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REPRESENTATION OF SCHRÖDINGER OPERATOR OF A FREE PARTICLE VIA SHORT-TIME FOURIER TRANSFORM AND ITS APPLICATIONS

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Abstract. We propose a new representation of the Schrödinger operator of a free particle by using the short-time Fourier transform and give its applications.

1. Introduction. We consider the Schrödinger equation of a free particle,

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, & (t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}^n, \end{cases} \tag{1}$$

where $i = \sqrt{-1}$, $u(t,x)$ is a complex valued function of $(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n$, $u_0(x)$ is a complex valued function of $x \in \boldsymbol{R}^n$, $\partial_t u = \partial u/\partial t$ and $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$.

When u_0 is a function in $\mathcal{S}(\boldsymbol{R}^n)$, the solution $u(t,x)$ of (1) can be written as

$$u(t,x) = (e^{\frac{1}{2}it\Delta}u_0)(x) = \mathcal{F}^{-1}_{\xi\to x}[e^{-\frac{1}{2}it|\xi|^2}\mathcal{F}u_0(\xi)](x)\,.$$

Here we use the notation $\mathcal{F}f(\xi) = \int_{\boldsymbol{R}^n} f(x)e^{-ix\cdot\xi}dx$ for the Fourier transform of f and $\mathcal{F}^{-1}f(x) = \int_{\boldsymbol{R}^n} f(\xi)e^{ix\cdot\xi}đ\xi$ with $đ\xi = (2\pi)^{-n}d\xi$ for the inverse Fourier transform of f.

The Schrödinger operator $e^{\frac{1}{2}it\Delta}$ and closely related operators such as

$$(e^{i|D|^\alpha}u_0)(x) = \mathcal{F}^{-1}_{\xi\to x}[e^{i|\xi|^\alpha}\mathcal{F}u_0(\xi)](x)\,, \quad \alpha \in \boldsymbol{R}\,,$$

have been studied extensively by many authors. Hörmander [8] has proved $e^{i|D|^2}$ is bounded on $L^p(\boldsymbol{R}^n)$ if and only if $p = 2$, and Miyachi [11] has proved the sharp endpoint L^p-Sobolev estimates for $e^{i|D|^\alpha}$, $\alpha > 1$. We also remark that $e^{i|D|^2}$ is bounded on the Besov space $\dot{B}_s^{p,q}(\boldsymbol{R}^n)$ or $B_s^{p,q}(\boldsymbol{R}^n)$ if and only if $p = 2$ (Mizuhara [13] and Li [10]). On the other hand, a recent work by Bényi, Gröchenig, Okoudjou and Rogers [1] has shown $e^{i|D|^\alpha}$, $0 \le \alpha \le 2$, is bounded on the modulation space $M^{p,q}$ for all $1 \le p, q \le \infty$, which means $e^{\frac{1}{2}it\Delta}$ preserves the $M^{p,q}$-norm (see the precise definition of $M^{p,q}$ in Section 2.2 below). For further developments in this direction we refer to Bényi-Okoudjou [2], Cordero-Nicola [3], Miyachi-Nicola-Rivetti-Tabacco-Tomita [12], Sugimoto [14], Wang-Zhao-Guo [17], Wang-Hudzik [16] and the references therein.

In this paper, we propose a new representation of the solution $u(t,x)$ by using the short-time Fourier transform and give its applications. More precisely, let φ_0 be a function in

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$\mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$ and suppose $\varphi(t,x) = (e^{\frac{1}{2}it\Delta}\varphi_0)(x)$, which solves the initial value problem (1) with initial data φ_0. Then we have

$$u(t,x) = \frac{1}{\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle} V^*_{\varphi(t,\cdot)[(y,\xi)\to x]}[e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0}u_0(y-\xi t, \xi)](x)\,, \tag{2}$$

where $V_{\varphi_0}u_0$ denotes the short-time Fourier transform of u_0 with respect to the window φ_0 and $V^*_{\varphi(t,\cdot)}$ denotes the (informal) adjoint operator of the short-time Fourier transform $V_{\varphi(t,\cdot)}$ for a fixed t, which are defined in Section 2.1.

