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REPRESENTATION OF ŁUKASIEWICZ AND POST ALGEBRAS BY CONTINUOUS FUNCTIONS

BY

ROBERTO CIGNOLI (BAHÍA BLANCA)

Introduction. The aim of this paper is to show that Łukasiewicz algebras can be isomorphically represented on some Łukasiewicz algebras of real continuous functions on a Boolean space (i. e., totally disconnected compact Hausdorff space), and to apply this representation to obtain some results of algebraic nature, namely a characterization of the injective n-valued Łukasiewicz algebras and a description of the free Post algebra of order n with an arbitrary non-void set of (free) generators.

n-valued Łukasiewicz algebras were introduced by Moisil in [11] (see also [12] and [13]). A systematic study from the standpoint of general algebra as well as reference to previous work can be found in the author's paper [5](1).

Post algebras of order n were introduced by Rosenbloom in [17], and further developed by Epstein [7], Traczyk [19]-[21] and Dwinger [6]. The connections between Łukasiewicz and Post algebras were established in [5].

In section 1 we briefly summarize the properties of Łukasiewicz and Post algebras needed for the understanding of this paper. The details can be found in [5].

In section 2 we give the main representation theorem, which is a generalization of a theorem of Epstein on Post algebras, and applying some standard techniques we obtain a "duality" between Łukasiewicz algebras

(1) It was shown by A. Rose that n-valued Łukasiewicz algebras are not matrices of the n-valued Łukasiewicz propositional calculus if $n \geqslant 5$ (see Introduction of [5]). For this reason, in [5] and in the Abstract of the present paper published in the Notices of the American Mathematical Society 17 (1970), p. 805, these algebras were called *Moisil algebras* of order n. The restoring of the original name is due to a suggestion of the Referee, who pointed out the existence of a great number of papers in which the name Łukasiewicz algebra was used, and that from the view point of applications of Łukasiewicz algebras the operation of implication is of a secondary importance.

and Boolean spaces similar to the duality between distributive lattices and Stone spaces. The best results are obtained for Post algebras; it is shown that the dual of the category of Post algebras of order n and homomorphisms is the category of Boolean spaces and continuous mappings. Some related results can be found in [8] and [9].

In section 3 we apply the above-mentioned representation theorem to characterize the injective n-valued Łukasiewicz algebras as the complete Post algebras of the same order, and in section 4 we study the free products of Post algebras, giving a description of the free Post algebra of order n with an arbitrary non-void set of (free) generators.

We freely use the notions of universal algebra, which can be found, for instance, in [2], Chapter VI, or [16], Chapter I, and the properties of Boolean algebras, mainly the duality theory, as can be found in [10].

Sections 1-3 of this paper are based on results contained in the author's doctoral thesis, submitted to the Universidad Nacional del Sur in October 1969. The author wishes to express his deep thanks to Professor A. Monteiro for his help and guidance.

1. n-valued Łukasiewicz algebras and Post algebras of order n.

1.1. Definition. An *n-valued Łukasiewicz* algebra (n an integer $\geqslant 2$) is a system $\langle A, 1, \vee, \wedge, \sim, s_1, \ldots, s_{n-1}\rangle$ such that $\langle A, 1, \vee, \wedge\rangle$ is a distributive lattice with unit 1 and $\sim, s_1, \ldots, s_{n-1}$ are unary operators defined on A fulfilling the following conditions:

M 1. $\sim\sim x = x$,
M 2. $\sim(x \vee y) = \sim x \wedge \sim y$,
L 1. $s_i(x \vee y) = s_i x \vee s_i y$,
L 2. $s_i x \vee \sim s_i x = 1$,
L 3. $s_i s_j x = s_j x$,
L 4. $s_i \sim x = \sim s_{n-i} x$,
L 5. $s_i x \vee s_{i+1} x = s_{i+1} x \ (1 \leqslant i \leqslant n-2)$,
L 6. $x \vee s_{n-1} x = s_{n-1} x$,
L 7. $(x \wedge \sim s_i x \wedge s_{i+1} y) \vee y = y \ (1 \leqslant i \leqslant n-2)$.

