Tohoku Math. J.
51 (1999), 13–20

REPRESENTATION OF FLAT LAGRANGIAN *H*-UMBILICAL SUBMANIFOLDS IN COMPLEX EUCLIDEAN SPACES

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(Received May 26, 1997)

Abstract. The author proved earlier that, a Lagrangian H-umbilical submanifold in complex Euclidean n-space with $n>2$ is either a complex extensor, a Lagrangian pseudo-sphere, or a flat Lagrangian H-umbilical submanifold. Explicit descriptions of complex extensors and of Lagrangian pseudo-spheres are given earlier. The purpose of this article is to complete the investigation of Lagrangian H-umbilical submanifolds in complex Euclidean spaces by establishing the explicit description of flat Lagrangian H-umbilical submanifolds in complex Euclidean spaces.

1. Statements of theorems. We follow the notation and definitions given in [2]. In order to establish the complete classification of Lagrangian H-umbilical submanifolds in $\boldsymbol{C}^n$ we need to introduce the notion of special Legendre curves as follows.

Let $z: I \to S^{2n-1} \subset \boldsymbol{C}^n$ be a unit speed Legendre curve in the unit hypersphere S^{2n-1} (centered at the origin), i.e., $z=z(s)$ is a unit speed curve in S^{2n-1} satisfying the condition: $\langle z'(s), iz(s)\rangle=0$ identically. Since $z=z(s)$ is a spherical unit speed curve, $\langle z(s), z'(s)\rangle=0$ identically. Hence, $z(s)$, $iz(s)$, $z'(s)$, $iz'(s)$ are orthonormal vector fields defined along the Legendre curve. Thus, there exist normal vector fields $P_3, \dots, P_n$ along the Legendre curve such that

$$z(s),\ iz(s),\ z'(s),\ iz'(s),\ P_3(s),\ iP_3(s), \dots, P_n(s),\ iP_n(s) \tag{1.1}$$

form an orthonormal frame field along the Legendre curve.

By taking the derivatives of $\langle z'(s), iz(s)\rangle=0$ and of $\langle z'(s), z(s)\rangle=0$, we obtain $\langle z'', iz\rangle=0$ and $\langle z'', z\rangle=-1$, respectively. Therefore, with respect to an orthonormal frame field chosen above, z'' can be expressed as

$$z''(s)=i\lambda(s)z'(s)-z(s)-\sum_{j=3}^{n} a_j(s)P_j(s)+\sum_{j=3}^{n} b_j(s)iP_j(s)\,, \tag{1.2}$$

for some real-valued functions $\lambda, a_3, \dots, a_n, b_3, \dots, b_n$. The Legendre curve $z=z(s)$ is called a *special Legendre curve* in S^{2n-1} if the expression (1.2) reduces to

$$z''(s)=i\lambda(s)z'(s)-z(s)-\sum_{j=3}^{n} a_j(s)P_j(s)\,, \tag{1.3}$$

for some parallel normal vector fields $P_3, \dots, P_n$ along the curve.

1991 *Mathematics Subject Classification*. Primary 53C40; Secondary 53C42, 53B25.

By a *Lagrangian cylinder* in C^n we mean a Lagrangian submanifold which is a cylinder over a curve whose rulings are $(n-1)$-planes parallel to a fixed $(n-1)$-plane.

The following result provides an explicit description of flat Lagrangian H-umbilical submanifolds in complex Euclidean spaces.

MAIN THEOREM. *Let $n\geq 2$ and $\lambda, b, a_3, \dots, a_n$ be n real-valued functions defined on an open interval I with λ nowhere zero and let $z: I\to S^{2n-1}\subset C^n$ be a special Legendre curve satisfying* (1.3). *Put*

$$f(t, u_2, \dots, u_n)=b(t)+u_2+\sum_{j=3}^{n} a_j(t)u_j\,. \tag{1.4}$$

Denote by $\hat{M}^n(0)$ the twisted product manifold $_fI\times E^{n-1}$ with twisted product metric given by

$$g=f^2dt^2+du_2^2+\cdots+du_n^2\,. \tag{1.5}$$

Then $\hat{M}^n(0)$ is a flat Riemannian n-manifold and

$$L(t, u_2, \dots, u_n)=u_2z(t)+\sum_{j=3}^{n} u_jP_j(t)+\int^t b(t)z'(t)dt \tag{1.6}$$

defines a Lagrangian H-umbilical isometric immersion $L: \hat{M}^n(0)\to C^n$.

