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REPRESENTATION THEOREMS FOR COMPLEMENTED ALGEBRAS

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Introduction. In this paper we obtain Hilbert space representations for as large a class of complemented algebras as possible. In [2] the same problem was considered for complemented B^*-algebras. It was shown then that if A is a topologically simple B^*-algebra, then a sufficient (subject to a dimension restriction) and a necessary condition that a complementor p be expressible in terms of a Hilbert space representation of A (or, equivalently, in the form $\boldsymbol{R}^p = (\boldsymbol{R}_l)^\#$ for some involution $\#$ in A) was that p be continuous. Throughout the present paper the same dimension restriction (that the algebra has no minimal left ideals of dimension less than three) will be imposed. In §5 a counterexample shows that it cannot be removed. The definition of continuity in [2] is not applicable to a general complemented algebra, but in §2 we give an alternative definition and show that this is an extension of the previous definition. In §3 we consider the case when A is a primitive Banach algebra. We obtain a faithful, continuous, strictly dense Hilbert space representation for A when endowed with a continuous complementor. Under this we identify A with a left ideal of $\boldsymbol{B}(\boldsymbol{H})$ that is closed under a norm that majorises the operator norm. We then show that this representation characterizes primitive Banach algebras with continuous complementors. In §4 we use the results of §3 to obtain a faithful, continuous Hilbert space representation for any semisimple Banach algebra A with a continuous complementor. Conversely, we show that, if any complemented algebra admits a representation of this form, then the complementor is continuous. We deduce that, if A is B^*, then a necessary and sufficient condition that its complementor be expressible in the form $\boldsymbol{R}^p=(\boldsymbol{R}_l)^\#$ is that it be continuous. This extends the result of [2]. In §5 we apply the results of §4 to show that the condition C_2 in the definition of a complementor cannot be relaxed.

1. **Preliminaries.** Throughout the paper A will denote a semisimple complex Banach algebra whose norm is $\| \ \|$; $\{I_\lambda : \lambda \in \Lambda\}$ is the set of all minimal closed two-sided ideals of A; R_A is the set of all closed right ideals of A and M_A the set of all minimal right ideals of A.

Following [8] we say that A is a *right complemented algebra* if there is a mapping $p: \boldsymbol{R} \to \boldsymbol{R}^p$ of R_A onto itself that has the following properties:

C_1: $\boldsymbol{R} \cap \boldsymbol{R}^p=(0)$ $(\boldsymbol{R} \in R_A)$;

C_2: $\boldsymbol{R}+\boldsymbol{R}^p=A$ $(\boldsymbol{R} \in R_A)$;

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C_3: $(R^p)^p = R$ $(R \in R_A)$;

C_4: if $R_1 \subset R_2$ then $R_1^p \supset R_2^p$ $(R_1, R_2 \in R_A)$.

The mapping p is called a *right complementor* on A. A *left complementor* and a *left complemented algebra* are defined analogously. Since we shall be exclusively concerned with right complemented algebras we shall refer to them as complemented algebras and to right complementors as complementors.

We recall that, from Lemma 5 and Theorem 4 of [8], if A is a complemented algebra, then A has a dense socle and is the direct topological sum of the family $\{I_\lambda : \lambda \in \Lambda\}$. In particular, if A is primitive, then it is topologically simple.

The notation we adopt is mostly that of [6]; however, we follow [3] in denoting by S_l, S_r respectively the left and right annihilators of a subset S of an algebra. Also we use cl ( ) to denote closure.

Some other specific points of notation should be mentioned here. Let X be a Banach space. Then $[x]$ will denote the one-dimensional linear subspace of X generated by a single element x of X. $B(X)$, $K(X)$ are the algebras of all bounded, compact (respectively) linear operators on X. The operator norms will always be denoted by $|\ |$. If B is a subalgebra of $B(X)$ then:

1. $S(R)$ is the smallest closed subspace of X that contains the range of each operator in a subset R of B;

2. $J_B(S)$ is the set of all elements in B whose range is contained in a subset S of X.

Suppose S and T are closed linear subspaces of X such that $S \oplus T = X$. Then by Theorem 4.8.D in [7] there is a bounded linear projection operator E on X such that $x = Ex + (1-E)x$ gives the unique decomposition of an element x of X into components Ex in S and $(1-E)x$ in T. We write $P(S, T)$ for E. This notation is used most frequently for the operator $P(R, R^p)$ on A.

The following set notations are adopted:

(1) if A, B are subsets of an algebra then $AB = \{ab : a \in A, b \in B\}$,

(2) if S is a subset of a space X and T a set of mappings of X into another space Y then $TS = \{ts : t \in T, s \in S\}$,

(3) if $a \to T_a$ is a representation of an algebra and A is a subset of the algebra then $T_A = \{T_a : a \in A\}$.

C will denote the complexes and Z the set of positive integers and zero.

2. **Complementors and continuous complementors on A, I_λ.** We first observe that since A is semisimple, Lemma 1 of [8] applies to A; thus $I_\lambda^p = (I_\lambda)_l = (I_\lambda)_r$ and, in particular, is a two-sided ideal of A. Also from the proof of Lemma 2.8.8 in [6] it can be seen that every minimal right (or left) ideal of A is contained in a minimal closed two-sided ideal of A. A third consequence of the semisimplicity of A is that $a \in \text{cl}(aA)$ for all a in A (Lemma 3 in [1]).

LEMMA 2.1. *If $R \in R_A$ and $R \subset I_\mu$ then $R^p \supset I_\lambda$ for all $\lambda \in \Lambda$, $\lambda \neq \mu$.*

Proof. Using C_4 it is clearly sufficient to show that $I_\mu^p \supset I_\lambda$ $(\mu \neq \lambda)$. Now $I_\mu^p \cap I_\lambda$

is a closed two-sided ideal of A that is contained in I_λ. It must, therefore, be (0) or I_λ. Suppose that there is some $\lambda \in \Lambda$ $(\lambda \neq \mu)$ such that $I_\mu^p \cap I_\lambda = (0)$. Then let x, x' be any elements of I_λ. Write $x = y + z$ where $y \in I_\mu$, $z \in I_\mu^p$. Then $yx' \in I_\mu \cap I_\lambda = (0)$ and $zx' \in I_\mu^p \cap I_\lambda = (0)$; therefore $xx' = 0$. Thus $(I_\lambda)^2 = (0)$ which contradicts the semisimplicity of A. This completes the proof.

We now give the definition of continuity of a complementor p. It is shown at the end of this section that if A is B^* then the present definition is equivalent to that of [2].

Definition. A sequence $\{R_n : n \in Z\} \subset M_A$ is *p-convergent* to R_0 if $P(R_n, R_n^p)$ converges uniformly to $P(R_0, R_0^p)$ (as $n \to \infty$) on any minimal left ideal of A. The complementor p is *continuous* if whenever $a_n A \in M_A$ $(n \in Z)$ and $a_n \to a_0$ as $n \to \infty$ then $\{a_n A\}$ is p-convergent to $a_0 A$.

