REPRESENTATIONS OF CERTAIN COMPACT SEMIGROUPS BY *HL*-SEMIGROUPS

BY

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Abstract. An *HL*-semigroup is defined to be a topological semigroup with the property that the Schützenberger group of each $\mathscr{H}$-class is a Lie group. The following problem is considered: Does a compact semigroup admit enough homomorphisms into *HL*-semigroups to separate points of S; or equivalently, is S isomorphic to a strict projective limit of *HL*-semigroups? An affirmative answer is given in the case that S is an irreducible semigroup. If S is irreducible and separable, it is shown that S admits enough homomorphisms into finite dimensional *HL*-semigroups to separate points of S.

Introduction. A semigroup analog of the theorem of Peter and Weyl is not in existence at the present time. Indeed, such a theory for compact semigroups closely paralleling that for compact groups is generally believed to be unfeasible. In this paper an alternate approach is considered. The alternative is to replace groups of nonsingular complex matrices by compact semigroups with the property that the Schützenberger group of each $\mathscr{H}$-class is a Lie group. Such semigroups are called *HL*-semigroups. The following question is considered: Given a compact semigroup S, do there exist enough homomorphisms of S into *HL*-semigroups to separate the points of S? Furthermore, can these homomorphisms be chosen so as to preserve the $\mathscr{H}$-class structure of S? We follow the current trend and call a homomorphism from S into an *HL*-semigroup T a representation of S by the *HL*-semigroup T.

The main result of this paper is that every irreducible semigroup admits enough $\mathscr{H}$-class separating representations by *HL*-semigroups to separate points in S. Moreover, if $S/\mathscr{H}$ is separable, then each of the *HL*-semigroups may be chosen to have finite dimension. §1 is devoted to preliminary results of a general nature. §2 deals with irreducible semigroups.

For the most part we will use the terminology and notation of [7]. All semigroups, homomorphisms, and isomorphisms will be in the category of compact semigroups and continuous homomorphisms. The authors are indebted to J. D. Lawson, Michael W. Mislove, and Eleanor Bailey for their useful comments and suggestions.

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1. **Representations by *HL*-semigroups.** The definition and properties of the Schützenberger group of an $\mathcal{H}$-class of a semigroup S may be found in [1], [2], or [7, §A.4]. The Schützenberger group of an $\mathcal{H}$-class acts as a topological transformation group on certain subsets of S and as a group of homomorphisms on other subsets. The assumption that the Schützenberger group be a Lie group should be valuable in investigating these actions.

DEFINITION 1.1. A compact semigroup S is an *HL-semigroup* if the Schützenberger group of each $\mathcal{H}$-class of S is a Lie group. If there exist enough representations of S by *HL*-semigroups to separate the points of S then S *has a complete system of representations*.

Certainly, S is an *HL*-semigroup if and only if S^1 is an *HL*-semigroup. Since each Schützenberger group of S is the homomorphic image of some closed subgroup of S^1 [7, A.4.7], it follows from standard results on Lie groups [11] that (i) S is an *HL*-semigroup if and only if (ii) each maximal group of S is a Lie group if and only if (iii) each closed subgroup of S is a Lie group. If ρ_1 and ρ_2 are closed congruences on S and $\rho = \rho_1 \cap \rho_2$, then S/ρ is isomorphic to a subsemigroup of $S/\rho_1 \times S/\rho_2$. Therefore, standard methods yield that a necessary and sufficient condition for S to admit a complete system of representations is that S be the strict projective limit of *HL*-semigroups. It is clear that if S has a complete system $\{\pi_i \colon S \to S_i\}_{i \in J}$ of representations we may assume if we like that each π_i is a surmorphism and that the system is directed in the sense that given any $i, j \in J$, there exists $k \in J$ with $k \geq i, j$ such that the following diagram may be completed:
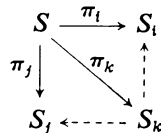
Equivalently, if ρ_i is the congruence on S induced by π_i, we may assume that the congruences $\{\rho_i\}_{i \in J}$ are directed by set inclusion.

