Research Article

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Representations by degenerate Daehee polynomials

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Abstract: In this paper, we consider the problem of representing any polynomial in terms of the degenerate Daehee polynomials and more generally of the higher-order degenerate Daehee polynomials. We derive explicit formulas with the help of umbral calculus and illustrate our results with some examples.

Keywords: degenerate Daehee polynomial, higher-order degenerate Daehee polynomial, umbral calculus

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1 Introduction and preliminaries

The aim of this paper is to derive formulas (see Theorem 3.1) expressing any polynomial in terms of the degenerate Daehee polynomials (see (1.12)) with the help of umbral calculus and to illustrate our results with some examples (see Chapter 6). This can be generalized to the higher-order degenerate Bernoulli polynomials (see (1.13)). Indeed, we deduce formulas (see Theorems 4.1) for representing any polynomial in terms of the higher-order degenerate Daehee polynomials again by using umbral calculus. Letting $\lambda \to 0$, we obtain formulas (see Remarks 3.2 and 4.2) for expressing any polynomial in terms of the Daehee polynomials (see (1.10)) and of the higher-order Daehee polynomials (see (1.11)). These formulas are also illustrated in Chapter 5. The contribution of this paper is the derivation of such formulas that, we think, have many potential applications.

Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k B_k(x)$, where $B_k(x)$ are the Bernoulli polynomials (see (1.3)). Then, it is known (see [1]) that

$$a_k = \frac{1}{k!}\int_0^1 p^{(k)}(x)\mathrm{d}x, \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{1.1}$$

The following identity (see [1,2]) is obtained by applying the formula in (1.1) to the polynomial $p(x) = \sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)B_{n-k}(x)$ and after slight modification:

$$\sum_{k=1}^{n-1}\frac{1}{2k(2n-2k)}B_{2k}(x)B_{2n-2k}(x) + \frac{2}{2n-1}B_1(x)B_{2n-1}(x) \tag{1.2}$$

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$$= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k}\binom{2n}{2k}B_{2k}B_{2n-2k}(x) + \frac{1}{n}H_{2n-1}B_{2n}(x) + \frac{2}{2n-1}B_1(x)B_{2n-1},$$

where $n \geq 2$ and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Letting $x = 0$ and $x = \frac{1}{2}$ in (1.2), respectively, give a slight variant of Miki's identity and the Faber-Pandharipande-Zagier (FPZ) identity. Here, it should be emphasized that the other proofs of Miki's (see [3–5]) and FPZ identities (see [6,7]) are quite involved, while our proofs of Miki's and FPZ identities follow from the simple formula in (1.1) involving only derivatives and integrals of the given polynomials.

Analogous formulas to (1.1) can be obtained for the representations by Euler, Frobenius-Euler, ordered Bell and Genocchi polynomials. Many interesting identities have been derived by using these formulas (see [1,8–14] and references therein). The list in the references is far from being exhaustive. However, the interested reader can easily find more related papers in the literature. Also, we should mention here that there are other ways of obtaining the same result as the one in (1.2). One of them is to use Fourier series expansion of the function obtained by extending by periodicity 1 of the polynomial function restricted to the interval $[0, 1)$ (see [2,15,16]).

The outline of this paper is as follows. In Section 1, we recall some necessary facts that are needed throughout this paper. In Section 2, we go over umbral calculus briefly. In Section 3, we derive formulas expressing any polynomial in terms of the degenerate Daehee polynomials. In Section 4, we derive formulas representing any polynomial in terms of the higher-order degenerate Daehee polynomials. In Section 5, we illustrate our results with examples of representation by the Daehee polynomials. In Section 6, we illustrate our results with examples of representation by the degenerate Daehee polynomials. Finally, we conclude our paper in Section 7.

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. We observe that $B_n(x) = \sum_{j=0}^{n}\binom{n}{j}B_{n-j}x^j$, $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$, and $B_n(x+1) - B_n(x) = nx^{n-1}$. The first few terms of B_n are given by:

$$B_0 = 1,\ \ B_1 = -\frac{1}{2},\ \ B_2 = \frac{1}{6},\ \ B_4 = -\frac{1}{30},\ \ B_6 = \frac{1}{42},\ \ B_8 = -\frac{1}{30},\ \ B_{10} = \frac{5}{66},\ \ B_{12} = -\frac{691}{2730},\ \ldots;$$
$$B_{2k+1} = 0,\ \ (k \geq 1).$$

More generally, for any nonnegative integer r, the Bernoulli polynomials $B_n^{(r)}(x)$ of order r are given by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.4}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r. We observe that $B_n^{(r)}(x) = \sum_{j=0}^{n}\binom{n}{j}B_{n-j}^{(r)}x^j$, $\frac{d}{dx}B_n^{(r)}(x) = nB_{n-1}^{(r)}(x)$, $B_n^{(r)}(x+1) - B_n^{(r)}(x) = nB_{n-1}^{(r-1)}(x)$.

The Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.5}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. We observe that $E_n(x) = \sum_{j=0}^{n}\binom{n}{j}E_{n-j}x^j$, $\frac{d}{dx}E_n(x) = nE_{n-1}(x)$, $E_n(x+1) + E_n(x) = 2x^n$. The first few terms of E_n are given by:

$$E_0 = 1,\ \ E_1 = -\frac{1}{2},\ \ E_3 = \frac{1}{4},\ \ E_5 = -\frac{1}{2},\ \ E_7 = \frac{17}{8},\ \ E_9 = -\frac{31}{2},\ \ldots;\ \ E_{2k} = 0,\ \ (k \geq 1).$$

The Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers. We observe that $G_n(x) = \sum_{j=0}^{n}\binom{n}{j}G_{n-j}x^j$, $\frac{d}{dx}G_n(x) = nG_{n-1}(x)$, $G_n(x+1) + G_n(x) = 2nx^{n-1}$, and $\deg G_n(x) = n-1$, for $n \geq 1$. The first few terms of G_n are given by:

$$G_0 = 0,\quad G_1 = 1,\quad G_2 = -1,\quad G_4 = 1,\quad G_6 = -3,\quad G_8 = 17,\quad G_{10} = -155$$
$$G_{12} = 2073, \ldots;\quad G_{2k+1} = 0,\quad (k \geq 1).$$

