REPRESENTATIONS OF HECKE ALGEBRAS OF FINITE GROUPS WITH BN-PAIRS OF CLASSICAL TYPE

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ABSTRACT

Let G be a finite group with BN-pair and Coxeter system (W, R). Let A be the generic ring corresponding to (W, R) in the sense of Tits, defined over the polynomial ring $D = Q[u_r, r \in R]$. Let k be any field of characteristic zero. For the homomorphism $\phi : D \longrightarrow k$ defined by $\phi(u_r) = q_r$, q_r the index parameters of G, the specialzed algebra $A_{\phi,k}$ is isomorphic to the Hecke algebra $H_k(G, B)$ of G with respect to a Borel subgroup B of G, while for the specialization defined by $\phi(u_r) = 1$, $r \in R$, $A_{\phi,k}$ is isomorphic to the group algebra kW. As the Hecke algebra $H_k(G, B)$ affords the induced representation 1_B^G, the irreducible representations of G appearing in 1_B^G can be obtained from the representations of $H_k(G, B)$.

In this thesis, we obtain all the irreducible representations, defined over the quotient field of D, of the generic ring corresponding to a Coxeter system of classical type. The method employed involves a generalization of Young's construction of the semi-normal matrix representations of the symmetric group.

We also obtain an explicit formula for the generic degree of these representations in terms of the hook lengths of Young diagrams. Thus the degrees of all the irreducible constituents of 1_B^G are obtained for the families of Chevalley groups $A_\ell(q)$, $B_\ell(q)$, $A^1_{2\ell}(q^2)$, $A^1_{2\ell-1}(q^2)$, $D^1_\ell(q^2)$ and for $D_\ell(q)$, ℓ odd. Also, most of the degrees are obtained for $D_\ell(q)$, ℓ even.

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INTRODUCTION

The results in this thesis are concerned with the irreducible complex representations of a finite group G with BN-pair and Weyl group W of classical type, which appear in the induced permutation representation 1_B^G from a Borel subgroup B of G. These representations were constructed by Steinberg in [16] for $GL(n, q)$, parametrized by partition of n and he showed an elegant formula to hold for their degrees in terms of the hook lengths of a Young diagram. Some special representations of the generic ring of a Coxeter system and the degrees of the corresponding irreducible constituents of 1_B^G were also obtained by Kilmoyer (cf. [6]).

In this thesis we construct explicitly all the irreducible representations of the generic ring corresponding to a Coxeter system of classical type, i.e. of type A_n, B_n $(n \geq 2)$ and D_n $(n \geq 4)$, which specialize to irreducible representations of the Hecke algebra $H_C(G, B)$ affording the induced representation 1_B^G. The method employed involves a generalization of Young's construction of the semi-normal matrix representations of the symmetric group. This construction also enables us to compute the degrees of the irreducible constituents of 1_B^G. In the case that the Weyl group of G is of type B_n, we obtain a formula for the degrees determined by the hook lengths of pairs of Young diagrams comparable to Steinberg's formula for $GL(n, q)$. Indeed, Steinberg's formula is recovered as a special case.

Here is a survey of the contents of this thesis. Chapter 1 contains the necessary preliminaries about the representation theory of the symmetric group. In Chapter 2 the generic ring corresponding to a Coxeter system is introduced. The irreducible representations of the generic ring $\mathfrak{A}(B_n)$ of a Coxeter system of type B_n, defined over the polynomial ring $Q[x, y]$ are constructed in Section 2.2. They are parametrized by pairs of partitions (α), (β) with $|\alpha| + |\beta| = n$ and are rational representations, i.e. they are defined over $K = Q(x, y)$. The representations of the generic rings $\mathfrak{A}(A_n)$ and $\mathfrak{A}(D_n)$, defined over $Q[x]$, corresponding to Coxeter systems of type A_n and D_n are obtained in Section 2.3 as corollaries by considering appropriate specialized algebras of $\mathfrak{A}(B_n)$. For the specialization $x \longrightarrow 1$, the representation of $\mathfrak{A}(A_n)$ obtained specialize to give the semi-normal matrix representations of the symmetric group.

Chapter 3 is concerned with the degrees of the irreducible constituents of 1_B^G. It is first shown in Section 3.1 that the representations obtained are of parabolic type, i.e. they appear with multiplicity one in some permutation representation 1_P^G, where P is a parabolic subgroup of G. The generic degree d_χ of an irreducible character χ of the generic algebra is introduced, such that the degree of the corresponding irreducible character of G is obtained by specializing d_χ. In Section 3.2 an interesting induction formula is derived for d_χ, where χ is an irreducible character of $\mathfrak{A}(B_n)$. This formula, and lengthy computations, enables us to prove in Section 3.4 an explicit formula for d_χ as a rational function in the indeterminates x and y, in terms of the hook lengths of pairs of Young diagrams. The generic degrees of all the irreducible characters of $\mathfrak{A}(A_n)$ and almost all the irreducible characters of $\mathfrak{A}(D_n)$ are obtained as corollary.

CHAPTER 1

PRELIMINARIES

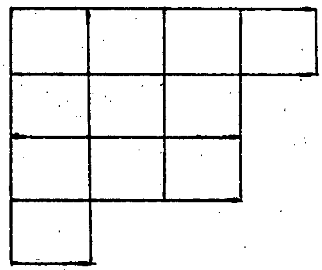
1.1. PARTITIONS AND TABLEAUX

A *partition* of n is an ordered set of positive numbers $(\alpha) = (\alpha_1, \ldots, \alpha_k)$ such that

$$n = \alpha_1 + \ldots + \alpha_k , \quad \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k ,$$

k arbitrary. Such a partition (α) is said to have k *parts* and its *length*, $|\alpha|$, is n. Thus $(4, 3, 3, 1)$ is a partition of 11. It may also be written $(4, 3^2, 1)$ and a similar notation will be used elsewhere.

We represent (α) by a *Young diagram*, $D(\alpha)$, having α_1 squares in the first row, α_2 squares in the second row and so on, the j^{th} squares of the rows making a column. $D(\alpha)$ is said to have *shape* (α). Thus



is the Young diagram of shape $(4, 3^2, 1)$. The square in the i^{th} row and j^{th} column of $D(\alpha)$ is said to have *coordinates* (i,j) and is called the (i,j)-*square*.

Let α_i' denote the number of squares appearing in the i^{th} column of $D(\alpha)$. The Young diagram $D(\alpha')$, obtained by interchanging the rows and columns of $D(\alpha)$, is called the *conjugate* of $D(\alpha)$ and the partition $(\alpha') = (\alpha_1', \ldots, \alpha_s')$ of n is called the *conjugate* of (α) .

The n *letters* $\underline{1}, \ldots, \underline{n}$ may be arranged in the squares of $D(\alpha)$ in $n!$ ways. Each such arrangement is called a *tableaux of shape* (α). A tableaux is called a *standard tableaux* if the letters in every row increase from left to right and in every column from top to bottom. Thus the tableaux (ii) and (iii) are standard tableaux while (i) is not. The tableaux of

1	4	5
2	3	

(i)

1	3	5
2	4	

(ii)

1	2	3
4	5	

(iii)

Figure 1.

shape (α) with the n letters arranged in consecutive order in the rows, starting with the first square in the first row is called the *canonical tableaux* of shape (α) . The tableaux (iii) above is the canonical tableaux of shape (3, 2).

The number, f^α, of standard tableaux of shape (α) is determined as follows (see [12], p.44). The (i,j)-square of $D(\alpha)$ determines the (i,j)-*hook* consisting of the (i,j)-square along with the $\alpha_i - i$ squares to the right in the i^{th} row and the $\alpha_j' - j$ squares below in the j^{th} column. Thus the length of the (i,j)-hook is

(1.1.1) $$h_{ij} = (\alpha_i - i) + (\alpha_j' - j) + 1.$$

Then

(1.1.2) $$f^{\alpha} = n! \Big/ \prod_{i,j} h_{ij} .$$

A *double partition* of n is an ordered pair of partitions $(\mu) = (\alpha, \beta)$ with $|\alpha| + |\beta| = n$. If $(\alpha) = (\alpha_1, \ldots, \alpha_s)$ and $(\beta) = (\beta_1, \ldots, \beta_t)$, we write the double partition (μ) as $(\mu) = (\mu_1, \ldots, \mu_{s+t})$ where $\mu_i = \alpha_i$, $1 \le i \le s$ and $\mu_j = \beta_i$ where $j > s$, $j = s+i$. We allow either (α) or (β) to be a partition of n in the above, i.e. let (0) denote the *empty partition*. For (α) a partition of n, $((\alpha), (0))$ and $((0), (\alpha))$ are distinct double partitions of n. We represent (μ) by an ordered pair of Young diagrams $D(\mu) = (D(\alpha), D(\beta))$, called the *Young diagram of shape* (μ). $D(\mu)$ is considered to have $s+t$ rows, where the i^{th} row of $D(\alpha)$ is the i^{th} row of $D(\mu)$ and the j^{th} row of $D(\beta)$ is the $s+j^{th}$ row of $D(\mu)$. The Young diagrams of shape $((\alpha), (0))$ and $((0), (\beta))$ are taken to be $(-, D(\alpha))$ and $(D(\alpha), -)$. The squares of $D(\mu)$ are identified by their coordinates in the diagrams $D(\alpha)$ and $D(\beta)$. Thus the square of $D(\mu)$ which is in the i^{th} row and j^{th} column of $D(\alpha)$ (resp. $D(\beta)$) is called the *(i,j)-square of* $D(\alpha)$ (resp. $D(\beta)$) and has *coordinates* (i,j). Hence distinct squares of $D(\mu)$ can have the same coordinates (for instance, the first square in the first row of $D(\mu)$ and the first square in the $s+1^{st}$ row of $D(\mu)$ both have coordinates $(1,1)$, where s is as above).

A *tableaux of shape* $(\mu) = (\alpha, \beta)$ is any arrangement of the letters $\underline{1}, \ldots, \underline{n}$ in $D(\mu)$. Thus a tableaux T^{μ} is an ordered pair (T^{α}, T^{β}) where, for complementary subsets K and L of $\{1, \ldots, n\}$ with $|K| = |\alpha|$, $|L| = |\beta|$, T^{α} denotes any arrangement of the letters of K in

$D(\alpha)$ and T^{β} denotes any arrangement of the letters of L in $D(\beta)$. The tableaux T^{μ} is a *standard tableaux* if the arrangement of the letters is in increasing order in the rows and columns of both T^{α} and T^{β}. Thus

3	5	13
4	12	
7		

1	6	10	11
2	8		
9			

Figure 2.

is a standard tableaux of shape $((3, 2, 1), (4, 2, 1))$. The *canonical tableaux* of shape (μ) is the tableaux where the letters are arranged consecutively in the rows of $D(\mu)$, starting with the first square in the first row of $D(\mu)$. The number, f^{μ}, of standard tableaux of shape $(\mu) = (\alpha, \beta)$ is

$$f^{\mu} = \binom{n}{k} f^{\alpha} f^{\beta}, \qquad k = |\alpha| , \tag{1.1.3}$$

where $f^{(0)} = 1$.

We order the standard tableaux of a given shape as follows :

DEFINITION (1.1.4) *Let* T_1^{μ} , T_2^{μ} *denote standard tableaux of shape* (μ). *We say* T_1^{μ} *precedes* T_2^{μ} *if the letters* $\underline{n}$, $\underline{n-1}$, ..., $\underline{n-r+1}$ *appear in the same row in both tableaux but the letter* $\underline{n-r}$ *appears in a lower row in* T_1^{μ} *than in* T_2^{μ} . *The enumeration of the standard tableaux according to their* *ordering is called the last letter sequence.*

Thus in the last letter sequence all tableaux which have the letter n in the last row precede those which have n in the second to the last row. These latter tableaux precede those which have n in the third to the last row and so on. Those tableaux which have n in the same row are arranged by the same scheme according to the position of the letter n-1 and so on. It is evident that the canonical tableaux is the first tableaux in this ordering.

We give an example of this ordering. For the double partition ((2, 1), (2)), there are 20 standard tableaux. Arranged according to the last letter sequence they are,

12	13	12	13	23	14	14	24	12	13
3	2	4	4	4	2	3	3	5	5
45	45	35	25	15	35	25	15	34	24

23	14	24	34	15	15	25	15	25	35
5	5	5	5	2	3	3	4	4	4
14	23	13	12	34	24	14	23	13	12

Figure 3.

