

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

*Representations and Invariants of the
Classical Groups*

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CONTENTS

Preface	xiii
1 Classical Groups as Linear Algebraic Groups	1
1.1 Linear Algebraic Groups	1
1. Definitions and Examples	
2. Regular Functions	
3. Representations	
4. Connected Groups	
5. Subgroups and Homomorphisms	
6. Group Structures on Affine Varieties	
1.2 Lie Algebra of an Algebraic Group	17
1. Left-Invariant Vector Fields	
2. Lie Algebras of the Classical Groups	
3. Differential of a Representation	
4. The Adjoint Representation	
1.3 Jordan Decomposition	34
1. Nilpotent and Unipotent Matrices	
2. Semisimple One-Parameter Groups	
3. Jordan–Chevalley Decomposition	
1.4 Real Forms of Classical Groups	41
1. Algebraic Groups as Lie Groups	
2. Real Forms	
3. Compact Forms	
4. Quaternionic Unitary Group	
5. Quaternionic General Linear Group	
1.5 Notes	49
2 Basic Structure of Classical Groups	50
2.1 Semisimple and Unipotent Elements	50
1. Conjugacy of Maximal Tori	
2. Unipotent Generators	
2.2 Irreducible Representations of $SL(2, \mathbb{C})$	62
1. Representations of $\mathfrak{sl}(2, \mathbb{C})$	
2. Representations of $SL(2, \mathbb{C})$	
2.3 The Adjoint Representation	67
1. Roots with respect to a Maximal Torus	
2. Commutation Relations of Root Spaces	
3. Structure of Classical Root Systems	
4. Irreducibility of the Adjoint Representation	
2.4 Reductivity of Classical Groups	84
1. Reductive Groups	
2. Casimir Operator	
3. Algebraic Proof of Complete Reducibility	
4. The Unitarian Trick	

2.5	Weyl Group and Weight Lattice	92
	1. Weyl Group 2. Root Reflections 3. Weight Lattice 4. Fundamental Weights and Dominant Weights	
2.6	Notes	109
3	Algebras and Representations	111
3.1	Representations of Associative Algebras	111
	1. Definitions and Examples 2. Schur's Lemma 3. Burnside's Theorem 4. Complete Reducibility	
3.2	Simple Associative Algebras	128
	1. Wedderburn's Theorem 2. Representations of $\text{End}(V)$	
3.3	Commutants and Characters	133
	1. Representations of Semisimple Algebras 2. Double Commutant Theorem 3. Characters	
3.4	Group Algebras of Finite Groups	147
	1. Structure of Group Algebras 2. Schur Orthogonality Relations 3. Fourier Inversion Formula 4. The Algebra of Central Functions	
3.5	Representations of Finite Groups	155
	1. Induced Representations 2. Characters of Induced Representations 3. Standard Representation of \mathfrak{S}_n 4. Representations of \mathfrak{S}_k on Tensors	
3.6	Notes	167
4	Polynomial and Tensor Invariants	168
4.1	Polynomial Invariants	169
	1. The Ring of Invariants 2. Invariant Polynomials for \mathfrak{S}_n	
4.2	Invariants for Classical Groups	180
	1. First Fundamental Theorem 2. Proof of a Basic Case 3. Invariant Polynomials as Tensors	
4.3	Tensor Invariants	190
	1. Tensor Invariants for $\text{GL}(V)$ 2. Tensor Invariants for $\text{O}(V)$ and $\text{Sp}(V)$	
4.4	Polynomial FFT for Classical Groups	198
	1. Proof of Polynomial FFT for $\text{GL}(V)$ 2. Proof of Polynomial FFT for $\text{O}(V)$ and $\text{Sp}(V)$	
4.5	Some Applications of the FFT	200
	1. Skew Duality for Classical Groups 2. General Duality Theorem 3. A Duality Theorem for Weyl Algebras 4. $\text{GL}(n) - \text{GL}(k)$ Howe Duality 5. $\text{O}(n) - \mathfrak{sp}(k)$ Howe Duality 6. $\text{Sp}(n) - \mathfrak{so}(2k)$ Howe Duality 7. Capelli Identities	
4.6	Notes	226

