On the time-frequency representation of operators and generalized Gabor multiplier approximations

Monika Dörfler [a] Bruno Torrésani [b]

[a] *University of Applied Sciences, St. Pölten, Matthias Corvinus-Straße 15 3100 St. Pölten, Austria.*

[b] *Laboratoire d'Analyse, Topologie et Probabilités, Centre de Mathématique et d'Informatique, 39 rue Joliot-Curie, 13453 Marseille cedex 13, France*

Abstract

The problem of representation and approximation of linear operators by means of modification in the time-frequency domain is considered. Before turning to the discrete and sub-sampled case, a complete characterization of linear operators by means of a twisted convolution in the continuous time-frequency domain is suggested. Subsequently, existing results on approximation by time-frequency multipliers are reviewed. To overcome the limitations imposed by these multipliers, two more general constructions are proposed, termed multiple Gabor multipliers and Twisted Spline type functions. Conditions ensuring the existence of optimal multiple Gabor multipliers are given. As the constructions suggested in this paper are mainly based on the Weyl-Heisenberg group, twisted convolution plays a central role in the results described in this paper.

Key words: Operator approximation, spreading function, twisted convolution, Gabor multiplier, optimal multiplier

1 Introduction

The usual goal of time-frequency transforms is to provide efficient representations for functions or distributions, in terms of weighted sums of *atoms* which are well localized in both time and frequency domains. The analyzed function or distribution is then characterized by the corresponding time-frequency coefficients, from which it is synthesized by the *synthesis map*. Concrete applications can be found mostly in signal analysis and processing (see [1,2,22] and references therein), but recent works in different areas such as numerical analysis may also be mentioned (see for example [5,6] and references therein).

Preprint submitted to Elsevier *3 September 2007*

Time-frequency analysis of operators, by far less developed, has enjoyed increasing interest during the last few years. Efficient time-frequency operator representation remains very challenging, despite a number of theoretical results.

Starting from the time-frequency representations, both continuous and discrete, of functions, it is an obvious first guess for operator representation, to change the coefficients in the time-frequency domain before resynthesizing the signal. This idea is at the heart of so-called time-frequency multipliers, for which the modification of the coefficients obtained by the time-frequency transform is restrained to be multiplicative. This restraint naturally leads to a very restrictive, be it important, class of operators, which are well-described by these representations.

Here, we start from a completely different point of view: motivated by a complete characterization of linear operators in the continuous short-time Fourier transform domain via a twisted convolution, we generalize the idea of time-frequency multipliers in several directions. In a few words, a Gabor multiplier is characterized by an analysis window, a synthesis window and a bounded sequence, which we shall call a *mask*, or *time-frequency transfer function*. For a given analysis window, the synthesis window may be adapted to accommodate time-frequency shifts, if the operator under study involves such shifts. In addition, several synthesis windows may be introduced to improve the quality of the approximation. We develop a construction that provides, under suitable assumptions on the windows, the optimal family of masks for a given Hilbert-Schmidt operator and a given family of synthesis windows. On the other hand, if the analysis window and a mask are fixed, one or several synthesis window may be constructed that yield a good Gabor multiplier or multiple Gabor multiplier approximation of a given Hilbert-Schmidt operator. Although the straightforward generalization of the operator representation in the continuous time-frequency domain via a twisted convolution is not possible, it turns out that in both cases, we end up dealing with some discrete version of the twisted convolution. This stems from the fact that the constructions we present here respect the structure imposed by the Heisenberg group.

This paper is organized as follows. The next section gives a review of the time-frequency plane and the corresponding continuous and discrete transforms. We then introduce the concept of Gelfand triples, which will allow us to consider operators beyond the Hilbert-Schmidt frame-work. The section closes with the important statement on operator-representation in the time-frequency domain via twisted convolution with an operator's spreading function. Section 3 introduces time-frequency multipliers and gives a criterion about their ability to approximate linear operators. Section 4 represents the main new insights in generalizations of Gabor multipliers. We conclude with the prospect of generalizations and applications which will be presented in future work.

2 Operators from the Time-frequency point of view

Whenever one is interested in the frequency content of a signal or operator which is desired to be time-localized at the same time, one is naturally led to the notion of the time-frequeny plane, which, in turn, is closely related to the Weyl-Heisenberg group.

2.1 Preliminaries: the time-frequency plane

The starting point of our operator analysis is the so-called *spreading function operator representation*. This operator representation expresses linear operators as a sum (in a sense to be specified below) of time-frequency shifts $\pi(b,\nu) = M_\nu T_b$. Here, the translation and modulation operators are defined as

$$T_b f(t) = f(t-b)\ , \quad M_\nu f(t) = e^{2i\pi\nu t} f(t)\ , \quad f \in \mathbf{L}^2(\mathbb{R})\ .$$

These operators generate a group, called the Weyl-Heisenberg group

$$\mathbb{H} = \{(b,\nu,\varphi) \in \mathbb{R} \times \mathbb{R} \times [0,1]\}\ , \tag{1}$$

with group multiplication

$$(b,\nu,\varphi)(b',\nu',\varphi') = (b+b', \nu+\nu', \varphi+\varphi'-\nu' b)\ . \tag{2}$$

The specific quotient space $\mathbb{P} = \mathbb{H}/[0,1]$ of the Weyl-Heisenberg group is called *phase space*, or *time-frequency plane*, which plays a central role in the subsequent analysis. Details on the Weyl-Heisenberg group and the time-frequency plane may be found in [13,23]. In the current article, we shall limit ourselves to the basic irreducible unitary representation of $\mathbb{H}$ on $\mathbf{L}^2(\mathbb{R})$, denoted by π^o, and defined by

$$\pi^o(b,\nu,\varphi) = e^{2i\pi\varphi} M_\nu T_b\ , \tag{3}$$

and we also denote by $\pi(b,\nu) = \pi^o(b,\nu,0)$ the restriction to the phase space. We refer to [3] or [16, Chapter 9] for a more detailed analysis of this quotient operation.

The left-regular (and right-regular) representation(s) generally plays a central role in group representation theory. The Weyl-Heisenberg group being unimodular, its left and right regular representations coincide, we thus focus on the left-regular one, acting on $\mathbf{L}^2(\mathbb{H})$ and defined by

$$[L(b',\nu',\varphi')F](b,\nu,\varphi) = F(b-b',\nu-\nu',\varphi-\varphi'+b'(\nu-\nu')) \ . \tag{4}$$

Denote by μ the Haar measure. Given $F,G \in \mathbf{L}^2(\mathbb{H},d\mu)$, the associated (left) convolution product is the bounded function $F*G$, given by

$$(F*G)(b,\nu,\varphi) = \int_{\mathbb{H}} F(h)[L(b,\nu,\varphi)G](h)\, d\mu(h) \ . \tag{5}$$

After quotienting out the phase term, this yields the *twisted convolution* on $\mathbf{L}^2(\mathbb{P})$:

$$(F\natural G)(b,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(b',\nu')G(b-b',\nu-\nu')e^{-2i\pi b'(\nu-\nu')}\, db'd\nu' \ . \tag{6}$$

The twisted convolution, which admits a nice interpretation in terms of group plancherel theory [3] is non-commutative (which reflects the non-Abelianess of $\mathbb{H}$) but associative. It satisfies the usual Young inequalities, but is in some sense nicer than the usual convolution, since $\mathbf{L}^2(\mathbb{R}^2)\natural\mathbf{L}^2(\mathbb{R}^2) \subset \mathbf{L}^2(\mathbb{R}^2)$ (see [13] for details).

As explained in [18,19] (see also [14] for a review), the representation π^o is unitarily equivalent to a subrepresentation of the left regular representation. The representation coefficient is given by a variant of the *short time Fourier transform* (STFT), which we define next.

Definition 1 *Let $g \in \mathbf{L}^2(\mathbb{R})$, $g \neq 0$. The STFT of any $f \in \mathbf{L}^2(\mathbb{R})$ is the function on the phase space $\mathbb{P}$ defined by*

$$\mathscr{V}_g(b,\nu) = \langle f, \pi(b,\nu)g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g}(t-b)e^{-2i\pi\nu t}\, dt \ . \tag{7}$$

This STFT is obtained by quotienting out $[0,1]$ in the group transform

$$\mathscr{V}_g^o(b,\nu,\varphi) = \langle f, \pi^o(b,\nu,\varphi)g\rangle \ . \tag{8}$$

The integral transform $\mathscr{V}_g^o$ intertwines L and π^o, i.e. $L(h)\mathscr{V}_g^o = \mathscr{V}_g^o\pi^o(h)$ for all $h \in \mathbb{H}$. The latter relation still holds true (up to a phase factor) when π^o and $\mathscr{V}_g^o$ are replaced with π and $\mathscr{V}_g$ respectively.

It follows from the general theory of square-integrable representations that for any $g \in \mathbf{L}^2(\mathbb{R})$, $g \neq 0$, the transform $\mathscr{V}_g^o$ is (multiple of) an isometry $\mathbf{L}^2(\mathbb{R}) \to L^2(\mathbb{P})$, and thus left invertible by the adjoint transform (up to a constant factor). More precisely, given $h \in \mathbf{L}^2(\mathbb{R})$ such that $\langle g, h\rangle \neq 0$, one has for all $f \in \mathbf{L}^2(\mathbb{R})$

$$f = \frac{1}{\langle h, g\rangle} \int_{\mathbb{P}} \mathscr{V}_g f(b,\nu)\, \pi(b,\nu) h \, db d\nu \ . \tag{9}$$

We refer to [1,16] for more details on the STFT and signal processing applications.

