Representation theory of *q*-rook monoid algebras

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Abstract We show that, over an arbitrary field, *q*-rook monoid algebras are iterated inflations of Iwahori-Hecke algebras, and, in particular, are cellular. Furthermore we give an algebra decomposition which shows a *q*-rook monoid algebra is Morita equivalent to a direct sum of Iwahori-Hecke algebras. We state some of the consequences for the representation theory of *q*-rook monoid algebras.

Keywords Rook monoid · Cellular algebra · Iwahori-Hecke algebra · Representation

Introduction

The *rook monoid* is the monoid of all $n \times n$ matrices containing at most one entry equal to 1 in each row and column and zeros elsewhere, with the operation of matrix multiplication. It is also sometimes called the symmetric inverse semigroup. For any field K, we consider the rook monoid algebra R_n. This has been studied (mostly in the case where K has characteristic zero) by, amongst others, Munn [13], Grood [6] and Solomon [14]. More generally, Solomon [15] defined a *q*-analogue of the rook monoid algebra and its representation theory was investigated by various authors in [1, 7, 8].

  We shall take the approach suggested by Henke in her review of [6] on MathSciNet, where it is stated that R_n can be shown to be a cellular algebra using methods similar to those employed in [12]. Firstly we shall show that the *q*-rook monoid algebra is cellular, and in fact is an iterated inflation of Iwahori-Hecke algebras. Halverson and Ram (in [8]) already showed a connection between the *q*-rook monoid algebra and Hecke algebras—they showed that the *q*-rook monoid algebra is isomorphic to

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a quotient of the affine Hecke algebra of type A. We will find another connection. Our result will allow us to prove that the q-rook monoid algebra is actually Morita equivalent to a direct sum of Iwahori-Hecke algebras. This will allow us to reprove several results from the literature, as well as giving us some new results such as the block decomposition of these algebras.

Throughout, K will denote an arbitrary field unless otherwise stated. All tensor products are taken over K.

1. Background

1.1. Cellular algebras

Cellular algebras were originally defined by Graham and Lehrer in [5], who proved that many familiar algebras are cellular, and also showed that cellular algebras possess many good properties. Let us recall the equivalent basis-free definition by König and Xi (from [10]). Let K be a field, and A be a finite dimensional K-algebra with an *anti-involution* i, that is a K-linear anti-automorphism ($i(ab) = i(b)i(a)$) of order 2. A *cell ideal* of A is a 2-sided ideal, J, satisfying:

 (i) $i(J) = J$;
(ii) there exists Δ, a left ideal of A, $\Delta \subset J$ and there is an $A - A$-bimodule isomorphism $\alpha : J \cong \Delta \otimes i(\Delta)$ such that the following diagram commutes:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x \otimes y \mapsto i(y) \otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes i(\Delta) \end{array}$$

The algebra A is *cellular* if there is a decomposition of A as a vector space:

$$A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$$

for some n, with $i(J_j') = J_j'$, such that setting $J_j = J_j' \oplus J_{j+1}' \oplus \cdots \oplus J_n'$ gives a chain of 2-sided ideals of A:

$$0 = J_{n+1} \subset J_n \subset \cdots \subset J_1 = A$$

(each fixed by i), and, for each j, $J_j' = J_j/J_{j+1}$ is a cell ideal of A/J_{j+1} with respect to the anti-involution induced by i on the quotient.

The left ideals of A (that is left A-modules), $\Delta_1, \ldots, \Delta_n$ appearing in the definition are called the *cell modules*.

As an example of a cellular algebra, and for use later, we recall the Iwahori-Hecke algebra of the symmetric group. For $q \in K^\times$, the Hecke algebra of the symmetric group $\mathcal{H}_n(q)$ is defined to be the associative K-algebra with a 1 generated by $T_1, \ldots, T_{n-1}$

subject to the relations:

$$
\begin{aligned}
T_i^2 &= q \cdot 1 + (q-1)T_i && \text{for } 1 \le i \le n-1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \le i \le n-2, \\
T_i T_j &= T_j T_i && \text{when } |i-j| > 1.
\end{aligned}
$$

If we take $q = 1$ we obtain the group algebra of the symmetric group $K\mathcal{S}_n$. As is well-known, to each $w \in \mathcal{S}_n$ one can associate an element $T_w \in \mathcal{H}_n(q)$ to obtain a basis of $\mathcal{H}_n(q)$. Graham and Lehrer showed that $\mathcal{H}_n(q)$ (and, in particular, the symmetric group algebra) is a cellular algebra (see [5]), the anti-involution i sending a generator T_j to itself, and therefore sending a basis element T_w to $T_{w^{-1}}$ for $w \in \mathcal{S}_n$.

We shall use the following lemma of Xi to demonstrate the cellularity of the q-rook monoid algebra. The lemma formalizes the method applied by König and Xi in [12] to show that Brauer algebras are cellular, ideas which first appeared in their article [11].

Lemma 1 ([16], 3.3). *Let A be a K-algebra with an anti-involution i. Suppose that the following hold:*

(i) *As a vector space,*

$$
A = \bigoplus_{j=1}^{m} V_j \otimes V_j \otimes B_j
$$

and, for each j, V_j is a vector space and B_j is a cellular algebra with respect to anti-involution σ_j and cell chain $0 \subset J_{s_j}^{(j)} \subset \cdots \subset J_1^{(j)} = B_j$.

