslant 4$. Then, in Γ, the intervals α_1, α_3 are disjoint. The interval α_2 meets both α_1 and α_3, but no interval α_j, with $j > 3$, while these intervals α_j connect α_1 and α_3. This is impossible. Hence, G is acyclic.

Next, suppose that G contains an asteroidal triple (a_1, a_3, a_3). In Γ, the intervals $\alpha_1, \alpha_2, \alpha_3$ are mutually disjoint. So, without loss of generality, we may suppose that α_2 separates α_1 and α_3. Then α_2 meets the image in Γ of each path W_2 connecting a_1 and a_3, and so $a_2 \vee W_2$ for each choice

of W_2. This contradicts the definition of asteroidal triple. Hence, G is not asteroidal.

Conversely, let G be an acyclic graph which is not asteroidal. Then we shall prove, by using theorem 1, that G is representable. We distinguish two cases.

Case 1. G is extremal (this implies that G is connected). By theorem 1, it has a simplicial point. By lemma 6, if G is not a simplex, it even has two non-neighbouring simplicial points. But it cannot have three simplicial points a_1, a_2, a_3, no two of which are non-neighbouring points. For, by hypothesis, these points would be strongly simplicial, and so a_j, a_k could be connected by a path not meeting $S(a_i)$ ((i, j, k) any permutation of $(1, 2, 3)$), so that G would be asteroidal, against hypothesis.

Consequently, it suffices to prove the following

ASSERTION. *A graph G which is acyclic and extremal and which does not contain three non-neighbouring simplicial points, is representable.*

We shall do this by using induction on the number of points, say n. If $n = 1$, then the assertion is trivially true. Now take $n > 1$ and suppose that the assertion holds for graphs with less than n points.

Let a be simplicial point of G and let S_1 be the subsimplex consisting of those points of $S(a)$ which are simplicial points of G.

Put

$$S_2 = S(a) \backslash S_1, \quad G_1 = G \backslash S_1, \quad G_2 = G \backslash S(a) = G_1 \backslash S_2.$$

Then $G_1 = G_2 \cup S_2$ is connected and acyclic. We investigate its simplicial points. First, let $b \in G_2$ be a simplicial point of G_1. Then, since not $b \vee S_1$, $[S(b)]_G = [S(b)]_{G_1}$. Consequently, b is a simplicial point of G. Now take two arbitrary points c_1, c_2 in $G_1 \backslash S(b)$. Since b is a strongly simplicial point of G, there exists a path W in $G \backslash S(b)$ connecting the points c_1, c_2. By lemma 2, there is an irreducible path W' which is a subgraph of W and which connects c_1, c_2. This path cannot contain a point of S_1.

Hence, it is contained in $G_1 \backslash S(b)$. It follows that b is a strongly simplicial point of G_1.

On the other hand, a simplicial point of G which belongs to G_2 is also a simplicial point of G_1.

Next, let a point $d \in S_2$ be a simplicial point of G_1. Write $S_1(d) = [S(d)]_{G_1}$. Let $G_1 \backslash S_1(d)$ have k components $C_1, \ldots, C_k$ $(k \geqslant 0)$. For each i, the graph $C_i \cup S_1(d)$ is not a simplex (because of $C_i \neq \emptyset$) and so, by lemma 6, $C_i \cup S_1(d)$ has a simplicial point $e_i \notin S_1(d)$. Then $e_i \notin S_2$. It is easy to see that

$$[S(e_i)]_{C_i \cup S_1(d)} = [S(e_i)]_{G_1} = [S(e_i)]_G.$$

Hence, e_i is a simplicial point of G $(i = 1, 2, \ldots, k)$. Then, by our hypotheses, we must have $k = 0$ or 1, i.e. d is a strongly simplicial point of G_1 (note that it may happen that $k = 0$, i.e. $G_1 = S_1(d)$).

Combining the results obtained so far, we see that G_1 is extremal. It is also acyclic. Further, it has simplicial points outside S_2; moreover, the simplicial points of G_1 belonging to $G_1 \backslash S_2$ form a simplex. It follows from lemma 6 that G_1 has a simplicial point $d \in S_2$ [6]. Also, by our induction hypothesis, there is a model Γ_1 of G_1. In this model, d is represented by an endinterval δ, on account of lemma 7. Let it be a left-hand enditerval. Since S_2 is contained in $S_1(d)$ and $S_1(d)$ is a simplex, we can produce to the left, in Γ_1, the intervals corresponding with S_2; this does not give rise to new overlappings. We can do this and add an interval a in such a way that a meets exactly the intervals of S_2. Representing every point of S_1 by such an interval a we get a model of G.