Finally we define the notion of axial distance.

DEFINITION (1.1.5) *For squares* A *and* B *in a Young diagram* $D(\mu)$ *with coordinates* (i,j) *and* (s,t) *respectively, define the axial distance,* ρ, *from* A *to* B *to be*

$$\rho = (t-s) - (j-i)$$

Axial distance has a simple graphical interpretation. Suppose the squares A and B are in the same diagram of $D(\mu) = (D(\alpha), D(\beta))$. Starting from A, proceed by any rectangular route one square at a time until B is reached. Counting +1 for each step made upwards or to the right and -1 for each step made downwards or to the left, the resultant number of steps made is the axial distance from A to B. For squares belonging to distinct diagrams, axial distance is the distance of any rectangular route, counted as above, in the diagram obtained by superimposing $D(\beta)$ upon $D(\alpha)$.

Finally

DEFINITION (1.1.6) *The axial distance from the letter* p *to the letter* q *in a tableaux* T^{μ} *is the axial distance from the square of* T^{μ} *in which* p *appears to the square of* T^{μ} *in which* q *appears.*

Thus in Figure 2 the axial distance from 4 to 13 is 3, the axial distance from 13 to 4 is -3, the axial distance from 6 to 3 is -1 and the axial distance from 9 to 7 is 0.

1.2. THE SEMI-NORMAL REPRESENTATIONS OF THE SYMMETRIC GROUP

We briefly describe the irreducible semi-normal representations of the symmetric group S_n on n letters. The conjugacy classes of S_n are parametrised by the partitions of n. In ([18]) Young constructed for each partition (α) of n an irreducible representation $[\alpha]$ of S_n of degree f^α by constructing primitive idempotents, the "natural idempotents", in the group algebra QS_n from the standard tableaux of shape (α). The distinctive feature of these representations is that they are integral, i.e., matrix representations afforded by the minimal left ideals generated by these idemptoents have entries in Z. In a subsequent paper ([20]) Young constructed an equivalent form of these representations by means of the "semi-normal idempotents". While the corresponding matrix representations are not integral, Young showed an elegant construction to hold for the matrices of the transpositions (i-1, i) by means of the standard tableaux. For a tableaux T^α of shape (α), let $(i-1, i)T^\alpha$ denote the tableaux obtained by interchanging the letters <u>i-1</u> and <u>i</u> in T. If T^α is a standard tableaux and the letters <u>i-1</u> and <u>i</u> do not occur either in the same row or column of T^α, then $(i-1, i)T^\alpha$ is again a standard tableaux. Young's fundamental theorem giving the semi-normal form of the representations of of S_n can now stated as follows.

THEOREM (1.2.1) *Let* $T_1^\alpha, \ldots, T_f^\alpha$, $f = f^\alpha$, *be the arrangement of the standard tableaux of shape* (α) *according to the last letter sequence. To construct the* $f \times f$ *matrix representing* $(i-1, i)$ *in the irreducible representation* $[\alpha]$ *of* S_n *corresponding to* (α), *place*

(i) 1 *in the* p,p^{th} *entry where* T_p^α *has* <u>i-1</u> *and* <u>i</u> *in the same row,*

(ii) -1 *in the* p,p^{th} *entry where* T_p^α *has* <u>i-1</u> *and* <u>i</u> *in the same column,*

(iii) *the matrix*

$$\begin{pmatrix} -\rho & \rho+1 \\ 1-\rho & \rho \end{pmatrix} \begin{matrix} \longleftarrow \textit{row } p \\ \longleftarrow \textit{row } q \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \textit{column } p & \textit{column } q \end{matrix}$$

in the p,p^{th}, p,q^{th}, q,p^{th} *and* q,q^{th} *entries where* $p < q$, $T_q^\alpha = (i-1, i)T_p^\alpha$ *and* $1/\rho$ *is the axial distance* (*see* (1.1.6)) *from* <u>i</u> *to* <u>i-1</u> *in* T_p^α ,

(iv) *zeros elsewhere.*

The importance of the semi-normal form is that it provides an inductive construction of the irreducible representations of S_n and moreover is defined in terms of the generators and relations of S_n.

Young's fundamental theorem can be extended to yield representations of S_n corresponding to double partitions $(\mu) = (\alpha, \beta)$ of n . If (α) is a partition of k and (β) is a partition of ℓ with $k+\ell = n$, let $[\alpha]\cdot[\beta]$ denote the representation of S_n induced from the direct product representation $[\alpha] \times [\beta]$ of the subgroup $S_k \times S_\ell$ of S_n . The representation $[\alpha]\cdot[\beta]$ is called the *outer product representation* of $[\alpha]$ and $[\beta]$. For a standard tableaux $T^\mu = (T^\alpha, T^\beta)$ of shape $(\mu) = (\alpha, \beta)$, if the letters $\underline{i-1}$ and $\underline{i}$ do not occur either in the same row or column of T^{α} or T^{β}, $(i-1, i)T^{\mu}$ is again a standard tableaux. Arranging the standard tableaux of shape (μ) according to the last letter sequence, we have (see [12], p.54)

THEOREM (1.2.2) *To construct the matrices presenting* $(i-1, i)$ *in the outer product representation* $[\alpha]\cdot[\beta]$ *of* S_n, *apply the construction given in* Theorem (1.2.1) *to the standard tableaux* (T^{α}, T^{β}) *of shape* (α, β), *setting* $\rho = 0$ *in* (iii) *if the letters* $\underline{i-1}$ *and* $\underline{i}$ *belong to distinct tableaux* T^{α}, T^{β}.

The *hyperoctahedral group* H_n of order $2^n n!$ is the group of signed permutations on n letters. It can be regarded as acting on an orthonormal basis $e_1, \ldots, e_n$ of R^n by means of permutations and sign changes. Denote the k^{th} sign change, $e_k \longrightarrow -e_k$ by $-(k)$. The set of transpositions $(i-1, i)$, $i = 2, \ldots, n$, and the first sign change, $-(1)$, generate H_n. In ([19]) Young showed the conjugacy classes of H_n to be parametrised by double partitions $(\mu) = (\alpha, \beta)$ of n and constructed for each double partition (μ) an irreducible representation $[\mu]$ of H_n of degree f^{μ} by constructing primitive idempotents in QH_n analogous to the "natural idempotent" of S_n. Young did not construct the analogous of the "semi-normal" idempotents for H_n. It is implicit in his work, however, that a "semi-normal form" can be constructed for the representations $[\mu]$ using Theorem (1.2.2). In particular, using the matrices for the transpositions $(i-1, i)$ in the outer product representation $[\alpha]\cdot[\beta]$, we need only construct a matrix for the first sign change $-(1)$ which satisfies the relations of H_n. It can readily be shown (see corollary (2.2.15)), using the inductive ordering provided by the last letter sequence on the standard tableaux of shape (α, β), that

THEOREM (1.2.3) *To construct the matrices representing* $(i-1, i)$ *in the irreducible representations* $[\mu] = [\alpha, \beta]$ *of* H, *apply* Theorem (1.2.2). *To construct the matrix representing* $-(1)$ *place*

(i) 1 *in the* p,p^{th} *entry if the letter* $\underline{1}$ *appears in the tableaux* T_p^{α} *of* $T_p^{\mu} = (T_p^{\alpha}, T_p^{\beta})$,

(ii) -1 *in the* p,p^{th} *entry if the letter* $\underline{1}$ *appears in the tableaux* T_p^{β} *of* $T_p^{\mu} = (T_p^{\alpha}, T_p^{\beta})$,

(iii) *zeros elsewhere*.

Thus, analogous to Theorem (1.2.1), the irreducible representations of H_n can be defined inductively and in terms of generators and relations.

CHAPTER 2

REPRESENTATION OF THE GENERIC RING CORRESPONDING TO A COXETER SYSTEM OF CLASSICAL TYPE

2.1. HECKE ALGEBRAS AND FINITE GROUPS WITH BN-PAIRS

Let B be a subgroup of a finite group G and let k be a field of characteristic zero. Set $e = |B|^{-1} \sum_{b \varepsilon B} b$ in the group algebra kG. Then e affords the 1-representation 1_B of B and the left kG-module kGe affords the induced representation 1_B^G.

DEFINITION (2.1.1) *The Hecke algebra* $H_k(G, B)$ *is the subalgebra of* kG *given by* $e(kG)e$.

The Hecke algebra acts on kGe by right multiplication and the action defines an isomorphism between $H_k(G, B)$ and the endomorphism algebra $\mathrm{END}_{kG}(kGe)$. The double coset sums $\sum_{x \varepsilon BgB} x$, $g \varepsilon G$, form a basis for $H_k(G, B)$ (see [14], Lemma 84).

In this thesis we will be concerned with finite groups G with BN-*pairs* of subgroups (B, N) satisfying the axioms of [17]. Then $H = B \cap N$ is a normal subgroup of N and the *Weyl group* $W = N/H$ has a presentation with a set of distinguished involutionary generators R and defining relations

(2.1.2)
$$r^2 = 1, \qquad r \varepsilon R,$$
$$(rs\ldots)_{n_{rs}} = (sr\ldots)_{n_{rs}}, \qquad r, s \varepsilon R, \ r \neq s,$$

where n_{rs} is the order of rs in W and $(xy\ldots)_m$ denotes a product of alternating x's and y's with m factors. The pair (W, R) is called a *Coxeter system*. The group G is said to be of *type* (W, R).

If $w \neq 1 \in W$, we denote by $\ell(w)$ the least length ℓ of all expressions

(2.1.3) $$w = r_1 \cdots r_\ell, \quad r_1, \ldots, r_\ell \in R .$$

(2.1.3) is called a reduced expression for w in R if $\ell = \ell(w)$.

There is a bijection between the double cosets $B\backslash G/B$ and the elements $w \in W$ resulting in the *Bruhat decomposition* $G = \bigcup_{w\in W} BwB$. The structure of the Hecke algebra $H_k(G, B)$ of a finite group with a BN-pair with respect to a Borel subgroup B was shown in [9] and [11] to be as follows.

THEOREM (2.1.4) $H_k(G, B)$ *has k-basis* $\{S_w : w \in W\}$ *where*

$$S_w = |B|^{-1} \sum_{x\in BwB} x$$

with S_1 *the identity element. Multiplication is determined by the formula*

$$S_w S_r = S_{wr}, \quad r \in R, \ \ell(wr) > \ell(w),$$

$$S_w S_r = q_r S_{wr} + (q_r - 1) S_w, \quad r \in R, \ \ell(wr) < \ell(w)$$

where the $\{q_r, r \in R\}$ *are the index parameters*

(2.1.5) $$q_r = |B : (B \cap rBr)| .$$

For any reduced expression $w = r_1 \ldots r_\ell$ *for* w *in* R, $w \neq 1$

$$S_w = S_{r_1} \ldots S_{r_\ell} .$$

Thus $H_k(G, B)$ *is generated by* $\{S_r , r \in R\}$ *and has defining relations*

$$(2.1.6) \qquad \begin{aligned} S_r^{\,2} &= q_r S_1 + (q_r - 1) r , \qquad r \in R, \\ (S_r S_s \ldots)_{n_{rs}} &= (S_s S_r \ldots)_{n_{rs}} , \end{aligned}$$

where n_{rs} *is as in* (2.1.2).

Let (W, R) be a Coxeter system and let $\{u_r , r \in R\}$ be indeterminates over k, chosen such that $u_r = u_s$ if and only if r and s are conjugate in W. Let D be the polynomial ring $D = k[u_r : r \in R]$. Then there exists an associative D-algebra $\mathfrak{A}$ with identity, free basis $\{a_w , w \in W\}$ over D and multiplication determined by the formulas

$$(2.1.7) \qquad \begin{aligned} a_w a_r &= a_{wr} , \qquad r \in R, \ w \in W, \ \ell(wr) > \ell(w) , \\ a_w a_r &= u_r a_{wr} + (u_r - 1) a_w , \quad r \in R, \ w \in W, \ \ell(wr) < \ell(w), \end{aligned}$$

(see [2], p.55). The D-algebra $\mathfrak{A}$ is called the *generic ring* corresponding to the Coxeter system (W, R). Analogous to Theorem (2.1.4), the generic ring $\mathfrak{A}$ has a presentation with generators $\{a_r , r \in R\}$ and relations

$$(2.1.8) \qquad \begin{aligned} a_r^{\,2} &= u_r 1 + (u_r - 1) a_r , \qquad r \in R , \\ (a_r a_s \ldots)_{n_{rs}} &= (a_s a_r \ldots)_{n_{rs}} , \qquad r, s \in R, \ r \neq s \end{aligned}$$

with n_{rs} as in (2.1.2).