(ii) *The restriction of i to $V_j \otimes V_j \otimes B_j$ is (for $v, v' \in V_j$, $b \in B_j$):*

$$
v \otimes v' \otimes b \to v' \otimes v \otimes \sigma_j(b).
$$

(iii) *For each j, there is a bilinear form $\phi_j : V_j \otimes V_j \to B_j$ such that for all $v, v' \in V_j$:*

$$
\sigma_j(\phi_j(v', v)) = \phi_j(v, v').
$$

(iv) *Putting $J_t = \bigoplus_{j=t}^{m} V_j \otimes V_j \otimes B_j$, the multiplication in J_j / J_{j+1} is given by:*

$$
(v_1 \otimes v_1' \otimes b_1)(v_2 \otimes v_2' \otimes b_2) = v_1 \otimes v_2' \otimes b_1 \phi_j(v_1', v_2) b_2 \quad modulo \quad J_{j+1}
$$

for $v_1, v_1', v_2, v_2' \in V_j$, $b_1, b_2 \in B_j$.

(v) *For all j, l, we have that $V_j \otimes V_j \otimes J_l^{(j)} + J_{j+1}$ is an ideal in A.*

Then under these conditions A is a cellular algebra.

In this situation A is called an iterated inflation of $B_1, B_2, \ldots, B_m$. For more details on inflations see Section 3 of [12].

1.2. The q-rook monoid

For K a field of characteristic zero, $q \in K^\times$, Solomon defined the q-rook monoid algebra $\mathcal{I}_n(q)$ to be the associative K-algebra with identity 1 and generators $T_1, \ldots, T_{n-1}, N$ and defining relations:

$$
\begin{aligned}
T_i^2 &= q \cdot 1 + (q-1)T_i & &\text{for } 1 \le i \le n-1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & &\text{for } 1 \le i \le n-2, \\
T_i T_j &= T_j T_i & &\text{when } |i-j| > 1, \\
N^{i+1} T_i &= q N^{i+1} & &\text{for } 1 \le i \le n-1, \\
T_i N^{n-i+1} &= q N^{n-i+1} & &\text{for } 1 \le i \le n-1, \\
T_i N &= N T_{i+1} & &\text{for } 1 \le i \le n-2, \\
N T_1 T_2 \cdots T_{n-1} N &= q^{n-1} N.
\end{aligned}
$$

We will allow K to be an arbitrary field in the above definition.

Halverson, in [7], defines $R_n(q)$ to be the associative K-algebra with identity generated by $T_1, \ldots, T_{n-1}$ and $P_1, \ldots, P_n$ subject to the relations:

$$
\begin{aligned}
\text{(A1)} \quad & T_i^2 = q \cdot 1 + (q-1)T_i & &\text{for } 1 \le i \le n-1, \\
\text{(A2)} \quad & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & &\text{for } 1 \le i \le n-2, \\
\text{(A3)} \quad & T_i T_j = T_j T_i & &\text{when } |i-j| > 1, \\
\text{(A4)} \quad & T_i P_j = P_j T_i = q P_j & &\text{for } 1 \le i < j \le n, \\
\text{(A5)} \quad & T_i P_j = P_j T_i & &\text{for } 1 \le j < i \le n-1, \\
\text{(A6)} \quad & P_i^2 = P_i & &\text{for } 1 \le i \le n, \\
\text{(A7)} \quad & P_{i+1} = P_i T_i P_i - (q-1)P_i & &\text{for } 1 \le i \le n-1.
\end{aligned}
$$

Again Halverson is concerned only with the case where K has characteristic zero, but we will allow K to be any field in this definition.

Halverson proves [7, Corollary 2.2] that, over a field of characteristic zero, $\mathcal{I}_n(q) \cong R_n(q)$. We will show that this result extends to the general case below. In Section 2 of [7], Halverson gave a basis of $R_n(q)$ defined over a field of characteristic zero, which we shall now recall. Let us denote $\{1, 2, \ldots, n\}$ by $[n]$. For $A = \{a_1, \ldots, a_k\} \subseteq [n]$, with $a_1 < a_2 < \cdots < a_k$, we have $a_j \ge j$, so let:

$$
T_A = (T_{a_1-1} \cdots T_2 T_1)(T_{a_2-1} \cdots T_3 T_2) \cdots (T_{a_k-1} \cdots T_k),
$$

where the j-th factor is read as 1 if $a_j = j$. For $A, B \subseteq [n]$ with $|A| = |B| = k$, and $w \in \mathcal{S}_{\{k+1,\ldots,n\}}$, define:

$$
T_{(A,B,w)} = T_A P_k T_w T_B^{-1},
$$

where, when $k = 0$, we interpret $P_0 = 1$ so $T_{(\emptyset,\emptyset,w)} = T_w$.