1.2. We now assume that S is a compact semigroup with a complete system of representations $\{\pi_i \colon S \to S_i\}_{i \in J}$ with the property that π_i is a surmorphism for each $i \in J$. Let $f \colon S \to T$ be a surmorphism onto a compact semigroup T. We would like to determine if there is a complete system of representations for T. Let ρ_i, ρ denote the closed congruences on S induced by π_i, f, respectively, and let σ_i be the smallest closed congruence on S which contains both ρ_i and ρ. Let $T_i = S/\sigma_i$ and $\phi_i \colon S \to T_i$ be the natural homomorphism. Now define $\tau_i \colon T \to T_i$ by

$$\tau_i(f(x)) = \phi_i(x), \quad \text{where } x \in S.$$

It is clear that τ_i is a well-defined homomorphism onto T_i. Finally define $f_i \colon S_i \to T_i$ by

$$f_i(\pi_i(x)) = \phi_i(x), \quad \text{where } x \in S,$$

and note that f_i is a well-defined homomorphism onto T_i. We now have the following commutative diagram:

$$
\begin{array}{ccc}
S_i & \xrightarrow{\;f_i\;} & T_i \\
{\scriptstyle \pi_i}\Big\uparrow & {\scriptstyle \phi_i}\nearrow & \Big\uparrow{\scriptstyle \tau_i} \\
S & \xrightarrow[\;f\;]{} & T
\end{array}
$$

(1)

Each maximal group H in T_i is the homomorphic image under f_i of a closed subgroup in S_i, namely any maximal group in the minimal ideal of $f_i^{-1}(H)$, and hence is a Lie group. Consequently, T_i is an *HL*-semigroup.

PROPOSITION 1.3. *Using the notation of 1.2, the collection $\{\tau_i : T \to T_i\}$ is a complete system of representations for T if and only if $\bigcap_J \sigma_i = \rho$.*

Proof. Suppose $\bigcap_J \sigma_i = \rho$, and for $x, y \in S$ suppose $\tau_i(f(x)) = \tau_i(f(y))$ for every $i \in J$. Then $\phi_i(x) = \phi_i(y)$ for every $i \in J$ and hence $(x, y) \in \bigcap_J \sigma_i = \rho$. Then $f(x) = f(y)$, and since f is a surmorphism, it follows that the family $\{\tau_i\}$ separates points of T.

Now suppose that $\{\tau_i\}$ is a complete system of representations for T and let $(x, y) \in \bigcap_J \sigma_i$. Then $\tau_i(f(x)) = \tau_i(f(y))$ for every $i \in J$ and it must follow that $f(x) = f(y)$, that is, $(x, y) \in \rho$.

1.4. In the situation described in 1.2, it is not always the case that $\bigcap_J \sigma_i = \rho$, even if the ρ_i's are assumed to be descending. This is demonstrated by the following example due to Eleanor Bailey.

Let S denote the usual unit interval with multiplication defined by $xy = \min \{x, y\}$. It can be shown that any closed equivalence relation on S with connected equivalence classes is a closed congruence. Let ρ be the closed congruence on S whose classes are the closed intervals [1/3, 2/3], [1/9, 2/9], [7/9, 8/9], etc., (that is, those intervals used in the construction of the ternary Cantor set), and singleton sets $\{x\}$ if x is not in one of these intervals. Let ρ_1 be the closed congruence whose classes are the closed intervals [0, 1/3], [2/3, 1], and singletons $\{x\}$ if $1/3 < x < 2/3$. Let ρ_2 be the closed congruence whose classes are [0, 1/9], [2/9, 1/3], [2/3, 7/9], [8/9, 1], and singletons $\{x\}$ if x is not in one of these intervals. In general, ρ_i is the closed congruence whose classes are the closed intervals which are the components of the complement of the union of those intervals used in the ith stage of the construction of the ternary Cantor set and singletons $\{x\}$ if x is not in this complement. Then the collection $\{\rho_i\}_{i=1}^{\infty}$ is directed under $\supset$, and $\bigcap \rho_i = \Delta$ where Δ denotes the diagonal of $S \times S$. Hence, the family of natural maps $\{\pi_i : S \to S/\rho_i\}$ is a complete system of representations for S (of course, S is itself an *HL*-semigroup, but for the purposes of this example, this is irrelevant). However, it is easy to see that the smallest closed congruence σ_i containing $\rho_i \cup \rho$ is $S \times S$ for each i. Hence $\bigcap \sigma_i = S \times S \neq \rho$.