The STFT, being a continuous transform, is not well adapted for numerical calculations, and is conveniently replaced with the Gabor transform, which is a sampled version of it. To fix notation, we outline some steps of the Gabor frame theory and refer to [2,16] for a detailed account.

Definition 2 (Gabor transform) *Given $g \in \mathbf{L}^2(\mathbb{R})$ and two constants $a_0, \nu_0 \in \mathbb{R}^+$, the corresponding Gabor transform associates with any $f \in \mathbf{L}^2(\mathbb{R})$ the sequence of Gabor coefficients*

$$\mathscr{V}_g f(mb_0, n\nu_0) = \langle f, M_{n\nu_0} T_{mb_0} g\rangle = \langle f, g_{mn}\rangle \ , \tag{10}$$

where the functions $g_{mn} = M_{n\nu_0} T_{mb_0} g$ are the Gabor atoms associated to g and the lattice constants b_0, ν_0.

Whenever the Gabor atoms associated to g and the given lattice $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$ form a frame, [1] the Gabor transform is left invertible, and there exists $h \in \mathbf{L}^2(\mathbb{R})$ such that any $f \in \mathbf{L}^2(\mathbb{R})$ may be expanded as

$$f = \sum_{m,n} \mathscr{V}_g f(mb_0, n\nu_0) h_{mn} \ . \tag{11}$$

The status of the STFT as intertwining operator between the representations L and π^o of $\mathbb{H}$ does not have any simple counterpart when STFT is replaced

[1] The operator

$$S_g f = \sum_{m,n\in\mathbb{Z}^d} \langle f, M_{mb_0} T_{n\nu_0} g\rangle M_{mb_0} T_{n\nu_0} g$$

is the *frame operator* corresponding to g and the lattice defined by (b_0, ν_0). If S_g is invertible, the family of time-frequency shifted atoms $M_{mb_0} T_{n\nu_0} g$, $m, n \in \mathbb{Z}$, is a *Gabor frame* for $L^2(\mathbb{R})$.

with Gabor transform. However, connections between Gabor representations of operators and twisted convolutions will appear below.

2.2 The Gelfand triple $(S_0, \mathbf{L}^2, S_0')$

We next set up a framework for the exact description of operators we are interested in. In fact, by their property of being compact operators, the Hilbert space of Hilbert-Schmidt operators turns out to be far too restrictive to contain most operators of practical interest, starting from the identity. Although the classical triple $(\mathscr{S}, \mathbf{L}^2, \mathscr{S}')$ might seem to be the appropriate choice of generalization, we prefer to resort to the Gelfand triple $(S_0, \mathbf{L}^2, S_0')$, which has proved to be more adapted to a time-frequency environment. Additionally, the Banach space property of S_0 guarantees a technically less elaborate account.

Definition 3 *Let $\mathscr{S}(\mathbb{R}^d)$ denote the Schwartz class. Fix a non-zero "window" function $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The space $S_0(\mathbb{R}^d)$ is given by*

$$S_0(\mathbb{R}^d) = \{f \in \mathbf{L}^2(\mathbb{R}^d) : \|f\|_{S_0} := \|\mathcal{V}_\varphi f\|_{L^1(\mathbb{R}^{2d})} < \infty\}.$$

The following proposition summarizes some properties of $S_0(\mathbb{R}^d)$ and its dual, the distribution space $S_0'(\mathbb{R}^d)$.

Proposition 1 *$S_0(\mathbb{R}^d)$ is a Banach space and densely embedded in $\mathbf{L}^2(\mathbb{R}^d)$. The definition of $S_0(\mathbb{R}^d)$ is independent of the window $\varphi \in \mathscr{S}(\mathbb{R}^d)$, and different choices of $\varphi \in \mathscr{S}(\mathbb{R})$ yield equivalent norms on $S_0(\mathbb{R}^d)$.*

By duality, $\mathbf{L}^2(\mathbb{R}^d)$ is densely and weak-continuously embedded in $S_0'(\mathbb{R}^d)$ and can also be characterized by the norm $\|f\|_{S_0'} = \|\mathcal{V}_\varphi f\|_{\mathbf{L}^\infty}$.*

In other words, the three spaces $(S_0(\mathbb{R}^d), \mathbf{L}^2(\mathbb{R}^d), S_0'(\mathbb{R}^d))$ is a special case of a *Gelfand triple* [15] or Rigged Hilbert space. For a proof, equivalent characterizations, and more results on S_0 we refer to [8,7,11].

Via an isomorphism between integral kernels in the Banach spaces S_0, S_0' and the operator spaces of bounded operators $S_0' \mapsto S_0$ and $S_0 \mapsto S_0'$, we obtain, together with the Hilbert space of Hilbert-Schmidt operators, a Gelfand triple of operator spaces, as follows. We denote by $\mathscr{B}$ the family of operators that are bounded $S_0' \to S_0$ and by $\mathscr{B}'$ the family of operators that are bounded $S_0 \to S_0'$. We have the following correspondence between these operator classes and their integral kernels κ:

$$H \in (\mathscr{B}, \mathscr{H}, \mathscr{B}') \longleftrightarrow \kappa_H \in (S_0(\mathbb{R}^d), \mathbf{L}^2(\mathbb{R}^d), S_0'(\mathbb{R}^d)) \ .$$

For all further details on the Gelfand triples just introduced, we again refer to [11], only mentioning here, that one of the reasons for investigating operator representation on the level of Gelfand triples instead of just a Hilbert space framework is the fact, that S_0' contains distributions such as the Dirac functionals, Shah distributions, pure frequencies or just time-frequency shifts!

Subsequently, we will usually assume that the analysis and synthesis windows g, h are in S_0. This is a rather mild condition, which has almost become the canonical choice in Gabor analysis, for many good reasons. Among others, this choice guarantees a beautiful correspondence between the ℓ^p-spaces and corresponding modulation space [16], which in the ℓ^2-case means, that an S_0-window is automatically a Bessel-atom for arbitrary lattices - a property which is useful and far from true for $\mathbf{L}^2$-functions.

2.3 The spreading function representation, and its connections to short time Fourier representation

The so-called *spreading function* representation expresses operators in $(\mathscr{B}, \mathscr{H}, \mathscr{B}')$ as a sum of time-frequency shifts. More precisely, one has (see [16, Chapter 9]):

Theorem 1 *Let $H \in (\mathscr{B}, \mathscr{H}, \mathscr{B}')$; then there exists $\eta_H \in (\mathscr{S}_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), \mathscr{S}_0'(\mathbb{R}^2))$ such that*

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_H(b,\nu)\pi(b,\nu)\, db d\nu \ . \tag{12}$$

For $H \in \mathcal{H}$, the correspondence $H \leftrightarrow \eta_H$ is isometric, i.e. $\|H\|_{\mathscr{H}} = \|\eta_H\|_{\mathbf{L}^2(\mathbb{P})}$.

Remark 1 For $H \in \mathscr{B}$, the decomposition given in (12) is absolutely convergent, whereas, for $H \in \mathscr{B}'$, it holds in the weak sense of bilinear forms on S_0. When $\eta_H \in \mathbf{L}^2(\mathbb{P})$, H is a Hilbert-Schmidt operator, and the above integral is defined as a Bochner integral.

The spreading function is intimately related to the integral kernel $\kappa = \kappa_H$ of H via

$$\eta_H(b,\nu) = \int_{-\infty}^{\infty} \kappa_H(t, t-b) e^{-2i\pi\nu t}\, dt \tag{13}$$

$$\kappa_H(t,s) = \int_{-\infty}^{\infty} \eta_H(t-s,\nu) e^{2i\pi\nu t}\, d\nu \tag{14}$$

The spreading function is also related to the Kohn-Nirenberg symbol σ_H of H via a symplectic Fourier transform

$$\eta_H(b,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_H(t,\xi)e^{2i\pi(\nu t-\xi b)}\,dtd\xi \ , \tag{15}$$

and to its Weyl symbol [13] via a similar transformation. Notice that all these transforms, being isometries for for Hilbert-Schmidt operators, extend to unitary Gelfand triple isomorphisms, see [11].

The spreading function representation of operators provides an interesting time-frequency implementation for operators, stated in the following proposition. It turns out to be closely connected to the tools described in the previous section, in particular twisted convolution and STFT.