Write $\Omega_k = \{(A, B, w) : A, B \subseteq [n], |A| = |B| = k, w \in \mathcal{S}_{\{k+1,\ldots,n\}}\}$ and $\Omega = \bigcup_{k=0}^n \Omega_k$. Halverson proves that, when the characteristic of K is zero, the follow-

ing set is a basis of $R_n(q)$:

$$\{T_{(A,B,w)} : (A, B, w) \in \Omega\}. \tag{1}$$

Proposition 2 ([7, Corollary 2.2] over a field of characteristic zero). *Over an arbitrary field* $\mathcal{I}_n(q) \cong R_n(q)$. *Moreover* $\{T_{(A,B,w)} : (A, B, w) \in \Omega\}$ *forms a basis of* $R_n(q)$.

Proof: We shall see that Halverson's proof of this in [7] generalises to this situation. Halverson proves that the set in (1) spans $R_n(q)$ in [7, Theorem 2.1] and this proof is readily seen to hold for an arbitrary field. So the dimension of $R_n(q)$ is at most $|\Omega|$. He shows that one can define a surjection from $R_n(q)$ to $\mathcal{I}_n(q)$ (see [7, Lemma 1.4]) and this is valid over any field. Thus $\dim_K(\mathcal{I}_n(q)) \leq \dim_K(R_n(q)) \leq |\Omega|$. To get equality, Halverson appeals to Solomon's proof that the dimension of $\mathcal{I}_n(q)$ is at least $|\Omega|$ (see [15, Remark 2.28]). The argument here is that if there is an algebra $\mathcal{I}'$ of dimension $|\Omega|$ which satisfies the defining relations of $\mathcal{I}_n(q)$, then $\mathcal{I}'$ is a homomorphic image of $\mathcal{I}_n(q)$ and so $\dim_K(\mathcal{I}_n(q)) \geq \dim_K(\mathcal{I}') = |\Omega|$. Solomon defines such an algebra ([15], 2.33, 2.34) and shows that it can be defined for any field K. This therefore means that Halverson's proof that $\mathcal{I}_n(q) \cong R_n(q)$ is valid for the algebras defined over an arbitrary field, and also shows that (1) is a basis of $R_n(q)$. $\qquad\square$

From now on we shall use Halverson's presentation, and write $R_n(q)$ for the q-rook monoid algebra.

The q-rook monoid algebra is indeed a q-analogue of the rook monoid, as one sees by setting $q = 1$ where we obtain the rook monoid algebra R_n. The generators T_i become the permutation matrices associated to the transpositions $(i, i + 1)$, and P_i becomes $E_{i+1,i+1} + \cdots + E_{n,n}$ (where $E_{i,j}$ denotes the elementary matrix with a 1 in position i, j). If we set $q = 1$ in Solomon's presentation then the generator N should be identified with $E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}$.

Let us record a few obvious statements which can be deduced from the relations:

$$T_i^{-1} = (q^{-1} - 1) \cdot 1 + q^{-1} T_i, \tag{2}$$

$$P_i P_j = P_j \quad \text{if } i \leq j, \tag{3}$$

$$P_i T_i^{-1} P_i = q^{-1} P_{i+1}. \tag{4}$$

The subalgebra of $R_n(q)$ generated by $T_1, T_2, \ldots, T_{n-1}$ is isomorphic to $\mathcal{H}_n(q)$. So for $w \in \mathcal{S}_n$ one can associate an element $T_w \in R_n(q)$ in the usual way.

2. A new basis

2.1. The anti-involution

As we saw above, part of the definition of a cellular algebra is the existence of an anti-involution on the algebra. We define an anti-involution, that is an anti-automorphism

i satisfying $i^2 = \mathrm{id}$, on $R_n(q)$ by extending:

$$i : R_n(q) \to R_n(q)$$
$$i(T_j) = T_j \quad \text{for } 1 \le j \le n - 1$$
$$i(P_1) = P_1.$$

From Eq. (2) we see $i(T_j^{-1}) = T_j^{-1}$, and thus from Eq. (4) we obtain $i(P_j) = P_j$ for $1 \le j \le n$.

2.2. A chain of ideals

For $1 \le k \le n$, let us write J_k to denote the 2-sided ideal of $R_n(q)$ generated by P_k, (and for convenience let us write $J_0 = R_n(q)$ and $J_{n+1} = 0$). Then this gives a chain of 2-sided ideals:

$$0 = J_{n+1} \subset J_n \subset J_{n-1} \subset \cdots \subset J_1 \subset J_0 = R_n(q). \tag{5}$$

Solomon also uses these ideals (see [15], (1.10)), where they are written Θ_k.

2.3. A new basis

We shall use a slightly different basis of $R_n(q)$ than that defined by Halverson, and recalled above. For $(A, B, w) \in \Omega_k, k = 0, 1, \ldots, n$, define:

$$\tilde{T}_{(A,B,w)} := i\left(T_A^{-1}\right) P_k T_w T_B^{-1}.$$

Proposition 3. *The set $\{\tilde{T}_{(A,B,w)} : (A, B, w) \in \Omega\}$ is a basis of $R_n(q)$.*