Theorem 2.2. *Every closed right (or left) ideal of I_λ is also a closed right (left) ideal of A. A complementor p_λ can be induced in I_λ by $R^{p_\lambda} = R^p \cap I_\lambda$ $(R \in R_{I_\lambda})$. Further, p is continuous if and only if p_λ is continuous for all λ in Λ.*

Proof. Let R be any closed right ideal of I_λ. Then

$$RA = R(I_\lambda + I_\lambda^p) = R(I_\lambda + (I_\lambda)_r) = RI_\lambda \subset R.$$

The proof for a left ideal of I_λ is similar. It is now possible to define p_λ and the verification that it is a complementor is easy.

Now suppose that each p_λ is continuous. Let $\{R_n : n \in Z\} \subset M_A$. For each n let I_n be the minimal closed two-sided ideal of A that contains R_n. Suppose there exist elements $a_n \in R_n$ such that $a_n \to a_0 \neq 0$ as $n \to \infty$. We show first that there is some integer N such that, for all $n > N$, $R_n \subset I_0$. Suppose the contrary if possible. Then, taking a subsequence if necessary, we have $a_n \in A$, $a_n \to a_0$ $(\neq 0)$ as $n \to \infty$, and $a_n I_0 = I_0 a_n = (0)$. Let $t \in I_0$: $\|ta_0\| = \|t(a_0 - a_n)\| \leqq \|t\| \, \|a_0 - a_n\|$ and, letting $n \to \infty$, we see that $ta_0 = 0$. Thus $I_0 a_0 = (0)$ and hence by the topological simplicity of I_0 we have $a_0 = 0$ which is the required contradiction. Thus we can assume that $R_n \subset I_0$ for all n in Z. Then, by hypothesis $\{R_n\}$ is p_0-convergent to R_0. However, for any $R \in M_A$, with $R \subset I_0$ and any minimal left ideal L of A there are two possibilities (let I_λ be the minimal closed two-sided ideal of A that contains L):

(1) $I_0 \neq I_\lambda$: then $P(R, R^p) = 0$ on L since $L \subset I_\lambda \subset R^p$,

(2) $I_0 = I_\lambda$: then $P(R, R^p) = P(R, R^{p_0})$ on L.

Thus $\{R_n\}$ p-convergent to R_0 is equivalent to $\{R_n\}$ p_0-convergent to R_0. Therefore $\{R_n\}$ is p-convergent to R_0 and so p is continuous.

The converse is now easily proved since $\{R_n\} \subset M_{I_\lambda}$ implies $\{R_n\} \subset M_A$.

Theorem 2.3. *Let $R \in R_A$. Then $R = \mathrm{cl}\,(\sum R \cap I_\lambda : \lambda \in \Lambda)$, and*

$$R^p = \mathrm{cl}\left(\sum (R \cap I_\lambda)^{p_\lambda} : \lambda \in \Lambda\right).$$

Proof. Let a be any element of R, and let $\varepsilon > 0$. Then, since $a \in \mathrm{cl}\,(aA)$, there is an element b of A such that $\|a - ab\| < \varepsilon/2$. Also since $A = \mathrm{cl}\,(\sum I_\lambda : \lambda \in \Lambda)$ there exists

$\{\lambda_i : i=1, 2, \ldots, n\} \subset \Lambda$ and elements b_i of I_{λ_i} such that $\|b-(b_1+\cdots+b_n)\| < \varepsilon/(2\|a\|)$. Now $\|a-a(b_1+\cdots+b_n)\| < \varepsilon$. Thus

$$R \subset \mathrm{cl}\left(\sum RI_\lambda : \lambda \in \Lambda\right) \subset \mathrm{cl}\left(\sum R \cap I_\lambda : \lambda \in \Lambda\right).$$

Conversely, $R \supset R \cap I_\lambda$ for all λ in Λ and so $R \supset \mathrm{cl}\,(\sum R \cap I_\lambda : \lambda \in \Lambda)$.

Now, applying this result to R^p, we have

$$\begin{aligned} R^p &= \mathrm{cl}\left(\sum R^p \cap I_\lambda : \lambda \in \Lambda\right) \subset \mathrm{cl}\left(\sum (R \cap I_\lambda)^p \cap I_\lambda : \lambda \in \Lambda\right) \\ &= \mathrm{cl}\left(\sum (R \cap I_\lambda)^{p_\lambda} : \lambda \in \Lambda\right). \end{aligned}$$

Also, since $R = \mathrm{cl}\,(\sum R \cap I_\lambda : \lambda \in \Lambda)$ and $I_\lambda^p \supset I_\mu$ $(\mu \neq \lambda)$, it is easily seen that $R \subset \mathrm{cl}\,((R \cap I_\lambda) + I_\lambda^p)$. Therefore

$$\begin{aligned} R^p \supset (\mathrm{cl}\,((R \cap I_\lambda) + I_\lambda^p))^p &= (R \cap I_\lambda)^p \cap I_\lambda \qquad \text{(by Lemma 2 in [1])} \\ &= (R \cap I_\lambda)^{p_\lambda}. \end{aligned}$$

Hence $R^p \supset \mathrm{cl}\,(\sum (R \cap I_\lambda)^{p_\lambda} : \lambda \in \Lambda)$ and, combining this with the previous inclusion, we have equality.

Before specializing to the topologically simple case in the next section we justify our present definition of a continuous complementor by showing that, if A is B^*, it is equivalent to the definition given in [2].

We recall the concepts of [2] using the notation of this paper. Suppose that A is B^*. Then a minimal idempotent f of A is said to be a p-projection if multiplication on the left by f is $P((fA), (fA)^p)$. Every minimal right ideal of A contains a unique p-projection as well as a unique hermitian idempotent which corresponds to the special case when p is the natural complementor $R \to (R_l)^*$. This gives rise to the p-derived mapping P from the set E of all hermitian minimal idempotents onto the set F of all p-projections in A $(Pe=f : fA=eA)$. In [2] the complementor p was said to be continuous if P was continuous with respect to the relative metric topologies induced in E, F by the norm in A.

In the special case when $A = K(H)$ for some Hilbert space H then, for any x in H, e_x, f_x denoted the unique elements of E, F respectively that are contained in $J_A([x])$. Thus multiplication on the left by f_x corresponds to $P(J_A([x]), (J_A([x]))^p)$ and by e_x corresponds to $P(J_A([x]), J_A(x^\perp)) = P(J_A([x]), (J_A([x]))_l^*)$. Also $e_x = x \otimes x/(x, x)$.

This definition is essentially a relative one in that it presupposes the existence of a natural (continuous) complementor $R \to (R_l)^*$. Accordingly we shall refer to it as relative continuity. In the next theorem we show that the two definitions do in fact coincide.

THEOREM 2.4. *If A is B^* then p is continuous if and only if it is relatively continuous.*

Proof. Suppose that p is continuous and that $e_0 \in E$. If P is not continuous at e_0 then there are elements e_n of E and $k>0$ such that

(i) $\|e_n - e_0\| < 1/n, \qquad \|P(e_0) - P(e_n)\| > k \qquad (n = 1, 2, \ldots).$

It is easy to verify that e_0, e_n are contained in the same minimal closed two-sided ideal $\boldsymbol{I}$ of A. Also $P(e_0), P(e_n) \in \boldsymbol{I}$. Now since any minimal left ideal of $\boldsymbol{I}$ is a minimal left ideal of A and the left regular representation of $\boldsymbol{I}$ on any minimal left ideal is an isometry, the continuity of p contradicts (i).