Proposition 2 *Let H be in $(\mathscr{B},\mathscr{H},\mathscr{B}')$, and let $\eta = \eta_H$ be its spreading function in $(S_0(\mathbb{R}^2),\mathbf{L}^2(\mathbb{R}^2),S_0'(\mathbb{R}^2))$. Let $g,h\in S_0(\mathbb{R})$ be such that $\langle g,h\rangle = 1$. Then H may be realized as a twisted convolution in the time-frequency domain: for all $f\in(S_0(\mathbb{R}),\mathbf{L}^2(\mathbb{R}),S_0'(\mathbb{R}))$,*

$$Hf = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\eta_H \natural \mathscr{V}_g f)\,(b,\nu) M_\nu T_b h\, dbd\nu \ . \tag{16}$$

Proof: The proposition follows from the spreading function representation (12) of H, combined with the STFT representation (9) of f. A change of variables yields (16). ♠

Remark 2 If $f\in\mathbf{L}^2(\mathbb{R})$ and $H\in\mathscr{H}$, then $\mathscr{V}_g f$, $\mathscr{V}_g Hf$ and $\eta_H\in\mathbf{L}^2(\mathbb{R}^2)$, which is in accordance with the fact that $\mathbf{L}^2\natural\mathbf{L}^2\subseteq\mathbf{L}^2$.
If $f\in S_0(\mathbb{R})$, hence $\mathscr{V}_g f\in\mathbf{L}^1(\mathbb{R}^2)$, then H may be in $\mathscr{B}'$, such that $\eta_H\in S_0'(\mathbb{R}^2)$, whereas, as Hf can only assumed to be in S_0', $\mathscr{V}_g Hf\in\mathbf{L}^\infty$. This leads to the inclusion $S_0'\natural\mathbf{L}^1\subseteq\mathbf{L}^\infty$, which may easily be verified directly.
On the other hand, if $f\in S_0'(\mathbb{R})$, i.e., $\mathscr{V}_g f\in\mathbf{L}^\infty(\mathbb{R}^2)$, and H in $\mathscr{B}$, such that $\eta_H\in S_0(\mathbb{R}^2)$, and, as Hf is in $S_0(\mathbb{R})$, $\mathscr{V}_g Hf\in\mathbf{L}^1(\mathbb{R}^2)$. Here, this leads to the conclusion that we have, at least for $f\in S_0'(\mathbb{R})$:

$$S_0\natural\mathscr{V}_g f\subseteq S_0. \tag{17}$$

Although it is known that $\mathscr{V}_g f$ is not only $\mathbf{L}^\infty(\mathbb{R}^2)$, but also in the Amalgam space $W(\mathcal{F}\mathbf{L}^1,\ell^\infty)$ for $f\in S_0'(\mathbb{R})$ and $g\in S_0(\mathbb{R})$,[7], it is not clear, whether (17) also holds for functions $F\in W(\mathcal{F}\mathbf{L}^1,\ell^\infty)$, which are not on the range of $S_0'(\mathbb{R})$ under $\mathscr{V}_g$. This and other interesting open questions concerning the twisted convolution of function spaces are currently under investigation [2].

Remark 3 Notice that Proposition 2 implies that the range of $\mathscr{V}_g$ is invariant under left twisted convolution. Notice also that this is no longer true if the left twisted convolution is replaced with the right twisted convolution. Indeed, in such a case, one has

$$\mathscr{V}_g f \natural \eta_H = \mathscr{V}_{H^* g} f \ .$$

Hence, one has the following simple rule: left twisted convolution on the STFT amounts to acting on the analyzed function f, while right twisted convolution on the STFT amounts to acting on the analysis window g. It is worth noticing that in such a case, applying $\mathscr{V}_g^*$ to $\mathscr{V}_g f \natural \eta_H$ yields the analyzed function f, up to some (possibly vanishing) constant factor.

Remark 4 Notice also that twisted convolution in the phase space is associated with the true translation structure. Indeed, time-frequency shifts take the form of twisted convolutions with a Dirac distribution on $\mathbb{P}$:

$$\delta_{b_0,\nu_0} \natural \mathscr{V}_g f = \mathscr{V}_g M_{\nu_0} T_{b_0} f \ .$$

This corresponds to the usage of engineers, who use to "adjust the phases" after shifting STFT coefficients.

2.4 Finite STFT and matrix representations

Similar developments can be made in the finite case, i.e. for performing time-frequency analysis of vectors and matrices. Indeed, the STFT may be developed as well in the Hermitian space $\mathbb{C}^N$, using the finite version of the Weyl-Heisenberg group

$$\mathbb{H}_N = \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N \ , \tag{18}$$

with group law

$$(m, n, \varphi)(m', n', \varphi') = (m + m', n + n', \varphi + \varphi' - n'b) \ , \tag{19}$$

all operations being understood modulo N. Defining periodic discrete translation and modulation operators by

$$T_m x[k] = x[(k - m)[\mathrm{mod}\ N]] \ , \quad M_n x[k] = e^{2i\pi kn/N} x[k] \ ,$$

[2] H. Feichtinger and F. Luef. Twisted convolution properties for Wiener amalgam spaces. *In preparation*, 2007.

the time-frequency shifts M_nT_n, $m,n=0,\dots N-1$ form a projective representation of $\mathbb{H}_N$ on $\mathbb{C}^N$.

The corresponding STFT is defined as before. Given an analysis window $g\in\mathbb{C}^N$, the corresponding STFT associates to any $x\in\mathbb{C}^N$ the vector $\mathscr{V}_g x\in\mathbb{C}^{N\times N}$ defined by

$$\mathscr{V}_g x[m,n]=\sum_{k=0}^{N-1} x[k]e^{-2i\pi nk/N}\overline{g}[k-m]\ , \tag{20}$$

all sums and differences being understood modulo N.

Homomorphisms of $\mathbb{C}^N$ also admit a spreading function decomposition, which takes the following simple form: the family of operators

$$\{M_nT_m,\ m,n\in\mathbb{Z}_N\}$$

is an orthonormal basis of $\mathrm{Hom}(\mathbb{C}^N)$. Every $H\in\mathrm{Hom}(\mathbb{C}^N)$ is characterized by a spreading function $\eta_H\in\mathbb{C}^{N^2}$, such that

$$H=\sum_{m,n=0}^{N-1}\eta_H[m,n]M_nT_m\ .$$

The counterpart of Proposition 2 in this context is as follows. Let $H\in\mathrm{Hom}(\mathbb{C}^N)$, and let η_H be its spreading function. Let $g\in\mathbb{C}^N$ be a unit norm vector. Then H may be realized as a twisted convolution in the time-frequency domain: for all $x\in\mathbb{C}^N$,

$$Hx=\sum_{m,n}\left(\eta_H\natural\mathscr{V}_g x\right)[m,n]M_nT_ng\ ,$$

where the discrete twisted convolution is defined as

$$F\natural G[m,n]=\sum_{m',n'=0}^{N-1}F[m',n']G[m-m',n-n']e^{-2i\pi m(n-n')/N}\ ,\quad F,G\in\mathbb{C}^N\ .$$

Remark 5 Note that, naturally, in the finite discrete case, all the function spaces introduced in the former section coincide.

3 Time-Frequency multipliers

Section 2 has shown the close connection between the spreading function representation of Hilbert-Schmidt operators and the short time Fourier transform.

However, the twisted convolution representation is generally of poor practical interest in the continuous case, because it does not discretize well. Even in the finite case, it relies on the full STFT on $\mathbb{C}^N$, which represents vectors with N^2 STFT coefficients, which may be far too large in practice, and sub-sampling is not possible in a straightforward way.

Time-frequency (in particular Gabor) multipliers represent a valuable alternative for time-frequency operator representation (see [12,20] and references therein for reviews). We analyze below the connections between these representations and the spreading function, and point out some limitations, before turning to generalizations.

3.1 Definitions and main properties

Let $g, h \in S_0(\mathbb{R})$ be such that $\langle g, h\rangle = 1$, let $\mathbf{m} \in L^\infty(\mathbb{R}^2)$, and define the STFT multiplier $\mathbb{M}_{\mathbf{m};g,h}$ by

$$\mathbb{M}_{\mathbf{m};g,h} f = \int_{\mathbb{P}} \mathbf{m}(b,\nu)\mathscr{V}_g f(b,\nu)\,\pi(b,\nu)h\,dbd\nu, \tag{21}$$

This clearly defines a bounded operator on $(S_0(\mathbb{R}), \mathbf{L}^2(\mathbb{R}), S_0'(\mathbb{R}))$. The function $\mathbf{m}$ is usually called in the mathematics literature the upper symbol of the operator. In more signal processing oriented terms, $\mathbf{m}$ is also called the time-frequency transfer function, or the mask of the multiplier.

Similarly, given lattice constants $b_0, \nu_0 \in \mathbb{R}^+$, set $\pi_{mn} = \pi(mb_0, n\nu_0) = M_{n\nu_0}T_{mb_0}$. Then, for $\mathbf{m} \in \ell^\infty(\mathbb{Z}^2)$, the corresponding Gabor multiplier is defined as

$$\mathbb{M}^G_{\mathbf{m};g,h} f = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0)\,\pi_{mn}h\ . \tag{22}$$

Note that the definition of time-frequency multipliers can of course be given for $g, h \in \mathbf{L}^2(\mathbb{R})$, many nice properties only apply with additional assumptions on the windows. Abstract properties of such multipliers have been studied extensively, and we refer to [12] for a review. One may show for example that, whenever the windows g and h are at least in S_0, if $\mathbf{m}$ belongs to $\mathbf{L}^2(\mathbb{P})$ (or $\ell^2(\mathbb{Z}^2)$) then the corresponding multiplier is an Hilbert-Schmidt operator and maps $S_0'(\mathbb{R})$ to $\mathbf{L}^2(\mathbb{R})$.

The spreading function of time-frequency multipliers may be computed explicitly.