Proof: We shall see that Halverson's basis elements $T_{(A,B,w)}$ can be expressed as linear combinations of the elements $\tilde{T}_{(C,D,w')}$. We use induction on the length of T_A. If the length is zero then $A = \{1, 2, \ldots, k\}$ and $T_{(A,B,w)} = \tilde{T}_{(A,B,w)}$. Otherwise write $T_{(A,B,w)}$ as $T_j T_{(A',B,w)}$, where $j = a_i - 1$ if $\{a_1, \ldots, a_{i-1}\} = \{1, \ldots, i - 1\}$ but $a_i \neq i$, and $A' = \{1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_k\}$. The inductive hypothesis tells us we can express $T_{(A',B,w)}$ in the required form so it remains to check that $T_j \tilde{T}_{(C,D,u)}$, or equivalently (using the expression for inverses)
$T_j^{-1} \tilde{T}_{(C,D,u)}$ can be written as a linear combination of elements $\tilde{T}_{(C,D,w')}$. To do this we refer to the proof of Theorem 2.1 in [7]. Here Halverson shows that $P_k T_C^{-1} T_j^{-1}$ can be written as a linear combination of elements $T_{u'} P_k T_C^{-1}$ (for $u' \in \mathcal{S}_{\{k+1,\ldots,n\}}$). But then applying the anti-involution i shows that $T_j^{-1} i(T_C^{-1}) P_k$ is a linear combination of elements $i(T_C^{-1}) P_k T_{u'^{-1}}$. So $T_j^{-1} \tilde{T}_{(C,D,u)}$ can be written as a sum of elements $i(T_C^{-1}) P_k (T_{u'^{-1}} T_u) T_D^{-1}$, which can be written in the required form. $\qquad\square$

Clearly the images of the basis elements $\tilde{T}_{(A,B,w)}$ for $(A, B, w) \in \Omega_k$ form a basis of J_k/J_{k+1}, for $0 \leq k \leq n$. Observe also that $i(\tilde{T}_{(A,B,w)}) = \tilde{T}_{(B,A,w^{-1})}$.

3. Basis calculations

Our aim is to show that $R_n(q)$ is an iterated inflation of Hecke algebras, coming from the chain of ideals (5). To do this we shall need to understand how to multiply elements within a layer J_k/J_{k+1}, where $0 \leq k \leq n$. Suppose $(A, B, w), (C, D, w') \in \Omega_k$. Then (A5) implies that $P_k T_w = T_w P_k$, hence:

$$\tilde{T}_{(A,B,w)}\tilde{T}_{(C,D,w')} = i\left(T_A^{-1}\right)P_k T_w T_B^{-1} i\left(T_C^{-1}\right)P_k T_{w'} T_D^{-1}$$
$$= i\left(T_A^{-1}\right)T_w X_{B,C} T_{w'} T_D^{-1},$$

where

$$X_{B,C} = P_k T_B^{-1} i\left(T_C^{-1}\right) P_k.$$

Suppose that $B = \{b_1, \ldots, b_k\}$, $C = \{c_1 \ldots, c_k\}$ where $b_1 < b_2 < \cdots < b_k$ and $c_1 < c_2 < \cdots < c_k$. Observe that by the relation (A5) along with Eq. (3), we can write $X_{B,C}$ in the following way:

$$X_{B,C} = P_k\left(T_k^{-1} T_{k+1}^{-1} \cdots T_{b_k-1}^{-1}\right)P_{k-1}\left(T_{k-1}^{-1} T_k^{-1} \cdots T_{b_{k-1}-1}^{-1}\right)P_{k-2} \cdots$$
$$\cdots P_1\left(T_1^{-1} T_2^{-1} \cdots T_{b_1-1}^{-1}\right)\left(T_{c_1-1}^{-1} \cdots T_2^{-1} T_1^{-1}\right)P_1 \cdots$$
$$\cdots P_{k-2}\left(T_{c_{k-1}-1}^{-1} \cdots T_k^{-1} T_{k-1}^{-1}\right)P_{k-1}\left(T_{c_k-1}^{-1} \cdots T_{k+1}^{-1} T_k^{-1}\right)P_k.$$

Lemma 4. *Suppose $b, c > i$ and $b \neq c$. Then:*

$$P_i T_i^{-1} T_{i+1}^{-1} \cdots T_{b-1}^{-1} T_{c-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-1} P_i$$
$$= \begin{cases} q^{-1} T_{c-1}^{-1} \cdots T_{i+1}^{-1} P_{i+1} T_{i+1}^{-1} \cdots T_b^{-1} & \text{if } b < c \\ q^{-1} T_c^{-1} \cdots T_{i+1}^{-1} P_{i+1} T_{i+1}^{-1} \cdots T_{b-1}^{-1} & \text{if } b > c. \end{cases}$$

Proof: Consider the case $b < c$. By (A3) and (A5),

$$P_i T_i^{-1} T_{i+1}^{-1} \cdots T_{b-1}^{-1} T_{c-1}^{-1} \cdots T_{b+1}^{-1} T_b^{-1} T_{b-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-1} P_i$$
$$= T_{c-1}^{-1} \cdots T_{b+1}^{-1} P_i T_i^{-1} T_{i+1}^{-1} \cdots T_{b-1}^{-1} T_b^{-1} T_{b-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-1} P_i.$$