The Hecke algebra $H_k(G, B)$ can be compared with the group algebra kW as follows. Let L be any field of characteristic zero and $\phi : D \longrightarrow L$ a homomorphism. Consider L as a D-module by setting

$$d \cdot \lambda = \phi(d)\lambda , \quad d \in D, \ \lambda \in L .$$

Then the specialized algebra

(2.1.9) $$\mathfrak{A}_{\phi,L} = L \otimes_D \mathfrak{A}$$

is an algebra over L with basis $\{a_{w\phi} = 1 \otimes a_w\}$, generators $\{a_{r\phi}, r \in R\}$ and defining relations obtained from (2.1.8) by applying ϕ. Thus if $\phi : D \longrightarrow k$ is defined by $\phi(\mu_r) = q_r$, $r \in R$, q_r the index parameters (2.1.5), then

(2.1.10) $$\mathfrak{A}_{\phi,k} \cong H_k(G, B)$$

while if $\phi_o : D \longrightarrow k$ is defined by $\phi_o(\mu_r) = 1$, for all $r \in R$, then

(2.1.11) $$\mathfrak{A}_{\phi_o,k} \cong kW .$$

We say the Coxeter system (W, R) is of *classical type* if W is of type A_n, B_n, $n \geq 2$ or D_n, $n \geq 4$. In this chapter, we will determine the irreducible representations of the generic ring corresponding to a Coxeter system of classical type and by means of the appropriate specialized algebras the irreducible representations of the Hecke algebras $H_k(G, B)$ of groups with BN-pair of classical type.

2.2. THE REPRESENTATIONS OF $\mathfrak{A}^K(B_n)$

If a Coxeter system (W, R) is of type B_n, $n \geq 2$, $W(B_n)$ is isomorphic to the Hyperoctohedral group, the group of signed permutations on n-letters (see 1.2). Thus $W(B_n)$ has a presentation with generators $R = \{w_1, \ldots, w_n\}$ where $w_i = (i-1, i)$, $i = 2, \ldots, n$, and $w_1 = -(1)$, the first sign change and relations

$$w_i^2 = 1 ,$$

$$w_1w_2w_1w_2 = w_2w_1w_2w_1 ,$$

$$w_iw_{i+1}w_i = w_{i+1}w_iw_{i+1} , \quad i = 2, \ldots, n-1 ;$$

$$w_iw_j = w_jw_i , \quad |i-j| > 1$$

(see [4]). Furthermore the set of generators R is partitioned into 2 sets under conjugation; namely, w_i is conjugate to w_j for $i, j \geq 2$ while the negative one-cycle w_1 is not conjugate to any w_j, $j \geq 2$.

For the Coxeter system $(W(B_n), R)$ taken as above, we take the generic ring $\mathfrak{A}(B_n)$ to be defined over the polynomial ring $D = Q[x,y]$, x, y indeterminates over Q . It has a presentation with generators $a_{w_i} = a_i$, $w_i \in R$, and relations

B1 $$a_1^2 = y1 + (y-1)a_1 ,$$

B2 $$a_i^2 = x1 + (x-1)a_i , \quad i = 2, \ldots, n ;$$

B3 $$a_1a_2a_1a_2 = a_2a_1a_2a_1 ,$$

B4 $$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \quad i = 2, \ldots, n-1 ;$$

B5 $$a_i a_j = a_j a_i, \quad |i-j| > 1 .$$

We depart from the notations of (2.1) strictly for notational convenience, i.e., we switch from $Q(u_1, u_2)$ to $Q(x, y)$ to avoid carrying around subscripts.

We now construct for each double partition $(\mu) = (\alpha, \beta)$ of n, $n \geq 2$, a K-representation of $^K(B_n) = K \otimes_D (B_n)$, $K = Q(x, y)$. The method involves the construction of $f^\mu \times f^\mu$ matrices over K for each of the generators a_i of $\mathfrak{A}^K(B_n)$ in a manner analogous to the construction of the matrices of the transpositions $(i-1, i)$ for the outer product representation $[\alpha]\cdot[\beta]$ of S_n.

For any integer k, let

$$\Delta(k, y) = x^k y + 1 .$$

Denote by $M(k, y)$ the 2×2 matrix

(2.2.1) $$M(k, y) = \frac{1}{\Delta(k, y)} \begin{pmatrix} (x-1) & \Delta(k+1, y) \\ x\Delta(k-1, y) & x^k y(x-1) \end{pmatrix}$$

Then trace $M(k, y) = (x-1)$, det $M(k, y) = -x$, so the characteristic polynomial of $M(k, y)$ gives

(2.2.2) $$M(k, y)^2 = xI + (x-1)M(k, y),$$

I the 2×2 identity matrix.

For $k \geq 1$, let

$$\Delta(k, -1) = \sum_{i=0}^{k-1} x^i .$$

Denote by $M(k, -1)$, $k \geq 2$, the 2×2 matrix

$$M(k, -1) = \frac{1}{\Delta(k, -1)} \begin{pmatrix} -1 & \Delta(k+1, -1) \\ x\Delta(k-1, 1) & x^k \end{pmatrix} . \tag{2.2.3}$$

As $M(k, -1)$ is obtained from $M(k, y)$ by setting $y = -1$, (2.2.1) shows

$$M(k, -1)^2 = xI + (x-1)M(k, -1) \tag{2.2.4}$$

Denote by $D(z, w)$ the 2×2 diagonal matrix

$$D(z, w) = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} .$$

Then

$$D(z, -1)^2 = zI + (z-1)D(z, -1) . \tag{2.2.5}$$

In what follows, we employ the definitions and notations of (1.1) in regards to double partitions, standard tableaux, and axial distance.

DEFINITION (2.2.6) *Let* $(\mu) = (\alpha, \beta)$ *be a double partition of* n *and let* $T_1^\mu, \ldots, T_f^\mu$, $f = f^\mu$ *be the ordering of the standard tableaux of shape* (μ) *according to the last letter sequence. Construct* $f \times f$ *matrices* $M^\mu(i)$, $i = 1, \ldots, n$, *over* $K = Q(x, y)$ *as follows :*

(1) *Construct* $M^\mu(1)$ *by placing*

(i) y *in the* p,p^{th} *entry if the letter* <u>1</u> *appears in* T_p^{α} *of* $T_p^{\mu} = (T_p^{\alpha}, T_p^{\beta})$,

(ii) -1 *in the* p,p^{th} *entry if the letter* <u>1</u> *appears in* T_p^{β} *of* $T_p^{\mu} = (T_p^{\alpha}, T_p^{\beta})$,

(iii) *zeros elsewhere* .

(2) *Construct* $M^{\mu}(i)$, $i = 2, \ldots, n$, *by placing*

(i) x *in the* p,p^{th} *entry if the letters* <u>i-1</u> *and* <u>i</u> *appear in the same row of* T_p^{α} *or* T_p^{β} *of* $T_p^{\mu} = (T_p^{\alpha}, T_p^{\beta})$,

(ii) -1 *in the* p,p^{th} *entry if the letters* <u>i-1</u> *and* <u>i</u> *appear in the same column of* T_p^{α} *or* T_p^{β} *of* T_p^{μ} ;

(iii) *The matrix* $M(k, -1)$ *in the* p,p^{th}, p,q^{th}, q,p^{th} *and* q,q^{th} *entries corresponding to the tableaux* T_p^{μ} *and* T_q^{μ} *where*

(a) $p < q$, $(i-1, i)T_p^{\mu} = T_q^{\mu}$ *and the letters* <u>i-1</u> *and* <u>i</u> *appear either both in* T_p^{α} *or* T_p^{β} *of* T_p^{μ} ,

(b) k *is the axial distance from* <u>i</u> *to* <u>i-1</u> *in* T_p^{μ} ;

(iv) *The matrix* $M(k, y)$ *in the* p,p^{th}, p,q^{th}, q,p^{th} *and* q,q^{th} *entries corresponding to the tableaux* T_p^{μ} *and* T_q^{μ} *where*

(a) $p < q$, $(i-1, i)T_p^{\mu} = T_q^{\mu}$ *and the letters* <u>i-1</u> *and* <u>i</u> *appear in different tableaux of* T_p^{μ} ,

(b) k *is the axial distance from* <u>i</u> *to* <u>i-1</u> *in* T_p^{μ} ;

(v) *zeros elsewhere.*

Let V_μ denote the free Q-module generated by $t_1, \ldots, t_f$, $f = f^\mu$ corresponding to the standard tableaux $T_1^\mu, \ldots, T_f^\mu$ of shape (μ) ordered according to the last letter sequence. For any field L of characteristic zero set $V_\mu^L = V_\mu \otimes L$. The corresponding basis elements $t_i \otimes 1$ of V_μ^L will be denoted simply by t_i. Set $K = Q(x, y)$. Define linear operators Z_i^μ, $i = 1, \ldots, n$, on V_μ^K such that the matrix of Z_i^μ with respect to the basis $\{t_1, \ldots, t_f\}$ of V_μ^K is given by $M^\mu(i)$.

THEOREM (2.2.7) *Let* $K = Q(x, y)$ *and let* $\mathfrak{A}^K(B_n)$ *denote the generic ring of the Coxeter system* $(W(B_n), R)$ *as before. Let* (μ) *be a double partition of* n, $n \geq 2$. *Then the* K-*linear map*

$$\pi^\mu : \mathfrak{A}^K(B_n) \longrightarrow \mathrm{END}\ (V_\mu^K)$$

defined by $\pi^\mu(a_i) = Z_i^\mu$ *is a representation of* $\mathfrak{A}^K(B_n)$.

PROOF : We need to show the relations (B1-B5) are satisfied with Z_i^μ in place of a_i. We argue by induction on n. For $n = 2$ it is a case by case verification. The double partitions $((2), (0))$, $((0), (2))$, $((1)^2, (0))$ and $((0), (1)^2)$ are clearly seen to yield the well known one-dimensional representations of $\mathfrak{A}^K(B_2)$ ([6], 10). For the double partition $((1), (1))$ there are two standard tableaux, $\begin{matrix}\boxed{1}\\ \boxed{2}\end{matrix}$ and $\begin{matrix}\boxed{2}\\ \boxed{1}\end{matrix}$. From (2.2.6) $M^{((1), (1))}(1) = D(y, -1)$ and $M^{((1), (1))}(2) = M(0, y)$. Direct computation verifies the relation

$$M(0, y)D(y, -1)M(0, y)D(y, -1) = D(y, -1)M(0, y)D(y, -1)M(0, y) .$$

Thus the relations (B1-B3) are satisfied with Z_1^μ and Z_2^μ in place of a_1 and a_2 by the above computation, (2.2.2) and (2.2.5).

Now let $(\mu) = (\mu_1, \ldots, \mu_s)$ be a double partition of n. Deletion of the letter $\underline{n}$ from a standard tableaux automaticly yields a standard tableaux involving $\underline{n-1}$ letters. In fact deletion of $\underline{n}$ from all standard tableaux having $\underline{n}$ at the end of the i^{th} row will yield all standard tableaux of shape $(\mu_1, \ldots, \mu_i-1, \ldots, \mu_s)$. Denoting this partition by (μ_i-) and using the fact that all standard tableaux with $\underline{n}$ in the i^{th} row precede all tableaux with $\underline{n}$ in the j^{th} row for $i > j$ when ordered according to the last letter sequence, we have

$$V_\mu^K = V_{(\mu_s-)}^K \oplus \ldots \oplus V_{(\mu_1-)}^K \tag{2.2.8}$$

and the corresponding matrix block form

$$M^\mu(i) = M^{(\mu_s-)}(i) \dotplus \ldots \dotplus M^{(\mu_1-)}(i)\ , \qquad i < n$$

as by (2.2.6), $M^\mu(i)$ depends only on the letters $\underline{i-1}$ and $\underline{i}$. It is understood that (μ_i-) is taken to equal zero if $\underline{n}$ cannot appear in the i^{th} row and the above summation, here and elsewhere, will be taken over those (μ_i-) which are non-zero. By the induction hypothesis it therefore suffices to check the relations (B1-B5) as they pertain to Z_n^μ.