Conversely, up to rigid motions of C^n, locally every flat Lagrangian H-umbilical submanifold in C^n without totally geodesic points is either a Lagrangian cylinder over a curve or a Lagrangian submanifold obtained in the way described above.

Clearly, every unit speed Legendre curve in S^3 is special. The following result shows that special Legendre curves in S^{2n-1} do exist abundantly for $n\geq 3$.

EXISTENCE THEOREM. *Let n be an integer ≥ 2. Then, for any given $n-1$ real-valued functions $\lambda, a_3, \dots, a_n$ defined on an open interval I with λ nowhere zero, there exists a special Legendre curve $z: I\to S^{2n-1}\subset C^n$ which satisfies* (1.3) *for some parallel orthonormal normal vector fields $P_3, \dots, P_n$ along the curve z.*

2. Proof of the main theorem. Let $\lambda, b, a_3, \dots, a_n$ be n functions defined on an open interval I with λ nowhere zero and let $z: I\to S^{2n-1}\subset C^n$ be a special Legendre curve satisfying (1.3) for some parallel orthonormal normal vector fields $P_3, \dots, P_n$ defined along the Legendre curve. Then, from the definition of parallel normal vector fields, we have

$$P_j'(t)=\eta_j(t)z'(t)\,, \qquad j=3, \dots, n\,, \tag{2.1}$$

for some functions $\eta_3, \dots, \eta_n$.

Let $L=L(t, u_2, \dots, u_n)$ be given by (1.6). Then, by taking the partial derivatives of L with respect to $t, u_2, \dots, u_n$, we get respectively

$$
\begin{aligned}
&L_t = u_2 z'(t) + \sum_{j=3}^{n} u_j P_j'(t) + b(t) z'(t)\,, \\
&L_{u_2} = z(t)\,, \\
&L_{u_j} = P_j(t)\,, \qquad j = 3, \ldots, n\,.
\end{aligned} \tag{2.2}
$$

From (2.2) and the definition of special Legendre curves we find

$$\langle L_t, L_{u_j} \rangle = 0\,, \quad \langle L_{u_j}, L_{u_k} \rangle = \delta_{jk}\,, \qquad j, k = 2, \ldots, n\,. \tag{2.3}$$

Since $z'(t)$ and $P_j(t)$ are perpendicular, (2.1) yields

$$P_j'(t) = a_j(t) z'(t)\,, \qquad j = 3, \ldots, n\,. \tag{2.4}$$

Combining (2.2) and (2.4) we get

$$L_t = f z'(t)\,. \tag{2.5}$$

(1.4), (1.5), (2.3) and (2.5) imply that $L = L(t, u_2, \ldots, u_n)$ is an isometric immersion of $\hat{M}^n(0)$ in $\boldsymbol{C}^n$. Moreover, from the definition of special Legendre curves, L is Lagrangian.

Using (1.3), (2.2), (2.5) and the definition of special Legendre curves, we find

$$L_{tt} = f_t z'(t) + f z''(t)\,, \quad L_{tu_j} = a_j(t) z'(t)\,, \quad L_{u_j u_k} = 0\,, \qquad j, k = 2, \ldots, n\,. \tag{2.6}$$

Applying (1.3), (2.2), (2.4), (2.5) and (2.6), we obtain

$$h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \lambda(t) J\left(\frac{\partial}{\partial t}\right), \quad h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_j}\right) = h\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = 0\,, \qquad j, k = 2, \ldots, n\,,$$

which implies that $L : \hat{M}^n(0) \to \boldsymbol{C}^n$ is Lagrangian H-umbilical.