Now use (A2) to write $T_{b-1}^{-1}T_b^{-1}T_{b-1}^{-1} = T_b^{-1}T_{b-1}^{-1}T_b^{-1}$, and then apply (A3) and (A5) again to get equality with:

$$T_{c-1}^{-1}\cdots T_{b+1}^{-1}T_b^{-1}P_iT_i^{-1}T_{i+1}^{-1}\cdots T_{b-2}^{-1}T_{b-1}^{-1}T_{b-2}^{-1}\cdots T_{i+1}^{-1}T_i^{-1}P_iT_b^{-1}.$$

Repeat this argument until we reach:

$$T_{c-1}^{-1}\cdots T_{i+1}^{-1}P_iT_i^{-1}P_iT_{i+1}^{-1}\cdots T_b^{-1} = q^{-1}T_{c-1}^{-1}\cdots T_{i+1}^{-1}P_{i+1}T_{i+1}^{-1}\cdots T_b^{-1}$$

by Eq. (4). The other case is analogous.   □

Lemma 5. *Suppose* $b_i < \cdots < b_k$ *and* $c_i < \cdots < c_k$ *with* $b_i \neq c_i$. *Then:*

$$P_k\Big(T_k^{-1}\cdots T_{b_k-1}^{-1}\Big)P_{k-1}\cdots P_i\Big(T_i^{-1}\cdots T_{b_i-1}^{-1}\Big)\Big(T_{c_i-1}^{-1}\cdots T_i^{-1}\Big)P_i\cdots P_{k-1}\Big(T_{c_k-1}^{-1}\cdots T_k^{-1}\Big)P_k$$

lies in J_{k+1}.

Proof: Consider the case $c_i < b_i$ (as the other case is analogous). Use Lemma 4 to replace

$$P_iT_i^{-1}\cdots T_{b_i-1}^{-1}T_{c_i-1}^{-1}\cdots T_i^{-1}P_i$$

by $q^{-1}T_{c_i}^{-1}\cdots T_{i+1}^{-1}P_{i+1}T_{i+1}^{-1}\cdots T_{b_i-1}^{-1}$. Now consider the subexpression

$$P_{i+1}T_{i+1}^{-1}\cdots T_{b_{i+1}-1}^{-1}T_{c_i}^{-1}\cdots T_{i+1}^{-1}P_{i+1}.$$

As $c_i < b_i < b_{i+1}$, we have $c_i < b_{i+1} - 1$, so we can use Lemma 4 again and replace the subexpression by $q^{-1}T_{c_i+1}^{-1}\cdots T_{i+2}^{-1}P_{i+2}T_{i+2}^{-1}\cdots T_{b_{i+1}-1}^{-1}$. Keep repeating this step until we have shown that:

$$P_k\Big(T_k^{-1}\cdots T_{b_k-1}^{-1}\Big)P_{k-1}\cdots P_i\Big(T_i^{-1}\cdots T_{b_i-1}^{-1}\Big)\Big(T_{c_i-1}^{-1}\cdots T_i^{-1}\Big)P_i\cdots P_{k-1}\Big(T_{c_k-1}^{-1}\cdots T_k^{-1}\Big)P_k$$
$$= T_{c_i+k-i}^{-1}\cdots T_{k+1}^{-1}P_{k+1}u$$

for some element $u \in R_n(q)$.   □

Lemma 6. *For* $a \geq 0$,

$$P_kT_k^{-1}T_{k+1}^{-1}\cdots T_{k+a}^{-1}T_{k+a}^{-1}\cdots T_{k+1}^{-1}T_k^{-1}P_k$$

$$= q^{-(a+1)}P_k + q^{-(a+2)}(1-q)P_{k+1} + (1-q)\sum_{j=1}^{a}q^{-a+j-2}T_{k+j}^{-1}\cdots T_{k+1}^{-1}P_{k+1}T_{k+1}^{-1}\cdots T_{k+j}^{-1}.$$

Proof: When $a = 0$ we have

$$P_k(T_k^{-1})^2 P_k = P_k(q^{-1} \cdot 1 + (1-q)q^{-1}T_k^{-1})P_k = q^{-1}P_k + q^{-2}(1-q)P_{k+1}.$$

For $a \geq 1$, substitute $(T_{k+a+1}^{-1})^2 = q^{-1} \cdot 1 + (1-q)q^{-1}T_{k+a+1}^{-1}$ into the following: $P_k T_k^{-1} T_{k+1}^{-1} \cdots T_{k+a}^{-1} (T_{k+a+1}^{-1})^2 T_{k+a}^{-1} \cdots T_{k+1}^{-1} T_k^{-1} P_k$. Use the inductive hypothesis for the first term, and use Lemma 4 for the second. The result follows by induction. □

Lemma 7. *Let $B = \{b_1, b_2, \ldots, b_k\}$ with $b_1 < \cdots < b_k$. Then $X_{B,B}$ is equal to*

$$q^{-(b_1-1)-\cdots-(b_k-k)} P_k$$

plus terms of the form

$$T_j^{-1} T_{j-1}^{-1} \cdots T_{k+1}^{-1} P_{k+1} u_j P_{k+1} T_{k+1}^{-1} \cdots T_{j-1}^{-1} T_j^{-1},$$

for $k \leq j \leq b_k - 1$, where each $u_j \in R_n(q)$ (and we read this as $P_{k+1} u_k P_{k+1}$ for $j = k$).