Conversely, suppose that p is relatively continuous. Let $\{a_nA : n \in Z\} \subset M_A$ be such that $a_n \to a_0$ as $n \to \infty$. Let $\boldsymbol{I}_0$ be the minimal closed two-sided ideal of A that contains a_0; then there is an integer N such that $a_n \in \boldsymbol{I}_0$ for all $n > N$. (See proof of Theorem 2.2.) Also for $n > N$:

$$P(a_nA, (a_nA)^p)/\boldsymbol{I}_\lambda = 0 \quad (\boldsymbol{I}_\lambda \neq \boldsymbol{I}_0), \qquad P(a_nA, (a_nA)^p)/\boldsymbol{I}_0 = P(a_nA, (a_nA)^{p_0}).$$

Thus it is sufficient to prove the result in the topologically simple case. Suppose that $A = \boldsymbol{K}(\boldsymbol{H})$ where $\boldsymbol{H}$ is a Hilbert space with inner product (,) and norm $\| \ \|$. Suppose that $a_n = x_n \otimes y_n$ $(n \in Z)$ and $a_n \to a_0 \neq 0$ as $n \to \infty$ $(x_n, y_n \in \boldsymbol{H})$. We may clearly assume that for all n in Z, $\|x_n\| = 1$ and $|(y_0, y_n)| = (y_0, y_n)$. We wish to show that

$$P(a_nA, (a_nA)^p) - P(a_0A, (a_0A)^p)$$

tends uniformly to zero on any minimal left ideal of A. This is certainly the case if $\|f_{x_n} - f_{x_0}\| \to 0$ as $n \to \infty$; by the relative continuity of p the latter will follow if $\|e_{x_n} - e_{x_0}\| \to 0$ as $n \to \infty$.

We prove this. Since $\|a_n - a_0\| \to 0$, we have

$$\|(x_n \otimes y_n)y_0 - (x_0 \otimes y_0)y_0\| = \|(y_0, y_n)x_n - (y_0, y_0)x_0\| \to 0.$$

Therefore

$$(y_0, y_n) - (y_0, y_0) = \|(y_0, y_n)x_n\| - \|(y_0, y_0)x_0\| \to 0.$$

Now

$$(y_0, y_0)\|x_n - x_0\| \leqq |(y_0, y_0 - y_n)| \cdot \|x_n\| + \|(y_0, y_n)x_n - (y_0, y_0)x_0\| \to 0.$$

Thus $\|x_n - x_0\| \to 0$ and hence $\|e_{x_n} - e_{x_0}\| = \|x_n \otimes x_n - x_0 \otimes x_0\| \to 0$ as $n \to \infty$.

3. **The primitive case.** In this section A will denote a primitive Banach algebra with a complementor p. Since A has a dense socle it is topologically simple. (See the discussion at the beginning of §2 in [1].)

Let $\boldsymbol{L} = A\boldsymbol{e}$ be a fixed minimal left ideal of A and e a fixed minimal idempotent. Following [1], for any closed right ideal $\boldsymbol{R}$ of A and any closed linear subspace $\boldsymbol{S}$ of $\boldsymbol{L}$, we write

$$S(\boldsymbol{R}) = \boldsymbol{R} \cap \boldsymbol{L} = \mathrm{cl}\,(\boldsymbol{RL}) = \boldsymbol{RL}, \qquad J(\boldsymbol{S}) = \mathrm{cl}\,(\boldsymbol{S}A) = \{a \in A : a\boldsymbol{L} \subset \boldsymbol{S}\}.$$

Definition. Suppose $\boldsymbol{X}$ is a Banach space and $\boldsymbol{S} \to \boldsymbol{S}^q$ a map of the set S of all closed linear subspaces of $\boldsymbol{X}$ onto itself that has the following properties:

L_1: $\boldsymbol{S} \cap \boldsymbol{S}^q = (0)$ $(\boldsymbol{S} \in S)$;

L_2: $\boldsymbol{S} + \boldsymbol{S}^q = \boldsymbol{X}$ $(\boldsymbol{S} \in S)$;

L_3: $(\boldsymbol{S}^q)^q = \boldsymbol{S}$ $(\boldsymbol{S} \in S)$;

L_4: if $\boldsymbol{S}_1 \subset \boldsymbol{S}_2$ then $\boldsymbol{S}_1^q \supset \boldsymbol{S}_2^q$ $(\boldsymbol{S}_1, \boldsymbol{S}_2 \in S)$.

Then $q: \boldsymbol{S} \to \boldsymbol{S}^q$ is a *linear space complementor* (LSC) on $\boldsymbol{X}$. The LSC is continuous if $P([x_n], [x_n]^q)$ converges uniformly to $P([x_0], [x_0]^q)$ whenever $\{x_n : n \in Z\} \subset \boldsymbol{X}$ and $x_n \to x_0 \neq 0$.

Now it can be shown using [8] (in particular Lemma 6 and corollary to Lemma 10) that the maps J, S are one-to-one, that J is the inverse of S, and hence that $\boldsymbol{S} \to \boldsymbol{S}^q = S((J(\boldsymbol{S}))^p)$ is an LSC on $\boldsymbol{L}$. (See the discussion at the beginning of §2 in [1].)

THEOREM 3.1. *The map q is an* LSC *on $\boldsymbol{L}$. If p is continuous then q is continuous.*

Proof. The first part is discussed above. Now suppose that p is continuous. Let $\{x_n\}$ be a sequence in $\boldsymbol{L}$ that converges to a nonzero element x_0 of $\boldsymbol{L}$. Then for n in Z, $x_n\boldsymbol{A} \in M_A$. Let E_n be $P((x_n\boldsymbol{A}), (x_n\boldsymbol{A})^p)$. Since $x_n \in \text{cl}\,(x_n\boldsymbol{A}) = x_n\boldsymbol{A}$ and p is continuous, $\{E_n\}$ converges uniformly to E_0 on $\boldsymbol{L}$. Now for any z in $\boldsymbol{L}$:

$$E_n z + (1-E_n)z = z = ze = E_n ze + (1-E_n)ze$$

and here $E_n z$, $E_n ze \in x_n\boldsymbol{A}$ and $(1-E_n)z$, $(1-E_n)ze \in (x_n\boldsymbol{A})^p$. Thus, by C_2, $E_n z = E_n ze$, and so $E_n z \in x_n\boldsymbol{A} \cap \boldsymbol{L} = S(x_n\boldsymbol{A}) = [x_n]$. Similarly $(1-E_n)z \in [x_n]^q$. Therefore, $E_n/\boldsymbol{L} = P([x_n], [x_n]^q)$. It is now immediate that q is continuous.

COROLLARY. *An inner product $(\ ,\ )$ can be induced into $\boldsymbol{L}$. With this inner product $\boldsymbol{L}$ becomes a Hilbert space with norm equivalent to its original norm.*

Proof. This is an immediate consequence of Theorem 2 of [4].

We now consider the algebra $\boldsymbol{F} = \boldsymbol{K}(\boldsymbol{L})$. By the above corollary $\boldsymbol{F}$, under an equivalent norm, is a dual B^*-algebra. Via the left regular representation on $\boldsymbol{L}$, A can be imbedded in $\boldsymbol{F}$. We now show that $\boldsymbol{F}$ has a complementor p_e that is a natural extension of p.