For any nonzero real number λ, the degenerate exponentials are given by

$$\begin{aligned} e_\lambda^x(t) &= (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \\ e_\lambda(t) = e_\lambda^1(t) &= (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}. \end{aligned} \tag{1.7}$$

Here, we recall that the λ-falling factorials are given by

$$(x)_{0,\lambda} = 1,\quad (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda),\quad (n \geq 1). \tag{1.8}$$

Especially, $(x)_n = (x)_{n,1}$ are called the falling factorials and hence given by

$$(x)_0 = 1,\quad (x)_n = x(x-1)\cdots(x-n+1),\quad (n \geq 1). \tag{1.9}$$

The compositional inverse of $e_\lambda(t)$ is called the degenerate logarithm and given by

$$\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1),$$

which satisfies $e_\lambda(\log_\lambda(t)) = \log_\lambda(e_\lambda(t)) = t$.

Note here that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$, $\lim_{\lambda\to 0}\log_\lambda(t) = \log(t)$.

Recall that the Daehee polynomials $D_n(x)$ are given by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x = 0$, $D_n = D_n(0)$ are the Daehee numbers.

More generally, for any nonnegative integer r, the Daehee polynomials $D_n^{(r)}(x)$ of order r are given by

$$\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.11}$$

When $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are the Daehee numbers of order r.

The degenerate Daehee polynomials $D_{n,\lambda}(x)$ are defined by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.12}$$

which are degenerate versions of the Daehee polynomials in (1.10). For $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the degenerate Daehee numbers and introduced in [7] (see also [14]).

More generally, for any nonnegative integer r, the degenerate Daehee polynomials $D_{n,\lambda}^{(r)}(x)$ of order r are defined by

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \tag{1.13}$$

which are degenerate versions of the Daehe polynomials of order r in (1.11). We remark that $D_{n,\lambda}(x) \to D_n(x)$, and $D_{n,\lambda}^{(r)}(x) \to D_n^{(r)}(x)$, as λ tends to 0.

We recall some notations and facts about forward differences. Let f be any complex-valued function of the real variable x. Then, for any real number a, the forward difference Δ_a is given by

$$\Delta_a f(x) = f(x+a) - f(x). \tag{1.14}$$

If $a = 1$, then we let

$$\Delta f(x) = \Delta_1 f(x) = f(x+1) - f(x). \tag{1.15}$$

In general, the nth oder forward differences are given by

$$\Delta_a^n f(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x+ia). \tag{1.16}$$

For $a = 1$, we have

$$\Delta^n f(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x+i). \tag{1.17}$$

Finally, we recall that the Stirling numbers of the second kind $S_2(n, k)$ can be given by means of

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}. \tag{1.18}$$

2 Review of umbral calculus

Here, we will briefly go over very basic facts about umbral calculus. For more details on this, we recommend the reader to refer to [3, 20, 22]. Let $\mathbb{C}$ be the field of complex numbers. Then, $\mathcal{F}$ denotes the algebra of formal power series in t over $\mathbb{C}$, given by

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\},$$

and $\mathbb{P} = \mathbb{C}[x]$ indicates the algebra of polynomials in x with coefficients in $\mathbb{C}$.

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. If $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, then the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \quad \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex number.

For $f(t) \in \mathcal{F}$ with $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, we define the linear functional on $\mathbb{P}$ by

$$\langle f(t)|x^k\rangle = a_k. \tag{2.1}$$

From (2.1), we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad (n, k \geq 0),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Some remarkable linear functionals are as follows:

$$\begin{aligned} \langle e^{yt}|p(x)\rangle &= p(y), \\ \langle e^{yt} - 1|p(x)\rangle &= p(y) - p(0), \\ \left\langle \frac{e^{yt}-1}{t} \;\middle|\; p(x) \right\rangle &= \int_0^y p(u)\,\mathrm{d}u. \end{aligned} \tag{2.2}$$

Let

$$f_L(t) = \sum_{k=0}^{\infty} \langle L|x^k\rangle \frac{t^k}{k!}. \tag{2.3}$$

Then, by (2.1) and (2.3), we obtain

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle.$$

That is, $f_L(t) = L$. In addition, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$.

Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra and the umbral calculus is the study of umbral algebra. For each nonnegative integer k, the differential operator t^k on $\mathbb{P}$ is defined by

$$t^k x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.4}$$

Extending (2.4) linearly, any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

gives the differential operator on $\mathbb{P}$ defined by

$$f(t)x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}, \quad (n \ge 0). \tag{2.5}$$

It should be observed that, for any formal power series $f(t)$ and any polynomial $p(x)$, we have

$$\langle f(t)|p(x)\rangle = \langle 1|f(t)p(x)\rangle = f(t)p(x)|_{x=0}. \tag{2.6}$$

Here, we note that an element $f(t)$ of $\mathcal{F}$ is a formal power series, a linear functional, and a differential operator. Some notable differential operators are as follows:

$$\begin{aligned} &e^{yt}p(x) = p(x+y), \\ &(e^{yt}-1)p(x) = p(x+y) - p(x) = \Delta_y p(x), \\ &\frac{e^{yt}-1}{t}p(x) = \int_x^{x+y} p(u)\mathrm{d}u. \end{aligned} \tag{2.7}$$

The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then $f(t)$ is called a delta series.

For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) of polynomials such that

$$\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}, \quad (n, k \ge 0). \tag{2.8}$$

The sequence $s_n(x)$ is said to be the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. We observe from (2.8) that

$$s_n(x) = \frac{1}{g(t)} p_n(x), \tag{2.9}$$

where $p_n(x) = g(t)s_n(x) \sim (1, f(t))$.