As usual, an n-valued Łukasiewicz algebra is denoted simply by A.

We recall that the element $0 = \sim 1$ is the zero of the lattice A. It is convenient to define the operators s_0 and s_n by

$$s_0 x = 0 \text{ and } s_n x = 1 \text{ for any } x \text{ in } A. \tag{1.1}$$

An important property is the following determination principle, introduced by Moisil to define n-valued Łukasiewicz algebras instead of equations L6 and L7:

$$\text{if } s_i x = s_i y \text{ for } i = 1, 2, \ldots, n-1, \text{ then } x = y. \tag{1.2}$$

The Boolean algebra of all complemented elements of A is denoted by $B(A)$; we have

(1.3) $x \in B(A)$ if and only if $s_i x = x$ $(i = 1, \ldots, n-1)$.

Furthermore, if $x \in B(A)$, the complement of x is $\sim x$.

L_n denotes the n-valued Łukasiewicz algebra formed by the fractions $j/n-1$, $j = 0, 1, \ldots, n-1$, with the natural lattice operations and $\sim$ and s_i defined as follows:

$$\sim (j/n-1) = 1-(j/n-1),$$

$$s_i(j/n-1) = \begin{cases} 0 & \text{if } i+j < n \\ 1 & \text{if } i+j \geqslant n. \end{cases} \quad (i = 1, \ldots, n-1; j = 0, 1, \ldots, n-1)$$

Let U be an ultrafilter of the Boolean algebra $B(A)$. Then $U_i = s_i^{-1}(U)$, $i = 1, \ldots, n-1$, are prime filters of the lattice A and $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_{n-1}$. Furthermore, if P is any prime filter of A, there exists a unique ultrafilter U of $B(A)$ and an i $(1 \leqslant i \leqslant n-1)$ such that $P = U_i$.

For any ultrafilter U of $B(A)$ we define

(1.4) $h_U(x) = n-j/n-1$ if and only if $x \in U_j - U_{j-1}$, $j = 1, 2, \ldots, n$, where $U_0 = \emptyset$ and $U_n = A$.

$h_U\colon A \to L_n$ is a homomorphism and the correspondence $U \mapsto h_U$ is a bijection between the ultrafilters of $B(A)$ and the homomorphisms of A into L_n.

Condition (1.4) implies that

(1.5) $h_U(x) \geqslant n-j/n-1$ if and only if $s_j x \in U$.

For, if $h_U(x) \geqslant n-j/n-1$, then $h_U(x) = n-i/n-j$ with $i \leqslant j$. From (1.4) and the definition of the U_i it follows that $s_i x \in U$, but since $s_i x \leqslant s_j x$ and U is a filter, $s_j x \in U$. Conversely, if $s_j x \in U$, then $s_i x \in U$ for any $i \geqslant j$, so $x \notin U_i - U_{i-1}$ if $i \geqslant j+1$. Hence $h_U(x) = n-i/n-1$ implies $i \leqslant j$.

The following lemma is a consequence of the results of [5] on the congruences in Łukasiewicz algebras, but we give the direct proof:

1.2. LEMMA. *If $h_U(x) = h_U(y)$ for any ultrafilter U of $B(A)$, then $x = y$.*

Proof. If $h_U(x) = h_U(y)$, then it follows at once from (1.5) that $s_i x \in U$ if and only if $s_i y \in U$ for $i = 1, 2, \ldots, n-1$. Therefore, if $h_U(x) = h_U(y)$ for any ultrafilter U of $B(A)$, we must have $s_i x = s_i y$ for $i = 1, 2, \ldots, n-1$ and it follows by (1.2) that $x = y$.

The connection between Łukasiewicz and Post algebras is given in the following

1.3. THEOREM ([5], 8.6). *A is a Post algebra of order n if and only if* 1° *A is an n-valued Łukasiewicz algebra and* 2° *A has $n-2$ elements $e_1, e_2, \ldots, e_{n-2}$ satisfying the following conditions:*

$$s_i e_j = \begin{cases} 0 & \text{if } i+j < n, \\ 1 & \text{if } i+j \geqslant n. \end{cases} \tag{1.6}$$

It is convenient to define $e_0 = 0$ and $e_{n-1} = 1$.