THEOREM 3.2. *A complementor p_e is induced in $\boldsymbol{F}$ by p. If p is continuous then p_e is continuous.*

Proof. For any $\boldsymbol{R} \in R_F$ define $\boldsymbol{R}^{p_e}$ to be $J_F((S(\boldsymbol{R}))^q)$. It is clear that p_e maps R_F into itself and satisfies C_1, C_4. Now for any $\boldsymbol{R} \in R_F$ and a in $\boldsymbol{F}$ write P for $P(S(\boldsymbol{R}), (S(\boldsymbol{R}))^q)$. Then $a = \lim_{n\to\infty} a_n$ where $\{a_n\}$ is a sequence of operators of finite rank on $\boldsymbol{L}$. Thus $Pa_n \in \boldsymbol{F}$. Also $\{Pa_n\}$ is Cauchy and so converges to an element b of $\boldsymbol{F}$. From Theorem 18 in [3] $Pa_n \in \boldsymbol{R}$, and, since $\boldsymbol{R}$ is closed, $b \in \boldsymbol{R}$. Now by a similar argument $\{(1-P)a_n\}$ converges to an element c of $\boldsymbol{R}^{p_e}$. Then $a = b + c \in \boldsymbol{R} + \boldsymbol{R}^{p_e}$. Since a and $\boldsymbol{R}$ were arbitrary, this proves C_2. Finally, by another application of Theorem 18 in [3], $\boldsymbol{R} = J_F(S(\boldsymbol{R}))$ $(\boldsymbol{R} \in R_F)$ and thus p_e satisfies C_3.

Now suppose that p is continuous. Let $\{a_n\}$ be a sequence of operators of rank one on $\boldsymbol{L}$ such that $a_n \to a_0 \neq 0$ as $n \to \infty$. Then we wish to show that $\{a_n\boldsymbol{F}\}$ is p_e-convergent to $a_0\boldsymbol{F}$. Every maximal right ideal in $\boldsymbol{F}$ is modular (since $\boldsymbol{F}$ is dual) and so by Corollary 3.3 in [2] every minimal right ideal in $\boldsymbol{F}$ contains a unique p_e-projection. Let f_n be the p_e-projection satisfying $f_n\boldsymbol{F} = a_n\boldsymbol{F}$ $(n \in Z)$. Then it is clearly sufficient to show that $|f_n - f_0| \to 0$ as $n \to \infty$. However, in the verification of C_2 for p_e it was shown that for $a \in \boldsymbol{F}$, $\boldsymbol{R} \in R_F$

$$a = Pa + (1-P)a, \qquad P = P(S(\boldsymbol{R}), (S(\boldsymbol{R}))^q)$$

is the decomposition of a into its components in $\boldsymbol{R}$, $\boldsymbol{R}^{p_e}$. Since we know this decomposition to be unique we may deduce that f_n is $P(a_n\boldsymbol{L}, (a_n\boldsymbol{L})^q)$. Now by Theorem 3.1, q is continuous and so $|f_n - f_0| \to 0$ as $n \to \infty$ and the proof is complete.

Thus when p is continuous $\boldsymbol{F}$ is, under an equivalent norm, a dual B^*-algebra with a continuous complementor p_e. Thus the work of [2] is applicable to $\boldsymbol{F}$. We recall some notation of [2].

NOTATION. Let $\boldsymbol{H}$ be a Hilbert space, $\boldsymbol{F}$ be $\boldsymbol{K}(\boldsymbol{H})$, and $\langle\ ,\ \rangle$ any equivalent inner product in $\boldsymbol{H}$ (i.e., one that gives rise to a norm that is equivalent to the given norm in $\boldsymbol{H}$). Then $p_{\langle\ \rangle}$ denotes the complementor $\boldsymbol{R} \to J_F((S(\boldsymbol{R})^{\perp_{\langle\ \rangle}}) = (\boldsymbol{R}_l)^{*_{\langle\ \rangle}}$ (where $^{\perp}{}_{\langle\ \rangle}$, $^{*}{}_{\langle\ \rangle}$ denote respectively the orthogonal complement and the adjoint with respect to $\langle\ ,\ \rangle$). For the proof that this a complementor and that the two expressions for it are the same see Corollary 4.3 in [2].

Now Theorem 6.11 of [2] gives the following.

THEOREM 3.3. *If p is continuous and the dimension of $\boldsymbol{L}$ is at least three then an equivalent inner product $\langle\ ,\ \rangle$ can be introduced into $\boldsymbol{L}$ such that $p_e = p_{\langle\ \rangle}$. The* LSC *$q$ in $\boldsymbol{L}$ corresponds to orthogonal complementation with respect to $\langle\ ,\ \rangle$.*

We can now prove the main result of this section.

THEOREM 3.4. *Let A be a primitive Banach algebra with a continuous complementor p and with no left ideals of dimension less than three. Then A has a faithful, continuous, strictly dense representation $a \to T_a$ on a Hilbert space $\boldsymbol{H}$. Also:*

(1) $\boldsymbol{R}^p = \{a \in A : T_a\boldsymbol{H} \perp T_{\boldsymbol{R}}\boldsymbol{H}\}$ $(\boldsymbol{R} \in R_A)$;

(2) *the socle of A consists of all elements of A whose image is of finite rank on $\boldsymbol{H}$. This image is generated by the set of all operators $x \otimes y$ where x ranges through $\boldsymbol{H}$ and y ranges through a dense subspace $\boldsymbol{H}_0$ of $\boldsymbol{H}$;*

(3) *T_A is a left ideal of $\boldsymbol{B}(\boldsymbol{H})$.*

Proof. Let $\boldsymbol{L} = Ae$ be a given minimal left ideal of A (e is a minimal idempotent) and $\langle\ ,\ \rangle$ the inner product induced in $\boldsymbol{L}$ by Theorem 3.3. Let $\boldsymbol{H}$ be the resultant Hilbert space and let $a \to T_a$ be the representation of A on $\boldsymbol{H}$ corresponding to the left regular representation of A on $\boldsymbol{L}$. Since A is primitive and $\boldsymbol{L}$ is minimal $a \to T_a$ is faithful, continuous, and strictly dense. Properties (1), (2) and that $ET_A \subset T_A$ for any orthogonal projection E on $\boldsymbol{H}$ are proved exactly as in Theorem 1 of [1] (for Theorem 3.3 gives $(\boldsymbol{RL})^{\perp} = (\boldsymbol{RL})^q = (\boldsymbol{R}^p\boldsymbol{L})$ for any $\boldsymbol{R}$ in R_A; the infinite dimensionality of A was used in [1] solely to establish this). By Theorem 1 in [5] any element of $\boldsymbol{B}(\boldsymbol{H})$ is a finite linear combination of orthogonal projections; (3) is now immediate.

Note. We say that the inner product $\langle\ ,\ \rangle$ in $\boldsymbol{L}$ represents p.