Case 2. G is not extremal. We may suppose that each proper subgraph of G is representable. Let a be a weakly simplicial point of G. Put $S = S(a)\backslash\{a\}$ and denote the components of $G\backslash S(a)$ by $C_1, \ldots, C_k$ $(k \geqslant 2)$. It is convenient to call a point $c \in G\backslash S(a)$ a *full neighbour* of S and to write $c\,\bar{v}\,S$, if we have $c\,v\,b$ for each point $b \in S$.

We shall apply the induction hypothesis in two different ways.

We first consider some trivial cases. Let Γ_1 be a model of $G\backslash\{a\}$ and let δ be the intersection of the intervals corresponding with points of S. If no C_i contains a point c with $c\,\bar{v}\,S$, then δ is not met by other intervals, and so a model Γ is obtained by adding to Γ_1 an interval $a = \delta$. If, on the other hand, there is an index i, such that each point $c \in C_i$ is a full neighbour of S, then we argue as follows.

In Γ_1, each interval γ of C_i meets δ. Further, these intervals form a connected model, and they do not intersect other intervals of $G\backslash S(a)$. We can diminish arbitrarily the dimensions of the submodel of C_i in Γ_1. We can do this and add an interval a to Γ_1 in such a way that we obtain a model of G.

It follows that we may restrict ourselves to the case that there is an index i, such that *some points of C_i are full neighbours of S and some points are not.* We put

$$G_1 = G\backslash\{a\}\,, \qquad G_2 = S(a) \cup C_i\,.$$

By induction, there are models Γ_1, Γ_2 of G_1, G_2 respectively. First, consider Γ_2. Since a is a strongly simplicial point of G_2, a is an endinterval of Γ_1, say a right-hand endinterval. There is an interval γ in Γ_2 which does not meet δ as there is a point in C_i which is not a full neighbour of S, we choose one for which $r(\gamma)$ is minimal. Let Σ_1 be the set of intervals which correspond with points of S and which meet γ (Σ_1 may be empty), and let $\Sigma(a)$ be the submodel of $S(a)$ in Γ_2. Then each interval set Γ_2^*

[6] Note that, by our hypotheses, $G\backslash S(a) = G_1\backslash S$ is not empty, so that G_1 has at least two points.

obtained from Γ_2 by producing arbitrarily to the left one or more intervals of Σ_1 and arbitrarily to the right one or more intervals of $\Sigma(a)$, is again a model of G_2. For this does not cause new overlappings, because γ and a are endintervals.

Next, consider Γ_1. It contains some model of $G_2\backslash\{a\}$; let γ', δ' be the intervals in Γ_1 corresponding with the intervals γ, δ, respectively, in Γ_2. Then γ', δ' do not meet; without loss of generality we may suppose that $r(\gamma') < l(\delta')$. On the real line, where Γ_1 is situated, we choose an interval ξ such that each interval of C_i is wholly contained in ξ and that each interval of each C_j $(j \neq i)$ falls outside ξ. Then $r(\xi)$ and $l(\xi)$ only belong to intervals of $S(a)$, and each interval of S intersects ξ, as there is a point in C_i which is a full neighbour of S. We prove that $l(\xi)$ can only belong to intervals of S_1 (corresponding with Σ_1).

Let $b \in S(a)\backslash S_1$ and let β, β' be the corresponding intervals in Γ_2, Γ_1 respectively. Then β does not meet γ. Hence, β' does not meet γ'. But it meets δ'. Hence, we have $l(\beta') \geqslant r(\gamma')$ and so $l(\xi) \notin \beta'$.

We can now construct a model of G in the following way. Take Γ_1, remove the part $\Gamma_1 \cap \xi$ and then insert the model Γ_2; produce to the left those Σ_1 which in Γ_1 contain $l(\xi)$ and to the right those intervals of $\Sigma(a)$ which in Γ_1 contain $r(\xi)$.

This proves the assertion and thus completes the proof of the theorem.