Observe that L_n is a Post algebra of order n, where $e_j = j/n-1$, $j = 0, 1, \ldots, n-1$.

From this theorem it follows that Post algebras of order n can be considered as algebras of the form

$$\langle A, \vee, \wedge, \sim, s_1, \ldots, s_{n-1}, e_1, \ldots, e_{n-1}\rangle,$$

where $\vee$ and $\wedge$ are binary operations, $\sim$ and $s_1, \ldots, s_{n-1}$ are unary operations, and $e_1, \ldots, e_{n-1}$ are constants, satisfying the postulates of Definition 1.1 together with (1.6). Observe that Post algebras of order m and n are not similar if $m \neq n$. So, following the standard definitions of universal algebra, we shall consider only homomorphisms between Post algebras of the same order. This point of view is not followed by other authors (see [19], Definition 4.1, and [6], p. 471) who define homomorphisms between Post algebras of arbitrary order. But these are not proper homomorphisms in the algebraic sense: for example, they do not preserve the $\sim$ operation, as can be easily seen using the algebras L_n. The same remarks apply, of course, to Łukasiewicz algebras.

Let us remark that Theorem 1.2 gives an equational characterization of Post algebras of order n. Other equational characterization was given by Traczyk in [20].

If A, A' are Post algebras of order n, a function $h: A \to A'$ is called a *Łukasiewicz homomorphism* if it is a homomorphism of A into A' considered as Łukasiewicz algebras. The following lemma is an immediate consequence of (1.6):

1.4. LEMMA. *Let A and A' be Post algebras of order n. A function $h: A \to A'$ is a homomorphism if and only if it is a Łukasiewicz homomorphism. Furthermore, if A is a Post algebra of order n, A' an n-valued Łukasiewicz algebra and $h: A \to A'$ a Łukasiewicz homomorphism, then A' is also a Post algebra of order n and h is a homomorphism.*

Finally, we recall that any element x of a Post algebra A of order n can be written in the form

$$x = (s_{n-1}x \wedge e_1) \vee \ldots \vee (s_2 x \wedge e_{n-2}) \wedge s_1 x. \tag{1.7}$$

2. Representation by continuous functions. If X is a set, we denote by L_n^X the set of all functions from X into L_n. Since n-valued Łukasiewicz algebras are equationaly characterizable, it follows that L_n^X becomes an n-valued Łukasiewicz algebra if we define the operations pointwise.

From now on, X will denote a *Boolean space* (i. e., a totally disconnected Hausdorff compact space), $B(X)$ — the Boolean algebra of all clopen sets of X, $C_n(X)$ — the set of all continuous functions from X into L_n considered as a subspace of the real numbers (i. e., L_n with the discrete topology), and $K(X)$ — the subset of $C_n(X)$ of functions with values 0 or 1.

$C_n(X)$ is contained in L_n^X, and since the elements of $C_n(X)$ are real continuous functions, it follows at once that $C_n(X)$ is closed under the operations $\vee$, $\wedge$ and $\sim$ (see the definition of L_n). On the other hand, it is easy to see that if f is in $C_n(X)$, then $s_i f \in K(X)$ and

$$(s_i f)^{-1}(\{0\}) = \bigcup_{i<j} f^{-1}(\{n-j/n-1\}),$$

$$(s_i f)^{-1}(\{1\}) = \bigcup_{i\geqslant j} f^{-1}(\{n-j/n-1\}).$$

Hence $C_n(X)$ is also closed under the s_i ($i = 1, \ldots, n-1$), and it is, consequently, a subalgebra of L_n^X.

Therefore, $C_n(X)$ is an n-valued Łukasiewicz algebra. Furthermore, $K(X) = B\big(C_n(X)\big)$ as follows at once from the definitions.