By using the representation (2), we have the following propositions.

PROPOSITION 1.1. *Let $1 \le p, q \le \infty$. Suppose $\varphi_0 \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$ and $\varphi(t,x) = (e^{\frac{1}{2}it\Delta}\varphi_0)(x)$, which solves the initial value problem* (1) *with initial data φ_0. Then*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi_0}} \tag{3}$$

holds for $u_0 \in M^{p,q}(\boldsymbol{R}^n)$.

REMARK 1.2. We note that the norms on the left-hand and right-hand side of (3) are measured by different windows. Moreover, by putting $e^{-\frac{1}{2}is\Delta}\varphi_0$, $s \in \boldsymbol{R}$, into φ_0 in the equality (3), we have

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t-s,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi(-s,\cdot)}}\,, \tag{4}$$

where φ is the solution of (1) with initial data φ_0.

PROPOSITION 1.3. *Let $1 \le p, q \le \infty$ and $\varphi_0 \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$. Then there exists a positive constant C such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(s,\cdot)}} \le C(1+|t|)^{n/2}\|u_0\|_{M^{p,q}_{\varphi(s,\cdot)}} \tag{5}$$

for $u_0 \in \mathcal{S}(\boldsymbol{R}^n)$ and $t, s \in \boldsymbol{R}$.

PROPOSITION 1.4. *Let $2 \le p \le \infty$, $1 \le q \le \infty$ and $\varphi_0 \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$. Then there exists a positive constant C such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(s,\cdot)}} \le C(1+|t|)^{-n(1/2-1/p)}\|u_0\|_{M^{p',q}_{\varphi(s,\cdot)}} \tag{6}$$

for $u_0 \in \mathcal{S}(\boldsymbol{R}^n)$ and $t, s \in \boldsymbol{R}$ with $1/p + 1/p' = 1$.

PROPOSITION 1.5. *Let $2 \le p \le \infty$, $1 \le q \le \infty$ and $\varphi_0 \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$. Then there exists a positive constant C such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(s,\cdot)}} \le C(1+|t|)^{n(1/2-1/p)}\|u_0\|_{M^{p,q}_{\varphi(s,\cdot)}} \tag{7}$$

for $u_0 \in \mathcal{S}(\boldsymbol{R}^n)$ and $t, s \in \boldsymbol{R}$.

REMARK 1.6. By taking $s = 0$ in each of the estimates (5), (6) and (7), we have the estimates due to Bényi-Gröchenig-Okoudjou-Rogers [1] and Wang-Hudzik [16].

The paper is organized as follows. In Section 2, we recall the definitions and basic properties of the short-time Fourier transform and the modulation spaces. In Section 3, we prove the representation (2). In Section 4, we prove Propositions 1.1 through 1.5. Finally, in Section 5, we give a local well-posedness result for the nonlinear Schrödinger equations with Cauchy data in the modulation spaces $M^{p,1}$.

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2. Preliminaries. Throughout this paper the letter C denotes a constant which may be different in each occasion.

2.1. The short-time Fourier transform. We recall the definitions of the short-time Fourier transform and its adjoint operator. Let $f \in \mathcal{S}'(\boldsymbol{R}^n)$ and $\phi \in \mathcal{S}(\boldsymbol{R}^n)$. Then the short-time Fourier transform $V_\phi f$ of f with respect to the window ϕ is defined by

$$V_\phi f(x,\xi) = \langle f(y), \phi(y-x)e^{iy\cdot\xi}\rangle = \int_{\boldsymbol{R}^n} f(y)\overline{\phi(y-x)}e^{-iy\cdot\xi}dy\,. \tag{8}$$