5	Highest Weight Theory	228
5.1	Irreducible Representations of Classical Groups	228
	1. Extreme Vectors and Highest Weights 2. Commuting Algebra and n -Invariant Vectors 3. Fundamental Representations 4. Cartan Product 5. Weights of Irreducible Representations 6. Lowest Weights and Dual Representations 7. Symplectic and Orthogonal Representations	
5.2	Some Applications	248
	1. Irreducible Representations of $GL(V)$ 2. Irreducible Representations of $O(V)$ 3. Spherical Harmonics 4. $GL(k) - GL(n)$ Duality 5. Decomposition of $S(S^2(V))$ under $GL(V)$ 6. Decomposition of $S(\wedge^2(V))$ under $GL(V)$ 7. Second Fundamental Theorems	
5.3	Notes	268
6	Spinors	269
6.1	Clifford Algebras	269
	1. Construction of $Cliff(V)$ 2. Spaces of Spinors 3. Structure of $Cliff(V)$	
6.2	Spin Representations of Orthogonal Lie Algebras	279
	1. Embedding $\mathfrak{so}(V)$ in $Cliff(V)$ 2. Spin Representations	
6.3	Spin Groups	284
	1. Action of $O(V)$ on $Cliff(V)$ 2. Algebraically Simply Connected Groups	
6.4	Real Forms of $Spin(n, \mathbb{C})$	291
	1. Real Forms of Vector Spaces and Algebras 2. Real Forms of Clifford Algebras 3. Real Forms of $Pin(n)$ and $Spin(n)$	
6.5	Notes	294
7	Cohomology and Characters	296
7.1	Character and Dimension Formulas	296
	1. Weyl Character Formula 2. Weyl Dimension Formula 3. Commutant Character Formulas	
7.2	Lie Algebra Cohomology	309
	1. Cochain Complex 2. Cohomology Spaces 3. Cohomology Exact Sequences 4. The Koszul Complex 5. Cohomology of Enveloping Algebras	
7.3	Algebraic Approach to Weyl Character Formula	324
	1. Casimir Identity on Cohomology 2. Weyl Group and Sets of Positive Roots 3. Expansion of an Invariant 4. Kostant's Lemma 5. Kostant's Theorem 6. Algebraic Proof of Weyl Character Formula	

7.4	Analytic Approach to Weyl Character Formula	337
	1. Semisimple Conjugacy Classes 2. Maximal Compact Torus	
	3. Weyl Integral Formula 4. Fourier Expansions of Skew	
	Functions 5. Analytic Proof of Weyl Character Formula	
7.5	Notes	347
8	Branching Laws	349
8.1	Branching for Classical Groups	349
	1. Statement of Branching Laws 2. Branching Patterns and	
	Weight Multiplicities	
8.2	Branching Laws from Weyl Character Formula	356
	1. Partition Functions 2. Kostant Multiplicity Formulas	
8.3	Proofs of Classical Branching Laws	359
	1. Restriction from $GL(n)$ to $GL(n - 1)$ 2. Restriction from	
	$Spin(2n + 1)$ to $Spin(2n)$ 3. Restriction from $Spin(2n)$ to	
	$Spin(2n - 1)$ 4. Restriction from $Sp(n)$ to $Sp(n - 1)$	
8.4	Notes	370
9	Tensor Representations of $GL(V)$	372
9.1	Schur Duality	372
	1. Duality between $GL(n)$ and \mathfrak{S}_k 2. Characters of \mathfrak{S}_k	
	3. Frobenius Formula	
9.2	Dual Reductive Pairs	384
	1. Seesaw Pairs 2. Reciprocity Laws 3. Schur Duality and	
	$GL(k)$ - $GL(n)$ Duality	
9.3	Young Symmetrizers and Weyl Modules	392
	1. Tableaux and Symmetrizers 2. Weyl Modules 3. Standard	
	Tableaux 4. Projections onto Isotypic Components	
9.4	Notes	404
10	Tensor Representations of $O(V)$ and $Sp(V)$	406
10.1	Commuting Algebras on Tensor Spaces	406
	1. Centralizer Algebra 2. Generators and Relations	
10.2	Decomposition of Harmonic Tensors	416
	1. Harmonic Tensors 2. Harmonic Extreme Tensors	
	3. Decomposition of Harmonics for $Sp(V)$ 4. Decomposition	
	of Harmonics for $O(2l + 1)$ 5. Decomposition of Harmonics	
	for $O(2l)$	
10.3	Decomposition of Tensor Spaces	433
	1. Partially Harmonic Tensors 2. Proof of Partial Harmonic	
	Decomposition 3. Decomposition in the Stable Range	