We observe that in the preceding example, the natural homomorphism $f : S \to S/\rho$ does not separate $\mathcal{H}$-classes of S (which, in this case, are the points

of S). It will be shown, in fact, that it is the lack of this $\mathcal{H}$-class separating property which makes the above counterexample possible.

DEFINITION 1.5. Let S be a compact semigroup, $f: S \to T$ a homomorphism, and ρ the closed congruence on S induced by f. If $\rho \subset \mathcal{H}$, then f is said to be $\mathcal{H}$-*class separating*.

If S is a compact semigroup and $f: S \to T$ is a homomorphism, then f can be extended to a homomorphism from S^1 to T^1 by defining (if necessary) $f(1)$ to be the identity element of T^1. Whenever necessary it will be assumed that this extension has been made.

The symbol $\leqq$ will be used to denote the $\mathcal{H}$-ordering on a semigroup S, that is $x \leqq y$ if and only if $x \in yS^1 \cap S^1y$. The semigroup S is totally $\mathcal{H}$-ordered if for $x, y \in S$, then either $x \leqq y$ or $y \leqq x$. If S is a compact totally $\mathcal{H}$-ordered semigroup then it follows from Theorem A. 3.20 of [7] that $\mathcal{H}$ is a congruence.

PROPOSITION 1.6. *Let S be a compact totally $\mathcal{H}$-ordered semigroup and let f be an $\mathcal{H}$-class separating homomorphism from S onto T. Let π be a surmorphism from S onto S' and let ρ and ρ_π denote the congruences on S induced by f and π, respectively. Then*

$$\mathscr{C} = \{(x, y) \in S \times S \mid (\pi(x), \pi(y)) \in (\pi \times \pi)\rho\}$$

is the smallest closed congruence on S containing both ρ and ρ_π.

Proof. It is easy to see that $\mathscr{C}$ is a closed, reflexive subset of $S \times S$ satisfying $\mathscr{C}\Delta \cup \Delta\mathscr{C} \subseteq \mathscr{C}$. We will first show that $\mathscr{C}$ is transitive and hence a closed congruence on S.

Let (x, y), $(y, z) \in \mathscr{C}$. Then there exist elements (x', y'), $(y'', z') \in \rho$ such that

$$(\pi(x), \pi(y)) = (\pi(x'), \pi(y')) \quad \text{and} \quad (\pi(y), \pi(z)) = (\pi(y''), \pi(z')).$$

Since S is totally $\mathcal{H}$-ordered, we may assume that $y' \in S^1y''$, so there exists an element $w \in S^1$ such that $y' = wy''$. Since f is $\mathcal{H}$-class separating and $(y'', z') \in \rho$, then $(y'', z') \in \mathcal{H}$, and $z' = y''u$ for some $u \in S^1$. Then

(2)
$$\pi(wz') = \pi(wy''u) = \pi(y'u) = \pi(y')\pi(u)$$
$$= \pi(y'')\pi(u) = \pi(y''u) = \pi(z') = \pi(z).$$

Also, $(y'', z') \in \rho$ implies that

$$(wy'', wz') = (y', wz') \in \rho.$$

But $(x', y') \in \rho$ and so the transitivity of ρ gives $(x', wz') \in \rho$. This together with (2) gives

$$(\pi(x), \pi(z)) = (\pi(x'), \pi(wz')),$$

and $(x', wz') \in \rho$. Therefore $(x, z) \in \mathscr{C}$ which establishes the transitivity of $\mathscr{C}$.

Now let $(x, y) \in \mathscr{C}$. We will show that $(x, y) \in \sigma$ where σ denotes the smallest closed congruence on S containing ρ and ρ_π. Let $(x', y') \in \rho$ such that

$$(\pi(x), \pi(y)) = (\pi(x'), \pi(y')).$$

Then (x, x'), $(y, y') \in \rho_\pi$; and $(x', y') \in \rho$, so (x, x'), (y, y'), and (x', y') are all in σ. It follows from the symmetry and transitivity of σ that $(x, y) \in \sigma$.