2.1. LEMMA. *If $\hat{A}$ is a subalgebra of $C_n(X)$ such that $K(X) \subseteq \hat{A}$, then $K(X) = B(\hat{A})$ and X is homeomorphic to the Stone space of $B(\hat{A})$.*

Proof. It follows from (1.3) that if A' is a subalgebra of an n-valued Łukasiewicz algebra A, then $B(A') = A' \cap B(A)$, and the first part of the lemma follows at once from this remark. The second is an easy consequence of the fact that X is homeomorphic to the Stone space of any Boolean algebra isomorphic to $K(X)$.

2.2. LEMMA. *Let $\hat{A}$ be a subalgebra of $C_n(X)$ such that $K(X) \subseteq \hat{A}$. Then $\hat{A} = C_n(X)$ if and only if $\hat{A}$ contains the constant functions f_j of value $j/n-1, j = 1, 2, \ldots, n-2$.*

Proof. It is clear that $f_j \in C_n(X)$ and $s_i f_j = 0$ if $i+j < n$ and $s_i f_j = 1$ if $i+j \geqslant n$ ($j = 1, 2, \ldots, n-2; i = 1, 2, \ldots, n-1$). So, by Theorem 1.3, $C_n(X)$ is a Post algebra of order n (we remark that this fact was already proved by Epstein in [7]). From property (1.7), any $f \in C_n(X)$ can be written in the form

$$f = (s_{n-1} f \wedge f_1) \vee \cdots \vee (s_2 f \wedge f_{n-2}) \vee s_1 f.$$

Since $s_i f \in K(X) \subseteq \hat{A}$, and $\hat{A}$ is a lattice, it follows that $f_j \in \hat{A}$ for $j = 1, 2, \ldots, n-2$ implies that $C_n(X) = \hat{A}$.

Let A be an n-valued Łukasiewicz algebra and X the Stone space of $B(A)$. We denote by λ the Stone isomorphism between $B(A)$ and

$B(X)$, i. e.

(2.1) $$\lambda(b) = \{U \in X : b \in U\}, \quad b \in B(A).$$

If $x \in A$, we define the function $\hat{x} : X \to L_n$ by the formula

(2.2) $$\hat{x}(U) = h_U(x).$$

It follows from (1.4) that $\hat{x}(U) = n-j/n-1$ if and only if $s_j x \in U$ and $s_{j-1} x \notin U$ $(1 \leqslant j \leqslant n)$, that is, if and only if $U \in \lambda(s_j x) - \lambda(s_{j-1} x)$.

Therefore,

$$\hat{x}^{-1}(\{n-j/n-1\}) = \lambda(s_j x) - \lambda(s_{j-1} x), \quad 1 \leqslant j \leqslant n,$$

and this shows that $\hat{x}$ is in $C_n(X)$.

The function $\hat{x}$ can be called the *Fourier transform* of the element x.

2.3. THEOREM. *For any n-valued Łukasiewicz algebra A (with more than one element), there exists a Boolean space X, unique up to homeomorphisms, such that 1° A is isomorphic to a subalgebra $\hat{A}$ of $C_n(X)$ and 2° $K(X) \subseteq \hat{A}$. A is a Post algebra if and only if $\hat{A} = C_n(X)$.*

The space X is called the *Boolean spectrum* of A.

Proof. Let X be the Stone space of the Boolean algebra $B(A)$. Since the $h_U : A \to L_n$ are homomorphisms, it follows that the correspondence $x \mapsto \hat{x}$ is a homomorphism from A into $C_n(X)$ and actually, by Lemma 1.2, an isomorphism between A and $\hat{A} = \{\hat{x} : x \in A\}$.

If $k \in K(X)$, then $k^{-1}(\{1\}) \in B(X)$; hence there exists $b \in B(A)$ such that $\lambda(b) = k^{-1}(\{1\})$. Since $s_1 b = \ldots = s_{n-1} b = b$, we have $\hat{b}(U) = 1$ if $b \in U$ and $= 0$ if $b \notin U$, which implies that $\hat{b} = k$. So $K(X) \subseteq \hat{A}$.

The uniqueness of X follows at once from Lemma 2.1.

Finally, if A is a Post algebra, it is easy to see that $\hat{e}_j = f_j$, so $f_j \in \hat{A}$ and, by Lemma 2.2, $\hat{A} = C_n(X)$. The converse follows from Lemma 1.4 and the fact that $C_n(X)$ is a Post algebra.