6. Structure of non-representable graphs. In this section we follow the original idea of Professor de Groot of determining a minimal set of graphs with the property that any graph is representable if and only if it does not contain a graph of this set. It turned out that a complete set with this property is given by figure 5; there, in each diagram, except III_n, we have indicated the three points which constitute an asteroidal triple.

In other words, we have the following

THEOREM 4. *A graph G is representable, if and only if it does not contain a subgraph which is one of the graphs* I, II, III_n, IV_n, V_n [7].

Theorem 4 gives a less elegant characterization of representable graphs than theorem 3. But it lies deeper, as in this theorem the various types of non-representable graphs are analysed. Actually, the proof of theorem 4 will be based on theorem 3.

Proof of theorem 4. We leave it to the reader to check that the graphs I, II, IV_n, V_n are all asteroidal [8], and hence also, that the condition is necessary. It remains to show that if G is not representable,

[7] Of course, it is understood that no junctions are present which are indicated in the diagrams.

[8] The cycle III^n is asteroidal for $n \geqslant 6$.

then G contains one of the subgraphs listed above. So by theorem 3 the proof of the theorem will be completed if we can deduce the following

ASSERTION. *Let G be a graph with the following properties:*

(1) *G is acyclic;*

(2) *G is asteroidal;*

(3) *G is minimal, i.e. no proper subgraph is asteroidal.*

Then G is one of the graph I, II, IV_n, V_n.

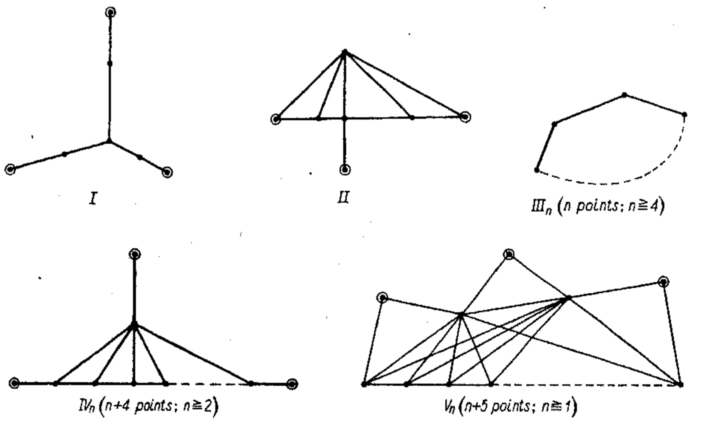

Fig. 5

Let G have the properties (1)-(3). Let (a_1, a_2, a_3) be an asteroidal triple and let W_1, W_2, W_3 be three paths such that

$$\left.\begin{array}{l}(\alpha)\ W_i \text{ connects the two points } a_j\ (j \neq i)\\ (\beta)\ a_i \text{ is not a neighbour of } W_i\end{array}\right\}\ (i = 1, 2, 3)\,.$$

In virtue of lemma 2, we may suppose that *the paths W_i are irreducible.* Further, it follows from (3) that we have

$$W_1 \cup W_2 \cup W_3 = G\,.$$

If $i \neq j$, then W_i contains only one point $\neq a_j$ of $S(a_j)$, as W_i is irreducible. Hence, $S(a_j)$ contains at most two points $\neq a_j$. If there are two, say a_j' and a_j'', then we must have $a_j' \nu a_j''$. For the three paths W_i constitute a cycle, in which a_j', a_j, a_j'' are successive points, $a_j' \neq a_j''$ and a_j has no other neighbours than a_j', a_j''. Then an application of lemma 1 learns that $a_j' \nu a_j''$.

We now distinguish some cases.

Case 1. Each a_i has two neighbours. Let the W_i be given by

$$W_1 = a_2 b_2 \ldots c_3 a_3\,, \quad W_2 = a_3 b_3 \ldots c_1 a_1\,, \quad W_3 = a_1 b_1 \ldots c_2 a_2\,.$$

We do not exclude that $b_2 = c_3$ or $b_3 = c_1$ or $b_1 = c_2$. If this occurs then the corresponding path W_i is called *short.*

We cannot have $b_1 = c_1$ as otherwise the third point of $S(a_1)$ would not occur in any W_i. Hence, $b_1 \neq c_1$. Similarly, $b_2 \neq c_2$, $b_3 \neq c_3$. Further, the points b_i are mutually distinct, and also the points c_i, because of $b_i, c_i \notin W_i$ $(i = 1, 2, 3)$. So we have the situation of figure 6. Note that the paths W_i may have interior points in common.