Lemma 1 *(1) The spreading function of the STFT multiplier $\mathbb{M}_{\mathbf{m};g,h}$ is given* *by*

$$\eta_{\mathbb{M}_{\mathbf{m};g,h}}(b,\nu) = \mathscr{M}(b,\nu)\mathscr{V}_g h(b,\nu)\ , \tag{23}$$

where $\mathscr{M}$ *is the symplectic Fourier transform of the transfer function* $\mathbf{m}$

$$\mathscr{M}(t,\xi) = \int_{\mathbb{P}} \mathbf{m}(b,\nu)e^{2i\pi(\nu t - \xi b)}\, db d\nu\ .$$

(2) The spreading function of the Gabor multiplier $\mathbb{M}^G_{\mathbf{m};g,h}$ *is given by*

$$\eta^G_{\mathbb{M}_{\mathbf{m};g,h}}(b,\nu) = \mathscr{M}^{(d)}(b,\nu)\mathscr{V}_g h(b,\nu)\ , \tag{24}$$

where the $({\nu_0}^{-1}, {b_0}^{-1})$*-periodic function* $\mathscr{M}^{(d)}$ *is the symplectic Fourier transform of the transfer function* $\mathbf{m}$

$$\mathscr{M}^{(d)}(t,\xi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathbf{m}(m,n)e^{2i\pi(n\nu_0 t - m b_0 \xi)}\ .$$

Proof : We prove the result in the STFT case, the other case is obtained following the same lines and replacing integrals with sums. Assume that $\mathbf{m} \in \mathbf{L}^2(\mathbb{P})$, and set $\mathbb{M} = \mathbb{M}_{\mathbf{m};g,h}$. Let us first write, for $f \in \mathbf{L}^2(\mathbb{R})$

$$\mathbb{M}f(t) = \int e^{2i\pi\nu(t-s)}\overline{g}(s-b)h(t-b)\mathbf{m}(b,\nu)f(s)\, db d\nu ds\ ,$$

so that the integral kernel of $\mathbb{M}$ takes the form

$$\kappa(t,s) = \int_{\mathbb{P}} e^{2i\pi\nu(t-s)}\overline{g}(s-b)h(t-b)\mathbf{m}(b,\nu)\, db d\nu\ .$$

The spreading function is then obtained by Fourier transformation

$$\begin{aligned}
\eta(b,\nu) &= \int_{-\infty}^{\infty} \kappa(t,t-b)e^{-2i\pi\nu t}\, dt \\
&= \int e^{2i\pi\xi b}\overline{g}(t-b-x)h(t-x)\mathbf{m}(x,\xi)e^{-2i\pi\nu t}\, dt dx d\xi \\
&= \int_{\mathbb{P}} e^{2i\pi(\xi b - \nu x)}\mathbf{m}(x,\xi)\, dx d\xi \int_{-\infty}^{\infty} h(s)\overline{g}(s-b)e^{-2i\pi\nu s}\, ds \\
&= \mathscr{M}(b,\nu)\ \mathscr{V}_g h(b,\nu)\ ,
\end{aligned}$$

which concludes the proof. A similar calculation yields the same expression in the Gabor case.
By virtue of that fact that for $g,h \in S_0$, $\mathscr{V}_g h$ is certainly in $\mathbf{L}^1(\mathbb{P})$ and even in the Wiener amalgam space $W(C,\mathbf{L}^1)$, hence in particular continuous, the

expressions for the spreading function given in the lemma are always well-defined. ♠

Remark 6 These expressions are easily generalized to Gabor frames associated to arbitrary lattices $\Lambda \subset \mathbb{R}^2$. In such situations, the spreading function takes a similar form, and involves some discrete symplectic Fourier transform of the transfer function $\mathbf{m}$, which is in that case a $\Lambda^\perp$-periodic function, $\Lambda^\perp$ being the dual lattice of Λ.

Notice that as a consequence of Theorem 1, one then has the following "intertwining property"

$$\mathscr{V}_g \mathbb{M}_{\mathbf{m};g,h} f = (\mathscr{M}\, \mathscr{V}_g h) \natural \mathscr{V}_g f \ .$$

Remark 7 It follows from the above calculations that the spreading functions of STFT and Gabor multipliers must have specific forms, and that not any Hilbert-Schmidt operator may be implemented as Gabor multiplier. For example, let us assume that the analysis and synthesis windows have been chosen, and let η be the spreading function of the operator under consideration.

- In the STFT case, if the analysis and synthesis windows are fixed, the decay of the spreading function has to be fast enough (at least as fast as the decay of $\mathscr{V}_g h$) to ensure the boundedness of the quotient $\mathscr{M} = \eta / \mathscr{V}_g h$. Such considerations have led to the introduction of the notion of *underspread* operators [21] whose spreading function is compactly supported in a domain of small enough area.
- In the Gabor case, the periodicity of $\mathscr{M}^{(d)}$ imposes extra constraints on the spreading function η.

3.2 Approximation by Gabor multipliers

The possibility of approximating operators by Gabor multipliers in Hilbert-Schmidt sense depends on the properties of the rank one operators associated with time-frequency shifted copies of the analysis and synthesis windows.

Let $g, h \in S_0(\mathbb{R})$ be such that $\langle g, h \rangle = 1$. Let $\lambda = (b_1, \nu_1) \in \mathbb{P}$, and consider the rank one operator (oblique projection) P_λ defined by

$$P_\lambda f = g_\lambda^* \otimes \rangle h_\lambda f = \langle f, g_\lambda \rangle h_\lambda \ , \quad f \in (S_0(\mathbb{R}), \mathbf{L}^2(\mathbb{R}, S_0'(\mathbb{R}) \ . \tag{25}$$

Direct calculations show that

Lemma 2 *The kernel of P_λ is given by*

$$\kappa_{P_\lambda}(t, s) = \overline{g}_\lambda(s) h_\lambda(t) \ , \tag{26}$$

and its spreading function reads

$$\eta_{P_\lambda}(b,\nu) = e^{2i\pi(\nu_1 b - b_1\nu)}\mathscr{V}_g h(b,\nu)\ . \tag{27}$$

The following result characterizes the situations for which time-frequency rank one operators form a Riesz sequence, in which case the best approximation by a Hilbert-Schmidt operator is well-defined.

Proposition 3 *Let $g,h\in\mathbf{L}^2(\mathbb{R})$, with $\langle g,h\rangle\neq 0$, let $b_0,\nu_0\in\mathbb{R}^+$, and set*

$$\mathcal{U}(t,\xi) = \sum_{k,\ell=-\infty}^{\infty}\left|\mathscr{V}_g h\left(t+\frac{k}{\nu_0},\xi+\frac{\ell}{b_0}\right)\right|^2\ . \tag{28}$$

The family $\{P_{mb_0,n\nu_0},\ m,n\in\mathbb{Z}\}$ is a Riesz sequence if and only if there exist real constants $0<A\leq B<\infty$ such that for almost all (t,ξ),

$$0<A\leq\mathcal{U}(t,\xi)\leq B<\infty\ . \tag{29}$$

We call this condition the $\mathcal{U}$ condition.

Proof : The family $\{P_{mb_0,n\nu_0},\ m,n\in\mathbb{Z}\}$ is a Riesz sequence if there exist constants $0<A\leq B<\infty$ such that for all $c\in\ell^2(\Lambda)$,

$$A\|c\|^2\leq\left\|\sum_{\lambda\in\Lambda}c_\lambda P_\lambda\right\|^2\leq B\|c\|^2\ .$$

We have that

$$\begin{aligned}\left\|\sum_{\lambda\in\Lambda}c_\lambda P_\lambda\right\|^2 &= \sum_{\lambda,\mu}c_\lambda\overline{c}_\mu Tr(P_\lambda P_\mu^*)\\ &= \sum_{\lambda,\mu}c_\lambda\overline{c}_\mu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\kappa_\lambda(t,s)\overline{\kappa_\mu}(t,s)\,dtds\\ &= \sum_{\lambda,\mu}c_\lambda\overline{c}_\mu\overline{\langle g_\lambda,g_\mu\rangle}\langle h_\lambda,h_\mu\rangle\end{aligned}$$

Taking $\lambda=(b,\nu)$ and $\mu=(b',\nu')$, a direct calculation yields

$$\langle g_\lambda,g_\mu\rangle = e^{2i\pi(\nu-\nu')b}\mathscr{V}_g g(b-b',\nu-\nu')\ ,$$

so that

$$\left\|\sum_{\lambda\in\Lambda}c_\lambda P_\lambda\right\|^2 = \sum_{(b,\nu),(b',\nu')\in\Lambda}c_{b\nu}\overline{c}_{b'\nu'}\,U(b-b',\nu-\nu')\ ,$$

where we have set

$$U(b,\nu) = \overline{\mathscr{V}_g g}(b,\nu)\mathscr{V}_h h(b,\nu) \ .$$

Specializing to the lattice $\Lambda = \mathbb{Z}b_0 \times \mathbb{Z}\nu_0$, denote by $\mathcal{U}$ the discrete symplectic Fourier transform of the sequence $\{U(mb_0, n\nu_0),\ m,n \in \mathbb{Z}\}$ of samples of U, i.e.

$$\mathcal{U}(t,\xi) = \sum_{m,n} U(mb_0, n\nu_0) e^{2i\pi(n\nu_0 t - mb_0\xi)} \ .$$

Notice that this defines a ${\nu_0}^{-1} \times {b_0}^{-1}$-periodic function. From the inverse discrete symplectic Fourier transform, we obtain

$$\left\|\sum_{\lambda\in\Lambda} c_\lambda P_\lambda\right\|^2 = b_0\nu_0 \int_{\Box} |\mathcal{C}(t,\xi)|^2 \mathcal{U}(t,\xi)\, d\xi dt \ ,$$

where $\mathcal{C}$ is the discrete symplectic Fourier transform of c

$$\mathcal{C}(t,\xi) = \sum_{m,n} c_{mn} e^{2i\pi(n\nu_0 t - mb_0\xi)} \ ,$$

and $\Box$ is the fundamental domain of the adjoint lattice $\Lambda^\perp$

$$\Box = [0, {\nu_0}^{-1}[\times[0, {b_0}^{-1}[\ .$$

From the above calculations we deduce that $\mathcal{U}(t,\xi)$ must be bounded below by a constant $A > 0$. for all t,ξ.
Hence, we have the following intermediate result: the family $\{P_{mb_0,n\nu_0},\ m,n \in \mathbb{Z}\}$ is a Riesz sequence if and only if

$$0 < A \le \mathcal{U}(t,\xi) \le B < \infty \ , \qquad \text{for almost all } (t,\xi) \tag{30}$$

for some positive constants $A < B$. Let us now compute

$$\begin{aligned}
\hat{U}(\xi,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\mathscr{V}_g g}(b,\nu)\mathscr{V}_h h(b,\nu) e^{-2i\pi(\nu t + b\xi)}\, db d\nu \\
&= \int h(x)\overline{g}(y)\overline{h}(x-b)g(y-b)e^{2i\pi\nu(x-y)}e^{-2i\pi(\nu t + b\xi)}\, db d\nu dx dy \\
&= \int h(x)\overline{h}(x-b)\overline{g}(x+t)g(x+t-b)e^{-2i\pi\xi(x-y)} dx dy \\
&= |\mathscr{V}_g h(-t,\xi)|^2
\end{aligned}$$

Finally, we obtain the result by applying the Poisson summation formula

$$\mathcal{U}(t,\xi) = \sum_{k,\ell} |\mathscr{V}_g h(t + k/\nu_0, \xi + \ell/b_0)|^2 \ ,$$

which concludes the proof. ♠

Remark 8 The intermediate result (30) has already appeared before, see [9].