THEOREM 3.5. *Let H be a Hilbert space and A a strictly dense subalgebra of $K(H)$. Suppose that A is a Banach algebra under a norm $\| \ \|$ that majorises the operator norm $| \ |$. Then A has a continuous complementor if it is a left ideal of $B(H)$.*

Proof. Theorem 3 in [1] states that a complementor p is defined in A by $R^p = J_A((S(R))^{\perp})$ $(R \in R_A)$. We show that p is continuous. Let $\{R_n : n \in Z\} \subset M_A$ and suppose there are elements $a_n \in R_n$ such that, in A, $a_n \to a_0 \neq 0$ as $n \to \infty$. Let E_n be $P(R_n, R_n^p)$. Since $\| \ \|$ majorises $| \ |$, $|a_n - a_0| \to 0$. Let F be $K(H)$; then the natural complementor $R \to (R_l)^*$ in F is continuous (by Theorem 2.4 and the definition of relative continuity). Let S_n be $S(R_n)$ and P_n be $P(S_n, S_n^{\perp})$. As in the proof of Theorem 3.2, it can be verified that left multiplication by P_n is $P(a_nF, (a_nF)_l^*)$ and $P(a_nA, (a_nA)^p)$. By the continuity of $R \to (R_l)^*$ in F, $|P_n - P_0| \to 0$.

Now let L be any minimal left ideal of A. Then

$$L = \{x \otimes g : g \in H \text{ fixed}, x \in H \text{ variable}\}$$

and by Lemma 2.4.13 in [6] the map $x \otimes g \to x$ is a bicontinuous isomorphism of L onto H. Under this map $E_n(x \otimes g) \to P_n x$. It is thus clear that $\{E_n\}$ converges uniformly to E_0 on L. Thus p is continuous.

4. **General representation theorems.** Throughout this section A will denote a semisimple Banach algebra with a continuous complementor p. $\{I_\lambda : \lambda \in \Lambda\}$ is the set of all minimal closed two-sided ideals of A and p_λ denotes the continuous complementor induced in I_λ by p.

NOTATION. R_λ will denote $R \cap I_\lambda$ $(R \in R_A, \lambda \in \Lambda)$. E_R will denote $P(R, R^p)$ $(R \in R_A)$ and E_λ will denote E_{I_λ}.

THEOREM 4.1. *$E_{R_\lambda} = E_\lambda \cdot E_R$ and $E_\lambda(1 - E_R) = (1 - E_{R_\lambda})E_\lambda$ $(R \in R_A, \lambda \in \Lambda)$.*

Proof. Let $a \in A$. Then:

$$(1) \qquad E_\lambda a = E_\lambda(E_R a + (1 - E_R)a) = E_\lambda E_R a + E_\lambda(1 - E_R)a.$$

Also:

$$(2) \qquad E_\lambda a = E_{R_\lambda} E_\lambda a + (1 - E_{R_\lambda})E_\lambda a.$$

Now, by Lemma 2.1, $E_\lambda(\sum_{\mu \in \Lambda} R_\mu) = R_\lambda$ and by Theorem 2.3 $\sum_{\mu \in \Lambda} R_\mu$ is a dense subspace of $R = E_R A$. Thus, since E_λ is continuous and R_λ is closed, we have $E_\lambda(E_{R_\lambda}) = R_\lambda$. Similarly, $E_\lambda((1 - E_R)A) = R_\lambda^{p_\lambda}$. Therefore $E_\lambda E_R a$, $E_{R_\lambda} E_\lambda a \in R_\lambda$ and $E_\lambda(1 - E_R)a$, $(1 - E_{R_\lambda})E_\lambda a \in R_\lambda^{p_\lambda}$. Thus by C_1 for p, the expressions (1), (2) are identical. Hence $E_\lambda(1 - E_R)a = (1 - E_{R_\lambda})E_\lambda a$ and

$$E_\lambda E_R a = E_{R_\lambda} E_\lambda a = E_{R_\lambda}(a - (1 - E_\lambda)a) = E_{R_\lambda} a.$$

COROLLARY. *For each $\lambda \in \Lambda$ select U_λ such that $U_\lambda A \in M_A$, $U_\lambda A \subset I_\lambda$, and $U_\lambda = P(U_\lambda A, (U_\lambda A)^p)$. Then* $\sup \{|U_\lambda / I_\lambda| : \lambda \in \Lambda\} < \infty$.

Proof. Let $\boldsymbol{R}=\mathrm{cl}\,(\sum_{\lambda\in\Lambda} U_\lambda A)$; then $\boldsymbol{R}\in R_A$ and so $E_{\boldsymbol{R}}$ is a bounded linear operator on A. Now let $a\in \boldsymbol{I}_\lambda$; $a=E_{\boldsymbol{R}_\lambda}a+(1-E_{\boldsymbol{R}_\lambda})a$ and thus, in particular, $(1-E_{\boldsymbol{R}_\lambda})a\in \boldsymbol{I}_\lambda$. Therefore, $(1-E_{\boldsymbol{R}_\lambda})a\in \boldsymbol{I}_\lambda\cap \boldsymbol{R}_\lambda^p=\boldsymbol{R}_\lambda^{p_\lambda}\subset \boldsymbol{R}^p$. Since $E_{\boldsymbol{R}_\lambda}a\in \boldsymbol{R}_\lambda\subset \boldsymbol{R}$, C_1 for p gives $E_{\boldsymbol{R}_\lambda}a=E_{\boldsymbol{R}}a$ or $E_{\boldsymbol{R}}/\boldsymbol{I}_\lambda=E_{\boldsymbol{R}_\lambda}/\boldsymbol{I}_\lambda$.

Also, $\boldsymbol{R}_\lambda=E_{\boldsymbol{R}_\lambda}A=E_\lambda E_{\boldsymbol{R}}A=E_\lambda(\mathrm{cl}\,(\sum_{\lambda\in\Lambda} U_\mu A))=U_\lambda A$. Therefore $E_{\boldsymbol{R}_\lambda}=U_\lambda$. Hence $E_{\boldsymbol{R}}/\boldsymbol{I}_\lambda=U_\lambda/\boldsymbol{I}_\lambda$ and so $|U_\lambda/\boldsymbol{I}_\lambda|\leqq |E_{\boldsymbol{R}}|$.

The following lemma shows that the inner products representing the complementors p_λ can be chosen to be uniformly equivalent to the norms on minimal left ideals of $\boldsymbol{I}_\lambda$. This is essential for the construction of a representing Hilbert space for A.

LEMMA 4.3. *Suppose that A has no minimal left ideals of dimension less than three. Then we can select minimal left ideals $\boldsymbol{L}_\lambda$ in $\boldsymbol{I}_\lambda$ and induce inner products $\langle\ ,\ \rangle_\lambda$ in $\boldsymbol{L}_\lambda$ such that $\langle\ ,\ \rangle_\lambda$ represents p_λ and for some finite constant M*

$$\|x\|^2 \leqq \langle x, x\rangle_\lambda \leqq M^2\|x\|^2 \qquad (x\in \boldsymbol{L}_\lambda,\ \lambda\in\Lambda).$$

Proof. Since any minimal left ideal in $\boldsymbol{I}_\lambda$ is a minimal left ideal in A its dimension is at least three. So we may select minimal left ideals $\boldsymbol{L}_\lambda$ of $\boldsymbol{I}_\lambda$ and in them induce inner products $\langle\ ,\ \rangle_\lambda$ representing p_λ as in Theorem 3.3. The inner product norm $|\ |_\lambda$ is equivalent to the original norm in $\boldsymbol{L}_\lambda$ and, since $\langle\ ,\ \rangle_\lambda$ retains the above properties on multiplication by a positive real constant we may suppose

$$(1) \qquad \|x\| \leqq |x|_\lambda \ngeqq \sqrt{2}\|x\| \qquad (x\in \boldsymbol{L}_\lambda,\ \lambda\in\Lambda).$$