Conversely, assume that $L : M^n \to \boldsymbol{C}^n$ is a Lagrangian H-umbilical isometric immersion of a flat Riemannian n-manifold M^n into $\boldsymbol{C}^n$ without totally geodesic points. Since M is flat, the second fundamental form h of L satisfies (cf. [2])

$$h(e_1, e_1) = \phi J e_1\,, \quad h(e_1, e_j) = h(e_j, e_k) = 0\,, \qquad j, k = 2, \ldots, n\,, \tag{2.7}$$

for some nowhere zero function ϕ, with respect to some suitable orthonormal local frame field $e_1, \ldots, e_n$. Without loss of generality, we may assume $\phi > 0$.

From (2.7) and Codazzi's equation, we find

$$e_j \ln \phi = \omega_1^j(e_1)\,, \quad \omega_1^j(e_k) = 0\,, \qquad 2 \le j, k \le n\,. \tag{2.8}$$

Let $\mathscr{D}$ and $\mathscr{D}^\perp$ denote the distributions of M spanned by $\{e_1\}$ and $\{e_2, \ldots, e_n\}$, respectively. $\mathscr{D}$ is clearly integrable, since it is 1-dimensional. From (2.7) and (2.8) it follows that $\mathscr{D}^\perp$ is also integrable and the leaves of $\mathscr{D}^\perp$ are totally geodesic submanifolds of $\boldsymbol{C}^n$. Because $\mathscr{D}$ and $\mathscr{D}^\perp$ are both integrable and they are perpendicular, there exist local coordinates $\{x_1, x_2, \ldots, x_n\}$ such that $\partial/\partial x_1$ spans $\mathscr{D}$ and $\{\partial/\partial x_2, \ldots, \partial/\partial x_n\}$ spans $\mathscr{D}^\perp$. Since $\mathscr{D}$ is 1-dimensional, we may choose x_1 such that $\partial/\partial x_1 = \phi^{-1} e_1$.

With respect to $\partial/\partial x_1, \ldots, \partial/\partial x_n$, (2.7) becomes

$$h\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right)=J\left(\frac{\partial}{\partial x_1}\right), \quad h\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right)=h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)=0, \quad j, k=2, \ldots, n. \tag{2.9}$$

Let N^{n-1} be an integral submanifold of $\mathscr{D}^{\perp}$. Then N^{n-1} is a totally geodesic submanifold of $\boldsymbol{C}^n$. Thus, N^{n-1} is an open portion of a Euclidean $(n-1)$-space $\boldsymbol{E}^{n-1}$. Therefore, M is an open portion of the twisted product manifold ${}_fI\times\boldsymbol{E}^{n-1}$ with twisted product metric [1] (see also [4])

$$g=f^2dx_1^2+dx_2^2+dx_3^2+\cdots+dx_n^2, \tag{2.10}$$

where $f=\phi^{-1}$ and I is an open interval on which ϕ is defined. (2.10) implies

$$\begin{aligned} &\nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_1}=\frac{f_1}{f}\frac{\partial}{\partial x_1}-f\sum_{k=2}^{n} f_k\frac{\partial}{\partial x_k}, \\ &\nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_j}=\frac{f_j}{f}\frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j}\frac{\partial}{\partial x_k}=0, \end{aligned} \tag{2.11}$$

for $2\le j, k\le n$, where $f_i=\partial f/\partial x_i$, $i=1, \ldots, n$. Using (2.11) we obtain

$$R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_1}=f\sum_{k=2}^{n} f_{jk}\frac{\partial}{\partial x_k}, \qquad j=2, \ldots, n. \tag{2.12}$$

Since M is flat, (2.12) yields $f_{jk}=0$, $j, k=2, \ldots, n$. Therefore, f is given by

$$f=\beta(x_1)+\sum_{j=2}^{n}\alpha_j(x_1)x_j, \tag{2.13}$$

for some functions $\beta, \alpha_2, \ldots, \alpha_n$. By (2.13), (2.11) reduces to

$$\begin{aligned} &\nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_1}=\frac{1}{f}\left(\beta'(x_1)+\sum_{j=2}^{n}\alpha_j'(x_1)x_j\right)\frac{\partial}{\partial x_1}-f\sum_{k=2}^{n}\alpha_k\frac{\partial}{\partial x_k}, \\ &\nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_j}=\frac{\alpha_j}{f}\frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j}\frac{\partial}{\partial x_k}=0, \qquad j, k=2, \ldots, n. \end{aligned} \tag{2.14}$$