The matrix $M^\mu(n)$ from (2.2.6) is composed of the matrices $M(k, y)$ and $M(k, -1)$ centred about the diagonal along with diagonal entries x and -1. Thus the relation

$$(Z_n^\mu)^2 = xI + (x-1)Z_n^\mu$$

follows from (2.2.2), (2.2.4) and (2.2.5) .

Let $V_{i,j}$ denote the subspace of V^K_μ with basis $t^{i,j}_1, \ldots, t^{i,j}_{s_{i,j}}$, corresponding to the standard tableaux of shape (μ) with the letter $\underline{n}$ appearing in the i^{th} row and $\underline{n-1}$ appearing in the j^{th} row, the ordering of the basis taken according to the last letter sequence. Then

$$V^K_\mu = \bigoplus_{i,j} V_{i,j} ,$$

the summation taken over all allowable i, j such that $\underline{n}$ appears in row i and $\underline{n-1}$ appears in row j, and this decomposition is consistent with the last letter sequence arrangement of the basis of V^K_μ . Thus, whenever $\underline{n}$ and $\underline{n-1}$ are in distinct rows and columns, we have $V_{i,j} \simeq V_{j,i}$ as $\mathfrak{A}^K(B_{n-2})$-modules for $\underline{n}$ appearing in row i, $\underline{n-1}$ appearing in row j, where $W(B_{n-2}) = \langle w_1, \ldots, w_{n-2} \rangle$.

Suppose first that $\underline{n}$ and $\underline{n-1}$ appear in distinct rows and columns, in the tableaux corresponding to $V_{i,j}$; $\underline{n}$ in row i, $\underline{n-1}$ in row j . Then $\underline{n}$ appears in row j and $\underline{n-1}$ appears in row i in the tableaux corresponding to $V_{j,i}$ and the map $t^{i,j}_p \longrightarrow t^{j,i}_p$, $p = 1, \ldots, s_{i,j}=s_{j,i}$, gives an isomorphism $V_{i,j} \simeq V_{j,i}$ as ${}^K(B_{n-2})$-modules, as the configuration of the first n-2 letters in the tableaux corresponding to $t^{i,j}_p$ is the same as the configuration of the first n-2 letters in the tableaux corresponding to $t^{j,i}_p$. In particular the matrix of Z^μ_k , $k = 1, \ldots, n-2,$ on $V_{i,j} \oplus V_{j,i}$ is

$$S_k = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}$$

where A_k is the matrix of $\pi^{\mu}(a_k)$ on $V_{i,j}$. On the other hand, the matrix of Z_n^{μ} on $Kt_p^{i,j} \oplus Kt_p^{j,i}$ is, by (2.2.6), $M(\ell, y)$ or $M(\ell, -1)$, ℓ the axial distance from $\underline{n}$ to $\underline{n-1}$. Thus the matrix of Z_n^{μ} on $V_{i,j} \oplus V_{j,i}$ is

$$S_n = \begin{pmatrix} a_{11}I & a_{12}I \\ a_{21}I & a_{22}I \end{pmatrix}$$

where $(a_{ij}) = M(\ell, y)$ or $M(\ell, -1)$, I the $s_{i,j} \times s_{i,j}$ identity matrix. Then $S_nS_k = S_kS_n$ for $k = 1, \ldots, n-2$, and (B5) holds. The only other possibility is when $\underline{n}$ and $\underline{n-1}$ appear in the same row or column of the tableaux corresponding to $V_{i,j}$. But in this case the matrix of Z_n^{μ} on $V_{i,j}$ is the scalar matrix xI or $-I$ by (2.2.6) and thus commutes with Z_n^{μ} on $V_{i,j}$, $k = 1, \ldots, n-2$. This proves (B5) for all cases.

To check the relation (B4), we consider the restriction of Z_{n-1}^{μ} and Z_n^{μ} to subspaces with basis $\{t_i\}$ corresponding to all tableaux T_i^{μ} having a fixed arrangement of the first $n-3$ letters and all possible rearrangements of the letters $\underline{n-2}$, $\underline{n-1}$ and $\underline{n}$. Let

$$V_{\mu}^{k} = \bigoplus_{p} V_p$$

denote the corresponding decomposition of V_{μ}^{k}, the ordering of the basis of each V_p taken with respect to the last letter sequence. Then each V_p is invariant under Z^{μ}_{n-1} and Z^{μ}_{n} and it suffices to check (B4) for the various possible arrangements of the last 3 letters in a case by case basis.

In what follows, $M_p(i)$, $i = n$ or $n-1$, will denote the matrix of Z^{μ}_{i} on V_p.

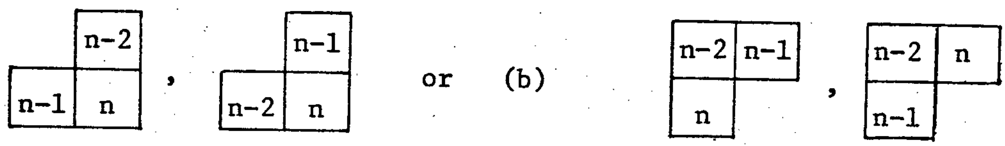
<u>Case 1</u> - the letters <u>n-2</u>, <u>n-1</u> and <u>n</u> in the same row or column. Then V_p is one dimensional and $M_p(n) = M_p(n-1) = x$ or -1 by (2.2.6). Thus

$$M_p(n)M_p(n-1)M_p(n) = M_p(n-1)M_p(n)M_p(n-1)$$

and (B4) is satisfied.

<u>Case 2</u> - the letters <u>n-2</u>, <u>n-1</u> and <u>n</u> in two adjacent rows and two adjacent columns of the same diagram. Then V_p is two-dimensional with basis elements corresponding to tableaux where the configuration of the last 3 letters is

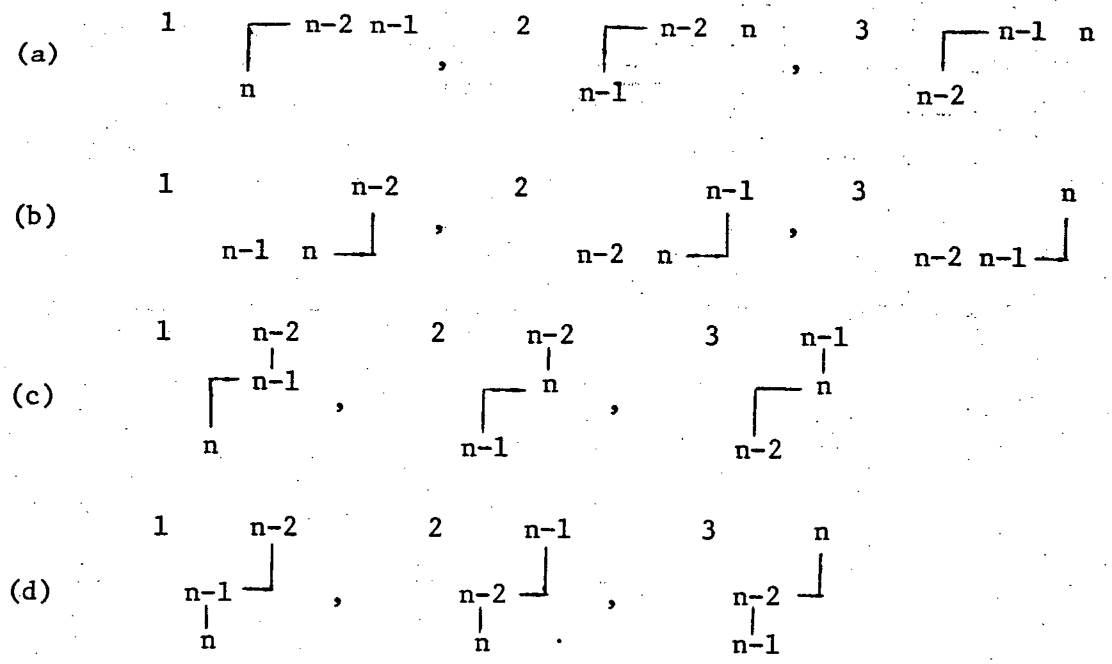
(a)



ordered according to the last letter sequence. Then by (2.2.6), $M_p(n-1) = M(2, -1)$ and $M_p(n) = D(x, -1)$ in (a) and $M_p(n-1) = D(x, -1)$ and $M_p(n) = M(2, -1)$ in (b). Thus (B4) is satisfied in both cases by direct verification of the relation

$$M(2, -1)D(x, -1)M(2, -1) = D(x, -1)M(2, -1)D(x, -1) .$$

Case 3 - the letters n-2, n-1, and n in two rows and three columns or three rows and two columns. Then V_p is three dimensional with basis elements corresponding to tableaux where the configuration of the last 3 letters is one of



ordered according to the last letter sequence. If we set

$$S_1 = \begin{pmatrix} c & \cdot & \cdot \\ \cdot & a_{11} & a_{12} \\ \cdot & a_{21} & a_{22} \end{pmatrix}, \qquad S_2 = \begin{pmatrix} b_{11} & b_{12} & \cdot \\ b_{21} & b_{22} & \cdot \\ \cdot & \cdot & c \end{pmatrix}$$

we have, by case (2.2.6), in case (a), $M_p(n-1) = S_1$ and $M_p(n) = S_2$ where $(a_{ij}) = M(d_1, \varepsilon)$, $(b_{ij}) = M(d_2, \varepsilon)$, $\varepsilon = y$ or -1, and $c = x$. Here d_1 is the axial distance from n-1 to n-2 in 2 and d_2 is the axial distance from $\underline{n}$ to $\underline{n-1}$ in 1 so that $d_2 = d_1 + 1$. In case (b), $M_p(n-1) = S_2$ and $M_p(n) = S_1$ with the same entries in S_i as in case (a). The analysis of (c) and (d) is similar except that now $c = -1$ in S_1 and S_2. Thus in all cases (B4) is satisfied by

LEMMA (2.2.9) *Let* S_1, S_2 *be as above and let* $(a_{ij}) = M(a, y)$, $(b_{ij}) = M(b, y)$. *Then for*

(i) $c = x$ *and* $b-1 = a$,

(ii) $c = -1$ *and* $b+1 = a$,

we have $S_1S_2S_1 = S_2S_1S_2$.

PROOF : Observe that $S_1S_2S_1 = S_2S_1S_2$ iff

(1) $$cb_{11}(c - b_{11}) = a_{11}b_{12}b_{21},$$

(2) $$ca_{22}(c - a_{22}) = b_{22}a_{12}a_{21},$$

(3) $$b_{ij}[a_{11}b_{22} + c(b_{11} - a_{11})] = 0, \quad i \neq j,$$

(4) $$a_{ij}[a_{11}b_{22} + c(a_{22} - b_{22})] = 0, \quad i \neq j,$$

(5) $$a_{11}b_{22}(b_{22} - a_{11}) = c(a_{12}a_{21} - b_{12}b_{21}).$$

For (1),

$$cb_{11}(c - b_{11}) - a_{11}b_{12}b_{21}$$

(2.2.10) $$= \frac{(x-1)}{\Delta(b, y)^2\Delta(a, y)}\left\{\Delta(a, y)[c^2\Delta(b, y) - c(x-1)] - x\Delta(b-1, y)\Delta(b+1, y)\right\}$$

Now for $c = x$,

$$c^2\Delta(b, y) - c(x-1) = x\Delta(b+1, y)$$

and for $c = -1$,

$$c^2\Delta(b, y) - c(x-1) = x\Delta(b-1, y) .$$

Hence (2.2.10) equals zero for $c = x$, $b+1 = a$ and for $c = -1$, $b-1 = a$. The relation (2) is entirely similar.