Suppose that it is not now possible to find M. Then there exists a sequence $\{\lambda_n\}\subset\Lambda$ such that $|\ |_{\lambda_n}\nless n\|\ \|$ on $\boldsymbol{L}_\lambda$. For convenience we shall now replace the suffices λ_n by n. Then there are elements x_n of $\boldsymbol{L}_n$ such that $\|x_n\|=1$, and $\langle x_n, x_n\rangle_n =k_n^2>n^2$. Also, by (1), there are elements z_n of $\boldsymbol{L}_n$ such that $\|z_n\|=1$, $\langle z_n, z_n\rangle_n\leqq 2$. Now write $z_n=\alpha_n x_n+x_n'$ where $\langle x_n, x_n'\rangle_n=0$, $\alpha_n\in C$. Then:

$$(2) \qquad \langle z_n, z_n\rangle_n = |\alpha_n|^2 k_n^2+\langle x_n', x_n'\rangle_n.$$

From (2) we deduce:

(3) $\langle x_n', x_n'\rangle_n\leqq 2$;

(4) $|\alpha_n|\leqq \sqrt{2}/k_n<\sqrt{2}/n$.

Also, since $\|z_n\|+|\alpha_n|\cdot\|x_n\|\geqq\|x_n'\|\geqq\|z_n\|-|\alpha_n|\cdot\|x_n\|$, we have from (4)

$$(5) \qquad 1+\sqrt{2}/n > \|x_n'\| > 1-\sqrt{2}/n.$$

We shall consider the subspace $[y_n]$ of $\boldsymbol{L}_n$ where $y_n=(1/k_n)x_n+x_n'$. Let $\boldsymbol{R}_n$ be $J_{I_n}([y_n])$. Then $\boldsymbol{R}_n\in M_{I_n}\subset M_A$; let U_n be $P(\boldsymbol{R}_n, \boldsymbol{R}_n^p)$. Then by the corollary to Theorem 4.1, there exists a finite constant M such that $|U_n/\boldsymbol{I}_n|<M$. However, $P([y_n], y_n^{\perp_n})$ is $(y_n\otimes_n y_n)/\langle y_n, y_n\rangle_n$. (We use $\perp_\lambda$, $\otimes_\lambda$ to denote orthogonality and tensor product respectively with respect to $\langle\ ,\ \rangle_\lambda$.) Also, from the proof of Theorem 3.1, $U_n/\boldsymbol{L}_n=P([y_n], y_n^{\perp_n})$. Therefore

$$(6) \qquad M\geqq|U_n/\boldsymbol{I}_n|\geqq|U_n/\boldsymbol{L}_n|\geqq\|U_n x_n\|=\left\|\frac{y_n\otimes_n y_n}{\langle y_n, y_n\rangle_n}x_n\right\|=\frac{|\langle x_n, y_n\rangle_n|}{\langle y_n, y_n\rangle_n}\|y_n\|.$$

However, from the definition of y_n and (3), (5) we have

(7) $\langle y_n, y_n\rangle_n = 1+\langle x_n', x_n'\rangle_n \leqq 3$;

(8) $\|y_n\| \geqq \|x_n'\| - \|(1/k_n)x_n\| > 1-\sqrt{2/n}-1/n$;

(9) $\langle x_n, y_n\rangle_n = k_n > n$.

Substituting (7), (8), (9) in (6): $M > (n/3)(1-\sqrt{2/n}-1/n)$ for all n which is the required contradiction.

THEOREM 4.4. *Let A be a semisimple Banach algebra with a continuous complementor p. Suppose that A has no minimal left ideals of dimension less than three. Then A has a faithful, continuous representation $a \to T_a$ on a Hilbert space H. For any closed right ideal R of A, $(T_R H)^\perp = \mathrm{cl}\,(T_{R^p}H)$; $R^p = \{a \in A : T_a H \perp T_R H\}$.*

Proof. Select minimal left ideals L_λ of I_λ and induce an inner product $\langle\ ,\ \rangle_\lambda$ in L_λ as in Lemma 4.3. Let H_λ be the resultant Hilbert space for each λ in Λ. We shall construct the direct sum of the representations $a \to T_a^\lambda$ of A on H_λ corresponding to the left regular representations of A on L_λ. Let $H = \sum^{(2)} H_\lambda$ in the notation of [6, p. 197]. For each a in A define T_a by $(T_a f)(\lambda) = T_a^\lambda f(\lambda)$ $(f \in H)$. Then $\|T_a^\lambda f(\lambda)\| \leqq \|a\|\ \|f(\lambda)\| \leqq \|a\|\ |f(\lambda)|_\lambda$, and thus

$$\sum_{\lambda\in\Lambda} |(T_a f)(\lambda)|^2 \leqq M^2 \sum_{\lambda\in\Lambda} \|(T_a f)(\lambda)\|^2 \leqq M^2 \sum_{\lambda\in\Lambda} \|a\|^2 |f(\lambda)|_\lambda^2 = M^2\|a\|^2|f|^2.$$

Therefore, $T_a f \in H$ for all f in H and a in A and so the direct sum of the representations is defined and $|T_a| \leqq M\|a\|$ $(a \in A)$, so that the representation is continuous.

Let $R \in R_A$. Then $R = \mathrm{cl}\,(\sum_{\lambda\in\Lambda} R_\lambda)$, $R^p = \mathrm{cl}\,(\sum_{\lambda\in\Lambda} R_\lambda^{p_\lambda})$ and, since the representation is continuous, $\{h \in T_a H; a \in \sum_{\lambda\in\Lambda} R_\lambda\}$ and $\{h \in T_a H : a \in \sum_{\lambda\in\Lambda} R_\lambda^{p_\lambda}\}$ are dense subspaces of $T_R H$, $T_{R^p}H$ respectively. Thus:

$$(T_R H)^\perp = \left\{h \in T_a H : a \in \sum_{\lambda\in\Lambda} R_\lambda\right\}^\perp = \{h : h(\lambda) \in S(R_\lambda)^{\perp_\lambda}\} = \{h : h(\lambda) \in S(R_\lambda^{p_\lambda})\}$$
$$= \mathrm{cl}\left\{h \in T_a H : a \in \sum_{\lambda\in\Lambda} R_\lambda^{p_\lambda}\right\} = \mathrm{cl}\,(T_{R^p}H).$$

Next, we show that the representation $a \to T_a$ is faithful. Let $a \in A$; then $a \in \mathrm{cl}\,(aA)$ which we shall denote by R. Then $R_\lambda = \mathrm{cl}\,((E_\lambda a)I_\lambda)$: in fact $(E_\lambda a)I_\lambda \subset E_\lambda E_R A = E_{R_\lambda} A = R_\lambda$ and so $\mathrm{cl}\,((E_\lambda a)I_\lambda) \subset R_\lambda$; conversely, given $\lambda \in \Lambda$, $b \in R_\lambda$, $\varepsilon > 0$, there exists $c \in A$ such that $\|b - ac\| < \varepsilon/|E_\lambda|$ and then $\|b - E_\lambda ac\| < \varepsilon$. Suppose $T_a = 0$; then $T^\lambda_{E_\lambda a} = T_a^\lambda = 0$ and, since $a \to T_a^\lambda$ is faithful on I_λ, we have $E_\lambda a = 0$. Therefore, $R_\lambda = \mathrm{cl}\,(E_\lambda a I_\lambda) = (0)$ and $R = \mathrm{cl}\,(\sum_{\lambda\in\Lambda} R_\lambda) = (0)$. It follows that $a = 0$.