Theorem 2.3 generalizes a result of Epstein [7], Theorem 16. The special case for $n = 3$ was partially announced in [4]. Observe that the $n = 2$ case is the well known result of Stone on the representation of Boolean algebras.

2.4. LEMMA. *Let A, A' be n-valued Łukasiewicz algebras, X and X' their Boolean spectra and $h : A \to A'$ a homomorphism. If h_1 is the restriction of h to $B(A)$, then $h_1 : B(A) \to B(A')$ is a Boolean homomorphism, and if we define, for any U in X', $h^*(U) = h_1^{-1}(U)$, then $h^* : X' \to X$ is a continuous function, called the dual transformation of h. Furthermore, the correspondence $h \mapsto h^*$ is one-to-one and if h is onto (one-to-one), then h^* is one-to-one (onto).*

Proof. Suppose $b \in B(A)$. Then $s_1(b) = b$ and $s_1 h(b) = h(s_1 b) = h(b)$ which implies that $h(b) \in B(A')$ (see (1.3)). Hence $h_1 : B(A) \to B(A')$, and it is easy to see that it is a Boolean homomorphism. The fact that $h^*: X' \to X$ is continuous is well known from the theory of Boolean algebras.

If $g: A \to A'$ is a homomorphism such that $h^* = g^*$, then we must have $h_1 = g_1$, so if $x \in A$, $s_i h(x) = h(s_i x) = h_1(s_i x) = g_1(s_i x) = g(s_i x) = s_i g(x)$ for $i = 1, \ldots, n-1$, and it follows by (1.2) that $h(x) = g(x)$, which proves that $h \mapsto h^*$ is one-to-one.

Finally, if h is onto (one-to-one), it follows that h_1 is onto (one-to-one), and by a well known result in the theory of Boolean algebras, h^* is one-to-one (onto).

2.5. Lemma. *Let A be an n-valued Łukasiewicz algebra and P a Post algebra of order n, X and Y their Boolean spectra, and $f: Y \to X$ a continuous function. Then there exists a (unique) homomorphism $h: A \to P$ such that $h^* = f$. Furthermore, if f is onto, then h is one-to-one.*

Proof. Let $\hat{A}$ be the subalgebra of $C_n(X)$ isomorphic to A. If we define $\hat{x}^* = \hat{x}f$ for any $\hat{x} \in \hat{A}$, then $\hat{x}^* \in C_n(Y)$ and, since the operations in $C_n(Y)$ are defined pointwise, it follows that the correspondence $\hat{x} \mapsto \hat{x}^*$ is a homomorphism. Therefore we obtain a homomorphism $h: A \to P$ if we define $h(x)$ as the element of P given by $\big(h(x)\big)\hat{} = \hat{x}^*$.

To see that $h^* = f$, note that the following conditions are equivalent for any U in Y and b in $B(A)$: 1° $b \in h^*(U) = h_1^{-1}(U)$; 2° $h_1(b) = h(b) \in U$; 3° $\big(h(b)\big)\hat{}(U) = 1$; 4° $\hat{b}\big(f(U)\big) = 1$ and 5° $b \in f(U)$.

Since $h_1: B(A) \to B(P)$ is the Boolean homomorphism dual to $f: Y \to X$, h_1 is one-to-one if f is onto, and this implies that h is one-to-one. For, if $x \neq y$, by (1.2) there exists an i such that $s_i x \neq s_i y$, and therefore $s_i h(x) = h(s_i x) = h_1(s_i x) \neq h_1(s_i y) = h(s_i y) = s_i h(y)$, so $h(x) \neq h(y)$.