Let F be a function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ and $\phi \in \mathcal{S}(\boldsymbol{R}^n)$. Then the adjoint operator V_ϕ^* of V_ϕ is defined by

$$V_\phi^* F(x) = \iint_{\boldsymbol{R}^{2n}} F(y,\xi)\phi(x-y)e^{ix\cdot\xi}dy\,đ\xi$$

with $đ\xi = (2\pi)^{-n}d\xi$. It is known that for $f \in \mathcal{S}'(\boldsymbol{R}^n)$ and $\phi \in \mathcal{S}(\boldsymbol{R}^n)$, $V_\phi f$ is a continuous function on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ and

$$|V_\phi f(x,\xi)| \le C(1+|x|+|\xi|)^N \quad \text{for all } (x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$$

for some constant $C, N \ge 0$ ([7, Theorem 11.2.3]). Moreover, for $\phi, \psi, \gamma \in \mathcal{S}(\boldsymbol{R}^n)$ satisfying $\langle\psi,\phi\rangle \ne 0$ and $\langle\gamma,\psi\rangle \ne 0$, we have the inversion formula

$$\frac{1}{\langle\psi,\phi\rangle}V_\psi^* V_\phi f = f\,, \quad f \in \mathcal{S}'(\boldsymbol{R}^n) \tag{9}$$

([7, Corollary 11.2.7]) and the pointwise inequality

$$|V_\phi f(x,\xi)| \le \frac{C}{|\langle\gamma,\psi\rangle|}(|V_\psi f| * |V_\phi\gamma|)(x,\xi)\,, \quad f \in \mathcal{S}'(\boldsymbol{R}^n)\,, \tag{10}$$

for all $(x,\xi) \in \boldsymbol{R}^{2n}$ ([7, Lemma 11.3.3]).

2.2. Modulation spaces. We recall the definition of modulation spaces $M^{p,q}$ which were introduced by Feichtinger [5] to measure smoothness of a function or a distribution in a way different from Besov spaces. Let $1 \le p, q \le \infty$ and $\phi \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$. Then the modulation space $M_\phi^{p,q}(\boldsymbol{R}^n) = M^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'(\boldsymbol{R}^n)$ such that the norm

$$\|f\|_{M_\phi^{p,q}} = \left(\int_{\boldsymbol{R}^n}\left(\int_{\boldsymbol{R}^n}|V_\phi f(x,\xi)|^p dx\right)^{q/p}d\xi\right)^{1/q} = \|V_\phi f(x,\xi)\|_{L_x^p L_\xi^q}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

The space $M_\phi^{p,q}(\boldsymbol{R}^n)$ is a Banach space, whose definition is independent of the choice of the window ϕ, i.e., $M_\phi^{p,q}(\boldsymbol{R}^n) = M_\psi^{p,q}(\boldsymbol{R}^n)$ for all $\phi, \psi \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$ ([5, Theorem 6.1]). This property is crucial in the sequel, since we choose a suitable window ϕ to estimate the modulation space norm. If $1 \le p, q < \infty$ then $\mathcal{S}(\boldsymbol{R}^n)$ is dense in $M^{p,q}$ ([5, Theorem 6.1]). We also note $L^2 = M^{2,2}$, and $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$ if $p_1 \le p_2, q_1 \le q_2$ ([5, Proposition 6.5]). Let us define by $\mathcal{M}^{p,q}(\boldsymbol{R}^n)$ the completion of $\mathcal{S}(\boldsymbol{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}}$. Then $\mathcal{M}^{p,q}(\boldsymbol{R}^n) = M^{p,q}(\boldsymbol{R}^n)$ for $1 \le p, q < \infty$. Moreover, the complex interpolation theory for these spaces reads as follows: Let $0 < \theta < 1$ and $1 \le p_i, q_i \le \infty, i = 1, 2$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, then $(\mathcal{M}^{p_1,q_1}, \mathcal{M}^{p_2,q_2})_{[\theta]} = \mathcal{M}^{p,q}$ ([5, Theorem 6.1], [15, Theorem 2.3]). We refer to [5] and [7] for more details.