Finally we show that if $T_a H \subset (T_R H)^\perp$ then $a \in R^p$. Given such an a, put $a = a_1 + a_2$ where $a_1 \in R$, $a_2 \in R^p$. Then for any h in H: $T_a h = T_{a_1}h + T_{a_2}h$. Thus, since $(T_R H)^\perp \supset T_{R^p}H$, $T_{a_1}h \in T_R H \cap (T_R H)^\perp = (0)$. Therefore, $T_{a_1} = 0$ and, since the representation is faithful, $a_1 = 0$ and $a = a_2 \in R^p$.

Note. In this case we cannot have T_A a left ideal of $B(H)$. Let μ, ρ be distinct elements of Λ. Let x, y be elements of H that satisfy $x(\rho) = 0$, $y(\rho) \neq 0$, $x(\mu) = y(\mu) \neq 0$

($=h$, say). Then for any a in A $(T_a x)(\rho)=T_a^\rho x(\rho)=0$. However, since T_{I_μ} contains all operators of finite rank on H, there exists b in I_μ such that $T_b^\mu = h \otimes_\mu h$. Consider $y \otimes y \cdot T_b x$:

$$(y \otimes y \cdot T_b x)(\rho) = (y, T_b x) y(\rho) = (y(\mu), T_b^\mu x(\mu)) \cdot y(\rho)$$
$$= (h, h)^2 \cdot y(\rho) \neq 0.$$

Therefore, there does not exist a in A such that $y \otimes y \cdot T_b = T_a$.

THEOREM 4.5. *Let A be a semisimple algebra of operators on a Hilbert space H that is a Banach algebra under a norm $\| \ \|$ that majorises the operator norm $| \ |$. Suppose $R \to R^p = J_A((RH)^\perp)$ $(R \in R_A)$ is a complementor on A. Then p is continuous.*

Proof. Let $\{I_\lambda : \lambda \in \Lambda\}$ be the set of all minimal closed-two sided ideals of A. Then, since A is complemented, $A = \text{cl}\,(\sum_{\lambda\in\Lambda} I_\lambda)$; also a complementor p_λ is induced in each I_λ. By Theorem 2.2 it is sufficient to show that each p_λ is continuous. Let $I_\lambda H = H_\lambda$, and for any set S in H let $S^{\perp_\lambda} = \{x \in H_\lambda : x \perp S\}$. If $a \in A$ and $aH \subset H_\lambda$ then $a \in I_\lambda$: in fact, put $a = a_1 + a_2$ where $a_1 \in I_\lambda$, $a_2 \in I_\lambda^p$, then $a_2 H \subset H_\lambda \cap H_\lambda^\perp = (0)$ and thus $a_2 = 0$. Let $R \in R_{I_\lambda}$; then

$$R^{p_\lambda} = R^p \cap I_\lambda = \{a : aH \subset (RH)^\perp, a \in I_\lambda\} = \{a : aH \subset (RH)^{\perp_\lambda}\}$$
$$= \{a : aH_\lambda \subset (RH_\lambda)^{\perp_\lambda}\}$$

and so, from the proof of Theorem 3.5, p_λ is continuous.

We now specialize to the B^*-case to extend Theorem 6.11 of [2].

THEOREM 4.6. *Let A be a B^*-algebra (with no left ideals of dimension less than three) and let p be any complementor in A. Then p is continuous if and only if there is a subsidiary involution $\#$ in A that satisfies $R^p = (R_l)^\#$ $(R \in R_A)$. If there is such an involution $\#$ then there is an equivalent norm $\| \ \|'$ in A that satisfies the B^* condition for $\#$.*

Proof. Suppose that p is continuous. Let L_λ be a minimal left ideal in I_λ and $\langle \ , \ \rangle_\lambda$ an inner product induced in L_λ as in Lemma 4.3. Let T_a, T_a^λ be the representations described in Theorem 4.4. Consider the restriction to I_λ of $a \to T_a^\lambda$. Since I_λ is B^* and hence its left regular representation on L_λ is isometric, we have

$$\|a\|/M < |T_a^\lambda/H_\lambda| < M\|a\| \qquad (a \in I_\lambda, \lambda \in \Lambda).$$

Also p_λ is represented by $\langle \ , \ \rangle_\lambda$ and it follows that $R^{p_\lambda} = (R_{l_\lambda})^{\#(\lambda)}$ $(R \in R_{I_\lambda})$ where $\#(\lambda)$ is defined in I_λ by:

$$T_{a^{\#(\lambda)}}^\lambda \text{ is the adjoint of } T_a^\lambda \text{ with respect to } \langle \ , \ \rangle_\lambda$$

and l_λ denotes the left annihilator in I_λ. Now:

$$\|a^{\#(\lambda)}\| < M|T_{a^{\#(\lambda)}}^\lambda/H_\lambda| = M|T_a^\lambda/H_\lambda| < M^2\|a\| \qquad (a \in I_\lambda, \lambda \in \Lambda).$$

Since A is B^*, A is isometrically $*$-isomorphic with $(\sum_{\lambda\in\Lambda} I_\lambda)_0$ [6, Theorem 4.10.14] and now since M is independent of λ we can define $\#$ on A by $(a^\#)(\lambda) = (a(\lambda))^{\#(\lambda)}$.

Also $a^{\#}a=0$ implies $a=0$, since in the proof of (iii) $\Rightarrow$ (ii) of [2, Theorem 6.11] it was shown that $(a(\lambda))^{\#(\lambda)}a(\lambda)=0$ implies $a(\lambda)=0$. Hence by Corollary 7.2 and Lemma 7.1 in [2]:

$$(\boldsymbol{R}_1)^{\#} = \Big(\sum_{\lambda\in\Lambda} (\boldsymbol{R}_\lambda)^{\#}_{l_\lambda}\Big)_0 = \Big(\sum_{\lambda\in\Lambda} (\boldsymbol{R}_\lambda)^{\#(\lambda)}_{l_\lambda}\Big)_0 = \Big(\sum_{\lambda\in\Lambda} \boldsymbol{R}_\lambda^{p_\lambda}\Big)_0 = \boldsymbol{R}^p \quad (\boldsymbol{R}\in R_A).$$

The converse and the remainder of the theorem are contained in Theorem 7.4 of [2].

The operators E_λ. It remains unknown whether in general the operators E_λ are uniformly bounded. This is true in the case when A is B^* ($|E_\lambda|=1$) and could have simplified slightly some of the proofs had we been interested only in the B^* case. In general, an equivalent question is whether each a in A may be expressed in the form $a=\lim_{n\to\infty}\sum_{r=1}^{r_n} E_{\lambda_r}a$ ($\{\lambda_r\}\subset\Lambda$).