Proof: We prove this by induction. When $k = 1$ this has the required form by Lemma 6. For $k > 1$, write $B' = \{b_1, \ldots, b_{k-1}\}$. Then

$$X_{B,B} = P_k(T_k^{-1} \cdots T_{b_k-1}^{-1}) X_{B',B'}(T_{b_k-1}^{-1} \cdots T_k^{-1}) P_k$$

which, by the inductive hypothesis, equals

$$q^{-(b_1-1)-\cdots-(b_{k-1}-k+1)} P_k(T_k^{-1} \cdots T_{b_k-1}^{-1})(T_{b_k-1}^{-1} \cdots T_k^{-1}) P_k,$$

plus terms of the form:

$$P_k(T_k^{-1} \cdots T_{b_k-1}^{-1}) T_j^{-1} \cdots T_k^{-1} P_k u_j P_k T_k^{-1} \cdots T_j^{-1}(T_{b_k-1}^{-1} \cdots T_k^{-1}) P_k,$$

for $k - 1 \leq j \leq b_{k-1} - 1 < b_k - 1$. Lemma 6 shows that the first term equals $q^{-(b_1-1)-\cdots-(b_k-k)} P_k$ plus terms of the required form. The other terms above can be written as:

$$P_k(T_k^{-1} \cdots T_j^{-1} T_{j+1}^{-1} T_j^{-1} \cdots T_k^{-1} P_k u_j' P_k T_k^{-1} \cdots T_j^{-1} T_{j+1}^{-1} T_j^{-1} \cdots T_k^{-1}) P_k$$

for some $u_j' \in R_n(q)$. Applying Lemma 4, this equals

$$q^{-2} T_{j+1}^{-1} T_j^{-1} \cdots T_{k+1}^{-1} P_{k+1} T_{k+1}^{-1} \cdots T_{j+1}^{-1} u_j' T_{j+1}^{-1} T_j^{-1} \cdots T_{k+1}^{-1} P_{k+1} T_{k+1}^{-1} \cdots T_{j+1}^{-1}.$$

Writing this as $T_{j+1}^{-1} T_j^{-1} \cdots T_{k+1}^{-1} P_{k+1} u_j'' P_{k+1} T_{k+1}^{-1} \cdots T_{j+1}^{-1}$, we see it has the required form.

□

Proposition 8. *Suppose* $|B| = |C| = k$. *Then modulo* J_{k+1},

$$
X_{B,C} = \begin{cases} q^{\alpha_B} P_k & \text{if } B = C \\ 0 & \text{if } B \neq C. \end{cases}
$$

where, if $B = \{b_1, b_2, \ldots, b_k\}$, $\alpha_B = -\sum_{i=1}^{k} b_i + \frac{k(k+1)}{2}$.

Proof: Lemma 7 deals with the case $B = C$. Now assume $B \neq C$. If $b_1 \neq c_1$ then $X_{B,C} \in J_{k+1}$ by Lemma 5. So assume $B' := \{b_1, \ldots, b_{i-1}\} = \{c_1, \ldots, c_{i-1}\}$ but $b_i \neq c_i$. Then

$$
X_{B,C} = P_k\left(T_k^{-1} \cdots T_{b_k-1}^{-1}\right) P_{k-1} \cdots P_i\left(T_i^{-1} \cdots T_{b_i-1}^{-1}\right) X_{B',B'}
$$
$$
\left(T_{c_i-1}^{-1} \cdots T_i^{-1}\right) P_i \cdots P_{k-1}\left(T_{c_k-1}^{-1} \cdots T_k^{-1}\right) P_k
$$

Use Lemma 7 to express $X_{B',B'}$, and substitute into the above. We obtain a term of a unit times:

$$
P_k\left(T_k^{-1} \cdots T_{b_k-1}^{-1}\right) P_{k-1} \cdots P_i\left(T_i^{-1} \cdots T_{b_i-1}^{-1}\right)\left(T_{c_i-1}^{-1} \cdots T_i^{-1}\right) P_i \cdots P_{k-1}\left(T_{c_k-1}^{-1} \cdots T_k^{-1}\right) P_k
$$

which lies in J_{k+1} by Lemma 5, and also terms of the form:

$$
P_k\left(T_k^{-1} \cdots T_{b_k-1}^{-1}\right) P_{k-1} \cdots P_i\left(T_i^{-1} \cdots T_{b_i-1}^{-1}\right)\left(T_j^{-1} \cdots T_i^{-1}\right) P_i w
$$

(where $i - 1 \leq j < b_{i-1} - 1 < b_i - 1$ and $w \in R_n(q)$). Now repeatedly apply Lemma 4, firstly replacing $P_i(T_i^{-1} \cdots T_{b_i-1}^{-1})(T_j^{-1} \cdots T_i^{-1}) P_i w$ by $T_{j+1}^{-1} \cdots T_{i+1}^{-1} P_{i+1} w'$, for some $w' \in R_n(q)$, until we arrive at $P_k(T_k^{-1} \cdots T_{b_k-1}^{-1})(T_{j'}^{-1} T_{j'+1}^{-1} \cdots T_k^{-1}) P_k \bar{w}$ for some $j' < b_k - 1$ and some $\bar{w} \in R_n(q)$. By Lemma 4, this lies in J_{k+1}. $\square$

4. Main results

Our calculations in the previous section have shown us how to multiply within a layer.