Similar results may be derived for non-square lattices Λ; the $\mathcal{U}$ condition then has to be replaced with an analogous condition, involving periodization with respect to the dual lattice $\Lambda^\perp$ of Λ.

It turns out, that the approximation of a given operator via a standard minimization process yields an expression, which is only well-defined if the $\mathcal{U}$ condition (29) holds.

Theorem 2 *Assume that $\mathscr{V}_g h$ and $b_0, \nu_0 \in \mathbb{R}^+$ are such that the $\mathcal{U}$ condition (29) is fulfilled. Then the best Gabor multiplier approximation (in Hilbert-Schmidt sense) of $H \in \mathscr{H}$ is defined by the time-frequency transfer function* $\mathbf{m}$ *whose discrete symplectic Fourier transform reads*

$$\mathscr{M}(b,\nu) = \frac{\sum_{k,\ell=-\infty}^{\infty} \overline{\mathscr{V}_g h}\,(b + k/\nu_0, \nu + \ell/b_0)\, \eta_H\,(b + k/\nu_0, \nu + \ell/b_0)}{\sum_{k,\ell=-\infty}^{\infty} |\mathscr{V}_g h\,(b + k/\nu_0, \nu + \ell/b_0)|^2} \tag{31}$$

<u>Proof:</u> Let us denote as before by $\square$ the rectangle $\square = [0, \nu_0{}^{-1}[\times[0, b_0{}^{-1}[$, and set $\mathscr{V} = \mathscr{V}_g h$ for simplicity of notations. The Hilbert-Schmidt optimization is equivalent to the problem

$$\min_{\mathscr{M}\in\mathbf{L}^2(\square)} \|\eta_H - \mathscr{M}\mathscr{V}\|^2 \ .$$

The latter squared norm may be written as

$$\begin{aligned}
\|\eta_H - \mathscr{M}\mathscr{V}\|^2 =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\eta_H(b,\nu) - \mathscr{M}(b,\nu)\mathscr{V}(b,\nu)|^2\, dbd\nu \\
=& \sum_{k,\ell=-\infty}^{\infty} \iint_\square |\eta_H(b + k/\nu_0, \nu + \ell/b_0) \mathscr{M}(b,\nu)\mathscr{V}(b + k/\nu_0, \nu + \ell/b_0)|^2\, dbd\nu \\
=& \iint_\square \Big[\sum_{k,\ell} |\eta_H(b + k/\nu_0, \nu + \ell/b_0)|^2 \\
&\quad - 2\Re\Big(\overline{\mathscr{M}}(b,\nu) \sum_{k,\ell} \eta_H(b + k/\nu_0, \nu + \ell/b_0)\overline{\mathscr{V}}(b + k/\nu_0, \nu + \ell/b_0)\Big) \\
&\quad + |\mathscr{M}(b,\nu)|^2 \sum_{k,\ell} |\mathscr{V}(b + k/\nu_0, \nu + \ell/b_0)|^2 \Big]
\end{aligned}$$

From this expression, the Euler-Lagrange equations may be obtained, which read

$$\mathscr{M}(b,\nu)\sum_{k,\ell}|\mathscr{V}(b+k/\nu_0,\nu+\ell/b_0)|^2=\sum_{k,\ell}\eta_H(b+k/\nu_0,\nu+\ell/b_0)\overline{\mathscr{V}}(b+k/\nu_0,\nu+\ell/b_0)\,,$$

and the result follows. ♠

Remark 9 Note that, although technically only defined for Hilbert-Schmidt operators, the approximation by Gabor multipliers can formally be extended to operators from $\mathscr{B}'$, see [12, Section 5.8]. In the finite discrete case which is of relevance in practice, this insight can be realized.

4 Generalizations: multiple Gabor multipliers and TST spreading functions

It has become clear in the last section that most operators cannot exactly be realized as a STFT or Gabor multiplier. Even if the analysis and synthesis windows as well as lattice constants have been carefully adjusted, so that the corresponding time-frequency projection operators form a Riesz sequence, they do not necessarily provide good approximations. An example for operators which are poorly represented by this class of multipliers are those with a spreading function that is not "well-concentrated". These are, in technical terms, overspread operators.

Guided by the desire to extend the good approximation quality that Gabor multipliers warrant for underspread operators to the above-mentioned class of their overspread counterparts, we introduce two classes of generalized TF-multipliers.

In the first approach, operators in $(\mathscr{B},\mathscr{H},\mathscr{B}')$ are approximated using a sum of Gabor multipliers, with various synthesis windows (a similar approach has been followed by D. Marelli [3].) This amounts to the use of more than one synthesis windows to account for the overspread character of the operator. The approximation is set up by Gabor multipliers with a fixed analysis window, synthesis windows with different (fixed) time-frequency localizations, and tunable masks.

In the second approach, a covariant spline type approximation of the spreading function of the operator is used. This ultimately leads to the adaptation of the synthesis windows to the properties of the operator of interest, with a fixed mask.

[3] Private communication

Coming back to the language of the previous section, we consider the problem of approximating a Hilbert-Schmidt operator, not by a single Gabor multiplier, but by a linear combination of such multipliers, using a fixed family of synthesis window.

Definition 4 (Multiple Gabor Multipliers) *Let $g \in S_0(\mathbb{R})$ and a family of reconstruction windows $h^{(j)} \in S_0(\mathbb{R})$ as well as corresponding time-frequency transfer functions $\mathbf{m}_j$ be given. Operators of the form*

$$\mathbb{M} = \sum_j \mathbb{M}^G_{\mathbf{m}_j;g,h^{(j)}} \ . \tag{32}$$

will be called Multiple Gabor Multipliers *(MGM for short).*

In the sequel, the discrete symplectic Fourier transforms of $\mathbf{m}_j$ will be denoted by $\mathscr{M}_j$, and the vector with $\mathscr{M}_j$ as coordinates will be denoted by $\mathscr{M}$. Theorem 2 may be extended to the situation of Multiple Gabor Multipliers as follows.

Proposition 4 *Let $g \in S_0(\mathbb{R})$ and $h^{(j)} \in S_0(\mathbb{R})$, $j = 1, \ldots J$ be such that for almost all b, ν, the matrix $\Gamma(b,\nu)$ defined by*

$$\Gamma(b,\nu)_{jj'} = \sum_{k,\ell} \overline{\mathscr{V}_g h^{(j)}}(b + k/\nu_0, \nu + \ell/b_0) \mathscr{V}_g h^{(j')}(b + k/\nu_0, \nu + \ell/b_0) \tag{33}$$

is invertible.

Let $H \in (\mathscr{B}, \mathscr{H}, \mathscr{B}')$ be an operator with spreading function $\eta \in (S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2))$. Then the functions $\mathscr{M}_j$ yielding optimal approximation of the form (32) may be obtained as

$$\mathscr{M} = \Gamma^{-1} \cdot \mathcal{B} \ , \tag{34}$$

where $\mathcal{B}$ is the vector whose entries read

$$\mathcal{B}_{j_0}(b,\nu) = \sum_{k,\ell} \eta(b + k/\nu_0, \nu + \ell/b_0) \overline{\mathscr{V}}_{j_0}(b + k/\nu_0, \nu + \ell/b_0). \tag{35}$$

Proof: The proof follows the lines of the Gabor multiplier case. The optimal approximation of the form (32), when it exists, is obtained by minimizing

$$\|\eta - \sum_j \mathscr{M}_j \mathscr{V}_j\|^2 = \sum_{k,\ell} \int_{\square} |\eta(b+k/\nu_0, \nu+\ell/b_0) - \sum_j \mathscr{M}_j(b,\nu)\mathscr{V}_j(b+k/\nu_0, \nu+\ell/b_0)|^2 \, dbd\nu$$

where one has set $\mathscr{V}_j = \mathscr{V}_g h^{(j)}$. Setting to zero the Gâteaux derivative with respect to $\overline{\mathscr{M}}_{j_0}$, we obtain the corresponding variational equation

$$\sum_j \mathscr{M}_j(b,\nu) \sum_{k,\ell} \mathscr{V}_j(b+k/\nu_0, \nu+\ell/b_0) \overline{\mathscr{V}}_{j_0}(b+k/\nu_0, \nu+\ell/b_0) = \mathcal{B}_{j_0}(b,\nu) \ ,$$

where $\mathcal{B}_j(b,\nu)$ are as defined in (35). Provided that the $\Gamma(b,\nu)$ matrices are invertible for almost b,ν, this implies that the functions $\mathscr{M}_j$ for optimal approximation of the form (32) may indeed be obtained as in (34). ♠.