Summing up, we have

2.6. Theorem. *Let A be an n-valued Łukasiewicz algebra and P a Post algebra of order n, X and Y their Boolean spectra. Then the correspondence $h \mapsto h^*$ establishes a bijection between the (Łukasiewicz) homomorphism $h: A \to P$ and the continuous functions from Y into X. Furthermore, h is one-to-one if and only if h^* is onto, and if h is onto, then h^* is one-to-one.*

2.7. Corollary. *Let P, P' be Post algebras of order n and X, X' their Boolean spectra. Then the correspondence $h \mapsto h^*$ establishes a bijection between the homomorphisms $h: P \to P'$ and the continuous functions from X' into X. Furthermore, h is one-to-one (onto) if and only if h^* is onto (one-to-one).*

Proof. Taking into account Lemma 1.4, we only need to prove that if h^* is one-to-one, then h is onto. But if h^* is one-to-one, then h_1 is onto, and by (1.7) this implies that also h is onto.

2.8. Corollary. *The dual of the category of Post algebras of order n and homomorphisms is isomorphic to the category of Boolean spaces and continuous functions, for any $n \geqslant 2$.*

The last result implies that the category of Post algebras of order n and homomorphisms is isomorphic to the category of Boolean algebras and Boolean homomorphisms, for any $n \geqslant 2$.

2.9. Remark. The correspondence between n-valued Łukasiewicz algebras and Boolean spectra is not one-to-one. In order to obtain a one-to-one correspondence between n-valued Łukasiewicz algebras and some topological spaces, Boolean spectra should be replaced by the prime spectra, i. e. the set of all prime filters with the Stone topology. A description of the dual of the category of n-valued Łukasiewicz algebras and homomorphisms in terms of the prime spectra is contained in the author's doctoral thesis, and will be published elsewhere (see also [8] and [9]).

2.10. Definition. Let P be a Post algebra of order n. An n-valued Łukasiewicz algebra A is said to be an *L-subalgebra* of P if it is a subalgebra of P considered as a Łukasiewicz algebra.

An immediate consequence of Theorem 2.3 is that any n-valued Łukasiewicz algebra is isomorphic to an L-subalgebra of a Post algebra of order n. This shows that the relation between Łukasiewicz and Post algebras is similar to that existing between distributive lattices and Boolean algebras: each distributive lattice can be embedded in a Boolean algebra. But it is well known that each distributive lattice can be embedded in a minimal Boolean algebra [15]. We shall see that an analogous theorem holds for Łukasiewicz and Post algebras, and we shall characterize the Post algebra $C_n(X)$ as a minimal extension. We begin by the following:

2.11. THEOREM. *Let A, A' be n-valued Łukasiewicz algebras, X, X' their Boolean spectra and $h: A \to A'$ a homomorphism. There exists a unique homomorphism $H: C_n(X) \to C_n(X')$ such that $H(\hat{x}) = \big(h(x)\big)\hat{}$ for any $x \in A$.*

Proof. Let h_1 be the restriction of h to $B(A)$, and $h_1^*: X' \to X$ the continuous transformation dual of h_1 (cf. Lemma 2.4). If $f \in C_n(X)$, we define $Hf = f^* = fh_1^*$. As in the proof of Lemma 2.5, we see that $H: C_n(X) \to C_n(X')$ is a homomorphism. If $x \in A$ and $U \in X'$, then $\big(H(\hat{x})\big)(U) = \hat{x}^*(U) = \hat{x}\big(h_1^*(U)\big)$. But $\hat{x}\big(h_1^*(U)\big) = \big(h(x)\big)\hat{}\,(U)$. For, taking into account (1.5), the following conditions are equivalent: 1. $\hat{x}\big(h_1^*(U)\big) \geqslant j/n-1$; 2. $s_j x \in h_1^*(U) = h_1^{-1}(U)$; 3. $h_1(s_j x) = h(s_j x) = s_j h(x) \in U$ and 4. $\big(h(x)\big)\hat{}\,(U) \geqslant j/n-1$. Therefore, $H(\hat{x}) = \big(h(x)\big)\hat{}$ for any $x \in A$.

If H and H' are homomorphisms from $C_n(X)$ into $C_n(X')$ such that $H(\hat{x}) = H'(\hat{x})$ for any $x \in A$, then $H = H'$. For, since $K(X) \subseteq \hat{A}$ and $K(X) = B\big(C_n(X)\big)$, it follows that $H_1 = H_1'$ and then $H = H'$ (cf. the proof of Lemma 2.4).