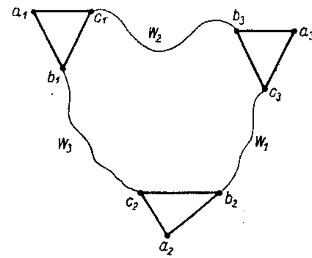

Fig. 6

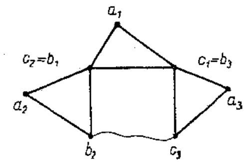

Fig. 7

We now prove that at least two paths W_i are short. Suppose that e.g. W_2 and W_3 are not short, and consider the point c_2. It is a point of the cycle $c_2 b_2 \ldots c_3 b_3 \ldots c_1 b_1 \ldots c_2$. If $c_2 \nu c_1$, then we may replace W_3 by $W_3' = a_1 c_1 c_2 a_2$. If $c_2 \nu d$ for some interior point $d \neq c_1, b_2$ of W_2 or W_1, then we may replace W_1 by $W_1' = a_2 c_2 d \ldots a_3$.

In both cases, G would not be minimal. Similarly, if $b_2 \nu e$ for some interior point $e \neq c_2$ of W_3 or W_2, G would not be minimal.

Hence,

$$S(c_2) \cap (W_1 \cup W_2) = b_2$$

and

$$S(b_2) \cap (W_2 \cup W_3) = c_2\,.$$

It is now easily verified that in the cycle $c_2 b_2 \ldots c_3 b_3 \ldots c_1 b_1 \ldots c_2$ none of the implications of lemma 1 holds. This contradiction proves that at least two paths are short.

So we may suppose that W_2 and W_3 are both short (fig. 7). Suppose that we do not have $b_1 \nu c_3$. Then $c_3 \neq b_2$, and then (a_1, a_2, c_3) is an asteroidal triple in $G \setminus \{a_3\}$. Hence, by (3), we must have $b_1 \nu c_3$. Then, by lemma 3, we have $b_1 \nu c$ for each point c of the irreducible path $b_2 \ldots c_3$. Similarly, $b_3 \nu c$ for each such point c.

Then G is of the form V_n (the case $n = 1$ occurs if $b_2 = c_3$).

Case 2. One of the points a_i has only one neighbour. Let a_1 be such a point and let b be its neighbour. Then W_2 and W_3 necessarily contain the point b. If not $b\,\nu\,W_1$, then (b, a_2, a_3) is an asteroidal triple of $G\backslash\{a_1\}$. So we have $b\,\nu\,W_1$. We now have to distinguish some subcases.

Case 2.1. b has $k \geqslant 2$ neighbours on W_1. Let c, c' be the first and the last neighbour respectively. To b and the part $c \ldots c'$ of the (irreducible) path W_1 we can apply lemma 3. We can also say that a_2, a_3 are not neighbours of b. It follows that G contains a graph IV_n. Hence, G is actually of the form IV_n.

Case 2.2. b has only one neighbour $c \neq a_1$. Then $c \in W_1$. Also, necessarily, $c \in W_2$, $c \in W_3$. Then we do not have $a_2\,\nu\,c$ or $a_3\,\nu\,c$. It follows that G is of the form I.

Case 2.3. b has exactly one neighbour $c_0 \in W_1$ and at least one neighbour $d_1 \notin W_1 \cup \{a_1\}$. We write $W_1 = c_{-k} \ldots c_0 \ldots c_l$, where $c_{-k} = a_2$, $c_l = a_3$, $k \geqslant 1$, $l \geqslant 1$. We may suppose that $d_1 \in W_3$.

W_3 have the form $W_3 = a_1 b d_1 \ldots c_{-p} c_{-p-1} \ldots c_{-k}$, where the point preceding c_{-p} is the last point of W_3 not belonging to W_1. Then $p > 0$, as $W_3 = a_1 b d_1 \ldots$ is irreducible and so $c_0 \notin W_3$ (see fig. 8).