3. Representation of the solution of free Schrödinger equation. In this section, we show that the solution $u(t,x)$ of (1) is represented by (2). Let $u(t,x)$ be the solution of (1) with $u(0,x) = u_0 \in \mathcal{S}(\boldsymbol{R}^n)$. Note that $u(t,x)$ is in $C^\infty(\boldsymbol{R}; \mathcal{S}(\boldsymbol{R}^n))$ in this case. Let $\varphi(t,x)$ be the solution of (1) with $\varphi(0,x) = \varphi_0(x) \in \mathcal{S}(\boldsymbol{R}^n) \setminus \{0\}$, which is used as a window function. Using integration by parts, we have

$$
\begin{aligned}
&V_{\varphi(t,\cdot)}\left(\frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi) \\
&= \frac{1}{2}\int_{\boldsymbol{R}^n} \overline{\varphi(t,y-x)}\Delta_y u(t,y) e^{-iy\cdot\xi}\,dy \\
&= \int_{\boldsymbol{R}^n} \overline{\frac{1}{2}\Delta_y \varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi}\,dy + \int_{\boldsymbol{R}^n} (-i\xi\cdot\nabla_y)\overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi}\,dy \\
&\quad -\frac{1}{2}|\xi|^2 \int_{\boldsymbol{R}^n} \overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi}\,dy \\
&= V_{\frac{1}{2}\Delta\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + \left(i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\,.
\end{aligned}
$$

Since $u(t,x)$ and $\varphi(t,x)$ are solutions of (1) and

$$
i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = V_{-i\partial_t\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + V_{\varphi(t,\cdot)}(i\partial_t u(t,\cdot))(x,\xi)
$$

is valid, we obtain

$$
\begin{aligned}
&i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + \left(i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) \\
&= V_{\varphi(t,\cdot)}\left(i\partial_t u(t,\cdot) + \frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi) - V_{[i\partial_t\varphi(t,\cdot)+\frac{1}{2}\Delta\varphi(t,\cdot)]}(u(t,\cdot))(x,\xi) \\
&= 0\,.
\end{aligned}
$$

Hence the initial value problem (1) is transformed via the short-time Fourier transform with window function $\varphi(t,x)$ to

$$
\begin{cases}
\left(i\partial_t + i\xi\cdot\nabla_x - \dfrac{1}{2}|\xi|^2\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = 0\,, \\
V_{\varphi(0,\cdot)}(u(0,\cdot))(x,\xi) = V_{\varphi_0}u_0(x,\xi)\,.
\end{cases}
\tag{11}
$$

It is easy to see that

$$
V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0}u_0(x-\xi t,\xi)
\tag{12}
$$

is the solution of (11). Applying the adjoint operator $V^*_{\varphi(t,\cdot)}$ of $V_{\varphi(t,\cdot)}$ to the both sides of (12), we have the representation (2) by the inversion formula (9). It is easy to check that the above argument is valid for $u_0(x) \in \mathcal{S}'(\boldsymbol{R}^n)$.

4. Proof of Propositions. In this section, we prove Propositions 1.1 through 1.5.

PROOF OF PROPOSITION 1.1. Taking $L^p_x L^q_\xi$ norm of the both sides of (12), we have

$$\begin{aligned}\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} &= \|V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\|_{L^p_x L^q_\xi} \\ &= \|e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0} u_0(x-\xi t,\xi)\|_{L^p_x L^q_\xi} \\ &= \|V_{\varphi_0} u_0(x,\xi)\|_{L^p_x L^q_\xi} \\ &= \|u_0\|_{M^{p,q}_{\varphi_0}}.\end{aligned}$$

□

PROOF OF PROPOSITION 1.3. By Remark 1.2, it suffices to prove

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t-s,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi(-s,\cdot)}} \le C(1+|t|)^{n/2}\|u_0\|_{M^{p,q}_{\varphi(t-s,\cdot)}},$$

where C is independent of t and s. We note that

$$V_{\varphi(-s,\cdot)}u_0 = V_{\varphi(-s,\cdot)}\left[\frac{1}{\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle} V^*_{\varphi(-s,\cdot)} V_{\varphi(t-s,\cdot)} u_0\right]$$

by the inversion formula (9) and $\langle\varphi(-s,\cdot),\varphi(t-s,\cdot)\rangle = \langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle$. Then, we have