LEMMA 1.7. *Let S be a compact totally $\mathscr{H}$-ordered semigroup and let $f\colon S \to T$ be an $\mathscr{H}$-class separating surmorphism. Then if S has a complete system of ($\mathscr{H}$-class separating) representations, T also has a complete system of ($\mathscr{H}$-class separating) representations.*

Proof. We will use the notation of 1.2, assume that the ρ_i's are descending, and show that $\bigcap \sigma_i = \rho$. The theorem will then follow by Proposition 1.3. Let $i \in J$ and, as in Proposition 1.6, define a subset $\mathscr{C}_i$ of $S \times S$ as follows:

$$\mathscr{C}_i = \{(x, y) \mid (\pi_i(x), \pi_i(y)) \in (\pi_i \times \pi_i)(\rho)\}.$$

Then, according to Proposition 1.6, $\mathscr{C}_i = \sigma_i$ for each i. We now proceed to show that $\bigcap \sigma_i = \rho$. Let $(x, y) \in \bigcap \sigma_i = \bigcap \mathscr{C}_i$. Then for each $i \in J$, $(\pi_i(x), \pi_i(y)) \in (\pi_i \times \pi_i)(\rho)$, so that the set

$$A_i = \{(z, w) \in \rho \mid (\pi_i(x), \pi_i(y)) = (\pi_i(z), \pi_i(w))\}$$

is nonempty. It is also easy to show that A_i is closed, and that if $\rho_i \subset \rho_j$, then $A_i \subset A_j$. Then, since $\{\rho_i\}_{i \in J}$ is directed under $\supset$, it follows that $\{A_i\}_{i \in J}$ is a descending family of nonempty compact subsets of ρ, and therefore $\bigcap_{i \in J} A_i \neq \square$. Let $(z, w) \in \bigcap A_i$. Then $(x, z) \in \bigcap \rho_i$ and $(y, w) \in \bigcap \rho_i$, but $\bigcap \rho_i = \Delta_S$; consequently $(x, y) = (z, w) \in \rho$. This concludes the proof of the first part of the theorem.

Suppose now that each of the representations π_i ($i \in J$), is $\mathscr{H}$-class separating. To show that each $\tau_i\colon T \to T_i$ (see diagram (1)) is $\mathscr{H}$-class separating, suppose $\tau_i(f(x)) = \tau_i(f(y))$. Then $(x, y) \in \sigma_i$, and so there exists $(x', y') \in \rho$ such that

$$(\pi_i(x), \pi_i(y)) = (\pi_i(x'), \pi_i(y')).$$

Since both π_i and f are $\mathscr{H}$-class separating, each of (x, x'), (x', y'), and (y', y) are in $\mathscr{H}$. Hence $(x, y) \in \mathscr{H}$, and so $f(x)$ and $f(y)$ are in the same $\mathscr{H}$-class of T. This concludes the proof.

DEFINITION 1.8. Let S be a compact semigroup and $f\colon S \to T$ a homomorphism. We will say that f is a *G-homomorphism* if $t \in T$ and $f^{-1}(t)$ nondegenerate imply the $\mathscr{H}$-class of t in T is a group.

If f is a G-homomorphism, then f is one-to-one on the set $S - f^{-1}(\mathscr{G})$ where $\mathscr{G}$ is the union of the subgroups of T.

LEMMA 1.9. *Let S be a compact totally $\mathscr{H}$-ordered semigroup and $f\colon S \to T$ a surmorphism such that f is a G-homomorphism. If S has a complete system of $\mathscr{H}$-class separating representations then so does T.*

Proof. We will again use the notation of 1.2, assume that the ρ_i's are descending, and show that $\bigcap \sigma_i = \rho$. The reader will find it useful to refer to diagram (1) in

1.2. Let $i \in J$. Since π_i is $\mathscr{H}$-class separating, we may apply Proposition 1.6 and conclude that

$$\sigma_i = \{(x, y) \mid (f(x), f(y)) \in (f \times f)(\rho_i)\}.$$

Let $(x, y) \in \sigma_i$. Then there are elements $x', y' \in S$ such that $(x', y') \in \rho_i$ and $(f(x), f(y)) = (f(x'), f(y'))$. We consider two cases.