In particular, if $s_n(x) \sim (g(t), t)$, then $p_n(x) = x^n$, and hence,

$$s_n(x) = \frac{1}{g(t)} x^n. \tag{2.10}$$

It is well known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\overline{f}(t))} e^{x\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k, \tag{2.11}$$

for all $x \in \mathbb{C}$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

Equations (2.12)–(2.14) are equivalent to the fact that $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$:

$$f(t)s_n(x) = ns_{n-1}(x), \quad (n \geq 0), \tag{2.12}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{2.13}$$

with $p_n(x) = g(t)s_n(x)$,

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\overline{f}(t))^{-1} \overline{f}(t)^j | x^n \rangle x^j. \tag{2.14}$$

Let $p_n(x)$, $q_n(x) = \sum_{k=0}^{n} q_{n,k} x^k$ be sequences of polynomials. Then, the umbral composition of $q_n(x)$ with $p_n(x)$ is defined to be the sequence

$$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} q_{n,k} p_k(x). \tag{2.15}$$

3 Representations by degenerate Daehee polynomials

Our interest here is to derive formulas expressing any polynomial in terms of the degenerate Daehee polynomials.

From (1.7), (1.9), and (1.11), we first observe that

$$D_{n,\lambda}(x) \sim \left(g(t) = \frac{\lambda f(t)}{e^{\lambda t} - 1} = \frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}, f(t) = e^t - 1 \right), \tag{3.1}$$

$$(x)_n \sim (1, f(t) = e^t - 1). \tag{3.2}$$

From (1.15), (2.7), (2.8), (2.12), (3.1), and (3.2), we note that

$$f(t)D_{n,\lambda}(x) = nD_{n-1,\lambda}(x) = (e^t - 1)D_{n,\lambda}(x) = \Delta D_{n,\lambda}(x), \tag{3.3}$$

$$f(t)(x)_n = n(x)_{n-1}, \tag{3.4}$$

$$g(t)D_{n,\lambda}(x) = (x)_n. \tag{3.5}$$

Now, we assume that $p(x) \in \mathbb{C}[x]$ has degree n, and write $p(x) = \sum_{k=0}^{n} a_k D_{k,\lambda}(x)$. Then, from (3.5), we have

$$g(t)p(x) = \sum_{k=0}^{n} a_k g(t) D_{k,\lambda}(x) = \sum_{k=0}^{n} a_k (x)_k. \tag{3.6}$$

For $k \geq 0$, from (3.4) and (3.6), we obtain

$$f(t)^k g(t)p(x) = f(t)^k \sum_{l=0}^{n} a_l (x)_{l,\lambda} = \sum_{l=k}^{n} l(l-1)\cdots(l-k+1) a_l (x)_{l-k,\lambda}. \tag{3.7}$$

Letting $x = 0$ in (3.7), we finally obtain

$$a_k = \frac{1}{k!} f(t)^k g(t) p(x)|_{x=0} = \frac{1}{k!} \langle g(t) f(t)^k | p(x) \rangle, \quad (k \geq 0). \tag{3.8}$$

Now, we want to find more explicit expressions for (3.8). As $\frac{\lambda t}{e^{\lambda t}-1} e^{xt} = \sum_{n=0}^{\infty} \lambda^n B_n\left(\frac{x}{\lambda}\right) \frac{t^n}{n!}$, we see from (2.10) that $\lambda^n B_n\left(\frac{x}{\lambda}\right) = \frac{\lambda t}{e^{\lambda t}-1} x^n$. To proceed further, we let $p(x) = \sum_{i=0}^{n} b_i x^i$.

From (2.7), (2.15), and (3.1), noting that $g(t) = \frac{e^t-1}{t} \frac{\lambda t}{e^{\lambda t}-1}$, we have

$$\begin{aligned} g(t) p(x) &= \frac{e^t - 1}{t} \frac{\lambda t}{e^{\lambda t} - 1} p(x) \\ &= \frac{e^t - 1}{t} \sum_{i=0}^{n} b_i \frac{\lambda t}{e^{\lambda t} - 1} x^i \\ &= \frac{e^t - 1}{t} \sum_{i=0}^{n} b_i \lambda^i B_i\left(\frac{x}{\lambda}\right) \\ &= \frac{e^t - 1}{t} p\left(\lambda \mathbf{B}\left(\frac{x}{\lambda}\right)\right) \\ &= \int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u, \end{aligned} \tag{3.9}$$

where $p\left(\lambda \mathbf{B}\left(\frac{x}{\lambda}\right)\right)$ denotes the umbral composition of $p(x)$ with $\lambda^i B_i\left(\frac{x}{\lambda}\right)$, that is, it is given by $p\left(\lambda \mathbf{B}\left(\frac{x}{\lambda}\right)\right) = \sum_{i=0}^{n} b_i \lambda^i B_i\left(\frac{x}{\lambda}\right)$.

We note from (3.5) and (3.9), in passing, that the following holds:

$$(x)_n = g(t) D_{n,\lambda}(x) = \int_x^{x+1} D_{n,\lambda}\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u.$$

From (2.7) and (3.9), we deduce

$$a_k = \frac{1}{k!} f(t)^k g(t) p(x)|_{x=0} = \frac{1}{k!} \Delta^k \left(\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \right) \Bigg|_{x=0}. \tag{3.10}$$

By making use of (1.17) and (3.10), an alternative expression of (3.10) is given by

$$a_k = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \int_i^{i+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u. \tag{3.11}$$

We obtain yet another expression from (1.18), (3.8), and (3.9), which is given by

$$\begin{aligned} a_k &= \frac{1}{k!} (e^t - 1)^k \left(\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \right) \Bigg|_{x=0} \\ &= \sum_{l=k}^{\infty} S_2(l, k) \frac{t^l}{l!} \left(\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \right) \Bigg|_{x=0} \\ &= \sum_{l=k}^{n} S_2(l, k) \frac{1}{l!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^l \left(\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \right) \Bigg|_{x=0}, \end{aligned} \tag{3.12}$$

where we need to note that $\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u$ has degree n.