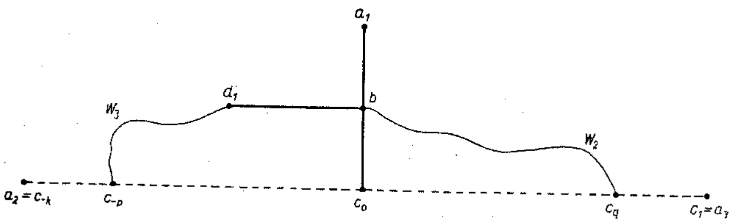

Fig. 8

Next, we show that W_3 does not contain a point d with $d\,\nu\,c_i$ for some $i > 0$ (then W_3 does not contain a point c_i, $i > 0$, either). Suppose that there was such an index i. Then $d \neq b$. Then, replacing W_1, W_2 successively by

$$W_1' = c_{-k} \ldots c_{-p} \ldots d c_i c_{i+1} \ldots c_l\,,$$

$$W_2' = a_1 b \ldots d c_i c_{i+1} \ldots c_k \quad \text{(which do not contain } c_0)\,,$$

we see that (a_1, a_2, a_3) would be an asteroidal triple in $G\backslash\{c_0\}$. This contradicts the requirement (3).

Having reached this point, let us consider the case that the part $c_1 \ldots c_l$ of W_1 has no neighbour $e \notin W_1$. If $l \geqslant 2$, then we have case 2.2, with a_1 replaced by a_3. If $l = 1$, then we apply lemma 1 to the cycle $c_0 b d_1 \ldots c_{-p} \ldots c_0$. Since not $b = c_{-1}$ or $b\,\nu\,c_{-j}$ $(j > 0)$ and c_0 has no neighbour c_i, $i < -1$, we find that c_0 has at least two neighbours b and d_1 on W_3. Then we have case 2.1, with a_1 replaced by a_3.

Consequently, we may suppose that some c_q, $q > 0$, has a neighbour $e \notin W_1$. Then $e \notin W_3$. Hence $e \in W_2$. Then $c_0 \notin W_2$, as otherwise $G\backslash\{e\}$ should be asteroidal. Then W_2 has the form $W_2 = a_1 b e_1 \ldots c_q c_{q+1} \ldots c_l$ and it does contain neither a point c_i, with $i < 0$, nor a neighbour of such a point. Further, we do not have $d' = e'$ or $d'\,\nu\,e'$ if $d' \in W_3$, $e' \in W_2$ and $d', e' \neq a_1, b$ (in the contrary case, $G\backslash\{c_0\}$ would be asteroidal).

We now apply lemma 1 to the cycles $c_0 b d_1 \ldots c_{-p} \ldots c_0$ and $c_0 b e_1 \ldots c_q \ldots c_0$. The consequence is that $c_0\,\nu\,d_1$ and $c_0\,\nu\,e_1$, whence, on account of (3), G must be of the form II.

This proves the assertion and so completes the proof of theorem 4.

A simple consequence of theorem 4 is the following

Corollary. *An acyclic graph with not more than five points is always representable.*

7. Numerical devices. In our final section we shall deal with a practical method by which we can decide whether a given graph G is representable. This method will be based on theorem 3. The treatment naturally splits up into two parts: we have to decide whether or not there are irreducible cycles in G and whether or not there are asteroidal triples in G.

A. Examination of cycles. We begin with a definition and a theorem.

Definition 6. Let G be a graph and let $a \in G$ be arbitrary. Let $C_1, C_2, \ldots, C_k$ $(k = k(a) \geqslant 1)$ be the components of $G\backslash S(a)$. Then, for each C_i, we denote by $S_i(a)$ the graph of points b with

$$b \neq a\,, \quad b \in S(a)\,, \quad b\,\nu\,C_i\,,$$

and call $S_i(a)$ a *substar* of $S(a)$.

Theorem 5. *A graph G is acyclic if and only if for each point $a \in G$ all substars $S_i(a)$ are simplices.*

Proof. First suppose that G is acyclic. Take any substar $S_i(a)$, and let b_1, b_2 be two distinct points of $S_i(a)$. Then there are points $c_1, c_2 \in C_i$ with $c_1\,\nu\,b_1$, $c_2\,\nu\,b_2$. Further, there is a path W in C_i connecting c_1, c_2. Then, by the definition of $S(a)$, we do not have $a\,\nu\,W$. Applying lemma 1 to the cycle $a b_1 c_1 \ldots c_2 b_2 a$ we find that $b_1\,\nu\,b_2$. It follows that $S_i(a)$ is a simplex.