Case (1). $(x, y) = (x', y')$. Then $(x, y) \in \rho_i$. We restrict our attention now to

Case (2). $(x, y) \neq (x', y')$; suppose $x \neq x'$. Then $f^{-1}(f(x))$ is nondegenerate and since f is a G-homomorphism, the $\mathscr{H}$-class $H(f(x))$ of $f(x)$ is a group. Since $(x', y') \in \rho_i$ and $\rho_i \subset \mathscr{H}$, then $(x', y') \in \mathscr{H}$ and it may be shown easily that $(f(x'), f(y')) \in \mathscr{H}$; so $H(f(x)) = H(f(y))$. Let $B = f^{-1}(H(f(x)))$.

Then B is a compact subsemigroup of S. Since S is totally $\mathscr{H}$-ordered, it is known that the minimal ideal $M(B)$ of B is a closed subgroup of B, (in fact, $M(B)$ is an $\mathscr{H}$-class of S). Let $M = M(B)$ and let e denote the identity of M. Now $\pi_i(M)$ is a closed subgroup of S_i with identity $\pi_i(e)$. Let

$$K(e) = \ker(f|M), \quad \text{and} \quad K_i(e) = \ker(\pi_i|M).$$

Since M is an ideal of B, $eB \subset M$. So, since $f(x) = f(x')$, we have $f(ex) = f(ex')$ and consequently $(ex)(ex')^{-1} \in K(e)$. Similarly, $(ey')(ey)^{-1} \in K(e)$. But $\pi_i(x') = \pi_i(y')$, so we have $(ex')(ey')^{-1} \in K_i(e)$. We conclude then that

$$(ex)(ey)^{-1} = (ex)(ex')^{-1}(ex')(ey')^{-1}(ey')(ey)^{-1} \in K(e)K_i(e)K(e).$$

But $K(e)$, $K_i(e)$ are normal subgroups of M, so

$$(ex)(ey)^{-1} \in K_i(e)K(e).$$

Reviewing the results of the preceding we have that for $(x, y) \in \sigma_i$, then either $(x, y) \in \rho_i$ (Case (1)), or there exists an idempotent $e \in S$ and closed normal subgroups $K_i(e)$ and $K(e)$ of $H(e)$ such that $(ex)(ey)^{-1} \in K_i(e)K(e)$, (Case (2)). (Note that e is independent of $i \in J$.) Let $(x, y) \in \bigcap \sigma_i$. If $(x, y) \in \rho_i$ for every $i \in J$, then $x = y$, and clearly $(x, y) \in \rho$. If $(x, y) \notin \rho_j$ for some $j \in J$, then for $\rho_i \subset \rho_j$, $(x, y) \notin \rho_i$, and so, since $(x, y) \in \bigcap \sigma_i$, we must have that Case (2) holds for all i such that $\rho_i \subset \rho_j$. There is no loss of generality then in assuming that Case (2) holds for all $i \in J$. It is easy to show that $\{K_i(e)\}_{i \in J}$ is a descending family of closed normal subgroups of $H(e)$ ($= M$), and that $\bigcap_{i \in J} K_i(e) = \{e\}$. Hence we conclude that

$$(ex)(ey)^{-1} \in K(e), \quad \text{or} \quad f(ex) = f(ey).$$

But $f(ex) = f(e)f(x) = f(x)$, since $f(e)$ is the identity of $H(f(x))$, and similarly $f(ey) = f(y)$. It follows that $(x, y) \in \rho$ and therefore $\bigcap \sigma_i = \rho$.

It now follows that the family of homomorphisms $\{\tau_i \colon T \to T_i\}$ constructed in 1.2 is a complete system of representations for T. It remains to be shown that each τ_i is $\mathscr{H}$-class separating. Let $i \in J$ and suppose $\tau_i(t) = \tau_i(s)$. Let $x, y \in S$ such that

$t = f(x)$ and $s = f(y)$. Then $(x, y) \in \sigma_i$ and according to Proposition 1.6, there exists $(x', y') \in \rho_i$ such that

$$(f(x'), f(y')) = (f(x), f(y)) = (t, s).$$

But $\rho_i \subset \mathscr{H}$, so $H(x') = H(y')$ and hence $H(t) = H(f(x')) = H(f(y')) = H(s)$. This concludes the proof.