Finally, from (3.10)–(3.12), and (3.8), we obtain the following theorem.

Theorem 3.1. *Let* $p(x) \in \mathbb{C}[x]$, *with* $\deg p(x) = n$. *Then, we have* $p(x) = \sum_{k=0}^{n} a_k D_{k,\lambda}(x)$, *where*

$$\begin{aligned} a_k &= \frac{1}{k!} f(t)^k g(t) p(x)|_{x=0} \\ &= \frac{1}{k!} \Delta^k \left(\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \right) \Bigg|_{x=0} \\ &= \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \int_i^{i+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \\ &= \sum_{l=k}^{n} S_2(l,k) \frac{1}{l!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \left(\int_x^{x+1} p\left(\lambda \mathbf{B}\left(\frac{u}{\lambda}\right)\right) \mathrm{d}u \right) \Bigg|_{x=0}, \quad \text{for} \quad k = 0, 1, \ldots, n, \end{aligned}$$

where $g(t) = \frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}$, $f(t) = e^t - 1$, *and* $p\left(\lambda \mathbf{B}\left(\frac{x}{\lambda}\right)\right)$ *denotes the umbral composition of* $p(x)$ *with* $\lambda^i B_i\left(\frac{x}{\lambda}\right)$.

Remark 3.2. Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k D_k(x)$. As λ tends to 0, $g(t) \to \frac{e^t - 1}{t}$, and $p\left(\lambda \mathbf{B}\left(\frac{x}{\lambda}\right)\right) \to p(x)$. Thus, we obtain the following result.

$$\begin{aligned} a_k &= \frac{1}{k!} \Delta^k \left(\int_x^{x+1} p(u) \mathrm{d}u \right) \Bigg|_{x=0} \\ &= \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \int_i^{i+1} p(u) \mathrm{d}u \\ &= \sum_{l=k}^{n} S_2(l,k) \frac{1}{l!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \left(\int_x^{x+1} p(u) \mathrm{d}u \right) \Bigg|_{x=0}, \quad \text{for} \quad k = 0, 1, \ldots, n. \end{aligned}$$

4 Representations by higher-order degenerate Daehee polynomials

Our interest here is to derive formulas expressing any polynomial in terms of the higher-order degenerate Daehee polynomials.

With $g(t) = \frac{\lambda f(t)}{e^{\lambda t} - 1} = \frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}$, $f(t) = e^t - 1$, from (1.11), we note that

$$D_{n,\lambda}^{(r)}(x) \sim (g(t)^r, f(t)), \tag{4.1}$$

$$(x)_n \sim (1, f(t)). \tag{4.2}$$

From (1.15), (2.7), (2.8), (2.12), (4.1), and (4.2), we note that

$$f(t) D_{n,\lambda}^{(r)}(x) = n D_{n-1,\lambda}^{(r)}(x) = (e^t - 1) D_{n,\lambda}^{(r)}(x) = \Delta D_{n,\lambda}^{(r)}(x), \tag{4.3}$$

$$f(t)(x)_n = n(x)_{n-1}, \tag{4.4}$$

$$g(t)^r D_{n,\lambda}^{(r)}(x) = (x)_n. \tag{4.5}$$

Now, we assume that $p(x) \in \mathbb{C}[x]$ has degree n, and write $p(x) = \sum_{k=0}^{n} a_k D_{k,\lambda}^{(r)}(x)$. Then, from (4.5), we have

$$g(t)^r p(x) = \sum_{k=0}^{n} a_k g(t)^r D_{k,\lambda}^{(r)}(x) = \sum_{k=0}^{n} a_k (x)_k. \tag{4.6}$$

For $k \geq 0$, from (4.4), we obtain

$$f(t)^k g(t)^r p(x) = f(t)^k \sum_{l=0}^{n} a_l (x)_l = \sum_{l=k}^{n} l(l-1)\cdots(l-k+1) a_l (x)_{l-k}. \tag{4.7}$$

Letting $x = 0$ in (4.7), we finally obtain

$$a_k = \frac{1}{k!} f(t)^k g(t)^r p(x)|_{x=0} = \frac{1}{k!} \langle g(t)^r f(t)^k | p(x) \rangle, \quad (k \geq 0). \tag{4.8}$$

This also follows from the observation $\langle g(t)^r f(t)^k | D_{l,\lambda}^{(r)}(x) \rangle = l!\ \delta_{l,k}$.

Now, we want to find more explicit expressions for (4.8). As $\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \lambda^n B_n^{(r)}\left(\frac{x}{\lambda}\right) \frac{t^n}{n!}$, we see from (2.10) that $\lambda^n B_n^{(r)}\left(\frac{x}{\lambda}\right) = \left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r x^n$. To proceed further, we let $p(x) = \sum_{i=0}^{n} b_i x^i$.

From (2.7), (2.15), and (4.1), noting that $g(t) = \frac{e^t-1}{t} \frac{\lambda t}{e^{\lambda t}-1}$, we have

$$\begin{aligned} g(t)^r p(x) &= \left(\frac{e^t-1}{t}\right)^r \left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r p(x) \\ &= \left(\frac{e^t-1}{t}\right)^r \sum_{i=0}^{n} b_i \left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r x^i \\ &= \left(\frac{e^t-1}{t}\right)^r \sum_{i=0}^{n} b_i \lambda^i B_i^{(r)}\left(\frac{x}{\lambda}\right) \\ &= \left(\frac{e^t-1}{t}\right)^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right) \\ &= I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right), \end{aligned} \tag{4.9}$$

where $p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)$ denotes the umbral composition of $p(x)$ with $\lambda^i B_i^{(r)}\left(\frac{x}{\lambda}\right)$, that is, it is given by $p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right) = \sum_{i=0}^{n} b_i \lambda^i B_i^{(r)}\left(\frac{x}{\lambda}\right)$, and I denotes the linear integral operator given by $q(x) \to \int_x^{x+1} q(x)\mathrm{d}x$.