For (3) and (4) we have

$$a_{11}b_{22} + c(b_{11} - a_{11})$$

$$= \frac{(x-1)}{\Delta(a, y)\Delta(b, y)} \left\{ x^b y(x-1) + c[\Delta(a, y) - \Delta(b, y)] \right\}$$

and

$$a_{11}b_{22} + c(a_{22} - b_{22})$$

$$= \frac{(x-1)}{\Delta(a, y)\Delta(b, y)} \left\{ x^b y(x-1) + c[x^a y\Delta(b, y) - x^b y\Delta(a, y)] \right\} .$$

But for $c = x$, $b-1 = a$ and $c = -1$, $b+1 = a$,

$$\Delta(a, y) - \Delta(b, y) = x^a y\Delta(b, y) - x^b y\Delta(a, y)$$

$$= x^b y(1-x) .$$

For (5), first note the useful factorization

$$\left(\frac{x\Delta(a-1, y)\Delta(a+1, y)}{\Delta(a, y)^2} \right) - \left(\frac{x\Delta(b-1, z)\Delta(b+1, z)}{\Delta(b, z)^2} \right) =$$

$$= \frac{(x-1)^2}{\Delta(a,\ y)^2\Delta(b,\ z)^2}\left\{ x^a y(x^b z + 1)^2 - x^b z(x^a y + 1)^2 \right\}$$

$$(2.2.11) \qquad = \frac{(x-1)^2(x^{a+b}yz - 1)(x^b z - x^a y)}{\Delta(a,\ y)^2\Delta(b,\ z)^2}$$

Now

$$\frac{1}{c}\, a_{11}b_{22}(b_{22} - a_{11})$$

$$(2.2.12) \qquad = \frac{(x-1)^2}{\Delta(b,\ y)^2\Delta(a,\ y)^2}\left(\frac{x^{b+1}y - x^b y}{c} \right)\left(x^{a+b}y^2 - 1 \right) .$$

But

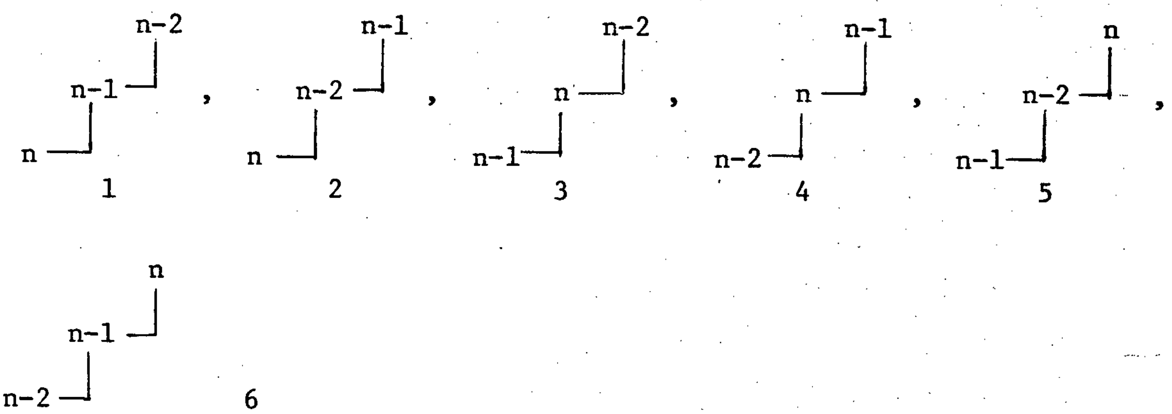
$$\frac{1}{c}\,[x^{b+1}y - x^b y] = x^b y - x^a y$$

for $c = x$, $b-1 = a$ and for $c = -1$, $b+1 = a$. So for both cases, (2.2.12) equals

$$a_{12}a_{21} - b_{12}b_{21}$$

using (2.2.11) with $z = y$.

<u>Case 4</u> - the letters <u>n-2</u>, <u>n-1</u> and <u>n</u> in three distinct rows and three distinct columns. Then V_p is 6-dimensional with basis elements corresponding to tableaux where the configuration of the last 3 letters is



ordered according to the last letter sequence. Let

$$S_1 = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b_{11} & b_{12} & \cdot & \cdot \\ \cdot & \cdot & b_{21} & b_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_{11} & c_{12} \\ \cdot & \cdot & \cdot & \cdot & c_{21} & c_{22} \end{pmatrix}, \qquad S_2 = \begin{pmatrix} c_{11} & \cdot & c_{12} & \cdot & \cdot & \cdot \\ \cdot & b_{11} & \cdot & \cdot & b_{12} & \cdot \\ c_{21} & \cdot & c_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_{11} & \cdot & a_{12} \\ \cdot & b_{21} & \cdot & \cdot & b_{22} & \cdot \\ \cdot & \cdot & \cdot & a_{21} & \cdot & a_{22} \end{pmatrix}$$

Then if all rows are in the same diagram we have by (2.2.6), $M_p(n-1) = S_1$ and $M_p(n) = S_2$ where $(a_{ij}) = M(d_1, -1)$, $(b_{ij}) = M(d_2, -1)$, and $(c_{ij}) = M(d_3, -1)$. Here d_1 is the axial distance from $\underline{n-1}$ to $\underline{n-2}$ in 1 , d_2 is the axial distance from $\underline{n-1}$ to $\underline{n-2}$ in 3 and d_3 is the axial distance from $\underline{n-1}$ to $\underline{n-2}$ in 5 so that $d_1 + d_3 = d_2$ and all $d_i \geq 2$. If two rows are in one diagram and the third in the second diagram, we assume, without loss of generality, the lowest box to be in the second diagram. Superimposing the second diagram upon the first again does not alter the relation $d_1 + d_3 = d_2$ except that now only $d_1 \geq 2$. In this case $M_p(n-1) = S_1$ and $M_p(n) = S_2$, where now $(a_{ij}) = M(d_1, -1)$, $(b_{ij}) = M(d_2, y)$ and $(c_{ij}) = M(d_3, y)$. Thus for both cases (B4) is satisfied by

LEMMA (2.2.13) *Let* S_1 *and* S_2 *be as above and let* $(a_{ij}) = M(s, w)$, $(b_{ij}) = M(p, y)$, $(c_{ij}) = M(t, z)$ *with* $s+t = p$. *Then for*

(i) $w = -1,\ y = z,\ s \geq 2$,

(ii) $z = -1,\ y = w,\ t \geq 2$,

(iii) $w = y = z = -1,\ s, t, p \geq 2$, *we have* $S_1S_2S_1 = S_2S_1S_2$.

PROOF : First, either (i) or (ii) clearly imply (iii). Now $S_1S_2S_1 = S_2S_1S_2$ iff

(1) $$a_{k,\ell}[c_{ii}a_{ii} + b_{ii}(a_{jj} - c_{ii})] = 0 , \quad k \neq \ell, \quad i \neq j ;$$

(2) $$c_{k,\ell}[c_{ii}a_{ii} + b_{ii}(c_{jj} - a_{ii})] = 0 , \quad k \neq \ell, \quad i \neq j ;$$

(3) $$b_{k,\ell}[c_{ii}a_{jj} - c_{ii}b_{jj} - a_{jj}b_{ii}] = 0, \quad k \neq \ell, \quad i \neq j ;$$

(4) $$a_{ii}c_{ii}(a_{ii} - c_{ii}) = b_{ii}(c_{ij}c_{ji} - a_{ij}a_{ji}) , \quad i \neq j ;$$

(5) $$a_{ii}b_{jj}(a_{ii} - b_{jj}) = c_{ji}(b_{ij}b_{ji} - a_{ij}a_{ji}) , \quad i \neq j ;$$

(6) $$c_{ii}b_{jj}(c_{ii} - b_{jj}) = a_{jj}(b_{ij}b_{ji} - c_{ij}c_{ji}) , \quad i \neq j ;$$

and as these relations are symmetric in the (a_{ij}) and (c_{ij}), it suffices to prove the lemma for (i).

For (1), (2), and (3), we observe that

$$E_1 = \Delta(s, -1) + x^s\Delta(t, y) - \Delta(p, y) = 0 ,$$

$$E_2 = x^t y\Delta(s, -1) - \Delta(t, y) + \Delta(p, y) = 0 .$$

Also set

$$D = \frac{(x-1)^2}{\Delta(p, y)\Delta(t, y)\Delta(s, -1)} .$$

We then obtain

$$c_{11}a_{11} + b_{11}(a_{22} - c_{11}) = - DE_1 = 0 ,$$

$$c_{22}a_{22} + b_{22}(a_{11} - c_{22}) = -x^p yDE_2 = 0 ,$$

$$c_{11}a_{11} + b_{11}(c_{22} - a_{11}) = DE_2 = 0 ,$$

$$c_{22}a_{22} + b_{22}(c_{11} - a_{22}) = x^p yDE_1 = 0 ,$$

$$c_{11}a_{22} - c_{11}b_{22} - a_{22}b_{11} = - x^s DE_2 = 0 ,$$

$$c_{22}a_{11} - c_{22}b_{11} - a_{11}b_{22} = - x^t yDE_1 = 0 .$$

For (4) we have

$$\frac{1}{b_{11}} a_{11}c_{11}(a_{11} - c_{11}) = \frac{1}{b_{22}} a_{22}c_{22}(a_{22} - c_{22}) =$$

$$= \frac{(x-1)^2(x^{s+t}y + 1)(x^t y + x^s)}{\Delta(s, -1)^2 \Delta(t, y)^2} = c_{12}c_{21} - a_{12}a_{21}$$

from (2.2.11), setting $a = t$, $b = s$, $z = -1$.

The relations (5) and (6) are handled in an entirely similar manner.

This completes the proof of the lemma and the proof of the theorem.

THEOREM (2.2.14) *Let* K *be as before. The representations* π^μ *of* $\mathfrak{A}^K(B_n)$ *are irreducible, pairwise inequivalent and are, up to isomorphism, a complete set of irreducible, inequivalent representations of* $\mathfrak{A}^K(B_n)$. *In particular* K *is a splitting field for* $\mathfrak{A}^K(B_n)$.

PROOF : By induction on n . For the representations of $\mathfrak{A}^K(B_2)$ it is a matter of direct computation to check irreducibility and inequivalence. Consideration of degrees shows a complete set of inequivalent representations is obtained. For $\mathfrak{A}^K(B_n)$ we employ the decomposition (2.2.8) afforded by the last letter sequence and the position of the letter $\underline{n}$ in a standard tableaux. Let (μ) be a double partition of n . The $\mathfrak{A}^K(B_n)$-module V_μ^K is either irreducible or (2.2.8) is the decomposition of V_μ^K into irreducible inequivalent $\mathfrak{A}^K(B_n)$ components, inequivalent because each of the double partitions (μ_i-) of $n-1$ is distinct. But for each pair (μ_i-), (μ_j-), $i \neq j$, there exists a tableaux T_p^μ with $\underline{n}$ in row i, $\underline{n-1}$ in row j and $(n-1, n)T_p^\mu = T_q^\mu$ a tableaux with $\underline{n}$ in row j and $\underline{n-1}$ in row i . Thus the action of $\pi^\mu(a_n)$ does not decompose with respect to the $V^{(\mu_i-)}$, $i = 1, \ldots, s+r$. Hence V_μ^K is irreducible. Furthermore the double partition (μ) is completely determined by the set of double partitions (μ_i-), $i = 1, \ldots, s+r$, of $n-1$. Thus by the induction hypothesis $V_\mu^K \simeq V_{\mu'}^K$, as $^K(B_n)$-modules implies $(\mu) = (\mu')$. From ([19]), $\sum_\mu (f^\mu)^2 = 2^n n!$, (μ) a double partition of n . Thus $\mathfrak{A}^K(B_n)$ is semisimple and as the π^μ are defined over K, K is a splitting field for $\mathfrak{A}^K(B_n)$. This completes the proof.