Combining (2.9), (2.14) and the formula of Gauss we obtain

$$L_{x_1x_1}=\frac{1}{f}(\beta'(x_1)+\sum_{j=2}^{n}\alpha_j'(x_1)x_j)L_{x_1}-f\sum_{k=2}^{n}\alpha_k L_{x_k}+iL_{x_1}, \tag{2.15}$$

$$L_{x_1x_j}=\frac{\alpha_j}{f}L_{x_1}, \tag{2.16}$$

$$L_{x_jx_k}=0, \qquad j, k=2, \ldots, n. \tag{2.17}$$

Integrating (2.17) yields

$$L=\sum_{j=2}^{n} P_j(x_1)x_j+D(x_1)\,, \tag{2.18}$$

for some C^n-valued functions $P_2, \dots, P_n$, D of x_1. Thus

$$L_{x_1}=\sum_{j=2}^{n} P_j'(x_1)x_j+D'(x_1)\,, \tag{2.19}$$

$$L_{x_j}=P_j(x_1)\,,\qquad j=2,\dots,n\,. \tag{2.20}$$

From (2.10) and (2.20), we know that $P_2, \dots, P_n$ are orthonormal tangent vector fields on M^n. By applying (2.16), (2.19) and (2.20), we obtain

$$\alpha_j(x_1)D'(x_1)=\beta(x_1)P_j'(x_1)\,, \tag{2.21}$$

$$\alpha_j(x_1)P_k'(x_1)=\alpha_k(x_1)P_j'(x_1)\,,\qquad j,k=2,\dots,n\,. \tag{2.22}$$

Case 1. $\alpha_2=\cdots=\alpha_n=0$. In this case, (2.21) yields $P_2'(x_1)=\cdots=P_n'(x_1)=0$, since $\beta\neq 0$ by (2.13). Hence, $P_2, \dots, P_n$ are constant vectors in C^n. Therefore, (2.18) becomes $L(x_1, \dots, x_n)=D(x_1)+\sum_{j=2}^{n} c_j x_j$, for some function $D=D(x_1)$ and orthonormal constant vectors $c_2, \dots, c_n\in C^n$. This means that L is a Lagrangian cylinder over the curve $D=D(x_1)$ whose ruling are $(n-1)$-planes parallel to the totally real $x_2\cdots x_n$-plane in C^n.

Case 2. At least one of $\alpha_2, \dots, \alpha_n$ is nonzero. In this case, without loss of generality, we may assume $\alpha_2\neq 0$. By making the following change of variables:

$$t=\int_0^{x_1}\alpha_2(x_1)dx_1\,,\quad u_2=x_2,\dots,u_n=x_n\,, \tag{2.23}$$

we obtain

$$g=\hat{f}^2dt^2+du_2^2+\cdots+du_n^2\,, \tag{2.24}$$

where

$$\hat{f}=b(t)+u_2+\sum_{j=3}^{n} a_j(t)u_j\,, \tag{2.25}$$

for some functions $b(t), a_3(t), \dots, a_n(t)$. From (2.9) and (2.23) we obtain

$$h\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\lambda(t)J\left(\frac{\partial}{\partial t}\right),\quad h\left(\frac{\partial}{\partial t},\frac{\partial}{\partial u_j}\right)=h\left(\frac{\partial}{\partial u_j},\frac{\partial}{\partial u_k}\right)=0\,,\qquad j,k=2,\dots,n\,. \tag{2.26}$$

where $\lambda=(\alpha_2)^{-1}$ is a function of t. By applying (2.11), (2.24), (2.25), (2.26) and the formula of Gauss, we get