We shall be particularly interested in the special case of h_j being time-frequency translates of a fixed window function, i.e.

$$h^{(j)}(t) = \pi(b_j, \nu_j)h(t) = e^{2i\pi\nu_j t} h(t-b_j) \ . \tag{36}$$

More specifically, if the reconstruction windows are TF-shifted versions of a single window h, where the TF-shifts are taken on the dual lattice of $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$, the matrix Γ turns out to enjoy quite a simple form. To fix some notation, let

$$\begin{aligned}\mathcal{A}_{mn}(b,\nu) = &\sum_{k,\ell} e^{2i\pi m[\nu-\ell/\nu_0]} \ \overline{\mathscr{V}}(b-k/\nu_0, \nu-\ell/b_0) \\ &\times \ \ \mathscr{V}(b-(k-m)/\nu_0, \nu-(\ell-n)/b_0),\end{aligned}$$

and introduce the right twisted convolution operator

$$K^\natural_{\mathcal{A}}(b,\nu) : \mathscr{M}(b,\nu) \to \mathscr{M}(b,\nu)\natural\mathcal{A}(b,\nu).$$

Theorem 3 *Let $g, h \in S_0$ as well as b_0, ν_0 be given. Furthermore, let $h^{(j)} = \pi(\frac{m}{\nu_0}, \frac{n}{b_0})h$. Then the variational equations read*

$$\mathscr{M}(b,\nu)\natural\mathcal{A}(b,\nu) = \mathcal{B}(b,\nu) \ . \tag{37}$$

Hence, if for all $b,\nu \in \mathbb{R}^2$, the discrete right twisted convolution operator $K^\natural_{\mathcal{A}}$ is invertible, the best MGM approximation of an Hilbert-Schmidt operator with spreading function η is given by the family of transfer functions

$$\mathscr{M}_{mn}(b,\nu) = \left[(K^\natural_A(b,\nu))^{-1}\mathcal{B}(b,\nu)\right]_{mn} \ ,$$

where $\mathcal{B}$ is given in (35).

Proof: First consider the case (36) where $h^{(j)} = \pi(b_j,\nu_j)h$, and set for simplicity $\mathscr{V} = \mathscr{V}_g h$. Then

$$\mathscr{V}_j(b,\nu) = e^{-2i\pi b_j(\nu-\nu_j)} \mathscr{V}_g h(b-b_j, \nu-\nu_j) \ ,$$

and the matrix Γ takes the form

$$\begin{aligned}\Gamma(b,\nu)_{jj'} = \sum_{k,\ell} &\exp\left(2i\pi[b_j(\nu+\ell/b_0-\nu_j) - b_{j'}(\nu+\ell/b_0-\nu_{j'})]\right)\\ &\times \overline{\mathscr{V}}(b-b_j+k/\nu_0, \nu-\nu_j+\ell/b_0)\\ &\times \mathscr{V}(b-b_{j'}+k/\nu_0, \nu-\nu_{j'}+\ell/b_0)\end{aligned}$$

Now assume that the sampling points (b_j, ν_j) are taken on the dual lattice, i.e. $b_j = m/\nu_0$, $b_{j'} = m'/\nu_0$, $\nu_j = n/b_0$ and $\nu_{j'} = n'/b_0$. Then the latter expression reads

$$\begin{aligned}\Gamma(b,\nu)_{mn;m'n'} &= \sum_{k,\ell} e^{2i\pi[m(\nu+(\ell-n)/b_0)/\nu_0 - m'(\nu+(\ell-n'/b_0)/\nu_0)]}\\ &\qquad\times \overline{\mathscr{V}}(b-(m-k)/\nu_0, \nu-(n-\ell)/b_0)\\ &\qquad\times \mathscr{V}(b-(m'-k)/\nu_0, \nu-(n'-\ell)/b_0)\\ &= e^{-2i\pi m'(n-n')/b_0\nu_0}\,\mathcal{A}_{m-m',n-n'}(b,\nu)\ ,\end{aligned}$$

with $\mathcal{A}$ as defined in (37). Therefore, in this situation, the variational equations gives (37), which concludes the proof. ♠

Notice that the invertibility of the right twisted convolution operator above is the direct generalization of the $\mathcal{U}$ condition (29). Actually, following the lines of the discussion in Section 3, one may prove the subsequent result.

Corollary 1 *The family of rank one projectors*

$$P_\lambda^{(j)} : f \in \mathbf{L}^2 \longmapsto \langle f, g_\lambda\rangle h_\lambda^{(j)}\ , \quad \lambda \in \Lambda^\perp, j = 1,\dots J$$

is a Riesz sequence if and only if the right twisted convolution operator $K_A^\natural$ is invertible.

Remark 10 It is quite obvious that the condition in the proposition directly generalizes Proposition 3. Whenever for each j the single system of projection operators defined by $g_\lambda^* \otimes h_\lambda^{(j)} f$, establishes a Riesz basis and the single systems do not overlap too much, an overall Riesz basis can be expected. Intuitively, and observing the result of Proposition 3, these considerations must lead to an invertible matrix Γ, which is diagonally dominant.

Example 1 As an example, we consider a situation, in which the approximation of a given operator which is not underspread, namely a convolution with a sinusoid, is performed by a Multiple Gabor multiplier of the following simple form. We use 5 synthesis windows, which are time- respectively frequency

Fig. 1. *Spectrogram of the five synthesis windows used in Example 1*

Fig. 2. *Approximation by Gabor multiplier and Multiple Gabor multiplier*

translates of the analysis window g on 5 points of the dual lattice:

$$(0,0),(0,1/b_0),(0,-1/b_0),(1/\nu_0,0),(-1/\nu_0,0)\ .$$

The spectrogram of the five windows, that is, $\mathscr{V}_g h^{(j)}, j = 1,\ldots,5$, is depicted in Figure 1. The sampling density of the Gabor family used for analysis was chosen to be critical. The approximation quality was compared to the approximation by a simple Gabor multiplier both at critical density and with redundancy 4.5. In both cases, the multiple Gabor multiplier model yields better results: the error ε defined by $\varepsilon = \|CC - M\|/\|CC\|$, where CC denotes the given convolution operator and M its approximation was 0.6525 for M the multiple Gabor multiplier approximation (with critical density), 0.9940 for the redundancy 1 case and 0.8473 for the redundancy 4.5 case of approximation by a simple Gabor multiplier. This approximation behavior can also be recognized when we take a look at the result of the operators applied to a random vector, shown in Figure 2. The dash dotted graph shows the result of true convolution, the dashed one the approximation by a Gabor multiplier and the solid line the result of the approximation by Multiple Gabor multiplier. Although this is a prototypical rather than an example of practical relevance, it shows, that multiple Gabor multiplier have the potential to better approximate overspread operators than Gabor multipliers.

4.2 Varying the synthesis window: TST spreading functions

In the approach described above, the synthesis windows of the MGMs were fixed functions, and the transfer functions were optimized. We now discuss a different approach, in which a single transfer function is given, and the MGM is defined by adapted synthesis windows. These rely on the notion of Twisted Spline Type functions.

Definition 5 (TST spreading functions) *Let ϕ be a given function from the function spaces $(S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2))$ and let b_1, ν_1 denote positive numbers. A spreading function $\eta = \eta_H$ of $H \in (\mathscr{B}, \mathscr{H}, \mathscr{B}')$, that may be written* *as*

$$\eta(b,\nu) = \sum_{k,\ell} \alpha_{k\ell}\phi(b-kb_1,\nu-\ell\nu_1)e^{-2i\pi(\nu-\ell\nu_1)kb_1} \tag{38}$$

will be called Twisted Spline Type *function (TST for short).*

Remark 11 Depending on the properties of the coefficient sequence α, we have the following function space membership of η:
If ϕ is in $S_0(\mathbb{R}^2)$, it suffices to require boundedness of α in order to obtain a spreading function in $S_0'(\mathbb{R}^2)$, with unconditional convergence of the sequence in the weak operator topology, see [17, Lemma 2.2]. Dually, if $\phi \in S_0'(\mathbb{R}^2)$ and $\alpha \in \ell^1$, the TST spreading function is again at least in $S_0'(\mathbb{R}^2)$. On the other hand, it is easy to see that for α in ℓ^1, η will again be in $S_0(\mathbb{R}^2)$, whenever ϕ is in $S_0(\mathbb{R}^2)$. Finally, for ℓ^2-sequences α, we obtain an $\mathbf{L}^2$-function η for $\phi \in \mathbf{L}^2(\mathbb{R}^2)$.

TST functions are nothing but spline type functions (following the terminology introduced in [9]), in which usual (Euclidean) translations are replaced with the natural (i.e. $\mathbb{H}$-covariant) translations on the phase space $\mathbb{P}$. This leads to the following property of operators associated with TST spreading functions.

Lemma 3 *Let $H \in (\mathscr{B}, \mathscr{H}, \mathscr{B}')$ be an operator associated with a TST spreading function $\eta \in (S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2))$ as in (38). Then*

$$H = \sum_{k,\ell} \alpha_{k\ell}\pi(kb_1,\ell\nu_1)H_\phi \ , \tag{39}$$

where H_ϕ is the linear operator with spreading function ϕ.