It is clear that the above representations of the generic ring yield representations of a wide variety of specialized algebras of $\mathfrak{A}^K(B_n)$. Specificly, set

$$P(B_n) = x \prod_{i=0}^{n-1} (x^i + y)(x^i y + 1)(1 + \ldots + x^i) \in D = Q[x, y] .$$

COROLLARY (2.2.15) *Let* L *be any field of characteristic zero,* $\phi : D \longrightarrow L$ *a homomorphism such that* $\phi(P(B_n)) \neq 0$. *Let* (μ) *be a double partition of* n *and let* $Z^{\mu}_{i\phi}$ *denote the linear operator on* V^{L}_{μ} *obtained by the substitution* $x \longrightarrow \phi(x)$, $y \longrightarrow \phi(y)$ *in the entry of* $M^{\mu}(i)$. *Then* $Z^{\mu}_{i\phi}$ *is well defined and the* L-*linear map*

$$\pi^{\mu}_{\phi,L} : \mathfrak{A}_{\phi,L}(B_n) \longrightarrow \text{END }(V^{L}_{\mu})$$

defined by $\pi^{\mu}_{\phi,L}(a_i) = Z^{\mu}_{i\phi}$ *is a representation of* $\mathfrak{A}_{\phi,L}(B_n)$. *The representations* $\{\pi^{\mu}_{\phi,L}\}$ *are a complete set of irreducible, inequivalent representations of* $\mathfrak{A}_{\phi,L}(B_n)$.

PROOF : If $\phi(P(B_n)) \neq 0$, (2.2.1) and (2.2.3) show the matrices $M(k, y)$ and $M(k, -1)$ are well defined under the substitution $x \longrightarrow \phi(x)$, $y \longrightarrow \phi(y)$ for $-n+1 \leq k \leq n-1$. It is clear from the definition that axial distance in a Young diagram corresponding to a double partition of n cannot exceed n-1 in absolute value. Thus by (2.2.6), Z^{μ}_{i} is well defined for all i. As $\mathfrak{A}_{\phi,L}$ has a presentation with generators $\{a_{i\phi}\}$ and relations obtained from (B1-B5) by applying ϕ, the proofs of Theorem (2.2.4) and (2.2.14) carry over to this case.

Let A be a separable algebra over a field L and let $\overline{L}$ be an algebraic closure of L. Define the numerical invariants of A to be the set of integers $\{n_i\}$ such that $A^{\overline{L}}$ is isomorphic to a direct sum of total matrix algebras

$$A^{\overline{L}} \simeq \bigoplus_{i} M_{n_i}(\overline{L}) .$$

Thus for ϕ defined as in Corollary (2.2.15) the algebras $\mathfrak{A}^K(B_n)$ and $\mathfrak{A}_{\phi,L}$ have the same numerical invariants. In particular Corollary (2.2.15) gives the well known result (see [1]) that for G a finite group with BN-pair with Coxeter system (W, R) of type B_n,

$$H_Q(G, B) \simeq QW .$$

Indeed in ([1]) this is shown to be the case for all Coxeter system with the possible exception of (W, R) of type E_7.

Finally, we remark that, for the specialization $\phi_o : D \longrightarrow Q$ defined by $\phi_o(x) = \phi_o(y) = 1$, the representations $\{\pi^{\mu}_{\phi_o,Q}\}$ are the irreducible representations of $W(B_n)$ given by Theorem (1.2.3).

2.3. THE REPRESENTATIONS OF $\mathfrak{A}^K(A_n)$ AND $\mathfrak{A}^K(D_n)$

We now obtain the representations of the generic ring of a Coxeter system of type A_n and D_n.

If (W, R) is a Coxeter system of type A_{n-1}, $W(A_{n-1})$ is isomorphic to the symmetric group S_n and we take the set R to be $\{w_2, \ldots, w_n\}$ where $w_i = (i-1, i)$, $i = 2, \ldots, n$. We take the generic ring $\mathfrak{A}(A_{n-1})$ to be defined over the polynomial ring $D = Q[x]$. It has a presentation with generators $a_{w_i} = a_i$, $i = 2, \ldots, n$, and relations (B2, B4, B5).

Set $K = Q(x)$. The representations of $\mathfrak{A}^K(A_{n-1}) = \mathfrak{A}(A_{n-1}) \otimes_D K$ are readily obtained from the results of the previous section. As the matrices $M(k, -1)$ are defined in $Q(x)$, (2.2.6) shows the matrices $M^{(\alpha, (0))}(i)$, $i = 2, \ldots, n$, are defined in $Q(x)$ and $Z_i^{(\alpha, (0))}$ can be regarded as a linear operator on $V^K_{(\alpha, (0))}$. Thus

THEOREM (2.3.1) *Let* α *be a partition of* n, $n \geq 2$ *and* $K = Q(x)$ *The* K-*linear map*

$$\pi^\alpha : \mathfrak{A}^K(A_{n-1}) \longrightarrow \mathrm{END}(V^K_{(\alpha, (0))})$$

defined by $\pi^\alpha(a_i) = Z_i^{(\alpha, (0))}$, $i = 2, \ldots, n$, *is a representation of* $\mathfrak{A}^K(A_{n-1})$. *The representations* $\{\pi^\alpha\}$, *are a complete set of irreducible, inequivalent representations of* $\mathfrak{A}^K(A_{n-1})$.

PROOF : Theorem (2.2.7) shows the $\{\pi^{\alpha}\}$ are representations of $\mathfrak{A}^{K}(A_{n-1})$. Irreducibility and inequivalence follows from Theorem (2.2.14) as the matrix of $Z_1^{(\alpha,\ (0))}$ on $V_{(\alpha,\ (1))}$ is the scalar matrix yI. As (see [19])

$$\sum_{\alpha} (f^{\alpha})^2 = n! , \quad \alpha \text{ a partition of } n ,$$

the $\{\pi^{\alpha}\}$ are a complete set of inequivalent representations and are absolutely irreducible.

The representations of the specialized algebras are handled entirely analogous to Corollary (2.2.15). Set

$$P(A_n) = x \sum_{i=1}^{n} (1 + \ldots + x^i) .$$

Then from the above and Corollary (2.2.15) we have

COROLLARY (2.3.2) *Let* L *be any field of characteristic zero,* $\phi : D = Q[x] \longrightarrow L$ *a homomorphism such that* $\phi(P(A_n)) \neq 0$. *Then for* (α) *a partition of* $n \geq 2$, *the linear operators* $Z_{i\phi}^{(\alpha,\ (0))}$, $i = 2, \ldots, n$, *are well defined and the* L-*linear maps*

$$\pi^{\alpha}_{\phi,L} : \mathfrak{A}_{\phi,L}(A_{n-1}) \longrightarrow \mathrm{END}(V^{L}_{(\alpha,\ (0))})$$

defined by $\pi^{\alpha}_{\phi,L}(a_{i\phi}) = Z_{i\phi}^{(\alpha,\ (0))}$ *is a representation of* $\mathfrak{A}_{\phi,L}(A_{n-1})$. *The* $\{\pi^{\alpha}_{\phi,L}\}$ *are a complete set of irreducible, inequivalent representations of* $\mathfrak{A}_{\phi,L}(A_{n-1})$.

Thus for ϕ as above, the algebras $\mathfrak{A}^K(A_n)$ and $\mathfrak{A}_{\phi,L}(A_n)$ have the same numerical invariants. We remark that for the specialization $x \longrightarrow 1$ the definitions of the matrices $M(k, -1)$ shows the semi-normal matrix representation of S_n is obtained (see Theorem (1.2.1)).

If (W, R) is a Coxeter system of type D_n, $n \geq 4$, $W(D_n)$ can be regarded as a subgroup of index 2 in $W(B_n)$; $W(D_n)$ acting on an orthonormal basis of R^n by means of permutations and even sign changes. A set of distinguished generators for $W(D_n)$ can be obtained from the set $\{w_1, \ldots, w_n\}$ of $W(B_n)$ given in section (2.1) by setting $\overline{w}_1 = w_1w_2w_1$ and taking the set R to be $\{\overline{w}_1, w_2, \ldots, w_n\}$ (see [4]).

Let $\phi : Q[x, y] \longrightarrow Q[x]$ be defined by $\phi(y) = 1$. Then the specialized ring $\mathfrak{A}_{\phi,Q[x]}(B_n)$ has basis $\{a_{w\phi}, w \in W(B_n)\}$ with relations obtained by applying ϕ to (B1-B5). In particular $(a_{1\phi})^2 = 1$. Set

$$\overline{a}_{1\phi} = a_{w_1w_2w_1\phi} .$$

As $w_1w_2w_1$ is reduced in $(W(B_n), R)$ we have $a_{w_1w_2w_1\phi} = a_{1\phi}a_{2\phi}a_{1\phi}$ by (2.1.8). Applying ϕ to (B1-B5) it is readily seen that

B'1 $$\overline{a}_{1\phi}^{\,2} = x1 + (x-1)\overline{a}_{1\phi} ,$$

B'2 $$\overline{a}_{1\phi}a_{3\phi}\overline{a}_{1\phi} = a_{3\phi}\overline{a}_{1\phi}a_{3\phi} ,$$

B'3 $$\overline{a}_{1\phi}a_{j\phi} = a_{j\phi}\overline{a}_{1\phi} , \quad j \neq 1, 3 .$$

As any reduced expression of $w \neq 1 \in W(D_n)$ in the generators $\{\overline{w}_1, w_2, \ldots, w_n\}$ is a reduced expression for w in the generators $\{w_1, \ldots, w_n\}$ of $W(B_n)$, the relations (B'1-B'3) show the subring of $\mathfrak{A}_{\phi,Q[x]}(B_n)$ generated by $\{\overline{a}_{1\phi}, a_{2\phi}, \ldots, a_{n\phi}\}$ has free basis $\{a_{w\phi}, w \in W(D_n)\}$. As all the generators $\{\overline{w}_1, w_2, \ldots, w_n\}$ are conjugate in $W(D_n)$, the subring generated by $\{\overline{a}_{1\phi}, a_{2\phi}, \ldots, a_{n\phi}\}$ is isomorphic to the generic ring of a Coxeter system of type D_n, $n \geq 4$. Denote this subring by $\mathfrak{A}(D_n)$.

Thus the representations of $\mathfrak{A}_{\phi,K}(B_n)$ given by Corollary (2.2.15) provide us with representations of $\mathfrak{A}^K(D_n)$. Young [19] showed the restrictions of the representations of $W(B_n)$ to $W(D_n)$ corresponding to a double partition (α, β) of n remain irreducible if $(\alpha) \neq (\beta)$ and decomposes into two irreducible components when $(\alpha) = (\beta)$. We show that this holds true in a "generic" sense.

Recall that a standard tableaux T for the double partition (α, β) of n is an ordered pair $T = (T^\alpha, T^\beta)$. Then the tableaux $T^* = (T^\beta, T^\alpha)$ is a standard tableaux of shape (β, α), called the *conjugate tableaux* of T. Moreover the map $T \longrightarrow T^*$ is a bijection from the standard tableaux of shape (α, β) to the standard tableaux of shape (β, α). Take $(\alpha) \neq (\beta)$. If $T_1, \ldots, T_p, \ldots, T_q, \ldots, T_f$, $f = f^{\alpha,\beta}$, is the arrangement of the standard tableaux of shape (α, β) according to the last letter sequence, order the tableaux of shape (β, α) according to the scheme; T^*_q

precedes T^*_p if T_p precedes T_q in the last letter sequence. Call this the *conjugate ordering* of the tableaux of shape (β, α).

Let I^*_n denote the $n \times n$ matrix

$$I^*_n = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$$

LEMMA (2.3.3) *Let* $M^{\alpha,\beta}_{\phi}(a)$ *denote the matrix of* $\pi^{\alpha,\beta}_{\phi}(a)$ *with respect to the basis* $\{t_i\}$ *of* $V^K_{\alpha,\beta}$ *ordered according to the last letter sequence,* $a \in \mathfrak{A}^K(D_n)$. *Then* $I^*_f M^{\alpha,\beta}_{\phi}(a) I^*_f$, $f = f^{\alpha,\beta}$, *is the matrix of* $\pi^{\beta,\alpha}_{\phi}(a)$ *with respect to the conjugate ordering of the basis* $\{t_i\}$ *of* $V^K_{\beta\ \alpha}$. *Thus the restrictions of the representations* $\pi^{\alpha,\beta}_{\phi}$ *and* $\pi^{\beta,\alpha}_{\phi}$ *to* $\mathfrak{A}^K(D_n)$ *are equivalent.*

PROOF : Let $T_1, \ldots, T_q, \ldots, T_f$ be the arrangement of the standard tableaux of shape (α, β) according to the last letter sequence. For fixed i, $i = 2, \ldots, n$, write $V^K_{\alpha,\beta}$ as the direct sum $V^K_{\alpha,\beta} = \oplus V_{p,q}$ of $Z^{\alpha,\beta}_{i\phi}$ invariant subspaces where $V_{p,p}$ is taken to have basis $\{t_p\}$ if $\underline{i-1}$ and $\underline{i}$ appear in the same row or column of T_p and $V_{p,q}$ has basis $\{t_p, t_q\}$ where $(i-1, i)T_p = T_q$, $p < q$ in the last letter sequence. Let $V^K_{\beta,\alpha} = \oplus V^*_{p,q}$ denote the corresponding decomposition of $V^K_{\beta,\alpha}$, where $V^*_{p,q}$ has basis $\{t^*_p, t^*_q\}$ corresponding to T^*_p, T^*_q, and where the ordering of t^*_p, t^*_q is taken with respect to the conjugate ordering. We need to show that if the matrix of $Z^{\alpha,\beta}_{i\phi}$ on $V_{p,q}$ is M, the matrix of $Z^{\alpha,\beta}_{i\phi}$ on $V^*_{p,q}$ is I^*MI^*. This is a simple case by case verification.