LEMMA 1.10. *Let S be a compact totally $\mathscr{H}$-ordered semigroup and let $f: S \to T$ be a homomorphism. Then there exists a compact semigroup R and homomorphisms $h: S \to R, g: R \to T$ such that* (i) *h is $\mathscr{H}$-class separating;* (ii) *g is a G-homomorphism; and* (iii) *$f = gh$.*

Proof. As previously observed, $\mathscr{H}$ is a congruence on S. Let ρ be the closed congruence on S induced by f, and let $R = S/(\rho \cap \mathscr{H})$ with $h: S \to R$ the natural map. Define $g: R \to T$ by $g(h(x)) = f(x)$. It is clear that g is a well-defined homomorphism. That (i) and (iii) of the theorem are satisfied is obvious. We must show that g is a G-homomorphism.

Suppose that $g(h(x)) = g(h(y))$ where $x, y \in S$ and $h(x) \neq h(y)$; then $f(x) = f(y)$. Since S is totally $\mathscr{H}$-ordered, either $x \leq y$ or $y \leq x$. Without loss of generality we assume $y \leq x$. Let $z_1, z_2 \in S^1$ such that $y = xz_1 = z_2x$. Then since $f(x) = f(y), f(yz_1) = f(xz_1) = f(y)$, and therefore if n is any positive integer, $f(yz_1^n) = f(y)$. There is an idempotent e in S such that e is a cluster point of the sequence $\{z_1^n\}$, and hence $f(ye) = f(y)$. Then

$$(3) \qquad f(x) = f(y) = f(ye) = f(y)f(e) = f(x)f(e).$$

Now either $e \leq x$ or $x \leq e$. But if $x \in S^1e$, then $xe = x$ and it follows from the fact that $\mathscr{H}$ is a congruence that

$$xH(e) \subset H(x).$$

It is known that $z_1e \in H(e)$ so that

$$ye = xz_1e \in xH(e) \subset H(x).$$

But since $y = z_2x$, then

$$ye = z_2xe = z_2x = y,$$

and we conclude that $y \in H(x)$. But then, since $f(x) = f(y)$, we would have that $(x, y) \in \rho \cap \mathscr{H}$, so that $h(x) = h(y)$ contrary to our assumption. It follows that $x \notin S^1e$ and so we must have $e \leq x$; hence $f(e) \leq f(x)$.

Now, $f(e) \leq f(x)$ implies $T^1f(e) \subset T^1f(x)$ which in turn implies $T^1f(e) \subset T^1f(x)f(e)$. The reverse inclusion, $T^1f(x)f(e) \subset T^1f(e)$ is obvious and hence, by Theorem A.3.20 of [7], $(f(e), f(x)f(e)) \in \mathscr{H}$. Finally, by (3), $f(x)f(e) = f(x)$ and it follows that $(f(e), f(x)) \in \mathscr{H}$. Therefore, $H(f(x))$ is a group and g is a G-homomorphism.

THEOREM 1.11. *Let S be a compact totally $\mathscr{H}$-ordered semigroup and let $f: S \to T$ be a surmorphism. If S has a complete system of $\mathscr{H}$-class separating representations, then so does T.*

Proof. The proof follows easily from Lemmas 1.7, 1.9, and 1.10, and the fact that a homomorphic image of a totally $\mathcal{H}$-ordered semigroup is totally $\mathcal{H}$-ordered.