<u>Proof:</u> We just have to compute

$$\begin{aligned} H &= \sum_{k,\ell} \alpha_{k\ell} \int_{\mathbb{P}} \phi(b',\nu')e^{-2i\pi\nu' kb_1} M_{\nu'+\ell\nu_1} T_{b'+kb_1}\, db'd\nu' \\ &= \sum_{k,\ell} \alpha_{k\ell} \int_{\mathbb{P}} \phi(b',\nu')e^{-2i\pi\nu' kb_1} M_{\ell\nu_1} M_{\nu'} T_{kb_1} T_{b'}\, db'd\nu' \\ &= \sum_{k,\ell} \alpha_{k\ell}\pi(kb_1,\ell\nu_1) \int_{\mathbb{P}} \phi(b',\nu') M_{\nu'} T_{b'}\, db'd\nu' \ , \end{aligned}$$

which is the desired result. ♠

In a next step we assume that the basic function ϕ entering in the composition of η is the spreading function of a Gabor multiplier (at least in an approximate sense). According to the discussion of Section 3, this essentially means that η

is sufficiently well concentrated in the time-frequency domain, and that $\eta/\mathscr{V}_g h$ possesses the desired periodicity properties.

We denote by $\mathbf{m} = \{\mathbf{m}(m,n),\ m,n \in \mathbb{Z}\}$ the corresponding time-frequency transfer function. Then for $f \in (S_0(\mathbb{R}), \mathbf{L}^2(\mathbb{R}), S_0'(\mathbb{R}))$, we have

$$H_\phi f = \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0) h_{mn}\ ,$$

for some $h \in S_0$. Hence

$$\begin{aligned} Hf =& \sum_{k,\ell} \alpha_{k\ell}\pi_{k\ell} \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0)\pi_{mn} h \\ =& \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0) \sum_{k,\ell} \alpha_{k\ell}\pi(kb_1, \ell\nu_1)\pi_{mn} h \end{aligned} \tag{40}$$

Based on this expression, we are going to pursue two different choices of the sampling-points $(kb_1, \ell\nu_1)$ in the TST representation of η. First, we assume that these sampling points are taken on the primal lattice Λ. The second choice of sampling points on the dual lattice leads to a completely different result.

4.2.1 Gabor twisters associated with the primal lattice

The subsequent construction has first been proposed in [3]. Assume that (b_1, ν_1) generate a sub lattice of Λ, i.e. $b_1 = rb_0$, and $\nu_1 = s\nu_0$. Then, based on (40), we may continue the calculation as

$$\begin{aligned} Hf =& \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0) \sum_{k,\ell} \alpha_{k\ell} e^{-2i\pi rsknb_0\nu_0} \pi_{rk+m, s\ell+n} h \\ =& \sum_{p,q} \widetilde{\mathscr{V}_g} f(p,q) h_{pq}\ . \end{aligned}$$

Here the new Gabor coefficients $\widetilde{\mathscr{V}_g} f(p,q)$ read

$$\widetilde{\mathscr{V}_g} f(p,q) = \sum_{k,\ell} \alpha_{k\ell} \mathscr{W}_{p-rk, q-s\ell} e^{-2i\pi rk(q-\ell)b_0\nu_0} = (\alpha \sharp \mathscr{W})_{pq} \tag{41}$$

and $\sharp$ denotes a downsampling followed by a twisted convolution, whereas the new coefficients $\mathscr{W}_{mn}$ are weighted copies of the original Gabor coefficients $\mathscr{V}_g f$:

$$\mathscr{W}_{mn} = \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0)\ . \tag{42}$$

In other words, Gabor coefficients of Hf may be obtained by the following two-steps procedure:

(1) Weight the Gabor coefficients of f using the time-frequency transfer function $\mathbf{m}$ of the Gabor multiplier H_ϕ.
(2) Evaluate the twisted convolution of the so-obtained weighted coefficients with the coefficients α of the TST expansion of the spreading function η of H.

Hence, in this situation, H may be realized as a Gabor twister, i.e. a (discrete) twisted convolution in the Gabor coefficient domain, after suitable weighting of Gabor coefficients.

Remark 12 Similar calculations may be made when b_1 and ν_1 generate a lattice containing Λ, which leads to quite the same results, with a subsampled version of the twisted convolution product.

Remark 13 An alternative model would be given by changing the definition of the TST spreading function as follows:

$$\eta(b,\nu) = \sum_{k,\ell} \alpha_{k\ell}\phi(b-kb_1, \nu-\ell\nu_1)e^{-2i\pi\ell\nu_1(b-kb_1)} \tag{43}$$

In a group-theoretical sense this modification corresponds to using the *right regular* action of Weyl-Heisenberg group to perform time-frequency shifts on ϕ rather than the *left regular* action.

Using the primal lattice for the sampling points, this modification leads to a completely analog procedure in reversed order: the Gabor coefficients of the resulting operator may be obtained by first evaluating the twisted convolution of the Gabor coefficients of f with the coefficients α of the TST expansion of the spreading function η and then weighting the so obtained coefficients with the time-frequency transfer function $\mathbf{m}$ of the Gabor multiplier H_ϕ.

4.2.2 Working on the dual lattice

The dual lattice $\Lambda^\perp$ turns out to be a natural choice for sampling the TST functions. The following theorem reflects this structural fact.

Theorem 4 *Let $b_0, \nu_0 \in \mathbb{R}^+$ generate the time-frequency lattice Λ, and let $\Lambda^\perp$ denote the dual lattice. Let $g, h \in S_0$ denote respectively Gabor analysis and synthesis windows, such that the $\mathcal{U}$ condition (29) is fulfilled. Let H denote the operator in $(\mathscr{B}, \mathscr{H}, \mathscr{B}')$ defined by the twisted spline type spreading function*

η as in (38), with $b_1, \nu_1 \in \mathbb{R}^+$.

(1) Assume that b_1 and ν_1 are multiple of the dual lattice constants. Then H is a Gabor multiplier, with analysis window g, synthesis window

$$\gamma = \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell\nu_1)h \ , \tag{44}$$

and transfer function

$$\mathbf{m}(m,n) = b_0\nu_0 \int_{\square} \mathscr{M}(b,\nu) e^{-2i\pi(n\nu_0 b - mb_0\nu)}\, dbd\nu \ , \tag{45}$$

with $\square$ the fundamental domain of the adjoint lattice $\Lambda^\perp$, and

$$\mathscr{M}(b,\nu) = \frac{\sum_{k,\ell=-\infty}^{\infty} \overline{\mathscr{V}_g h}\,(b+k/\nu_0, \nu+\ell/b_0)\, \phi\,(b+k/\nu_0, \nu+\ell/b_0)}{\sum_{k,\ell=-\infty}^{\infty} |\mathscr{V}_g h\,(b+k/\nu_0, \nu+\ell/b_0)|^2} \tag{46}$$

(2) Assume that the lattice generated by b_1 and ν_1 contains the dual lattice:

$$b_1 = \frac{1}{p\nu_0} \ , \qquad \nu_1 = \frac{1}{qb_0} \ . \tag{47}$$

Then H may be written as a finite sum of Gabor multipliers

$$Hf = \sum_{i=1}^{p}\sum_{j=1}^{q} \Big(\sum_{m\equiv i\,[\mathrm{mod}\, p]} \ \sum_{n\equiv j\,[\mathrm{mod}\, q]} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0)\pi_{mn}\Big)\gamma_{ij} \ , \tag{48}$$

with at most $p\cdot q$ different synthesis windows γ_{ij} and the transfer function given in (45) and (46).

<u>Proof:</u>
Let us first compute

$$\begin{aligned} Hf &= \sum_{k,\ell} \alpha_{k\ell}\pi_{k\ell} \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0) h_{mn} \\ &= \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0) \sum_{k,\ell} \alpha_{k\ell}\pi(kb_1, \ell\nu_1)\pi_{mn} h \\ &= \sum_{m,n} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0)\pi_{mn}\gamma_{mn} \ , \end{aligned}$$

where

$$\gamma_{mn} = \sum_{k,\ell} \alpha_{k\ell} e^{2i\pi[knb_0\nu_1 - \ell m\nu_0 b_1]}\pi(kb_1, \ell\nu_1)h \tag{49}$$

Now observe that if $(b_1, \nu_1) \in \Lambda^\perp$, one obviously has

$$\gamma_{mn} = \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell\nu_1) h = \gamma_{00} \ , m, n \in \mathbb{Z} \ ,$$

i.e. the above expression for Hf involves a single synthesis window $\gamma = \gamma_{00}$. Therefore, in this case, H takes the form of a standard Gabor multiplier, with fixed time-frequency transfer function, and a synthesis window prespribed by the coefficients in the TST expansion. This proves the first part of the theorem.

Let us now assume that the TST expansion of the spreading function is finer than the one prescribed by the lattice $\Lambda^\perp$, but nevertheless the lattice $\Lambda_1 = \mathbb{Z}b_1 \times \mathbb{Z}\nu_1$ contains $\Lambda^\perp$. In other words, there exist positive integers p, q such that (47) holds.
We then have

$$\gamma_{mn} = \sum_{k,\ell} \alpha_{k\ell} e^{2i\pi[\frac{knp-lmq}{pq}]} \pi(kb_1, \ell\nu_1) h \tag{50}$$

and it is readily seen that there are at most pq different synthesis windows γ_{ij},

$$\gamma_{ij} = \gamma_{m\,[\mathrm{mod}\,p], n\,[\mathrm{mod}\,q]} \ , i = 1, \ldots, p; \ j = 1, \ldots, q. \tag{51}$$

The operator H may hence be written as a sum of Gabor multipliers, with one prescribed time-frequency transfer function, which is sub-sampled on several sub-lattices of the lattice Λ:

$$\Lambda_{ij} = (pb_0 \cdot \mathbb{Z} + i \cdot b_0) \times (q\nu_0 \cdot \mathbb{Z} + j \cdot \nu_0), \ i = 0, \ldots, p-1; j = 0, \ldots, q-1,$$

and a single synthesis window per sub-lattice as given in (51). The resulting expression for H is hence as given in (48).