2.12. COROLLARY. *Let A be an n-valued Łukasiewicz algebra and X its Boolean spectrum. If P is a Post algebra of order n and $h: A \to P$*

a (Łukasiewicz) homomorphism, then there exists a unique homomorphism $f: C_n(X) \to P$ *such that* $f(\hat{x}) = h(x)$ *for any* $x \in A$.

Proof. Let Y be the Boolean spectrum of P. From Theorem 3.2 we know that $y \mapsto \hat{y}$ is a isomorphism of P onto $C_n(Y)$. Call $\varphi: C_n(Y) \to P$ the inverse isomorphism. From Theorem 2.11, there exists a unique homomorphism $H: C_n(X) \to C_n(Y)$ such that $H(\hat{x}) = (h(x))\hat{}$. If we define $f = \varphi H$, it follows that $f: C_n(X) \to P$ is a homomorphism and that $f(\hat{x}) = h(x)$ for any x in A. The uniqueness of f follows considering the restriction f_1 to $B(A)$.

3. Injective algebras. We recall that an n-valued Łukasiewicz algebra is said to be *complete* if it is a complete lattice. Since M1 and M2 imply that the conditions $x \leqslant y$ and $\sim y \leqslant \sim x$ are equivalent in any Łukasiewicz algebra, it follows at once that an n-valued Łukasiewicz algebra A is complete if and only if there exists $\bigcup_{i \in I} x_i$ for any family $\{x_i\}_{i \in I}$ of elements of A.

The following lemma is an immediate generalization of [7], Theorem 21, and [14], Lemma 1.1, and the proof will be omitted.

3.1. LEMMA. *If A is a complete n-valued Łukasiewicz algebra, then $B(A)$ is a complete Boolean algebra.*

3.2. Remark. In [7], Theorem 23, it is proved that the converse of Lemma 3.1 holds for Post algebras. In the author's doctoral thesis an example is given of a non-complete three-valued Łukasiewicz algebra A such that $B(A)$ is a complete Boolean algebra.

We need the following results:

3.3. THEOREM ([7], Theorem 26). *The normal completion P_N of a Post algebra P of order n is a Post algebra of order n. Furthermore, if $B_N(P)$ is the normal completion of $B(P)$, then $B_N(P) = B(P_N)$.*

3.4. COROLLARY. *Any n-valued Łukasiewicz algebra is isomorphic to an L-subalgebra of a complete Post algebra of order n.*

We recall that an n-valued Łukasiewicz algebra C is said to be *injective* if for any pair of n-valued Łukasiewicz algebras A, A' and any homomorphism $f: A' \to C$ and any monomorphism $g: A' \to A$, there exists a homomorphism $f: A \to C$ such that $fg = h$.

Using the same technique as in [14], p. 579 (cf. also [10], p. 141, and [1], Theorem 3.2), and taking into account Lemma 1.4, we can prove

3.5. LEMMA. *Any injective n-valued Łukasiewicz algebra is a complete Post algebra of order n.*

3.6. LEMMA. *Any complete Post algebra P of order n is injective in the category of n-valued Łukasiewicz algebras.*

Proof. Let A and A' be n-valued Łukasiewicz algebras, $g: A \to A'$ a monomorphism and $h: A' \to P$ a homomorphism. If X, X' and Y are the Boolean spectra of A, A' and P, respectively, it follows from Lemma 2.4 that the dual transformations $h^*: Y \to X'$ and $g^*: X' \to X$ are continuous and, moreover, that g^* is onto. Since P is complete, $B(P)$ is complete, so Y is the Stone space of a complete Boolean algebra. By dualizing the results of Halmos on injective Boolean algebras (see [10], p. 143), there exists a continuous map $f: Y \to X$ such that $h^*f = g^*$. Let $f^*: A \to P$ be the dual homomorphism (Lemma 2.5). To complete the proof, we need to show that $f^*g = h$. But f^*g is defined by the condition $\big(f^*(g(x))\big)^\wedge = \big(g(x)\big)^\wedge f$ for any $x \in A'$. Let $U \in Y$. Proposition (1.3) and the definitions of g^* and h^* imply that the following conditions are equivalent: 1. $\big(g(x)\big)^\wedge \big(f(U)\big) \geqslant j/n-1$; 2. $s_j g(x) = g(s_j x) = g_1(s_j x) \in f(U)$; 3. $s_j x \in g_1^{-1}\big(f(U)\big) = g^*\big(f(U)\big) = h^*\big(f(U)\big) = h_1^{-1}(U)$; 4. $h_1(s_j x) = h(s_j x) = s_j\big(h(x)\big) \in U$ and 5. $\big(h(x)\big)^\wedge (U) \geqslant j/n-1$.

