REPRESENTATIONS OF ALGEBRAIC GROUPS PRESERVING QUATERNION SKEW-HERMITIAN FORMS

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Introduction. Let K be an infinite perfect field of characteristic different from 2, let $\mathfrak{B}$ be a quaternion division algebra over K, and let $\xi \rightarrow \bar{\xi}$ denote the canonical involution of the first kind on $\mathfrak{B}$. Let V be a finite-dimensional right vector space over $\mathfrak{B}$.

A *quaternion skew-hermitian form* H over $\mathfrak{B}$ is a sesquilinear form on $V \times V$, i.e., H is a map from $V \times V$ to $\mathfrak{B}$ such that

(i) $H(x, y_1+y_2)=\overline{H(x, y_1)}+H(x, y_2)$ and $H(x, y\alpha)=H(x, y)\alpha$ for all x, y, y_1, y_2 in V and α in $\mathfrak{B}$;

(ii) $H(x, y)=-\overline{H(y, x)}$ for all x, y in V. Let $\{x_1, \cdots, x_n\}$ be a basis for V over $\mathfrak{B}$. We say that H is nondegenerate if the reduced norm in $M_n(\mathfrak{B})$ of the matrix $(H(x_i, x_j))$ is not zero. Associated to such a nondegenerate form H are 3 invariants, the dimension of V over $\mathfrak{B}$, $\dim_{\mathfrak{B}}(V)$, the discriminant of H, $\delta(H)$, and the Clifford algebra of H, $\mathfrak{C}$.

Let G be a simply connected semisimple algebraic group (in some $\mathrm{GL}(m, \overline{K})$) which is defined over K and let $\rho: G \rightarrow \mathrm{GL}(V/\mathfrak{B})$ be an absolutely irreducible representation of G defined over K into the group of all nonsingular $\mathfrak{B}$-linear endomorphisms of V. We shall assume that there is a nondegenerate quaternion skew-hermitian form H on V which is invariant with respect to $\rho(G)$.

The purpose of this paper is to describe the Clifford algebra of the invariant form H in terms of ρ, G, and the Steinberg group associated to G. In a previous paper, we have described $\dim_{\mathfrak{B}}(V)$ and $\delta(H)$ in such a way and have indicated how representations such as ρ arise [**2**, Theorem I.2]. The invariant $\gamma(G)$ plays an important role in our study and so we recall some of its properties in §1. In 2, we define the invariant $\mathfrak{C}$ using the representation ρ. Jacobson first constructed the Clifford algebra of a quaternion skew-hermitian form [**4**]. In this paper, however, we shall follow a method due to Satake [**6**]. We give some examples in §3 with special emphasis on the case where G is absolutely simple.

1. **The invariant $\gamma(G)$.** A connected semisimple algebraic group G_1 defined over K is said to be *K-quasi-split* (or of Steinberg type) if

Received by the editors April 17, 1969.

there is a Borel subgroup B in G_1 defined over K. Let $\mathfrak{G} = \mathrm{Gal}(\overline{K}/K)$ (where $\overline{K}$ denotes the algebraic closure of K). It is known that there is a quasi-split group G_1 defined over K which is isomorphic to G over $\overline{K}$. Furthermore, the isomorphism $f:G \to G_1$ can be chosen so that $(f^{-1})^\sigma \circ f = I_{g_\sigma}$ (for each $\sigma \in \mathfrak{G}$) where $g_\sigma \in G$ and $I_{g_\sigma}(h) = g_\sigma h g_\sigma^{-1}$ for all h in G. It follows that for each σ, τ in $\mathfrak{G}$, there is an element $c_{\sigma,\tau}$ in $Z(G)$, the center of G, such that $g_\sigma^\tau g_\tau = c_{\sigma,\tau} g_{\sigma\tau}$. The mapping $(\sigma, \tau) \to c_{\sigma,\tau}$ from $\mathfrak{G} \times \mathfrak{G}$ to $Z(G)$ is a 2-cocycle of $\mathfrak{G}$ in $Z(G)$ whose cohomology class $(c_{\sigma,\tau})$ is independent of G_1 and f. This class is denoted by $\gamma(G)$ and has been studied by Satake [5], [7].