We note from (4.5) and (4.9), in passing, that the following holds:

$$(x)_n = g(t)^r D_{n,\lambda}(x) = I^r D_{n,\lambda}\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right).$$

From (2.7) and (4.9), we deduce

$$a_k = \frac{1}{k!} f(t)^k g(t)^r p(x)|_{x=0} = \frac{1}{k!} \Delta^k \left(I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\right)\Bigg|_{x=0}. \tag{4.10}$$

By making use of (1.17) and (4.10), an alternative expression of (3.10) is given by

$$a_k = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\Bigg|_{x=i}. \tag{4.11}$$

We obtain yet another expression from (1.18), (4.8), and (4.9), which is given by

$$a_k = \frac{1}{k!} (e^t - 1)^k \left(I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\right)\Bigg|_{x=0} \tag{4.12}$$

$$= \sum_{l=k}^{\infty} S_2(l,k)\frac{t^l}{l!}\left(I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\right)\Big|_{x=0}$$
$$= \sum_{l=k}^{n} S_2(l,k)\frac{1}{l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l\left(I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\right)\Big|_{x=0},$$

where we need to observe that $I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)$ has degree n.

Finally, from (4.10)–(4.12) and (4.8), we obtain the following theorem.

Theorem 4.1. *Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Then, we have $p(x) = \sum_{k=0}^{n} a_k D_{k,\lambda}^{(r)}(x)$, where*

$$a_k = \frac{1}{k!}f(t)^k g(t)^r p(x)|_{x=0}$$
$$= \frac{1}{k!}\Delta^k\left(I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\right)\Big|_{x=0}$$
$$= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\Big|_{x=i}$$
$$= \sum_{l=k}^{n} S_2(l,k)\frac{1}{l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l\left(I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\right)\Big|_{x=0}, \quad \text{for} \quad k = 0, 1, \ldots, n,$$

where $g(t) = \frac{\lambda(e^t-1)}{e^{\lambda t}-1}$, $f(t) = e^t - 1$, $p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)$ indicates the umbral composition of $p(x)$ with $\lambda^i B_i^{(r)}\left(\frac{x}{\lambda}\right)$, and I denotes the linear integral operator given by $q(x) \to \int_x^{x+1} q(x)\mathrm{d}x$.

We observe that $I^r p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\Big|_{x=i} = \int_i^{i+1} I^{r-1} p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)\mathrm{d}x.$

Remark 4.2. Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k D_k^{(r)}(x)$. As λ tends to 0, $g(t) \to \frac{e^t-1}{t}$, and $p\left(\lambda \mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right) \to p(x)$. Thus, we obtain the following result.

$$a_k = \frac{1}{k!}\Delta^k(I^r p(x))\Big|_{x=0}$$
$$= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} I^r p(x)\Big|_{x=i}$$
$$= \sum_{l=k}^{n} S_2(l,k)\frac{1}{l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l (I^r p(x))\Big|_{x=0}, \quad \text{for} \quad k = 0, 1, \ldots, n.$$

We note that $I^r p(x)|_{x=i} = \int_i^{i+1} I^{r-1} p(x)\mathrm{d}x$.

5 Examples on representation by Daehee polynomials

Here, we illustrate our formulas in Remarks 3.2 and 4.2 with some examples.

(a) Let $p(x) = B_n(x) = \sum_{k=0}^{n} a_k D_k(x)$. Then, as $B_n(x+1) - B_n(x) = nx^{n-1}$, $\int_x^{x+1} B_n(u)\mathrm{d}u = x^n$, from Remark 3.2, we have

$$a_k = \frac{1}{k!}\Delta^k x^n\,|_{x=0} = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} i^n = S_2(n,k), \tag{5.1}$$

which are well known.

Thus, we obtain the following identity:

$$B_n(x) = \sum_{k=0}^{n} S_2(n,k) D_k(x).$$

Next, we let $p(x) = B_n(x) = \sum_{k=0}^{n} a_k D_k^{(r)}(x)$. Then, we first observe that

$$I^r B_n(x) = \frac{1}{(n+r)_r} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} B_{n+r}(x+i). \tag{5.2}$$

Now, by making use of Remark 4.2, we obtain

$$\begin{aligned} a_k &= \frac{1}{k!(n+r)_r} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \Delta^k B_{n+r}(x+i)|_{x=0} \\ &= \frac{1}{(n+r)_r} \sum_{l=k}^{n} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \binom{n+r}{l} S_2(l,k) B_{n+r-l}(i). \end{aligned} \tag{5.3}$$

Thus, we have the following:

$$\begin{aligned} B_n(x) &= \frac{1}{(n+r)_r} \sum_{k=0}^{n} \frac{1}{k!} \left\{ \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \Delta^k B_{n+r}(x+i)|_{x=0} \right\} D_k^{(r)}(x) \\ &= \frac{1}{(n+r)_r} \sum_{k=0}^{n} \left\{ \sum_{l=k}^{n} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \binom{n+r}{l} S_2(l,k) B_{n+r-l}(i) \right\} D_k^{(r)}(x). \end{aligned}$$

(b) Here, we consider $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x)$, $(n \geq 2)$. For this, we first recall from [12] that

$$p(x) = \frac{2}{n} \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} B_m(x) + \frac{2}{n} H_{n-1} B_n(x), \tag{5.4}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number and a slight modification of (5.4) gives the identity in (1.2). Let $p(x) = \sum_{k=0}^{n} a_k D_k(x)$. Then, we have

$$\begin{aligned} a_k &= \frac{2}{n} \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} \sum_{l=k}^{n} S_2(l,k) \binom{m}{l} \delta_{m,l} + \frac{2}{n} H_{n-1} \sum_{l=k}^{n} S_2(l,k) \binom{n}{l} \delta_{n,l} \\ &= \frac{2}{n} \sum_{m=k}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} S_2(m,k) + \frac{2}{n} H_{n-1} S_2(n,k), \end{aligned} \tag{5.5}$$

where we understand that the sum in (5.5) is zero for $k = n-1$ or n. Thus, we obtain the following identity:

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x) = \frac{2}{n} \sum_{k=0}^{n} \left\{ \sum_{m=k}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} S_2(m,k) + H_{n-1} S_2(n,k) \right\} D_k(x).$$

(c) In [12], it is shown that the following identity holds for $n \geq 2$:

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x) E_{n-k}(x) = -\frac{4}{n} \sum_{m=0}^{n} \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} E_{n-m+1} B_m(x), \tag{5.6}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number.