5. **Two counterexamples.**

1. C_2 cannot be replaced by cl $(\boldsymbol{R}+\boldsymbol{R}^p)=A$.

Let $\boldsymbol{H}$ be a three dimensional Hilbert space with an orthonormal basis $\{x, y, z\}$. Let the inner product in $\boldsymbol{H}$ be denoted by (,). Define an operator T_n on $\boldsymbol{H}$ by $T_n(\alpha x+\beta y+\gamma z)=n\alpha x+\beta y+\gamma z$ $(\alpha, \beta, \gamma\in C)$. Then T_n is a bounded, positive, hermitian operator on $\boldsymbol{H}$ and its inverse is defined and is an operator of the same kind. Thus we may define an equivalent inner product $\langle\ ,\ \rangle_n$ in $\boldsymbol{H}$ by $\langle u, v\rangle_n =(T_nu, v)$. Let A be $\boldsymbol{K}(\boldsymbol{H})$ and p_n be $p_{\langle\ \rangle_n}$. Now let $\boldsymbol{B}$ be the sum $(\sum_{i=1}^{\infty} A_i)_0$ of the countable collection $\{A_i\}$ where $A_i=A$ for all i. Then $\boldsymbol{B}$ is a dual B^*-algebra and so, by Lemma 7.1 in [2], $\boldsymbol{R}=(\sum_{i=1}^{\infty}\boldsymbol{R}_i)_0$ $(\boldsymbol{R}\in R_B)$ where $\boldsymbol{R}_i$ is the intersection of $\boldsymbol{R}$ with the image of A_i in $\boldsymbol{B}$. Define p on $\boldsymbol{B}$ by $\boldsymbol{R}^p=(\sum_{i=1}^{\infty}\boldsymbol{R}_i^{p_i})_0$. Then it can easily be verified that p satisfies C_1, C_3, C_4; also $\boldsymbol{R}+\boldsymbol{R}^p$ is dense in $\boldsymbol{B}$ $(\boldsymbol{R}\in R_B)$. Suppose that p is a complementor. Then since each p_i is continuous, p is continuous. Now it is clear that we may substitute $\boldsymbol{H}$ for $\boldsymbol{L}_\lambda$ in Lemma 4.3. Thus, taking the slightly stronger result in the proof of that lemma we see that if $[\ ,\ ]_n$ are inner products in $\boldsymbol{H}$, $[\ ,\ ]_n$ represents p_n and $\|h\|^2\leqq[h, h]_n\nleqq 2\|h\|^2$ $(h\in\boldsymbol{H})$ ($\|\ \|$ denotes the norm in $\boldsymbol{H}$), then there exists a finite constant M such that $[h, h]_n\leqq M\|h\|^2$ for all h in $\boldsymbol{H}$ and all integers n. Now $\langle\ ,\ \rangle_n$ is such a collection of inner products. However, $\langle x, x\rangle_n=n\|x\|^2$ which contradicts the existence of M. Therefore $\boldsymbol{B}$ is not complemented.

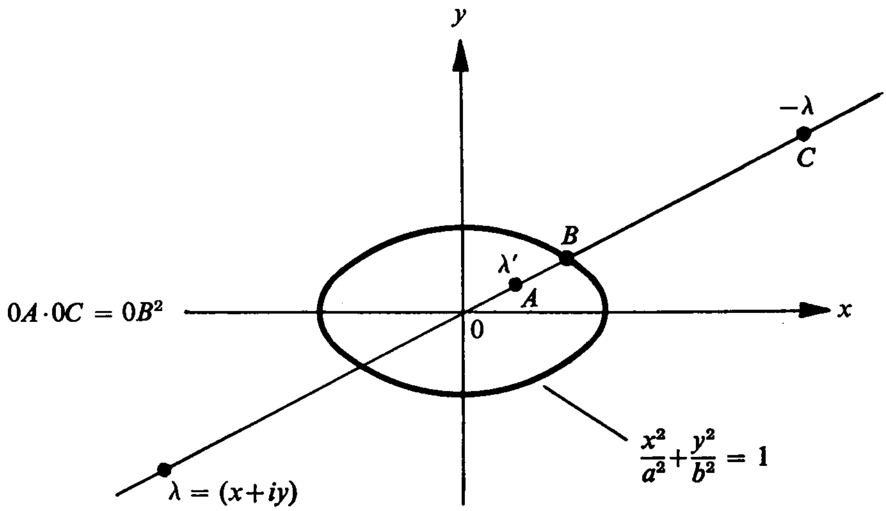
2. The dimension restriction in Theorems 3.3, 3.4, 4.4, 4.6 cannot be removed.

Let $\boldsymbol{H}$ be a two-dimensional Hilbert space and $\{x, y\}$ an orthonormal basis in $\boldsymbol{H}$. We wish to construct a continuous complementor p in $\boldsymbol{K}=\boldsymbol{K}(\boldsymbol{H})$ such that p is not $p_{\langle\ \rangle}$ for any inner product $\langle\ ,\ \rangle$ in $\boldsymbol{H}$. To do this it is clearly sufficient (from the method of Theorem 3.2) to construct a continuous LSC q in $\boldsymbol{H}$ that is not an orthogonal complementation with respect to any inner product.

Let $\lambda=re^{i\theta}\in C$, $\lambda\neq 0$, and let λ' be defined by

$$\lambda' = -e^{i\theta}(a^2\cos^2\theta+b^2\sin^2\theta)/r$$

where a, b are any two distinct positive reals. Then $\lambda \to \lambda'$ is a continuous, one-to-one, involutory map of $C-(0)$ onto $C-(0)$ and it has no fixed points. A geometrical interpretation of the map is as follows:



Also, as $\lambda \to 0$, $\lambda' \to \infty$ and as $\lambda \to \infty$, $\lambda' \to 0$.

Now q can be defined on $\boldsymbol{H}$ by:

$$[x]^q = [y], \quad [y]^q = [x], \quad \boldsymbol{H}^q = (0), (0)^q = \boldsymbol{H}, \quad [x+\lambda y]^q = [x+\lambda' y], \quad \lambda \neq 0.$$

It is easy to see that q is an LSC. Also:

$$f_x = x \otimes x, \qquad f_y = y \otimes y, \qquad f_{x+\lambda y} = (x+\lambda y) \otimes (x+\lambda^* y)/(x+\lambda^* y, x+\lambda y)$$
$$(\lambda \neq 0, \text{co } \lambda^* = -1/\lambda').$$

Then, since $\lambda \to \lambda'$ is continuous, $\lambda \to \lambda^*$ is continuous on $C-(0)$. As $\lambda \to 0$, $\lambda^* \to 0$, and as $\lambda \to \infty$, $\lambda^* \to \infty$. It is now clear that $f_{z_n} \to f_z$ when $z_n \to z$ and z_n, $z \in \boldsymbol{H}$; thus q is continuous.

Suppose that q is orthogonal complementation with respect to an inner product $\langle\ ,\ \rangle$ in $\boldsymbol{H}$. Then $\langle x, y\rangle = \langle x+\lambda y, x+\lambda' y\rangle = 0$. Therefore,

$$\langle x, x\rangle + \lambda \text{ co } \lambda' \langle y, y\rangle = 0. \tag{1}$$

Let $\lambda = re^{i\theta}$: then $\lambda \text{ co } \lambda' = -(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \neq$ constant. Thus (1) cannot be satisfied for all λ.

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