THEOREM. *Let G be a simply connected semisimple algebraic group defined over K and let $\rho:G \to \mathrm{GL}(V/\mathfrak{B})$ be an absolutely irreducible representation of G defined over K which preserves a quaternion skew-hermitian form H (defined over K). Let G_1 be the quasi-split group associated to G, let $f:G \to G_1$ be an isomorphism defined over $\overline{K}$ such that $(f^{-\sigma}) \circ f = I_{g_\sigma}$ where $g_\sigma \in G$, and let $\gamma(G) = (c_{\sigma,\tau})$. Then there exists an absolutely irreducible representation $\rho_1:G_1 \to \mathrm{GL}(V_1)$ defined over K which preserves a nondegenerate symmetric bilinear form S_1 (defined over K). Furthermore, the following conditions hold:*

(i) *There is an absolutely irreducible representation $M:\mathrm{End}(V/\mathfrak{B}) \to \mathrm{End}(V_1)$ defined over $\overline{K}$ such that $M(\rho(g)) = (\rho_1 \circ f)(g)$ for all $g \in G$.*

(ii) *The central simple division algebra $\mathfrak{B}$ is characterized by the property that $c(\mathfrak{B})$ (the Hasse invariant of $\mathfrak{B}$) $= ((\rho_1 \circ f)(c_{\sigma,\tau}))$.*

(iii) *The invariants of H are* $\dim_{\mathfrak{B}}(V) = \frac{1}{2} \dim V_1$ *and* $\delta(H) = \Delta(S_1)$.

This result is Theorem I.2 in [2]. However, the construction of ρ_1 and statements (i) and (ii) are due to Satake [7]. We shall later extend this theorem to include a description of $\mathfrak{C}$.

COROLLARY. *If G_1 is a split group, then* $\delta(H) = 1$.

PROOF. It follows from (ii) that $\rho_1(Z(G_1)) \neq \{1\}$. Hence, S_1 has maximal Witt index [3, Lemma 1.1]; this completes the proof.

REMARK. The invariant Δ is defined as follows: let $q = \dim V_1$ and let $\{e_1, \cdots, e_q\}$ be a K-rational basis of V_1. Then Δ is the equivalence class of $(-1)^{q(q-1)2} \det(S_1(e_i, e_j))$ in $K^*/(K^*)^2$ where K^* is the multiplicative group $K - \{0\}$.

2. **The invariant $\mathfrak{C}$.** We denote the Clifford algebra of S_1 by C and the algebra of "even elements" in C by C^+. Let $\mathrm{Spin}(V_1, S_1)$ denote the "spin group" of S_1 and let $\pi:\mathrm{Spin}(V_1, S_1) \to \mathrm{SO}(V_1, S_1)$ be the canonical homomorphism. It is well known that π is defined over K and has kernel $\{+1, -1\}$. Since G is simply connected, there is a

(polynomial) map $\rho_s:G\to\mathrm{Spin}(V_1, S_1)$ such that $\pi\circ\rho_s=\rho_1\circ f$. We put $A_\sigma=\rho_s(g_\sigma^{-1})$ and $B_\sigma=\pi(A_\sigma)=(\rho_1\circ f)(g_\sigma^{-1})$. It follows that $(\rho_1\circ f)^\sigma(g)=B_\sigma(\rho_1\circ f)(g)B_\sigma^{-1}$ for each $\sigma\in\mathfrak{G}$ and all $g\in G$; hence, since G is connected $\rho_s^\sigma(g)=A_\sigma\rho_s(g)A_\sigma^{-1}$ for each $\sigma\in\mathfrak{G}$ and all $g\in G$. From this we see that $A_\sigma^\tau A_\tau=\rho_s(c_{\sigma,\tau}^{-1})A_{\sigma\tau}$ for each σ, $\tau\in\mathfrak{G}$. The elements $z_{\sigma,\tau}=\rho_s(c_{\sigma,\tau}^{-1})$ are in the center of C^+ but may not be in the center of C.