Write $\sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x) E_{n-k}(x) = \sum_{k=0}^{n} a_k D_k(x)$.

By proceeding similarly to (b), we see that

$$\begin{aligned} a_k &= -\frac{4}{n} \sum_{m=0}^{n} \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} E_{n-m+1} \sum_{l=k}^{n} S_2(l,k) \binom{m}{l} \delta_{m,l} \\ &= -\frac{4}{n} \sum_{m=k}^{n} \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} E_{n-m+1} S_2(m,k). \end{aligned} \tag{5.7}$$

Thus, (5.7) implies the next identity:

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x) = -\frac{4}{n}\sum_{k=0}^{n}\left\{\sum_{m=k}^{n}\frac{\binom{n}{m}(H_{n-1}-H_{n-m})}{n-m+1}E_{n-m+1}S_2(m,k)\right\}D_k(x).$$

(d) In [16], it is proved that the following identity is valid for $n \geq 2$:

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)G_{n-k}(x) = -\frac{4}{n}\sum_{m=0}^{n-2}\binom{n}{m}\frac{G_{n-m}}{n-m}B_m(x). \tag{5.8}$$

Again, by proceeding analogously to (b), we can show that

$$a_k = -\frac{4}{n}\sum_{l=k}^{n-2}S_2(l,k)\binom{m}{l}\sum_{m=0}^{n-2}\binom{n}{m}\frac{G_{n-m}}{n-m}\delta_{m,l} = -\frac{4}{n}\sum_{m=k}^{n-2}\binom{n}{m}S_2(m,k)\frac{G_{n-m}}{n-m}. \tag{5.9}$$

Therefore, we obtain the following identity:

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)G_{n-k}(x) = -\frac{4}{n}\sum_{k=0}^{n-2}\left\{\sum_{m=k}^{n-2}\binom{n}{m}S_2(m,k)\frac{G_{n-m}}{n-m}\right\}D_k(x).$$

(e) Nielsen [2,19] also represented products of two Euler polynomials in terms of Bernoulli polynomials as follows:

$$E_m(x)E_n(x) = -2\sum_{r=1}^{m}\binom{m}{r}E_r\frac{B_{m+n-r+1}(x)}{m+n-r+1} - 2\sum_{s=1}^{n}\binom{n}{s}E_s\frac{B_{m+n-s+1}(x)}{m+n-s+1} + 2(-1)^{n+1}\frac{m!\,n!}{(m+n+1)!}E_{m+n+1}. \tag{5.10}$$

In the same way as (b), we can show that

$$\begin{aligned} a_k = {} & 2(-1)^{n+1}\frac{m!\,n!}{(m+n+1)!}E_{m+n+1}\delta_{k,0} - 2\sum_{r=1}^{m}\binom{m}{r}\frac{E_r}{m+n-r+1}S_2(m+n-r+1,k) \\ & - 2\sum_{s=1}^{n}\binom{n}{s}\frac{E_s}{m+n-s+1}S_2(m+n-s+1,k). \end{aligned} \tag{5.11}$$

Thus, we arrive at the next identity:

$$\begin{aligned} E_m(x)E_n(x) = {} & 2(-1)^{n+1}\frac{m!\,n!}{(m+n+1)!}E_{m+n+1} - 2\sum_{k=1}^{m+n}\sum_{r=1}^{m}\binom{m}{r}\frac{E_r}{m+n-r+1}S_2(m+n-r+1,k)D_k(x) \\ & - 2\sum_{k=1}^{m+n}\sum_{s=1}^{n}\binom{n}{s}\frac{E_s}{m+n-s+1}S_2(m+n-s+1,k)D_k(x). \end{aligned}$$

6 Examples on representation by degenerate Daehee polynomials

Here, we illustrate our formulas in Theorems 3.1 and 4.1.

(a) Let $p(x) = B_n(x) = \sum_{k=0}^{n}a_kD_{k,\lambda}(x)$. Then, as $B_n(x) = \sum_{j=0}^{n}\binom{n}{j}B_{n-j}x^j$, we have

$$\int_x^{x+1}B_n\left(\lambda\mathbf{B}\left(\frac{u}{\lambda}\right)\right)\mathrm{d}u = \sum_{j=0}^{n}\binom{n}{j}B_{n-j}\lambda^{j+1}\frac{1}{j+1}\left(B_{j+1}(\frac{x+1}{\lambda}) - B_{j+1}\left(\frac{x}{\lambda}\right)\right). \tag{6.1}$$

Thus, for $1 \leq l \leq n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}\int_{x}^{x+1}B_n\left(\lambda\mathbf{B}\left(\frac{u}{\lambda}\right)\right)\mathrm{d}u=\sum_{j=0}^{n}\binom{n}{j}B_{n-j}\lambda^{j}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-1}\left(B_j\left(\frac{x+1}{\lambda}\right)-B_j\left(\frac{x}{\lambda}\right)\right)$$
$$=\sum_{j=l}^{n}\binom{n}{j}B_{n-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{x+1}{\lambda}\right)-B_{j-l+1}\left(\frac{x}{\lambda}\right)\right). \tag{6.2}$$