Conversely, suppose that there is an irreducible cycle $c_1 c_2 \ldots c_k c_1$ $(k \geqslant 4)$ in G. Put $a = c_1$ and let C_i be the component of $G\backslash S(a)$ containing the point c_3. Then $S_i(a)$ contains c_2, c_k and so it is not a simplex.

Below we shall apply the following slightly different and less elegant proposition, the proof of which offers no difficulties.

Proposition. *A graph G is acyclic if, for some point $a \in G$, the substars $S_i(a)$ are simplices and the graph $G\backslash\{a\}$ is acyclic.*

In order to find out whether G is acyclic, one could now proceed along the following lines.

a) Choose arbitrarily $a \in G$ and determine the neighbours of a.

b) Determine the components $C_1, \ldots, C_k$ in the following way. Take any point $c_1 \in G \setminus S(a)$. Determine the neighbours of c_1 in $G \setminus S(a)$, say $c_2, \ldots, c_{k_1}$. Then take the neighbours of c_2 in $G \setminus S(a)$ which do not belong to the set $\{c_1, \ldots, c_{k_1}\}$, say $c_{k_1+1}, \ldots, c_{k_2}$. Then take c_3 and repeat the process until no new points are found. Then one component C_1 has been found. If $G \setminus S(a)$ contains a point $c \notin C_1$, then determine in the same way a second component C_2 of $G \setminus S(a)$ containing c. Repeat this until $G \setminus S(a)$ is exhausted.

c) For each component C_i determine the substar $S_i(a)$ by taking the points $b \in S(a)$ which have at least one neighbour in C_i.

d) Check whether $S_i(a)$ is a simplex.

e) Omit a and examine in the same way $G \setminus \{a\}$. Etc.

Let G, $S(a)$ and the C_i have n, m, n_i points respectively ($i = 1, \ldots, k$). Then the points a)-d) require at most n, $\sum n_i(n-m)$, $\sum (m-1)n_i$, $\frac{1}{2}m^2k$ operations successively. The sum of these numbers is $\leqslant$

$$n+(n-m)^2+m(n-m)+\tfrac{1}{2}m^2(n-m) \leqslant \tfrac{2}{27}n^3+\tfrac{4}{9}n^2+O(n),$$

the expression on the left attaining its maximum for $m \sim \frac{2}{3}n-1$. So the examination requires in the aggregate not more than about $\frac{1}{54}(n^4+10n^3)$ operations.

B. Examination of triples. First, we prove

THEOREM 6. *If G is acyclic and asteroidal, then it contains an asteroidal triple of simplicial points.*

Proof. Let (a_1, a_2, a_3) be an asteroidal triple and let W_3 be a path in $G \setminus S(a_3)$ connecting a_1 and a_2. We shall apply theorem 2, with $H = W_3$. We have $a_3 \notin S(H)$, so that $G \setminus S(H)$ is not empty. Let C be the component of $G \setminus S(H)$ which contains the point a_3.

By theorem 2, C contains a point a_3' which is a simplicial point of $C \cup S(H)$. It also is a simplicial point of G (confer the end of the proof of theorem 2). Further, $a_3' \notin S(H) = S(W_3)$, and a_3, a_3' can be connected by a path which does not meet $S(a_1)$ or $S(a_2)$. Hence, (a_1, a_2, a_3') is an asteroidal triple in G.

Repeating this procedure two times, we get an asteroidal triple (a_1', a_2', a_3') of simplicial points.

Let now G be an acyclic graph. Let Σ' be the set of its simplicial points. They form a certain number of non-neighbouring simplices in G (i.e. no simplex contains a neighbour of another simplex). From each simplex we select arbitrarily one point. Let Σ be the set of selected simplicial points.

Our method for deciding whether G is asteroidal or not now consists of the following stages.

a) For each point $a \in G$ examine whether $S(a)$ is a simplex.

b) Determine the simplices of which the set of simplicial points consists, in the following way. Take any simplicial point s_1 and determine its neighbours, say $s_2, \ldots, s_{k_1}$, among the other simplicial points. Repeat this process with a simplicial point $s \neq s_1, \ldots, s_{k_1}$. Then select a simplicial point in each component thus found; this gives a set Σ.

c) Construct a matrix $i(a,b)$ ($a, b \in \Sigma$; $i(a,b)$ a suitable positive integer) as follows. Take $a \in \Sigma$ arbitrarily. Determine the components $C_1, \ldots, C_k$ [9] of $G \setminus S(a)$ as in A, b). For each component C_j, put $i(a,b) = j$ for all $b \in \Sigma \cap C_j$.

d) Check whether for each triple (a, b, c) in Σ the following equations hold true:

$$i(a,b) = i(a,c), \quad i(b,c) = i(b,a), \quad i(c,a) = i(c,b)$$

(the graph G is asteroidal if and only if there is a triple (a, b, c) in Σ, such that the above equations hold).