$$\begin{aligned}&V_{\varphi(-s,\cdot)}u_0(x,\xi)\\ &= V_{\varphi(-s,\cdot)[y\to(x,\xi)]}\left[\frac{1}{\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle}\iint_{\boldsymbol{R}^{2n}} V_{\varphi(t-s,\cdot)}u_0(z,\eta)\varphi(-s,y-z)e^{iy\cdot\eta}dz\,d\!\!\!{}^-\eta\right](x,\xi)\\ &= \frac{1}{\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle}\iiint_{\boldsymbol{R}^{3n}} \overline{\varphi(-s,y-x)}e^{-iy\cdot\xi} V_{\varphi(t-s,\cdot)}u_0(z,\eta)\varphi(-s,y-z)e^{iy\cdot\eta}dy\,dz\,d\!\!\!{}^-\eta\\ &= \frac{1}{\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle}\iint_{\boldsymbol{R}^{2n}}\left(\int_{\boldsymbol{R}^n}\overline{\varphi(-s,y-x)}\varphi(-s,y-z)e^{-i(\xi-\eta)\cdot y}dy\right)\\ &\quad\times V_{\varphi(t-s,\cdot)}u_0(z,\eta)dz\,d\!\!\!{}^-\eta\\ &= \frac{1}{\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle}\iint_{\boldsymbol{R}^{2n}} V_{\varphi(-s,\cdot)}(\varphi(-s,\cdot))(x-z,\xi-\eta)e^{-i(\xi-\eta)\cdot z}V_{\varphi(t-s,\cdot)}u_0(z,\eta)dz\,d\!\!\!{}^-\eta\,.\end{aligned}$$

By Young's inequality, we have

$$\|u_0\|_{M^{p,q}_{\varphi(-s,\cdot)}} \le \frac{C}{|\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle|}\|\varphi(-s,\cdot)\|_{M^{1,1}_{\varphi(-s,\cdot)}}\|u_0\|_{M^{p,q}_{\varphi(t-s,\cdot)}}.$$

Since

$$|\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle| = |\langle\widehat{\varphi_0}, e^{-\frac{1}{2}it|\xi|^2}\widehat{\varphi_0}\rangle| = \left|\int_{\boldsymbol{R}^n} e^{\frac{1}{2}it|\xi|^2}|\widehat{\varphi_0}(\xi)|^2 d\xi\right|,$$

the stationary phase method yields that

$$|\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle| \sim C|\widehat{\varphi_0}(0)|^2|t|^{-n/2} \quad (\text{as } |t|\to\infty). \tag{13}$$

The fact that $\|\varphi(-s,\cdot)\|_{M^{1,1}_{\varphi(-s,\cdot)}} = \|\varphi_0\|_{M^{1,1}_{\varphi_0}}$ and (13) yield (5). □

PROOF OF PROPOSITION 1.4. Firstly, we prove (6) for $p=2$. By Remark 1.2, it suffices to show that $\|V_{\varphi(-s,\cdot)}u_0(x,\xi)\|_{L^2_xL^q_\xi} = \|V_{\varphi(t-s,\cdot)}u_0(x,\xi)\|_{L^2_xL^q_\xi}$. By Plancherel theorem, we have

$$\begin{aligned}
\|V_{\varphi(t-s,\cdot)}u_0(x,\xi)\|_{L^2_xL^q_\xi} &= \left\|\left\|\int_{\boldsymbol{R}^n} \overline{\varphi(t-s,y-x)}u_0(y)e^{-iy\cdot\xi}dy\right\|_{L^2_x}\right\|_{L^q_\xi}\\
&= \left\|\left\|\int_{\boldsymbol{R}^n} \overline{\widehat{\varphi}(t-s,\eta)}e^{-iy\cdot\eta}u_0(y)e^{-iy\cdot\xi}dy\right\|_{L^2_\eta}\right\|_{L^q_\xi}\\
&= \left\|\left\|\int_{\boldsymbol{R}^n} e^{\frac{1}{2}i(t-s)|\eta|^2}\overline{\widehat{\varphi_0}(\eta)}e^{-iy\cdot\eta}u_0(y)e^{-iy\cdot\xi}dy\right\|_{L^2_\eta}\right\|_{L^q_\xi}\\
&= \left\|\left\|\int_{\boldsymbol{R}^n} \overline{e^{\frac{1}{2}is|\eta|^2}\widehat{\varphi_0}(\eta)}e^{-iy\cdot\eta}u_0(y)e^{-iy\cdot\xi}dy\right\|_{L^2_\eta}\right\|_{L^q_\xi}\\
&= \left\|\left\|\int_{\boldsymbol{R}^n} \overline{\varphi(-s,y-x)}u_0(y)e^{-iy\cdot\xi}dy\right\|_{L^2_x}\right\|_{L^q_\xi}\\
&= \|V_{\varphi(-s,\cdot)}u_0(x,\xi)\|_{L^2_xL^q_\xi}\,.
\end{aligned}$$