THEOREM 1.12. *Let S be a compact semigroup with the property that $\mathcal{H}$ is a congruence, let G be a compact group, and let $f: S \to G$ be a homomorphism such that f restricted to any $\mathcal{H}$-class of S is an injection. Then S has a complete system of $\mathcal{H}$-class separating representations $\{\pi_i: S \to S_i\}_{i \in J}$ with the property that for each i, S_i is a subsemigroup of $S/\mathcal{H} \times G_i$ for some compact Lie group G_i.*

Proof. Let $\{\alpha_i: G \to G_i\}_{i \in J}$ be a complete system of representations for G where each G_i is a Lie group. For $i \in J$, define $\sigma_i \subset S \times S$ by

$$\sigma_i = \{(x, y) \mid (x, y) \in \mathcal{H} \text{ and } \alpha_i f(x) = \alpha_i f(y)\}.$$

It is easy to see that σ_i is a closed congruence on S. Let $S_i = S/\sigma_i$ and $\pi_i: S \to S_i$ be the natural homomorphism. Now let $x, y \in S$ with $x \neq y$. If $(x, y) \notin \mathcal{H}$, then $\pi_i(x) \neq \pi_i(y)$ for all $i \in J$. Suppose then that $(x, y) \in \mathcal{H}$. Then since f is one-to-one on the $\mathcal{H}$-classes of S, $f(x) \neq f(y)$, and there exists $i \in J$ such that $\alpha_i f(x) \neq \alpha_i f(y)$. Hence $\pi_i(x) \neq \pi_i(y)$ and therefore the system of homomorphisms $\{\pi_i\}_{i \in J}$ separates points of S. It is also clear from the definition of π_i that π_i is $\mathcal{H}$-class separating. We now show that each S_i is an HL-semigroup. Let H be a closed subgroup of S_i, and define $f_i: H \to G_i$ by

$$f_i(\pi_i(x)) = \alpha_i f(x), \qquad (x \in S, \pi_i(x) \in H).$$

Suppose that for $\pi_i(x), \pi_i(y) \in H$, we have $f_i \pi_i(x) = f_i \pi_i(y)$. Since $\pi_i(x)$ and $\pi_i(y)$ are in the same $\mathcal{H}$-class of S_i and π_i is $\mathcal{H}$-class separating, it follows that $(x, y) \in \mathcal{H}$. But also $\alpha_i f(x) = \alpha_i f(y)$, and so $(x, y) \in \sigma_i$; hence $\pi_i(x) = \pi_i(y)$. It follows that f_i is an injection, and since G_i is a Lie group, so is H. Hence, S_i is an HL-semigroup.

Since $\mathcal{H}$ is a congruence, $S/\mathcal{H}$ is a semigroup. Let $\phi: S \to S/\mathcal{H}$ be the natural map. Define $g_i: S_i \to S/\mathcal{H} \times G_i$ by

$$(4) \qquad\qquad g_i(\pi_i(x)) = (\phi(x), \alpha_i f(x)) \qquad (x \in S).$$

Then g_i is a well-defined homomorphism. Suppose that $g_i(\pi_i(x)) = g_i(\pi_i(y))$. Then $\phi(x) = \phi(y)$, so that $(x, y) \in \mathcal{H}$. But $\alpha_i f(x) = \alpha_i f(y)$ implies that $(x, y) \in \sigma_i$, so $\pi_i(x) = \pi_i(y)$. Hence g_i is an injection, and the proof is complete.

COROLLARY 1.13. *Let everything be as in 1.12, and suppose that $S/\mathcal{H}$ is finite dimensional. Then each of the HL-semigroups S_i $(i \in J)$ is finite dimensional.*

Proof. Since each of the compact Lie groups in question may be considered as a group of matrices over the complex numbers, it is finite dimensional relative to either inductive or cohomological dimension. The result follows from [10] for inductive dimension and from [6] for cohomological dimension.

COROLLARY 1.14. *Let everything be as in 1.12 and assume moreover that G is abelian. Then there are enough $\mathcal{H}$-class separating representations into the HL-*

semigroup $S/\mathscr{H} \times R/Z$ to separate points (where R/Z denotes the circle group of complex numbers having modulus 1).

Proof. This follows immediately from [11, §37] and Theorem 1.12.

2. Irreducible semigroups.

DEFINITION 2.1. A compact connected semigroup S with an identity element 1 is *irreducible* if S contains no proper compact connected semigroup which contains 1 and meets the minimal ideal of S.