Therefore, $\big(g(x)\big)^\wedge f = h(x)$ for any $x \in A'$, and $f^*g = h$.

Summing up Lemmas 3.5 and 3.6, we have

3.7. THEOREM. *An n-valued Łukasiewicz algebra is injective if and only if it is a complete Post algebra of order n.*

3.8. COROLLARY. *A Post algebra of order n is injective if and only if it is complete.*

Theorem 3.7 was first proved for $n = 3$ by Monteiro in [14], but without reference to Post algebras. We remark that the proof of Lemma 3.6 given in the author's doctoral thesis follows the same lines of Monteiro's and differs from the proof given here. Monteiro's method was also used in [3] to give a similar characterization of injective Moisil algebras, but without reference to Post algebras. Another proof is in [9].

3.9. COROLLARY. *Any n-valued Łukasiewicz algebra is isomorphic to an L-subalgebra of an injective n-valued Łukasiewicz algebra.*

4. Free products of Post algebras. A free product (Sikorski [18]) of a family $\{P_i\}_{i \in I}$ of Post algebras of order n consists of a Post algebra P of order n together with a family $\{g_i: P_i \to P\}_{i \in I}$ of monomorphisms satisfying the following two requirements: 1. $\bigcup_{i \in I} g_i(P_i)$ generates P and 2. if $\{h_i: P_i \to A\}_{i \in I}$ is a family of homomorphism, where A is a Post algebra of order n, then there exists a homomorphism $h: P \to A$ with $hg_i = h_i$ for $i \in I$.

4.1. THEOREM. *The free product of a family $\{P_i\}_{i \in I}$ of Post algebras of order n is isomorphic to $C_n(X)$, where X is the cartesian product of the Boolean spectra X_i of P_i, $i \in I$.*

Proof. Since the cartesian product of Boolean spaces is a Boolean space, it follows that $C_n(X)$ is a Post algebra of order n.

Let $P_i: X \to X_i$ be the projections, and $p_i^*: P_i \to C_n(X)$ the dual homomorphisms. Since the p_i are onto, it follows from Corollary 2.7 that the p_i^* are monomorphisms.

The definition of the product topology implies that $\{p_i^*(b) : b \in B(P_i)\}_{i \in I}$ is a family of Boolean generators of $B(C_n(X))$, therefore $C_n(X)$ is the only subalgebra of $C_n(X)$ containing $\bigcup_{i \in I} p_i^*(P_i)$ and, a fortiori, $C_n(X)$ is generated by the set $\bigcup_{i \in I} p_i^*(P_i)$.

To complete the proof, we only need to consider the properties of the cartesian product and apply Corollary 2.7.

It was shown in [5], Theorem 8.10, that the free Post algebra of order n with r generators (r finite cardinal > 0) is isomorphic to the direct product of n^r copies of the Post algebra L_n. In particular, the free Post algebra of order n with one free generator, wich we denote by F_n, is the direct product of n copies of L_n. Hence the Boolean spectrum of F_n is the discrete space with n points, which we denote by T_n.

Since the free Post algebra of order n with c generators (c cardinal > 0) is the free product of c copies of F_n ([18], p. 215), we have

4.2. THEOREM. *The free Post algebra of order n with c (free) generators (c cardinal > 0) is isomorphic to $C_n(T)$, where T is the cartesian product of c copies of the space T_n.*

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INSTITUTO DE MATEMATICA
UNIVERSIDAD NACIONAL DEL SUR

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