$$L_{tt}=\frac{1}{\hat{f}}\left(b'(t)+\sum_{j=3}^{n} a_j'(t)u_j\right)L_t-\hat{f}\sum_{k=2}^{n} a_k L_{u_k}+i\lambda L_t\,, \tag{2.27}$$

$$L_{tu_j}=\frac{a_j}{\hat{f}}L_t, \tag{2.28}$$

$$L_{u_ju_k}=0, \qquad j,k=2,\ldots,n, \tag{2.29}$$

where $a_2=1$. By solving (2.29), we find

$$L=\sum_{j=2}^{n} u_jP_j(t)+D(t), \tag{2.30}$$

for some $\boldsymbol{C}^n$-valued functions $P_2,\ldots,P_n$, D of t. Thus

$$L_t=\sum_{j=2}^{n} u_jP_j'(t)+D'(t), \quad L_{u_j}=P_j(t), \qquad j=2,\ldots,n. \tag{2.31}$$

(2.24) and (2.31) imply that $P_2,\ldots,P_n$ are orthonormal tangent vector fields on M^n. By applying (2.28) and (2.31), we obtain

$$D'(t)=b(t)P_2'(t), \quad P_k'(t)=a_k(t)P_2'(t), \qquad k=2,\ldots,n. \tag{2.32}$$

Substituting (2.32) into (2.31) yields

$$L_t=\hat{f}P_2'(t). \tag{2.33}$$

(2.24) and (2.33) imply that $P_2'(t)$ is a unit vector field.

If we put $z(t)=P_2(t)$, then $z=z(t)$ can be regarded as a unit speed spherical curve $z: I\to S^{2n-1}\subset \boldsymbol{C}^n$ defined on some open interval I. Since L is Lagrangian, (2.31) and (2.33) imply that $z=z(t)$ is a Legendre curve in S^{2n-1}. Moreover, by (2.31) and (2.32) we know that $z(t), iz(t), z'(t), iz'(t), P_3(t), iP_3(t),\ldots,P_n(t), iP_n(t)$ form an orthonormal frame field where $P_3,\ldots,P_n$ are parallel normal vector fields along the Legendre curve. Furthermore, (2.30) and (2.33) imply that, up to rigid motions of $\boldsymbol{C}^n$, L is given by

$$L(t,u_2,\ldots,u_n)=u_2z(t)+\sum_{k=3}^{n} u_kP_k(t)+\int^t b(t)z'(t)dt. \tag{2.34}$$

Finally, from (2.27), (2.31), (2.32) and (2.34), we know that $z=z(t)$ satisfies (1.3). Therefore, $z=z(t)$ in (2.34) is a special Legendre curve in S^{2n-1}. □

3. Proof of the existence theorem. Let $\lambda(t), a_3(t),\ldots,a_m(t)$ be $n-1$ functions of t defined on an open interval I with λ nowhere zero. Put

$$f(t,u_2,\ldots,u_n)=1+u_2+\sum_{j=3}^{n} a_j(t)u_j. \tag{3.1}$$

Consider the twisted product manifold $M^n(0)$ with twisted product metric

$$g=f^2dt^2+du_2^2+\cdots+du_n^2. \tag{3.2}$$

Then $M^n(0)$ is a flat Riemannian n-manifold. Define a symmetric bilinear form σ on $M^n(0)$ by

$$(3.3)\qquad \sigma\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\lambda\frac{\partial}{\partial t}\,,\quad \sigma\left(\frac{\partial}{\partial t},\frac{\partial}{\partial u_j}\right)=0\,,\quad \sigma\left(\frac{\partial}{\partial u_j},\frac{\partial}{\partial u_k}\right)=0\,,\qquad j,k=2,\dots,n\,.$$

Then $\langle\sigma(X, Y), Z\rangle$ is totally symmetric in X, Y and Z.

From (3.2) and (3.3) it follows that $(\nabla\sigma)(X, Y, Z)$ is totally symmetric in X, Y and Z and, moreover, σ and the Riemann curvature tensor R of M satisfy

$$(3.4)\qquad R(X, Y)Z=\sigma(\sigma(Y, Z), X)-\sigma(\sigma(X, Z), Y)\,.$$

Theorems A and B of [2] imply that, up to rigid motions of $\boldsymbol{C}^n$, there is a unique Lagrangian isometric immersion $L\colon \hat{M}^n(0)\to\boldsymbol{C}^n$ with second fundamental form given by $h=J\sigma$.