The concept of irreducible semigroups was formulated by Hunter and Rothman in [9], and important results were obtained by Hunter and Rothman in [9] and by Hunter in [8] using the assumption that the semigroups were either normal or abelian. It was later proved by Hofmann and Mostert, [7], that an irreducible semigroup must be abelian. It was also in [7] that the structure theory of irreducible semigroups was brought to its present, rather advanced state. Irreducible semigroups play a central role in the study of compact connected semigroups with an identity.

For the remainder of this section, S will denote an irreducible semigroup. We have already noted that S is abelian. It is also known that S is totally $\mathscr{H}$-ordered; in fact, it was shown in [9] (under the assumption that S is normal), and by different methods in [7, p. 143], that $S/\mathscr{H}$ is an I-semigroup. In addition, we will need the following result:

2.2. *If X is a compact totally ordered semilattice then there is an irreducible semigroup* Irr (X) *with the following properties*:

(i) *If e is the idempotent of $M(\mathrm{Irr}\,(X))$ and g is the Clifford-Miller endomorphism defined on* Irr (X) *by $g(x)=ex$, then g restricted to any $\mathscr{H}$-class of* Irr (X) *is an injection into the compact abelian group $M(\mathrm{Irr}\,(X))$.*

(ii) *If S is any irreducible semigroup with the property that the set of idempotents $E(S)$ of S is isomorphic to X, then there exists a homomorphism from* Irr (X) *onto S.*

Part (ii) of this result is stated on p. 143 of [7] but the proof given there is incorrect. However, Michael W. Mislove has discovered proofs for both parts and they will be submitted for publication in the near future.

LEMMA 2.3. *If X is a compact totally ordered semilattice then* Irr (X) *has a complete system of $\mathscr{H}$-class separating representations*

$$\{\pi_i \colon \mathrm{Irr}\,(X) \to \mathrm{Irr}\,(X)/\mathscr{H} \times R/Z\}_{i \in J}.$$

The space Irr $(X)/\mathscr{H} \times R/Z$ *has codimension two and hence there are enough two-dimensional $\mathscr{H}$-class separating representations of* Irr (X) *to separate points.*

Proof. The first part follows from Theorem 1.12, Corollary 1.14 and 2.2(i). The second part follows from [6, Theorem 6.5].

THEOREM 2.4. *Every irreducible semigroup S has a complete system of $\mathscr{H}$-class separating representations $\{\Psi_i\colon S \to S_i\}_{i \in J}$. Moreover, if $S/\mathscr{H}$ is separable, then each S_i may be chosen to have finite inductive dimension*([1]).

Proof. According to Lemma 2.3, if $X = E(S)$, there is a complete system of $\mathscr{H}$-class separating representations for Irr (X). Hence, by 2.2(ii) and Theorem 1.12, S has a complete system of $\mathscr{H}$-class separating representations. If $S/\mathscr{H}$ is separable, then it follows from Lemma 2.3, Theorem 1 of [3] and standard results in dimension theory [10] that each S_i may be chosen to have finite inductive dimension.

COROLLARY 2.5. *Any irreducible semigroup S is the strict projective limit of HL-semigroups. If $S/\mathscr{H}$ is separable, then each of the HL-semigroups may be chosen to have finite inductive dimension.*

COROLLARY 2.6. *Every cylindrical semigroup [7, B.2] has a complete system of $\mathscr{H}$-class separating representations.*

Proof. It is obvious that $\Sigma \times G$ is totally $\mathscr{H}$-ordered if Σ is the universal compact solenoidal semigroup and G is a compact group. According to Theorem 2.4, Σ admits a complete system of $\mathscr{H}$-class separating representations and it follows easily that $\Sigma \times G$ does also. The result now follows from Theorem 1.11.

3. **Remarks.** The previous section classified cylindrical semigroups and irreducible semigroups among those semigroups which have a complete system of representations by *HL*-semigroups. A natural question presents itself: what other compact semigroups have a complete system of representations by *HL*-semigroups? Also, it may seem desirable to require that these representations be $\mathscr{H}$-class separating. It is rather easy to show, for example, that a compact completely simple semigroup has a complete system of $\mathscr{H}$-class separating representations. Other likely candidates, for which the techniques of this paper may be applicable, are (abelian) hormi, [7, §B-5], and (abelian) Clifford semigroups.

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