Let ϑ_σ be the automorphism of C^+ given by $\vartheta_\sigma(\xi)=A_\sigma\xi A_\sigma^{-1}$ for each $\xi\in C^+$. Since the elements $z_{\sigma,\tau}$ are in the center of C^+, we have $\vartheta_\sigma^\tau\vartheta_\tau=\vartheta_{\sigma\tau}$ and, therefore, the mapping $\sigma\to\vartheta_\sigma$ is a 1-cocycle of $\mathfrak{G}$ in $\mathrm{Aut}(C^+)$ and gives rise to a K-form $\mathfrak{C}$. There is an isomorphism $h:\mathfrak{C}\to C^+$ such that $h^\sigma\circ h^{-1}=\vartheta_\sigma$ for each σ in $\mathfrak{G}$.

We set $K'=K(\Delta^{1/2})$ and $\mathfrak{G}'=\mathrm{Gal}(\overline{K}/K')$. From statement (i) in §1, it follows that dim $V_1\equiv 0 \pmod 2$. Hence, C is a central simple algebra over K and C^+ decomposes over K' into a direct sum of two central simple algebras C_1 and C_2 which are equivalent to C over K'. This decomposition gives rise to a decomposition $\mathfrak{C}=\mathfrak{C}_1+\mathfrak{C}_2$ of the algebra $\mathfrak{C}$ as a direct sum of two central simple algebras over K'.

We shall now determine the invariant $c(\mathfrak{C}_1)$ over K'. Let h_1: $C_1\to M(2^{n-1}, \overline{K})$ be an isomorphism of C_1 onto a full matrix algebra; the mapping h_1 is defined over $\overline{K}$. By the theorem of Skolem-Noether, $h_1^\sigma\circ h_1^{-1}=I_{M_\sigma}$ (for each σ in $\mathfrak{G}'$) where M_σ is a nonsingular $2^{n-1}\times 2^{n-1}$ matrix. It follows that $M_\sigma^\tau M_\tau=d_{\sigma,\tau}M_{\sigma\tau}$ where $d_{\sigma,\tau}$ is a diagonal matrix. The invariant $c(C_1)$ over K' is the cohomology class $(d_{\sigma,\tau})$ over K'.

If $\xi\in C^+$, we denote by ξ' the projection of ξ on C_1. The mapping $h_1\circ h$ gives an isomorphism of $\mathfrak{C}_1$ onto a full matrix algebra and $(h_1\circ h)^\sigma\cdot(h_1\circ h)^{-1}=I_{N_\sigma}$ where $N_\sigma=M_\sigma h_1(A_\sigma')$. From this it follows that $N_\sigma^\tau N_\tau=e_{\sigma,\tau}N_{\sigma\tau}$ where $e_{\sigma,\tau}=d_{\sigma,\tau}h_1(z_{\sigma,\tau}')$.

Let ω_1 and ω_2 be the "spin representations" of $\mathrm{Spin}(V_1, S_1)$. These representations come from the canonical representations of C^+ on the ideals C_1 and C_2. Hence, $h_1(z_{\sigma,\tau}')$ may be identified with $\omega_1(z_{\sigma,\tau})$.

REMARK. The representations $\omega_1\circ\rho_s$ and $\omega_2\circ\rho_s$ of G are, in general, *not* absolutely irreducible. However, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the positive weights relative to some ordering with multiplicities of $\rho_1\circ f$. Then it is well known that the highest weight of $\omega_1\circ\rho_s$ (resp. $\omega_2\circ\rho_s$) is $\frac{1}{2}(\lambda_1+\cdots+\lambda_{n-1}+\lambda_n)$ (resp. $\frac{1}{2}(\lambda_1+\cdots+\lambda_{n-1}-\lambda_n)$). Indeed, the weights of $\omega_1\circ\rho_s$ (resp. $\omega_2\circ\rho_s$) are $\frac{1}{2}(\pm\lambda_1\pm\cdots\pm\lambda_{n-1}\pm\lambda_n)$ with an even number (resp. odd number) of minus signs.