10.4	Invariant Theory and Knot Polynomials	446
	1. The Braid Relations 2. Orthogonal Invariants and the Yang–Baxter Equation 3. The Braid Group 4. The Jones Polynomial	
10.5	Notes	461
11	Algebraic Groups and Homogeneous Spaces	464
11.1	Structure of Algebraic Groups	465
	1. Quotient Groups 2. Commutative Algebraic Groups 3. Solvable and Semisimple Lie Algebras 4. Levi Decomposition of Lie Algebras 5. Unipotent Radical 6. Connected Algebraic Groups and Lie Groups	
11.2	Homogeneous Spaces	481
	1. G -Spaces and Orbits 2. Flag Manifolds 3. Involutions and Symmetric Spaces 4. Involutions of Classical Groups 5. Classical Symmetric Spaces	
11.3	Borel Subgroups	499
	1: Solvable Groups 2. Lie–Kolchin Theorem 3. Structure of Connected Solvable Groups 4. Conjugacy of Borel Subgroups 5. Centralizer of a Torus	
11.4	Further Properties of Real Forms	506
	1. Groups with a Compact Real Form 2. Polar Decomposition by a Compact Form	
11.5	Gauss Decomposition	512
	1. Gauss Decomposition of $GL(n, \mathbb{C})$ 2. Gauss Decomposition of an Algebraic Group 3. Gauss Decomposition for Real Forms	
11.6	Notes	517
12	Representations on Spaces of Regular Functions	518
12.1	Some General Results	518
	1. Isotypic Decomposition of $\text{Aff}(X)$ 2. Decomposition of $\text{Aff}(G)$ 3. Frobenius Reciprocity 4. Models for Irreducible Representations on Function Spaces	
12.2	Multiplicity-Free Spaces	526
	1. Multiplicity and B -Orbits 2. B -Eigenfunctions for Linear Actions 3. Branching from $GL(n)$ to $GL(n - 1)$	
12.3	Regular Functions on Symmetric Spaces	534
	1. Iwasawa Decomposition for Symmetric Spaces 2. Examples of Iwasawa Decompositions 3. Spherical Representations	
12.4	Separation of Variables for Isotropy Representations	553
	1. A Theorem of Kostant and Rallis 2. Some Theorems of Chevalley 3. Classical Examples 4. Some Results from	

Algebraic Geometry	5. Proof of the Kostant–Rallis Theorem	
	6. Some Remarks on the Proof	
12.5 Notes		576
A Algebraic Geometry		579
A.1 Affine Algebraic Sets		579
	1. Basic Properties 2. Zariski Topology 3. Products of Affine Sets 4. Principal Open Sets 5. Irreducible Components 6. Transcendence Degree and Dimension	
A.2 Maps of Algebraic Sets		591
	1. Rational Maps 2. Extensions of Homomorphisms 3. Image of a Dominant Map 4. Factorization of a Regular Map	
A.3 Tangent Spaces		597
	1. Tangent Space and Differentials of Maps 2. Vector Fields 3. Dimension 4. Differential Criterion for Dominance	
A.4 Projective and Quasiprojective Sets		604
	1. Basic Definitions 2. Products of Projective Sets 3. Regular Functions and Maps	
B Linear and Multilinear Algebra		612
B.1 Jordan Decomposition		612
	1. Primary Projections 2. Additive Jordan Decomposition 3. Multiplicative Jordan Decomposition	
B.2 Multilinear Algebra		615
	1. Bilinear Forms 2. Tensor Products 3. Symmetric Tensors 4. Alternating Tensors 5. Determinants and Gauss Decomposition 6. Pfaffians and Skew-Symmetric Matrices 7. Irreducibility of Determinants and Pfaffians	
C Associative Algebras and Lie Algebras		632
C.1 Some Associative Algebras		632
	1. Filtered and Graded Algebras 2. Tensor Algebra 3. Symmetric Algebra 4. Exterior Algebra	
C.2 Universal Enveloping Algebras		639
	1. Lie Algebras 2. Universal Cyclic Module 3. Poincaré–Birkhoff–Witt Theorem 4. Adjoint Representation of Enveloping Algebra	
D Manifolds and Lie Groups		648
D.1 C^∞ Manifolds		648
	1. Basic Definitions 2. Tangent Space 3. Differential Forms and Integration	

D.2 Lie Groups	660
1. Basic Definitions	
2. Lie Algebra of a Lie Group	
3. Homogeneous Spaces	
4. Integration on Lie Groups and Homogeneous Spaces	
Bibliography	673
Index	679