1. If $\underline{i-1}$ and $\underline{i}$ are in the same row or column of T_p, they are likewise in T_p^* and the lemma is shown for this case.

2. If $\underline{i-1}$ and $\underline{i}$ are in distinct rows and columns of the same tableaux T_p^α or T_p^β of $T_p = (T_p^\alpha, T_p^\beta)$, set $T_q = (i-1, i)T_p$ and take $p < q$. Then $T_p^* < T_q^*$ in the arrangement according to the last letter sequence while $T_q^* < T_p^*$ in the conjugate ordering. The axial distance, k, from $\underline{i}$ to $\underline{i-1}$ is the same in both T_p and T_p^*. Thus from (2.2.6) the matrix of $Z_{i\phi}^{\alpha,\beta}$ on $V_{p,q}$ is $M(k, -1)$ while the matrix of $Z_{i\phi}^{\beta,\alpha}$ on $V_{p,q}^*$ is $I^*M(k, -1)I^*$ as is required.

3. If $\underline{i-1}$ and $\underline{i}$ are in distinct tableaux of $T_p = (T_p^\alpha, T_p^\beta)$ set $T_p = (i-1, i)T_q$ and take $p < q$. Then $T_p^* < T_q^*$ in both the ordering according to the last letter sequence and the conjugate ordering. If k is the axial distance from $\underline{i}$ to $\underline{i-1}$ in T_p, $-k$ is the axial distance from $\underline{i}$ to $\underline{i-1}$ in T_p^*. Let $M(k, 1)$ denote the 2×2 matrix obtained from $M(k, y)$ under the substitution $y = 1$. Then (2.2.6) shows the matrix of $Z_{i\phi}^{\alpha,\beta}$ on $V_{p,q}$ is $M(k, 1)$ while the matrix of $Z_{i\phi}^{\beta,\alpha}$ on $V_{p,q}^*$ is $M(-k, 1)$. Direct computation verifies the relation

$$I^*M(k, 1)I^* = M(-k, 1) \tag{2.3.4}$$

as is required.

It remains to show the lemma for $M_\phi^{\alpha,\beta}(\overline{a}_{i\phi})$. As $Z_{1\phi}^{\alpha,\beta}$ acts on the basis $\{t_i\}$ of $V_{\alpha,\beta}^K$ by scalar multiplication, the decomposition of $V_{\alpha,\beta}^K$ into $Z_{2\phi}^{\alpha,\beta}$ invariant subspaces as above is valid for $Z_{1\phi}^{\alpha,\beta}Z_{2\phi}^{\alpha,\beta}Z_{1\phi}^{\alpha,\beta}$ as well. It is furthermore clear from (2.2.6) that the action of $Z_{1\phi}^{\alpha,\beta}Z_{2\phi}^{\alpha,\beta}Z_{1\phi}^{\alpha,\beta}$

differs from that of $Z_{2\phi}^{\alpha,\beta}$ only on the spaces $V_{p,q}$ where the letters $\underline{1}$ and $\underline{2}$ appear in distinct tableaux of $T_p = (T_p^{\alpha}, T_p^{\beta})$. In this case the matrix of $Z_{i\phi}^{\alpha,\beta}$ on $V_{p,q}$ and on $V^*_{p,q}$ is $D(1, -1)$. Using (2.3.4), a simple matrix calculation completes the proof for this case. This completes the proof of the lemma.

We define the *conjugate ordering* of the standard tableaux $T = (T_1^{\alpha}, T_2^{\alpha})$ of shape (α, α) as follows. Set

$$\mathcal{T}_i = \{T = (T_1^{\alpha}, T_2^{\alpha}) : \underline{n} \text{ appears in } T_i^{\alpha}\}, \quad i = 1, 2.$$

All standard tableaux belonging to $\mathcal{T}_2$ precede those belonging to $\mathcal{T}_1$ in the arrangement according to the last letter sequence. Rearrange the last $\frac{1}{2} f^{\alpha,\alpha}$ tableaux in the last letter sequence, i.e. those in $\mathcal{T}_1$, as follows; for T_1, T_2 in $\mathcal{T}_1$, T_1 precedes T_2 if T_2^* precedes T_1^* in the last letter sequence arrangement of the tableaux in $\mathcal{T}_2$.

LEMMA (2.3.5) *Let* $M_{\phi}^{\alpha,\alpha}(a)$ *denote the matrix of* $\pi_{\phi}^{\alpha,\alpha}(a)$ *on* $V_{\alpha,\alpha}^K$ *with respect to the conjugate ordering of the basis* $\{t_i\}$ *of* $V_{\alpha,\alpha}^K$, $a \in \mathcal{Cl}^K(D_n)$ *Set*

$$R_f = \left(\begin{array}{c|c} -I_{\frac{1}{2}f} & I^*_{\frac{1}{2}f} \\ \hline I^*_{\frac{1}{2}f} & I_{\frac{1}{2}f} \end{array}\right), \qquad f = f^{\alpha,\alpha}.$$

Then

$$(2.3.6) \qquad R_f M_\phi^{\alpha,\alpha}(a) R_f^{-1} = \left(\begin{array}{c|c} M_1(a) & 0 \\ \hline 0 & M_2(a) \end{array}\right),$$

PROOF : If $(\alpha, (\alpha_i-))$ is a double partition of $n-1$ contained in (α, α) then so is $((\alpha_i-), \alpha)$, (α_i-) as in the proof of Theorem (2.2.7). Thus we have the decomposition, in the conjugate ordering,

$$(2.3.7) \qquad V^K_{\alpha,\alpha} = V_{(\alpha,\ (\alpha_s-))} \oplus \cdots \oplus V_{(\alpha,\ (\alpha_1-))} \oplus V_{((\alpha_1-),\ \alpha)} \oplus \cdots \oplus V_{((\alpha_s-),\ \alpha)}$$

of $V^K_{\alpha,\alpha}$ as $\mathfrak{A}^K(D_{n-1})$-modules, $\mathfrak{A}^K(D_{n-1})$ generated by $\{\bar{a}_{1\phi}, a_{2\phi}, \ldots, a_{(n-1)\phi}\}$. Thus Lemma (2.3.3) shows that for $a \in \mathfrak{A}^K(D_{n-1})$, $M_\phi^{\alpha,\alpha}(a)$ is of the form

$$(2.3.8) \qquad \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I^*AI^* \end{array}\right)$$

which is easily seen to commute with R_f.

Hence we need to show (2.3.6) only for $M_\phi^{\alpha,\alpha}(a_{n\phi})$. If the letters <u>n-1</u> and <u>n</u> appear in the tableaux T_2^α of $T_p = (T_1^\alpha, T_2^\alpha) \in \mathcal{T}_2$, the proof of lemma (2.3.3) shows the matrix of $Z_{n\phi}^{\alpha,\alpha}$ on the subspaces with corresponding basis $\{t_p, t_p^*\}$ or $\{t_p, t_q, t_p^*, t_q^*\}$ if $(n-1, n)T_p = T_q$, $p \neq q$, is of the form (2.3.8) and the above reasoning applies. If the letters

<u>n-1</u> and <u>n</u> appear in distinct tableaux of $T_p = (T_1^\alpha, T_2^\alpha)$, with $T_p \in \mathcal{T}_2$, then $(n-1, n)T_p = T_q \in \mathcal{T}_1$. Thus $T_q^* \in \mathcal{T}_2$ and we can choose T_p such that $T_p < T_q^*$ in the last letter sequence arrangement of the tableaux belonging to $\mathcal{T}_2$. Then $T_p < T_q^* < T_q < T_p^*$ is the arrangement of the tableaux according to the conjugate ordering. Taking the same ordering of the corresponding basis, the matrix of $Z_{n\phi}^{\alpha,\alpha}$ on the subspace with basis $\{t_p, t_q^*, t_q, t_p^*\}$ is of the form

$$\begin{pmatrix} a_{11} & \cdot & a_{12} & \cdot \\ \cdot & a_{22} & \cdot & a_{21} \\ a_{21} & \cdot & a_{22} & \cdot \\ \cdot & a_{12} & \cdot & a_{11} \end{pmatrix}$$

where $\{a_{ij}\} = M(k, 1)$, $M(k, 1)$ defined as in lemma (2.3.3) and k the axial distance from <u>i</u> to <u>i-1</u> in T_p. A simple matrix calculation shows

$$R_f A R_f^{-1} = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) .$$

This completes the proof.

Let

$$V_{\alpha,\alpha}^K = {}_1V_{\alpha,\alpha}^K \oplus {}_2V_{\alpha,\alpha}^K$$

where, for basis elements t_p corresponding to tableaux $T_p \in \mathcal{T}_2$, ${}_1V_{\alpha,\alpha}^K$

has basis $\{t_p + t_p^*\}$ and ${}_2V^K_{\alpha,\alpha}$ has basis $\{t_p - t_p^*\}$. By Lemma (2.3.5) the K-linear maps

$$ {}_i\pi_\phi^{\alpha,\alpha} : \mathfrak{A}^K(D_n) \longrightarrow \mathrm{END}(V^K_{\alpha,\alpha}) $$

where the matrix of ${}_i\pi_\phi^{\alpha,\alpha}(a_w)$ with respect to the above basis is $M_i(a_w)$, $w \in W(D_n)$, are representations of $\mathfrak{A}^K(D_n)$.

THEOREM (2.3.9) *For double partitions* (α, β), $|\alpha| < |\beta|$, $(\alpha) \neq (\beta)$, *and* (α, α) *of* $n \geq 4$, *the representations* $\pi_\phi^{\alpha,\alpha}$ *and* ${}_i\pi_\phi^{\alpha,\alpha}$, $i = 1, 2$, *are a complete set of irreducible, inequivalent representations of* $\mathfrak{A}^K(D_n)$.

PROOF : By induction on n . For $n = 4$, it is a matter of direct verification. For $n > 4$, the induction assumption and the proof of Lemma (2.3.5) shows $\dim \mathrm{Hom}_{\mathfrak{A}}(V^K_{\alpha,\alpha}) = 2$, $\mathfrak{A} = \mathfrak{A}^K(D_n)$, $\mathrm{Hom}_{\mathfrak{A}}(V^K_{\alpha,\alpha})$ generated by I_f and I_f^* , $f = f^{\alpha,\alpha}$. Thus ${}_1\pi_\phi^{\alpha,\alpha}$ and ${}_2\pi_\phi^{\alpha,\alpha}$ are irreducible and inequivalent. The argument employed in Theorem (2.2.14) suffices for the irreducibility and inequivalence of the $\{\pi_\phi^{\alpha,\alpha}\}$. By (2.3.7)

$$ (2.3.10) \qquad {}_iV^K_{\alpha,\alpha} \simeq V^K_{((\alpha_s-),\,\alpha)} \oplus \dots \oplus V^K_{((\alpha_1-),\,\alpha)} , \qquad i = 1, 2, $$

as $\mathfrak{A}^K(D_{n-1})$-modules. Thus none of the $\pi_\phi^{\alpha,\beta}$ are equivalent to ${}_i\pi_\phi^{\alpha,\alpha}$, $i = 1, 2$. Finally, consideration of degrees using the formula given in Theorem (2.2.14) shows a complete set of inequivalent representations is obtained, and the representations are absolutely irreducible. Thus $K = Q(x)$ is a splitting field for $\mathfrak{A}^K(D_n)$. This completes the proof.