Now, from Theorem 3.1, (6.1), and (6.2), we obtain

$$a_k=\frac{1}{k!}\sum_{j=0}^{n}\binom{n}{j}\frac{\lambda^{j+1}}{j+1}B_{n-j}\Delta^k\left(B_{j+1}\left(\frac{x+1}{\lambda}\right)-B_{j+1}\left(\frac{x}{\lambda}\right)\right)\Bigg|_{x=0}$$
$$=\frac{1}{k!}\sum_{i=0}^{k}\sum_{j=0}^{n}(-1)^{k-i}\binom{k}{i}\binom{n}{j}\frac{\lambda^{j+1}}{j+1}B_{n-j}\left(B_{j+1}\left(\frac{i+1}{\lambda}\right)-B_{j+1}\left(\frac{i}{\lambda}\right)\right) \tag{6.3}$$
$$=\sum_{l=k}^{n}\sum_{j=l}^{n}S_2(l,k)\frac{1}{l!}\binom{n}{j}\lambda^{j-l+1}(j)_{l-1}B_{n-j}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right),$$

where we understand that $(j)_{-1}=\frac{1}{j+1}$.

Hence, from (6.3), we obtain the following identity:

$$B_n(x)=\sum_{k=0}^{n}\frac{1}{k!}\left\{\sum_{j=0}^{n}\binom{n}{j}\frac{\lambda^{j+1}}{j+1}B_{n-j}\Delta^k\left(B_{j+1}\left(\frac{x+1}{\lambda}\right)-B_{j+1}\left(\frac{x}{\lambda}\right)\right)\Bigg|_{x=0}\right\}D_{k,\lambda}(x)$$
$$=\sum_{k=0}^{n}\frac{1}{k!}\left\{\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{k-i}\binom{k}{i}\binom{n}{j}\frac{\lambda^{j+1}}{j+1}B_{n-j}\left(B_{j+1}\left(\frac{i+1}{\lambda}\right)-B_{j+1}\left(\frac{i}{\lambda}\right)\right)\right\}D_{k,\lambda}(x)$$
$$=\sum_{k=0}^{n}\left\{\sum_{l=k}^{n}\sum_{j=l}^{n}S_2(l,k)\frac{1}{l!}\binom{n}{j}\lambda^{j-l+1}(j)_{l-1}B_{n-j}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}D_{k,\lambda}(x).$$

Next, we let $p(x)=B_n(x)=\sum_{k=0}^{n}a_kD_{k,\lambda}^{(r)}(x)$. Then, we first observe that

$$I^rB_n\left(\lambda\mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)=I^r\sum_{j=0}^{n}\binom{n}{j}B_{n-j}\lambda^jB_j^{(r)}\left(\frac{x}{\lambda}\right)=\sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{r-i}\binom{n}{j}\binom{r}{i}\frac{\lambda^{j+r}}{(j+r)_r}B_{n-j}B_{j+r}^{(r)}\left(\frac{x+i}{\lambda}\right). \tag{6.4}$$

So, for l with $j+r\geq l$, we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}I^rB_n\left(\lambda\mathbf{B}^{(r)}\left(\frac{x}{\lambda}\right)\right)=\sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{r-i}\binom{n}{j}\binom{r}{i}\frac{\lambda^{j+r}}{(j+r)_r}B_{n-j}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}B_{j+r}^{(r)}\left(\frac{x+i}{\lambda}\right)$$
$$=\sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{r-i}\binom{n}{j}\binom{r}{i}\frac{\lambda^{j+r-l}}{(j+r)_r}B_{n-j}(j+r)_lB_{j+r-l}^{(r)}\left(\frac{x+i}{\lambda}\right). \tag{6.5}$$

Thus, from Theorem 4.1, (6.4), and (6.5), we have

$$B_n(x)=\sum_{k=0}^{n}\frac{1}{k!}\left\{\sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{r-i}\binom{n}{j}\binom{r}{i}\frac{\lambda^{j+r}}{(j+r)_r}B_{n-j}\Delta^kB_{j+r}^{(r)}\left(\frac{x+i}{\lambda}\right)\Bigg|_{x=0}\right\}D_{k,\lambda}^{(r)}(x)$$
$$=\sum_{k=0}^{n}\left\{\sum_{l=k}^{n}S_2(l,k)\frac{1}{l!}\sum_{j=\max\{0,l-r\}}^{n}\sum_{i=0}^{r}(-1)^{r-i}\binom{n}{j}\binom{r}{i}\frac{\lambda^{j+r-l}}{(j+r)_r}B_{n-j}(j+r)_lB_{j+r-l}^{(r)}\left(\frac{i}{\lambda}\right)\right\}D_{k,\lambda}^{(r)}(x).$$

(b) Let $p(x)=\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)B_{n-k}(x)$, for $n\geq 2$. As we stated earlier, it was shown in [12] that

$$p(x)=\frac{2}{n}\sum_{m=0}^{n-2}\frac{1}{n-m}\binom{n}{m}B_{n-m}B_m(x)+\frac{2}{n}H_{n-1}B_n(x), \tag{6.6}$$

where $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$ is the Harmonic number.

Write $p(x) = \sum_{k=0}^{n} a_k D_k(x)$. Then, from Theorem 4.1 and (6.6), we have

$$\begin{aligned} a_k = {} & \frac{2}{n}\left\{\sum_{l=k}^{n-2}\sum_{m=l}^{n-2}\sum_{j=l}^{m} S_2(l,k)\frac{1}{l!}\frac{1}{n-m}\binom{n}{m}\binom{m}{j} B_{n-m}B_{m-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\} \\ & + \frac{2}{n}H_{n-1}\sum_{l=k}^{n}\sum_{j=l}^{n} S_2(l,k)\frac{1}{l!}\binom{n}{j}B_{n-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right). \end{aligned} \tag{6.7}$$

Thus, from (6.7), we obtain

$$\begin{aligned} & \sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)B_{n-k}(x) \\ = {} & \frac{2}{n}\sum_{k=0}^{n}\left\{\sum_{l=k}^{n-2}\sum_{m=l}^{n-2}\sum_{j=l}^{m} S_2(l,k)\frac{1}{l!}\frac{1}{n-m}\binom{n}{m}\binom{m}{j}B_{n-m}B_{m-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right. \\ & \left. + H_{n-1}\sum_{l=k}^{n}\sum_{j=l}^{n} S_2(l,k)\frac{1}{l!}\binom{n}{j}B_{n-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}D_{k,\lambda}(x), \end{aligned}$$

where we understand that the triple sum in the parentheses is zero for $k = n-1$ or $k = n$, and $(j)_{-1} = \frac{1}{j+1}$.