We shall now state our results on the invariants $\mathfrak{C}_1$ and $\mathfrak{C}_2$. In doing so, we shall use the assumptions and notation of the theorem.

PROPOSITION. *The cohomology class $c(\mathfrak{C}_i)$ $(i=1, 2)$ over K' is given by the equation $c(\mathfrak{C}_i)=c(C_i)$ $(\omega_i\circ\rho_s(c_{\sigma,\tau}^{-1}))$.*

COROLLARY 1. *If G_1 is a split group, then* $c(\mathfrak{C}_i)=(\omega_i\circ\rho_s(c_{\sigma,\tau}^{-1}))$.

PROOF. As we saw in the corollary to the theorem (§1), the form S_1 has maximal Witt index and so $c(C_i)\sim 1$. This completes the proof.

COROLLARY 2 (JACOBSON). *The following relations on $\mathfrak{C}_1$ and $\mathfrak{C}_2$ hold over K':*

(i) *if* $n\equiv 0 \pmod 2$, *then* $\mathfrak{C}_j^2\sim 1$ $(j=1,2)$ *and* $\mathfrak{C}_1\otimes\mathfrak{C}_2\sim\mathfrak{B}$;
(ii) *if* $n\equiv 1 \pmod 2$, *then* $\mathfrak{C}_1\otimes\mathfrak{C}_2\sim 1$ *and* $\mathfrak{C}_j^2\sim\mathfrak{B}$ $(j=1,2)$.

PROOF. As before, let ω_i:Spin$(V_1,S_1)\to$GL(W_i) $(i=1,2)$ be the "spin representations" of Spin(V_1,S_1). We shall denote by ω_{ij} the representation $(\omega_i\circ\rho_s)\otimes(\omega_j\circ\rho_s)$ of G on $W_i\otimes W_j$ (for $i,j=1,2$). Since $c(C)^2=1$, it follows from the proposition that $c(\mathfrak{C}_i)c(\mathfrak{C}_j)=(\omega_{ij}(c_{\sigma,\tau}^{-1}))$ for $i,j=1,2$. In the rest of this proof, we use the notation of the remark preceding the proposition. Furthermore, we shall denote by λ_1 the highest weight of $\rho_1\circ f$.

(i) We shall assume that $n\equiv 0 \pmod 2$. Since $\frac{1}{2}(\lambda_1+\cdots+\lambda_n)$ and $-\frac{1}{2}(\lambda_1+\cdots\lambda_n)$ are weights of $\omega_1\circ\rho_s$, it follows that 0 is a weight of ω_{11}; hence, $\omega_{11}(Z(G))=\{1\}$ and $c(\mathfrak{C}_1)^2=1$. Similarly, $\frac{1}{2}(\lambda_1+\cdots+\lambda_n)$ is a weight of $\omega_1\circ\rho_s$ and $\frac{1}{2}(\lambda_1-\lambda_2-\cdots-\lambda_n)$ is a weight of $\omega_2\circ\rho_s$; hence, λ_1 is a weight of ω_{12} and so $(\omega_{12}(c_{\sigma,\tau}^{-1}))=(\lambda_1(c_{\sigma,\tau}^{-1}))=c(\mathfrak{B})$ by statement (ii) in the theorem. Therefore, $\mathfrak{C}_1\otimes\mathfrak{C}_2\sim\mathfrak{B}$ and the proof of (i) is finished.

(ii) We now assume that $n\equiv 1 \pmod 2$. Since $\frac{1}{2}(\lambda_1+\cdots+\lambda_n)$ and $\frac{1}{2}(\lambda_1-\lambda_2-\cdots-\lambda_n)$ are weights of $\omega_1\circ\rho_s$, it follows as before that $c(\mathfrak{C}_1)c(\mathfrak{C}_1)=(\lambda_1(c_{\sigma,\tau}^{-1}))=c(\mathfrak{B})$. Similarly, $\frac{1}{2}(\lambda_1+\cdots+\lambda_n)$ (resp. $-\frac{1}{2}(\lambda_1+\cdots+\lambda_n)$) is a weight of $\omega_1\circ\rho_s$ (resp. $\omega_2\circ\rho_s$) and so $c(\mathfrak{C}_1)c(\mathfrak{C}_2)=1$. This completes the proof of the corollary.