Theorem 9. *Let* $(A, B, w), (C, D, w') \in \Omega_k$. *Then, if* $B \neq C$,

$$
\tilde{T}_{(A,B,w)} \tilde{T}_{(C,D,w')} = 0 \, (mod \, J_{k+1})
$$

and, if $B = C$,

$$
\tilde{T}_{(A,B,w)} \tilde{T}_{(C,D,w')} = q^{\alpha_B} \sum_u \gamma_u \tilde{T}_{(A,D,u)} \, (mod \, J_{k+1})
$$

where $T_w T_{w'} = \sum_{u \in S_{\{k+1,\ldots,n\}}} \gamma_u T_u$ is the multiplication in $\mathcal{H}_{\{k+1,\ldots,n\}}(q)$, and, if $B = \{b_1, \ldots, b_k\}$, then $\alpha_B = -\sum_{i=1}^{k} b_i + \frac{k(k+1)}{2}$.

We can translate this into the language of inflations as follows. For $k = 0, 1, \ldots, n$, let V_k denote a K-vector space with basis labelled by subsets of $[n]$ of size k, that is $\{v_A : A \subseteq [n], |A| = k\}$. Now define a bilinear form $\phi : V_k \times V_k \to K$ by extending:

$$\phi(v_B, v_C) = \begin{cases} q^{\alpha_B} & \text{if } B = C, \\ 0 & \text{if } B \neq C. \end{cases}$$

where α_B is the coefficient appearing in Theorem 9 above

Corollary 10.

$$J_k/J_{k+1} \cong V_k \otimes V_k \otimes \mathcal{H}_{n-k}(q),$$

where the multiplication is given by:

$$(v_A \otimes v_B \otimes T_w)(v_C \otimes v_D \otimes T_{w'}) = v_A \otimes v_D \otimes (T_w \phi(v_B, v_C) T_{w'}).$$

The cellularity of the q-rook monoid algebra will now follow from Xi's statement (Lemma 1) once we verify that condition (v) of that lemma holds. This is done next.

Lemma 11. *If I is an ideal of $\mathcal{H}_{\{k+1,\ldots,n\}}(q)$ then $\tilde{I} = V_k \otimes V_k \otimes I + J_{k+1}$ is an ideal of $R_n(q)$.*

Proof: Take $i(T_A^{-1}) P_k y T_B^{-1} = i(T_A^{-1}) P_k y P_k T_B^{-1}$ where $|A| = |B| = k$ and $y \in I$ and consider right-multiplication by a basis element, $\tilde{T}_{(C,D,w)}$ (and the argument for left-multiplication is identical). By noting that $P_k T_B^{-1} \tilde{T}_{(C,D,w)} \in J_k$, it is sufficient to consider $i(T_A^{-1}) P_k y \tilde{T}_{(E,F,w')} = i(T_A^{-1}) y P_k \tilde{T}_{(E,F,w')}$ for $(E, F, w') \in \Omega_k$, whence we can apply Proposition 8 to see that the product is in $\tilde{I}$. $\qquad\square$

Corollary 12. *The q-rook monoid algebra $R_n(q)$ is an iterated inflation of the Hecke algebras: $\mathcal{H}_n(q), \mathcal{H}_{n-1}(q), \ldots, \mathcal{H}_1(q), \mathcal{H}_0(q) = K$. Thus $R_n(q)$ is a cellular algebra.*

The rook monoid algebra (the case $q = 1$) was already known to be a cellular algebra by work of East in [4]. Corollary 12 tells us that, as a vector space:

$$R_n(q) \cong \bigoplus_{k=0}^{n} (V_k \otimes V_k \otimes \mathcal{H}_{n-k}(q))$$

where $\mathcal{H}_0(q)$ is read as K. In fact this is more than just a vector space decomposition. To see this we use the following lemma of König and Xi.

Lemma 13 ([12], Lemma 7.1). *If A is a ring and J a 2-sided ideal in A then $J \subseteq A$ induces a ring decomposition $A \cong J \oplus (A/J)$ if and only if J has a unit element.*

We briefly recall the proof. One direction is trivial. For the other, assume there is some $e \in J$ such that $ex = xe = x$ for all $x \in J$. Then $J = eJe \subseteq eAe \subseteq J$, so $J = eAe$. Then $(1_A - e)Ae \subseteq (1_A - e)J = (1_A - e)eJ = 0$ and similarly $eA(1_A - e) = 0$. Hence $A = eAe \oplus (1_A - e)A(1_A - e)$.

Theorem 14. *As an algebra,*

$$R_n(q) \cong \bigoplus_{k=0}^{n} (V_k \otimes V_k \otimes \mathcal{H}_{n-k}(q))$$

where $\mathcal{H}_0(q)$ is read as K.