(c) Let $p(x) = \sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x)$, for $n \geq 2$. Then, as we saw earlier, it was proved in [12] that

$$p(x) = -\frac{4}{n}\sum_{m=0}^{n}\frac{\binom{n}{m}(H_{n-1}-H_{n-m})}{n-m+1}E_{n-m+1}B_m(x). \tag{6.8}$$

Write $p(x) = \sum_{k=0}^{n-2} a_k D_{k,\lambda}(x)$. Then, from Theorem 4.1 and (6.8), we can show that

$$a_k = -\frac{4}{n}\left\{\sum_{l=k}^{n}\sum_{m=l}^{n}\sum_{j=l}^{m} S_2(l,k)\frac{1}{l!}\frac{\binom{n}{m}(H_{n-1}-H_{n-m})}{n-m+1}E_{n-m+1}\binom{m}{j}B_{m-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}, \tag{6.9}$$

where we understand that $(j)_{-1} = \frac{1}{j+1}$.

Hence, from (6.9), we have

$$\begin{aligned} & \sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x) \\ = {} & -\frac{4}{n}\sum_{k=0}^{n}\left\{\sum_{l=k}^{n}\sum_{m=l}^{n}\sum_{j=l}^{m} S_2(l,k)\frac{1}{l!}\frac{\binom{n}{m}(H_{n-1}-H_{n-m})}{n-m+1}E_{n-m+1}\binom{m}{j}B_{m-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}D_{k,\lambda}(x). \end{aligned}$$

(d) Here, we consider $p(x) = \sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)G_{n-k}(x)$, for $n \geq 2$. As we mentioned earlier, it was shown in [16] that

$$p(x) = -\frac{4}{n}\sum_{m=0}^{n-2}\binom{n}{m}\frac{G_{n-m}}{n-m}B_m(x). \tag{6.10}$$

Write $p(x) = \sum_{k=0}^{n-2} a_k D_{k,\lambda}(x)$. Then, from Theorem 4.1 and (6.10), we obtain that

$$a_k = -\frac{4}{n}\left\{\sum_{l=k}^{n-2}\sum_{m=l}^{n-2}\sum_{j=l}^{m} S_2(l,k)\frac{1}{l!}\binom{n}{m}\frac{G_{n-m}}{n-m}\binom{m}{j}B_{m-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}, \tag{6.11}$$

where we understand that $(j)_{-1} = \frac{1}{j+1}$.

Thus, from (6.11), we obtain

$$\begin{aligned}&\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)G_{n-k}(x)\\&=-\frac{4}{n}\sum_{k=0}^{n-2}\left\{\sum_{l=k}^{n-2}\sum_{m=l}^{n-2}\sum_{j=l}^{m}S_2(l,k)\frac{1}{l!}\binom{n}{m}\frac{G_{n-m}}{n-m}\binom{m}{j}B_{m-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}D_{k,\lambda}(x).\end{aligned}$$

(e) As we mentioned earlier, it was shown (see [17,18]) that, for positive integers m and n, with $m+n\geq 2$, we have

$$B_m(x)B_n(x)=\sum_r\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}. \tag{6.12}$$

Then, from Theorem 4.1 and (6.12), we can show that

$$\begin{aligned}a_k&=(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}\delta_{k,0}+\sum_r\sum_{l=k}^{m+n}\sum_{j=l}^{m+n-2r}S_2(l,k)\frac{1}{l!}\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}\\&\quad\times\frac{B_{2r}}{m+n-2r}\binom{m+n-2r}{j}B_{m+n-2r-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right).\end{aligned} \tag{6.13}$$

Thus, form (6.13), we obtain

$$\begin{aligned}B_m(x)B_n(x)&=(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}+\sum_{k=0}^{m+n}\left\{\sum_r\sum_{l=k}^{m+n-2r}\sum_{j=l}^{m+n-2r}S_2(l,k)\frac{1}{l!}\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}\right.\\&\quad\left.\times\frac{B_{2r}}{m+n-2r}\binom{m+n-2r}{j}B_{m+n-2r-j}\lambda^{j-l+1}(j)_{l-1}\left(B_{j-l+1}\left(\frac{1}{\lambda}\right)-B_{j-l+1}\right)\right\}D_{k,\lambda}(x),\end{aligned}$$

where $(j)_{-1}=\frac{1}{j+1}$.

7 Conclusion

In this paper, we were interested in representing any polynomial in terms of the degenerate Daehee polynomials and of the higher-order degenerate Daehee polynomials. We were able to derive formulas for such representations with the help of umbral calculus. We showed that, by letting λ tends to zero, they give formulas for representations by the Daehee polynomials and by the higher-order Daehee polynomials. Further, we illustrated the formulas with some examples.

As we mentioned in Section 1, both Faber-Pandharipande-Zagier (FPZ) identity and a variant of Miki's identity follow from the one identity (see (1.2)) that can be derived from the formula (see (1.1)) involving only derivatives and integrals of the given polynomial, while all the other proofs are quite involved. We recall here that the FPZ identity was a conjectural relation between Hodge integrals in Gromov-Witten's theory. It should be stressed that our method is very useful and powerful, even though it is elementary.

It is one of our future research projects to continue to find formulas representing polynomials in terms of some specific special polynomials and to apply those in discovering some interesting identities.

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