3. **An example.** In this section, we shall assume that K is a field of characteristic 0 and that G is an absolutely simple algebraic group defined over K. If G is *not* of type A_n, B_n or D_n, then quaternion skew-hermitian representations cannot exist. For 0 is a weight of each orthogonal representation and, therefore $\rho(Z(G))=\{1\}$; statement (ii) of the theorem then cannot be satisfied. If G is of type B_n or if G is of type D_n or A_n and the quasi-split group associated to G is of Chevalley type (i.e., split) then we have the following description of the invariants associated to (V,H): $\delta(H)=1$ and $c(\mathfrak{C}_i)=(\omega_i\circ\rho_s(c_{\sigma,\tau}^{-1}))$.

It only remains to examine invariant symmetric bilinear forms on representations of quasi-split groups of type A_n and D_n. We have described these forms in an earlier paper [2, Theorem II.1]. Here, we shall only give a small extension of these results.

Let G be a simply connected semisimple Chevalley group defined over K. The automorphism group of G is the semidirect product of a finite group Θ and the inner automorphisms of G. The group Θ can be chosen so that each θ in Θ is defined over K.

Let $L=K(\alpha^{1/2})$ be a quadratic extension of K (where $\alpha\in K^*$) and let $\mathrm{Gal}(L/K)=\{1, \sigma\}$ where $\sigma(\alpha^{1/2})=-\alpha^{1/2}$. Let $\theta\in\Theta$ be such that $\theta^2=1$. The mapping of $\{1, \sigma\}$ to Θ defined by $1\to 1_G$ and $\sigma\to\theta$ is a 1-cocycle of Θ in $\mathrm{Aut}(G)$. Hence, there is a group G_1 defined over K (which is "split" over L) and an isomorphism $f:G_1\to G$ such that $f^\sigma\circ f^{-1}=\theta$. The group G_1 is quasi-split; each group of type D_n^2 arises in this way.

Let $\rho:G\to \mathrm{SO}(V, S)$ be an absolutely irreducible orthogonal representation of G defined over K such that $\rho\circ\theta\sim\rho$. We shall also assume that $\rho(Z(G))\neq\{1\}$. Then there is an absolutely irreducible orthogonal representation $\rho_1:G_1\to\mathrm{SO}(V_1, S_1)$ of G_1 defined over K such that $\rho_1\sim\rho\circ f$ [2, Theorem II.1.]. We shall sketch this construction. There exists an $A\in\mathrm{GL}(V, K)$ such that $A^2=1$, $A\rho(g)A^{-1}=\rho(\theta(g))$ for all $g\in G$, and ${}^tASA=S$. Hence, we may find a K-rational orthogonal basis of V such that in this basis $A=\mathrm{diag}(1, \cdots, 1, -1, \cdots, -1)$ (with, say, $r+1$'s). Let $S=\mathrm{diag}(\beta_1, \cdots, \beta_r, \beta_{r+1}, \cdots, \beta_n)$ in this basis. Then $S_1=(\beta_1, \cdots, \beta_r, \alpha\beta_{r+1}, \cdots, \alpha\beta_n)$. Since $\rho(Z(G))\neq\{1\}$, S has maximal Witt index and so $\Delta(S)=1$ and $c(S)=1$. It is then not hard to see that $\Delta(S_1)=\alpha^{n-r}$. If $\det(A)=-1$, then $K'=K(\Delta(S_1)^{1/2})=L$ and over L, $S_1=S$. *Hence, if* $\det(A)=-1$, *then* $c(S_1)\sim 1$ *over* K'.

The facts on quadratic forms that we have used may all be found in [1].

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