The expression for the transfer function is derived in analogy to the case discussed in Section 3. ♠

Remark 14 Let us observe that in this approximation, the time-frequency transfer function $\mathbf{m}$ is completely characterized by the function ϕ used in the TST expansion. The choice of ϕ therefore imposes a fixed mask for the multipliers that come into play in equation (48).

Example 2 We first consider a situation similar to the one described in Example 1. This means, that for a given primal lattice $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$, we assume the representation of a spreading function by 5 building blocks:

$$\eta(b,\nu) = \sum_{k=-1}^{1} \alpha_{k0}\phi(b - \frac{k}{\nu_0}, \nu) + \sum_{\ell=-1}^{1} \alpha_{0\ell}\phi(b, \nu - \frac{\ell}{b_0}).$$

In this case, we obtain a single Gabor multiplier with synthesis window

$$\gamma_{00} = \sum_{k=-1}^{1} \alpha_{k0}\pi(\frac{k}{\nu_0}, 0)h + \sum_{\ell=-1}^{1} \alpha_{0\ell}\pi(0, \frac{\ell}{b_0})h\ .$$

If we add the windows $\phi(b \pm \frac{1}{2\nu_0}, \nu \pm \frac{1}{2b_0})$ to the representation of η, we are now dealing with the finer lattice $\Lambda = \frac{1}{2\nu_0}\mathbb{Z} \times \frac{1}{2b_0}\mathbb{Z}$ and we obtain the sum of 4 Gabor multipliers with the following synthesis windows:

$$\begin{aligned}
\gamma_{00} &= \sum_{k=-1}^{1} \alpha_{k0}\pi\left(\frac{k}{2\nu_0}, 0\right) h + \sum_{\ell=-1}^{1} \alpha_{0\ell}\pi\left(0, \frac{\ell}{2b_0}\right) h, \\
\gamma_{01} &= \sum_{k=-1}^{1} \alpha_{k0}e^{\pi i k}\pi\left(\frac{k}{2\nu_0}, 0\right) h + \sum_{\ell=-1}^{1} \alpha_{0\ell}\pi\left(0, \frac{\ell}{2b_0}\right) h, \\
\gamma_{10} &= \sum_{k=-1}^{1} \alpha_{k0}\pi\left(\frac{k}{2\nu_0}, 0\right) h + \sum_{\ell=-1}^{1} e^{\pi i \ell}\alpha_{0\ell}\pi\left(0, \frac{\ell}{2b_0}\right) h\ , \\
\gamma_{11} &= \sum_{k=-1}^{1} \alpha_{k0}e^{\pi i k}\pi\left(\frac{k}{2\nu_0}, 0\right) h + \sum_{\ell=-1}^{1} \alpha_{0\ell}e^{-\pi i \ell}\pi\left(0, \frac{\ell}{2b_0}\right) h\ ,
\end{aligned}$$

and corresponding lattices: $\Lambda_{00} = 2\mathbb{Z}b_0 \times 2\mathbb{Z}\nu_0$, $\Lambda_{01} = 2\mathbb{Z}b_0 \times (2\mathbb{Z}+1)\nu_0$, $\Lambda_{10} = (2\mathbb{Z}+1)b_0 \times 2\mathbb{Z}\nu_0$, and $\Lambda_{11} = (2\mathbb{Z}+1)b_0 \times (2\mathbb{Z}+1)\nu_0$.

It is important to note, that in both cases described in Theorem 4 as well as the above example, the transfer function $\mathbf{m}$ can be calculated as the best approximation by a regular Gabor multiplier - a procedure, for which fast algorithms exist, see [10].

4.2.3 Riesz sequences of TST functions

The TST expansion for the spreading function may be written in the form

$$\eta = K\alpha\ ,$$

where $K : \ell^2 \to L^2$ is a discrete right twisted convolution operator. Then one has

$$\|\eta\|^2 = \langle K^* K\alpha, \alpha\rangle\ ,$$

and the properties of the TST expansion depend upon the spectral properties of the self-adjoint operator $K^*K : \ell^2 \to \ell^2$, which is a discrete right twisted convolution operator:

$$K^*K\alpha_{mn} = \sum_{m',n'} \alpha_{m'n'} G(m-m', n-n') e^{-2i\pi m' b_0 (n-n')\nu_0} = (\alpha \natural G)_{mn} \ , \quad (52)$$

with

$$G(k,\ell) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(b+kb_0, \nu+\ell\nu_0) \overline{\phi}(b,\nu) e^{2i\pi k b_0 \nu} \, db d\nu \ .$$

Proposition 5 *The TST functions ϕ_{mn} form a Riesz sequence if and only if the right twisted convolution operator K^*K in (52) bounded above, and below by a positive constant.*

Again, we observe that the problem of approximating a Hilbert-Schmidt operator using generalized Gabor multipliers leads to the question of invertibility of some twisted convolution operator. Quite little seems to be known currently on this problem. Criteria in the finite case have been given in [4].

5 Conclusions and Future Work

Inspired by the representation of operators by a twisted convolution with the operator's spreading function in the STFT-domain, we have derived several generalizations of the classical approximation of operators by time-frequency multipliers. Motivated by the desire to transfer the original result to the situation of discrete time-frequency transforms (Gabor transforms), we have achieved several results in which a discrete twisted convolution in the sampled Gabor-domain play a central role. Often the invertibility of a discrete twisted convolution operator is an important issue. To our knowledge, the problem of invertibility of such operators is unsolved.
Additional future work will, on the one hand, concern the numerical realization of the proposed approaches in the context of realistic examples, also see [3]. On the other hand, the main steps of the proposed approach can be performed on any locally compact abelian group. In particular the case of the affine group might yield interesting results for wavelet multipliers.
Finally, in future work, non-rectangular lattices will be considered more closely and the possibility of operator approximation in non-Hilbert space sense will provide interesting new results for both Gabor multiplier and generalized multipliers approximation.

References

[1] R. Carmona, W. Hwang, and B. Torrésani. *Practical Time-Frequency Analysis: continuous wavelet and Gabor transforms, with an implementation in S*, volume 9 of *Wavelet Analysis and its Applications.* Academic Press, San Diego, 1998.

[2] I. Daubechies. *Ten lectures on wavelets.* SIAM, Philadelphia, PA, 1992.

[3] M. Dörfler and B. Torrésani. Spreading function representation of operators and Gabor multiplier approximation. In *Sampling Theory and Applications (SAMPTA'07), Thessaloniki, June 2007*, 2007.

[4] Y. C. Eldar, E. Matusiak, and T. Werther. A constructive inversion framework for twisted convolution. *Monatshefte für Mathematik*, 150(4):297–308, 2007.

[5] H. Feichtinger and T. Strohmer, editors. *Gabor Analysis and Algorithms, Theory and Applications.* Applied and Numerical Harmonic Analysis. Birkhaüser, Boston, 1998.

[6] H. Feichtinger and T. Strohmer, editors. *Advances in Gabor Analysis.* Applied and Numerical Harmonic Analysis. Birkhaüser, Boston, 2003.

[7] H. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 123–170. Birkhäuser, Boston, 1998. Chap. 3.

[8] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92(4):269–289, 1981.

[9] H. G. Feichtinger. Spline type spaces in Gabor analysis. In D. Zhou, editor, *Wavelet analysis: twenty years' developments*, Singapore, 2002. World Scientific.

[10] H. G. Feichtinger, M. Hampejs, and G. Kracher. Approximation of matrices by Gabor multipliers. *IEEE Signal Proc. Letters*, 11(11):883– 886, 2004.

[11] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In *Gabor analysis and algorithms*, pages 233–266. Birkhäuser Boston, Boston, MA, 1998.

[12] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Boston, 2002. Birkhauser.

[13] G. Folland. *Harmonic Analysis in Phase Space.* Princeton University Press, Princeton, NJ, 1989.

[14] H. Führ. *Abstract harmonic analysis of continuous wavelet transforms.* Number 1863 in Lecture Notes in Mathematics. Springer Verlag, Berlin; Heidelberg; New York, NY, 2005.

[15] I. M. Gel'fand and N. Y. Vilenkin. *Generalized functions. Vol. 4*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]. Applications of harmonic analysis. Translated from the Russian by Amiel Feinstein.

[16] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhaüser, Boston, 2001.

[17] K. Gröchenig. Gabor frames without inequalities, 2007. Preprint.

[18] A. Grossmann, J. Morlet, and T. Paul. Transforms associated to square integrable group representations I: General results. *J. Math. Phys.*, 26:2473–2479, 1985.

[19] A. Grossmann, J. Morlet, and T. Paul. Transforms associated to square integrable group representations II: Examples. *Annales de l'Institut Henri Poincaré*, 45:293, 1986.

[20] F. Hlawatsch and G. Matz. Linear time-frequency filters. In B. Boashash, editor, *Time-Frequency Signal Analysis and Processing: A Comprehensive Reference*, page 466:475, Oxford (UK), 2003. Elsevier.

[21] W. Kozek. *Matched Weyl-Heisenberg Expansions of Nonstationary Environments*. PhD thesis, NuHAG, University of Vienna, 1996.

[22] S. Mallat. *A wavelet tour of signal processing*. Academic Press, 1998.

[23] W. Schempp. *Harmonic analysis on the Heisenberg nilpotent Lie group*, volume 147 of *Pitman Series*. J. Wiley, New York, 1986.