(3.1)–(3.3) and $h=J\sigma$ imply that L satisfies

$$(3.5)\qquad L_{tt}=\frac{1}{f}\sum_{j=3}^{n}a_j'(t)u_jL_t-f\sum_{k=2}^{n}a_kL_{u_k}+i\lambda L_t\,,$$

$$(3.6)\qquad L_{tu_j}=\frac{a_j}{f}L_t\,,\quad L_{u_ju_k}=0\,,\qquad j,k=2,\dots,n\,,$$

where $a_2=1$. Solving (3.6) as before yields

$$(3.7)\qquad L=\sum_{j=2}^{n}u_jP_j(t)+D(t)\,,$$

$$(3.8)\qquad L_t=fP_2'(t)\,,\quad L_{u_k}=P_k(t)\,,\quad D'(t)=P_2'(t)\,,\quad P_k'(t)=a_k(t)P_2'(t)\,,\qquad k=2,\dots,n\,,$$

for some $\boldsymbol{C}^n$-valued functions $P_3,\dots,P_n$, D.

From (3.2) and (3.8), it follows that $P_2'(t)$ is a unit vector field and, moreover, $P_2(t),\dots,P_n(t)$ are orthonormal vector fields. Put $z(t)=P_2(t)$. Then $z\colon I\to S^{2n-1}\subset\boldsymbol{C}^n$ is a unit speed curve defined on some open interval I. Since L is Lagrangian, (3.8) implies that $z(t), iz(t), z'(t), iz'(t), P_3(t), iP_3(t),\dots,P_n(t), iP_n(t)$ form an orthonormal frame field with $P_3,\dots,P_n$ being parallel orthonormal normal vector fields along z and $z=z(t)$ is a Legendre curve in S^{2n-1}. Finally, from (3.5) and (3.8), we conclude that $z=P_2$ is a special Legendre curve in S^{2n-1} satisfying (1.3) for some associated parallel normal vector fields $P_3,\dots,P_n$. □

4. Examples of special Legendre curves. Legendre curves in $S^3\subset\boldsymbol{C}^2$ are special Legendre curve automatically. Here, we provide some examples of special Legendre curves in $S^{2n-1}\subset\boldsymbol{C}^n$ for $n\geq 3$.

EXAMPLES. Let $\lambda, a_3,\dots,a_n$ be $n-1$ real numbers with $\lambda>0$. Put

$$\gamma = 1 + \sum_{j=3}^{n} a_j^2, \quad \mu = (\lambda^2 + 4\gamma)^{1/2} \tag{4.1}$$

$$\begin{aligned} z(s) = {} & \frac{\mu-\lambda}{2\mu\gamma}\left(\frac{2\gamma}{\mu-\lambda}, 1, a_3, \dots, a_n\right)e^{(\lambda+\mu)is/2} \\ & + \frac{\lambda+\mu}{2\mu\gamma}\left(-\frac{2\gamma}{\lambda+\mu}, 1, a_3, \dots, a_n\right)e^{(\lambda-\mu)is/2} - \frac{1}{\gamma}(0, 1-\gamma, a_3, \dots, a_n), \end{aligned} \tag{4.2}$$

$$c_3 = (0, a_3, -1, 0, \dots, 0), \dots, c_n = (0, a_n, 0, \dots, 0, -1). \tag{4.3}$$

Then, $z = z(s)$ is a (unit speed) special Legendre curve in $S^{2n-1} \subset \boldsymbol{C}^n$ satisfying

$$z''(s) = i\lambda z'(s) - z(s) - \sum_{j=3}^{n} a_j P_j(s), \tag{4.4}$$

where

$$P_j(s) = a_j z(s) - c_j, \qquad j = 3, \dots, n, \tag{4.5}$$

are the associated orthonormal parallel normal vector fields.

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