Proof: By Lemma 13, it is sufficient to find a unit element in each layer. In J_n we define $e_n = P_n$, which is certainly a unit in J_n. Now in $J_k/J_{k+1} \cong V_k \otimes V_k \otimes \mathcal{H}_{n-k}(q)$, define $e_k = \sum_A \frac{1}{\phi(v_A, v_A)} \tilde{T}_{(A,A,1)}$. The images of $\{\tilde{T}_{(B,C,w)} : (B, C, w) \in \Omega_k\}$ form a basis of J_k/J_{k+1}, and:

$$
\begin{aligned}
\tilde{T}_{(B,C,w)} e_k &= \sum_A \frac{1}{\phi(v_A, v_A)} \tilde{T}_{(B,C,w)} \tilde{T}_{(A,A,1)} \\
&= \frac{1}{\phi(v_C, v_C)} \tilde{T}_{(B,C,w)} \tilde{T}_{(C,C,1)} \ (mod \ J_{k+1}) \\
&= \tilde{T}_{(B,C,w)} \ (mod \ J_{k+1})
\end{aligned}
$$

by Corollary 10. Similarly $e_k \tilde{T}_{(B,C,w)} = \tilde{T}_{(B,C,w)} \ (mod \ J_{k+1})$, and the image of e_k in J_k/J_{k+1} is the required unit. $\qquad \square$

Solomon proves this for the rook monoid (that is $q = 1$ case) over a field of characteristic zero, in [14, Corollary 2.19], and it is originally a result of Munn. Although the assumption on the characteristic is required for later results in Solomon's paper, it seems not to be needed for the arguments leading up to Corollary 2.19, so his proof would also apply to a field of non-zero characteristic when $q = 1$.

We have the following corollary.

Corollary 15. *Over any field $R_n(q)$ is Morita equivalent to*

$$\mathcal{H}_n(q) \oplus \mathcal{H}_{n-1}(q) \oplus \cdots \oplus \mathcal{H}_1(q) \oplus K.$$

Proof: This follows from the theory of inflations in [12]. The bilinear form $\phi : V_k \times V_k \to K$ is non-singular, and thus (by [12], 3.4) we have a Morita equivalence between $V_k \otimes V_k \otimes \mathcal{H}_{n-k}(q)$ and $\mathcal{H}_{n-k}(q)$. $\qquad \square$

   This means that the representation theory of the q-rook monoid algebra is the same as that of the above direct sum of Hecke algebras. This allows us to answer many representation theoretic questions about $R_n(q)$.

- The cell modules of $R_n(q)$ are, as vector spaces, isomorphic to $S^\lambda \otimes V_l$ for $0 \leq l \leq n$ and λ a partition of $n - l$, where S^λ denotes the usual Specht module for $\mathcal{H}_{n-1}(q)$ (as defined in Section 4 of [2]). If $R_n(q)$ is semisimple then these form a complete set of non-isomorphic simple $R_n(q)$-modules. However, in general they need not be simple. Let e be the least positive integer such that

$$1 + q + q^2 + \cdots + q^{e-1} = 0 \tag{6}$$

  if such an integer exists, and put $e = \infty$ otherwise. Then the simple $R_n(q)$ modules are labelled by the e-regular partitions λ of $n - l$, $0 \leq l \leq n$. As vector spaces these are isomorphic to $D^\lambda \otimes V_l$ where D^λ is a simple $\mathcal{H}_{n-l}(q)$-module (as in [2] Sections 4 and 6.1). In particular this shows that the dimension of the simple $R_n(q)$-module labelled by λ equals $\binom{n}{l} \dim_K(D^\lambda)$.

- The block decomposition of $R_n(q)$ is given by the so-called Nakayama Conjecture (see Section 4 of [3]). Associated to a partition λ is its e-core, obtained by deleting all rim e-hooks from the Young diagram of λ (see [9], Section 2.7). If λ is a partition with e-core κ then $|\lambda| = |\kappa| + ew$ for some w which is called the e-weight of λ. Two cell modules for $R_n(q)$, $S^\lambda \otimes V_l$ and $S^\mu \otimes V_k$ lie in the same block if and only if λ and μ have the same e-cores and e-weights.

- $R_n(q)$ is a semisimple algebra if and only if $n < e$. This was proved by Halverson and Ram in the case when $K = \mathbb{C}$ (see [8], 2.21). A block of $R_n(q)$ is simple if and only if its e-weight equals zero. A block of $R_n(q)$ is of finite type if and only if its e-weight is at most 1.

- The decomposition matrix of a cellular algebra records the multiplicity of each simple module in each cell module. The decomposition matrix of $R_n(q)$ is just the block diagonal matrix containing the decomposition matrices of the Hecke algebras $\mathcal{H}_n(q), \mathcal{H}_{n-1}(q), \ldots, \mathcal{H}_1(q), K$.

- In particular, the rook monoid algebra R_n (that is the $q = 1$ case) is Morita equivalent to $K\mathcal{S}_n \oplus K\mathcal{S}_{n-1} \oplus \cdots \oplus K\mathcal{S}_1 \oplus K$. As noted in Henke's review of [6], the Morita equivalence of Corollary 15 in the case when $q = 1$ sends Specht modules for symmetric groups to the Specht modules constructed by Grood—these are just the cell modules of R_n, $S^\lambda \otimes V_l$ where S^λ is a Specht module of $K\mathcal{S}_{n-l}$ (as defined in Section 7.1 of [9]) for $l \in \{0, 1, \ldots, n\}$.

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