Let G and Σ consist of n and s points respectively. Then the total number of operations needed for the steps a)-d) is $\leqslant$

$$n \cdot \tfrac{1}{2}n^2 + \tfrac{1}{2}n^2 + s(n^2+sn) + \tfrac{1}{2}s^3 \leqslant 3n^3 + O(n^2).$$

Finally, we make the following remarks. In part B the restriction to the set Σ—with the proviso that we know already that G is acyclic—enables us to suppress the dimensions of the matrix $i(a,b)$ to be constructed. Further, in part A we can begin by omitting the points of Σ' (which have to be determined in B); for G is acyclic if $G \setminus \Sigma'$ is acyclic. Then the various stages have to be performed in the following order: B, a); A; B, b)-d). Then, apart from a term of order $O(n^2)$, the total number of operations needed can be estimated by

$$\tfrac{1}{54}(n-s)^4 + \tfrac{5}{27}(n-s)^3 + \tfrac{1}{2}n^3 + sn^2 + s^2n + \tfrac{1}{2}s^3,$$

which is not larger than $\frac{1}{54}n^4 + \frac{37}{54}n^3$ (note that the derivative with respect to s is negative if $n - s \geqslant 4n^{2/3}$).

We have the impression that in general the method exhibited here cannot be improved essentially.

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[9] The numbering of the components C_j is immaterial.

Reference

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Sur l'enfilage et la fixation des ensembles compacts

par

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§ 1. Relations générales. E étant un espace métrique, un ensemble $X \subset E$ sera dit *fixable dans* E [1] lorsque, pour tout $\varepsilon > 0$, il existe dans E une somme finie $F_\varepsilon = F_1 \cup F_2 \cup \ldots \cup F_{k(\varepsilon)}$ d'ensembles fermés tels que $\delta(F_i) < \varepsilon$ pour $i = 1, 2, \ldots, k(\varepsilon)$, $F_i \cap F_j = \emptyset$ pour $i \neq j$ et $F_\varepsilon \cap C \neq \emptyset$ pour toute composante C de X.

De plus, si pour $\varepsilon_n \downarrow 0$, c'est-à-dire pour toute suite $\{\varepsilon_n\}$ décroissante et convergente vers 0, les F_{ε_n} qui fixent X peuvent être choisis de manière qu'ils forment une suite descendante, j'appelle la fixation de X *monotone*.

Knaster appelle un ensemble $X \subset E$ *enfilable dans* E lorsque E contient un arc L tel que $L \cap C \neq \emptyset$ pour toute composante C de X.

J'appelle *réduit* de X tout ensemble $R \subset X$ tel que $R \cap C \neq \emptyset$ pour toute composante C de X. En outre, j'appelle *l'adduit* de X l'ensemble A de tous les points p de E tels que $(p) = \operatorname*{Lim}_{i\to\infty} C_i$ pour une suite $\{C_i\}$ de composantes de X. Ainsi défini, A est donc l'ensemble de tous les points de E qui sont des points-limites des suites de points appartenant à des composantes C_i de X telles que $\delta(C_i)$ tend à 0. On voit aussitôt qu'un adduit est toujours fermé, donc compact, pour des X compacts.

$\mathcal{E}^n$ désignera constamment l'espace euclidien de dimension $n \geqslant 1$.

THÉORÈME 1. *Les trois propriétés suivantes sont équivalentes pour les X compacts dans $\mathcal{E}^n$:*

(1) *l'existence d'une fixation monotone de X,*

(2) *l'existence dans X d'un réduit R compact de dimension* 0,

(3) *l'existence d'un enfilage de X.*

La démonstration de ce théorème se trouve dans mon travail [8], p. 14.

THÉORÈME 2. *Si un X (compact ou non) est fixable dans $\mathcal{E}^n$, son adduit A est vide ou de dimension* 0.

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[1] cf. Knaster [2], où l'on trouve une définition équivalente de cette notion par des ensembles ouverts.