Secondly, we prove (6) for $p=\infty$. By the same argument as in the proof of Proposition 1.3, we have

$$\begin{aligned}
&\|u_0\|_{M^{\infty,q}_{\varphi(-s,\cdot)}}\\
&= \left\|V_{\varphi(-s,\cdot)}\left[\frac{1}{\|\varphi(t-s,\cdot)\|^2_{L^2}}V^*_{\varphi(t-s,\cdot)}V_{\varphi(t-s,\cdot)}u_0(x,\xi)\right]\right\|_{L^\infty_xL^q_\xi}\\
&= \frac{1}{\|\varphi_0\|^2_{L^2}}\left\|\iint_{\boldsymbol{R}^{2n}} V_{\varphi(-s,\cdot)}(\varphi(t-s,\cdot))(x-z,\xi-\eta)e^{-iz\cdot(\xi-\eta)}V_{\varphi(t-s,\cdot)}u_0(z,\eta)dz\bar{d}\eta\right\|_{L^\infty_xL^q_\xi}\\
&\le \frac{C}{\|\varphi_0\|^2_{L^2}}\|V_{\varphi(-s,\cdot)}(\varphi(t-s,\cdot))(x,\xi)\|_{L^\infty_xL^1_\xi}\|V_{\varphi(t-s,\cdot)}u_0(x,\xi)\|_{L^1_xL^q_\xi}\\
&= \frac{C}{\|\varphi_0\|^2_{L^2}}\|\varphi(t-s,\cdot)\|_{M^{\infty,1}_{\varphi(-s,\cdot)}}\|u_0\|_{M^{1,q}_{\varphi(t-s,\cdot)}}\,.
\end{aligned}$$

Since

$$\begin{aligned}
V_{\varphi(-s,\cdot)}(\varphi(t-s,\cdot))(x,\xi) &= \int_{\boldsymbol{R}^n} \overline{\varphi(-s,y-x)}\varphi(t-s,y)e^{-iy\cdot\xi}dy\\
&= e^{-ix\cdot\xi}\int_{\boldsymbol{R}^n} \widehat{\varphi}(t-s,\eta)\overline{\widehat{\varphi}(-s,\eta-\xi)}e^{ix\cdot\eta}d\eta
\end{aligned}$$

$$= e^{-ix\cdot\xi}\int_{\boldsymbol{R}^n} e^{-\frac{1}{2}it|\eta|^2}\widehat{\varphi_0}(\eta)\overline{\widehat{\varphi_0}(\eta-\xi)}e^{ix\cdot\eta}d\eta\,,$$

the stationary phase method (see [6], [9]) yields that

$$\begin{aligned}&|V_{\varphi(-s,\cdot)}(\varphi(t-s,\cdot))(x,\xi)|\\ &\quad\le C|t|^{-n/2}\left|\widehat{\varphi_0}\Big(-\frac{x}{t}\Big)\widehat{\varphi_0}\Big(-\frac{x}{t}-\xi\Big)\right|\\ &\qquad+C|t|^{-n/2-1}\sum_{|\alpha|\le 2(1+n)}\int_{\boldsymbol{R}^n}\left|\Big(\frac{\partial}{\partial\eta}\Big)^{\alpha}\Big[\widehat{\varphi_0}(\eta)\overline{\widehat{\varphi_0}(\eta-\xi)}\Big]\right|d\eta\quad(\text{as } |t|\to\infty)\,.\end{aligned}$$

Hence we have (6) for $p=\infty$. By the complex interpolation method, we have (6) for all $1\le p\le\infty$. □

PROOF OF PROPOSITION 1.5. By using the complex interpolation method between $p=\infty$ for (5) and $p=2$ for (6), we have the conclusion. □

5. nonlinear Schrödinger equation. Next we consider the following initial value problem of the nonlinear Schrödinger equation,

$$\begin{cases} i\partial_t u+\dfrac{1}{2}\Delta u=f(u)\,,\\ u(0,x)=u_0(x)\,,\end{cases}\tag{14}$$

where $f(u)$ is a polynomial of u and $\overline{u}$ with $f(0)=0$.

The following result is already known, but we obtain it as a corollary of our representation (2).

PROPOSITION 5.1 (Bényi-Okoudjou [2]). *For $u_0\in M^{p,1}(\boldsymbol{R}^n)$ with $1\le p\le\infty$, there exists a positive constant T and a unique solution of* (14) *such that $u\in C([0,T];M^{p,1}(\boldsymbol{R}^n))$.*

PROOF. Using the representation (2) of the Schrödinger operator $e^{\frac{1}{2}it\Delta}$, we have the integral equation associated to (14),

$$\begin{aligned}&V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\\ &\quad=e^{-\frac{1}{2}it|\xi|^2}V_{\varphi_0}u_0(x-t\xi,\xi)+\int_0^t e^{-\frac{1}{2}i(t-s)|\xi|^2}V_{\varphi(s,\cdot)}[f(u)](x-(t-s)\xi,\xi)ds\,.\end{aligned}$$

We recall that $M^{p,1}$ is a Banach algebra, i.e., for $\phi\in\mathcal{S}(\boldsymbol{R}^n)\setminus\{0\}$, there exists a positive constant C such that

$$\|u_1u_2\|_{M_\phi^{p,1}}\le C\|u_1\|_{M_\phi^{p,1}}\|u_2\|_{M_\phi^{p,1}}\tag{15}$$

for all $u_1,u_2\in M^{p,1}(\boldsymbol{R}^n)$ ([2, Corollary 2.7], [17, Corollary 4.2]).

We define the mapping $F(u)$ from $C([0,T];M^{p,1}(\boldsymbol{R}^n))$ to itself by

$$F(u) = V^*_{\varphi(t,\cdot)} \times \left[e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0} u_0(x - t\xi, \xi) + \int_0^t e^{-\frac{1}{2}i(t-s)|\xi|^2} V_{\varphi(s,\cdot)}[f(u)](x - (t-s)\xi, \xi) ds \right].$$

Putting $A = \|u_0\|_{M^{p,1}_{\varphi_0}}$ and $X_T = C([0,T]; M^{p,1}(\boldsymbol{R}^n))$, we define a closed subset $X_{T,A}$ of $C([0,T]; M^{p,1}(\boldsymbol{R}^n))$ by

$$X_{T,A} = \left\{ u \in C([0,T]; M^{p,1}(\boldsymbol{R}^n)) \;\middle|\; \|u\|_{X_T} = \sup_{t\in[0,T]} \|u(t,\cdot)\|_{M^{p,1}_{\varphi(t,\cdot)}} \le 2A \right\}.$$

The mapping F is well defined on $X_{T,A}$ for small $T > 0$. In fact, the above fact (15) for multiplication on $M^{p,1}(\boldsymbol{R}^n)$ yields that

$$\|F(u)\|_{X_T} \le \|u_0\|_{M^{p,1}} + \int_0^T C(s) \widetilde{f}(\|u\|_{M^{p,1}_{\varphi(s,\cdot)}}) ds\,,$$

where $C(s)$ is a positive continuous function of s and $\widetilde{f}(u)$ is a polynomial of u and $\overline{u}$ which is made from $f(u)$ replacing all the coefficients to their absolute values. Hence we have

$$\|F(u)\|_{X_T} \le A + \widetilde{f}(A) C_1 T$$

with $C_1 = \sup_{s\in[0,T]} C(s)$, which implies $F(u) \in X_{T,A}$ for small $T > 0$.

The same argument as above yields that F is a contraction mapping from $X_{T,A}$ to itself for small $T > 0$. Picard's fixed point theorem for a contraction mapping on $X_{T,A}$ implies the conclusion. □

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