CHAPTER 3

DEGREES OF THE IRREDUCIBLE CONSTITUENTS OF 1_B^G

3.1. DEFINITIONS AND CHARACTERS OF PARABOLIC TYPE

In this chapter we give some results on the irreducible constituents of the induced representation 1_B^G of Borel subgroup B of a finite group G with BN-pair of classical type. The following theorem is basic to the study of these representations.

THEOREM (3.1.1) ([5]) *Let* $\overline{Q}$ *denote the algebraic closure of* Q. *Each irreducible* $\overline{Q}$*-character* χ *of* $H_{\overline{Q}}(G, B)$ *is the restriction to* $H_{\overline{Q}}(G, B)$ *of a unique irreducible* $\overline{Q}$*-character* ζ_χ *of* G, *such that* $(\zeta_\chi, 1_B^G) > 0$. *Moreover every irreducible constituent of* 1_B^G *is obtained in this way. The degree of* ζ_χ *is given by*

$$\text{(3.1.2)} \qquad \deg \zeta_\chi = |G : B| \deg \chi \left(\sum_{w \in W} (\text{ind } w)^{-1} \chi(S_w) \chi(\hat{S}_w) \right)^{-1}$$

where $\hat{S}_w$ *is the basis element of* $H_{\overline{Q}}(G, B)$ *corresponding to* w^{-1} *and*

$$\text{ind } w = |B : B \cap B^w| , \qquad w \in W, \quad B^w = w^{-1}Bw .$$

Let $\mathfrak{A}$ be the generic ring of a Coxeter system (W, R) defined over $D = Q[\mu_r, r \in R]$ as in (2.1) and let K be the quotient field of D, $\overline{K}$ the algebraic closure of K. It is clear from the relations (2.1.8) (see e.g. [6], lemma 2.7) that there exists a unique homomorphism $\nu : \mathfrak{A} \longrightarrow D$ such that $\nu(a_r) = \mu_r$, $r \in R$.

DEFINITION (3.1.3) *Let* χ *be an irreducible* $\overline{K}$*-character of* $\mathfrak{A}^{\overline{K}}$. *Set*

$$d_\chi = \left(\sum_{w\in W} \nu(a_w)\right) \deg \chi \left(\sum_{w\in W} \nu(a_w)^{-1}\chi(a_w)\chi(\hat{a}_w)\right)^{-1}$$

where $\hat{a}_w = a_{w^{-1}}$. *We call* d_χ *the generic degree associated with* χ.

Let M be an irreducible $\overline{K}$-matrix representation of $\mathfrak{A}^{\overline{K}}$. Let $M_{ij}(a)$ denote the i,j^{th} entry of $M(a)$, $a \in \mathfrak{A}^{\overline{K}}$. Thus M_{ij} is a function from $\mathfrak{A}^K$ to K. The ring $\mathfrak{A}^K$ is a symmetric algebra with dual basis $\{a_w\}$ and $\{\nu(a_w)^{-1}\hat{a}_w\}$ (see e.g. [8], Lemma 5.1.). Then from ([7], Lemma 62.8) and Schur's lemma, we have

$$(3.1.4) \qquad (M_{ij}, M_{rs}) = \sum_{w\in W} \nu(a_w)^{-1}M_{ij}(a_w)M_{rs}(\hat{a}_w) = C_M\delta_{is}\delta_{jr},$$

where δ is the Kronecker delta and if χ is the character of M,

$$(3.1.5) \qquad C_M = (\deg \chi)^{-1} \sum_{w\in W} \nu(a_w)^{-1}\chi(a_w)\chi(\hat{a}_w).$$

Now let $\mathfrak{A}$ be the generic ring of a Coxeter system (W, R) of classical type and let G be a finite group with BN-pair of type (W, R). Let $\phi : D \longrightarrow \overline{Q}$ be the homomorphism defined by $\phi(\mu_r) = q_r$, $r \in R$, q_r the index parameters (see (2.1.5)). Let $P = \ker f$ and let $\phi^* : D_P \longrightarrow \overline{Q}$ be the extension of ϕ to the ring of fractions D_P, regarded as a subring of K. Let χ be an irreducible character of $\mathfrak{A}^{\overline{K}}$. The results of chapter 2 (see also [6], Proposition 7.1) show $\chi(a_w) \in D_P$

for all $w \in W$ and the $\overline{Q}$-linear map $\chi_\phi : \mathfrak{A}_{\phi,\overline{Q}} \longrightarrow \overline{Q}$ defined by

(3.1.6) $$\chi_\phi(a_{w\phi}) = \phi^*(\chi(a_w))$$

is an irreducible character of $\mathfrak{A}_{\phi,\overline{Q}}$. The map $\chi \longrightarrow \chi_\phi$ is a bijection between the irreducible characters of $\mathfrak{A}^{\overline{K}}$ and those of $\mathfrak{A}_{\phi,\overline{Q}}$.

As $\mathfrak{A}_{,\overline{Q}} \cong H_{\overline{Q}}(G, B)$, we regard the specialized character χ_ϕ as an irreducible character of $H_{\overline{Q}}(G, B)$ and denote the corresponding irreducible constituent of 1_B^G in the sense of Theorem (3.1.1) by $\zeta_{\chi,\phi}$.

PROPOSITION (3.1.7) *With the notations as above, we have*

$$\phi^*(d_\chi) = \deg(\zeta_{\chi,\phi}) .$$

PROOF : From ([6], lemma 5.9), $\left(\sum_{w\in W} \nu(a_w)\right) = |G : B|$ and $\phi(\nu(a_w)) = \text{ind } w$. The statement now follows from (3.1.2) and (3.1.6) and the definition of d_χ.

In particular if $\phi_o : D \longrightarrow \overline{Q}$ is defined by $\phi_o(\mu_r) = 1$ for all $r \in R$, $\mathfrak{A}_{\phi_o,\overline{Q}} \cong \overline{Q}W$ and (3.1.7) becomes $\deg(\chi) = \deg(\zeta_{\chi,\phi_o})$.

We will evaluate d_χ for the irreducible character χ of the generic ring corresponding to a Coxeter system of classical type in the next section. We conclude this section with the following.

Let $J \subset R$ and let $W_J = \langle J\rangle$. J determines a *parabolic* subgroup $G_J = BW_JB$.

DEFINITION (3.1.8) *Let* ζ *be an irreducible character of* G *such that* $(\zeta, 1_B^G) > 0$. ζ *is said to be of parabolic type if* $(\zeta, 1_{G_J}^G) = 1$ *for some* $J \quad R$.

From the above there is a natural bijective correspondence $\zeta_{\chi,\phi} \longrightarrow \zeta_{\chi,\phi_o}$ between the irreducible $\overline{Q}$-characters $\zeta_{\chi,\phi}$ of G and the irreducible $\overline{Q}$-characters of W. In ([6], Theorem 7.2) it is shown that

(3.1.9) $$(\zeta_{\chi,\phi}, 1_{G_J}^G) = (\zeta_{\chi,\phi_o}, 1_{W_J}^W)$$

for all $J \subset R$. Thus to show the irreducible constituents of 1_B^G are of parabolic type it is enough to show it for the irreducible characters of the Weyl groups.

PROPOSITION (3.1.10) *Every irreducible character* χ *of* $W(A_n)$, $W(B_n)$, $n \geq 2$, *and* $W(D_n)$, $n \geq 4$, *is of parabolic type.*

PROOF : Let (α) be a partition of n. Let $R(\alpha)$ denote the group of row permutations of the canonical tableaux of shape (α). Then $R(\alpha)$ coincides with W_J for some $J \subset R$, R the set of distinguished generators for $W(A_{n-1}) \cong S_n$ given in (2.3). Order the partitions of n lexicographicly and let χ^α denote the character of the irreducible representation of $W(A_n)$ corresponding to (α). From ([12], pp.40-41)

(3.1.11) $$1_{R(\alpha)}^{S_n} = \chi^\alpha + \sum_{\beta>\alpha} m_{\alpha,\beta}\chi^\beta , \qquad m_{\alpha,\beta} \geq 0 .$$

Thus $(\chi^\alpha, 1_{R(\alpha)}^{S_n}) = 1$, which is well known.

For a double partition (α, β) of n, $(\alpha) = (\alpha_1, \ldots, \alpha_r)$, $\beta = (\beta_1, \ldots, \beta_s)$, let $(\alpha+\beta)$ denote the partition of n defined by $(\alpha+\beta) = (\alpha_1+\beta_1, \ldots, \alpha_t+\beta_t)$, $t = \max\{r,s\}$. Let $\chi^{[\alpha]\cdot[\beta]}$ denote the character of the outer product representation $[\alpha]\cdot[\beta]$ of S_n (see (1.2.2)). From ([12], Theorem 3.31),

$$\chi^{[\alpha]\cdot[\beta]} = \chi^{\alpha+\beta} + \sum_{\mu<\alpha+\beta} m_{\mu,\alpha+\beta}\chi^{\mu}, \qquad m_{\mu,\alpha+\beta} \geq 0 .$$

Then by (3.1.11)

$$(3.1.12) \qquad (\chi^{[\alpha]\cdot[\beta]}, 1_{R(\alpha+\beta)}^{S_n}) = (\chi^{\alpha+\beta}, \chi^{\alpha+\beta}) = 1 .$$

But by Theorem (2.2.15) the restriction of the irreducible representation $\pi_{\phi_o}^{\alpha,\beta}$ of $W(B_n)$ to S_n is the outer product representation $[\alpha]\cdot[\beta]$. Thus, letting $\chi^{\alpha,\beta}$ denote the character of $\pi_{\phi_o}^{\alpha,\beta}$, we have

$$(\chi^{\alpha,\beta}, 1_{R(\alpha+\beta)}^{W(B_n)}) = (\chi^{[\alpha]\cdot[\beta]}, 1_{R(\alpha+\beta)}^{S_n}) = 1$$

by (3.1.12) and Frobenius reciprocity. As $R(\alpha+\beta)$ is a parabolic subgroup of $W(B_n)$, the $\chi^{\alpha,\beta}$ are of parabolic type.

The representations of $W(D_n)$, $n \geq 4$, are handled similarly. If $(\alpha) \neq (\beta)$, $\chi^{\alpha,\beta}$ remains irreducible when restricted to $W(D_n)$ by As S_n is a subgroup of $W(D_n)$,

$$(\chi^{\alpha,\beta}, 1_{R(\alpha+\beta)}^{W(D_n)}) = (\chi^{[\alpha]\cdot[\beta]}, 1_{R(\alpha+\beta)}^{S_n}) = 1 .$$

If $(\alpha) = (\beta)$ the situation is only slightly more complicated. Set $J' = \{w_2, \ldots, w_{n-1}\} \subset R$, the distinguished generators of $W(D_n)$. Then $W_{J'} \cong S_{n-1}$. Let ${}_i\chi^{\alpha,\alpha}$ denote the character of ${}_i\pi^{\alpha,\alpha}_{\phi_o}$, $i = 1, 2$. From (2.3.7)

$$(3.1.13) \qquad {}_i\chi^{\alpha,\alpha}|_{W_{J'}} = \chi^{[\alpha]\cdot[\alpha_r-]} + \ldots + \chi^{[\alpha]\cdot[\alpha_1-]}, \quad i = 1, 2$$

where (α_j-) are the partitions of $n-1$ contained in $(\alpha) = (\alpha_1, \ldots, \alpha_r)$. Now

$$(\alpha) + (\alpha_r-) > (\alpha) + (\alpha_{r-1}-) > \ldots > (\alpha) + (\alpha_1-)$$

in the lexicographic order so

$$({}_i\chi^{\alpha,\alpha}, 1^{W(D_n)}_{R((\alpha)+(\alpha_r-))}) = ({}_i\chi^{\alpha,\alpha}|_{S_{n-1}}, 1^{S_{n-1}}_{R((\alpha)+(\alpha_r-))})$$

$$= (\chi^{(\alpha)+(\alpha_r-)}, \chi^{(\alpha)+(\alpha_r-)}), \; i = 1, 2$$

by (3.1.12), (3.1.13) and Frobenius reciprocity. Hence ${}_i\chi^{\alpha,\alpha}$, $i = 1, 2$, are of parabolic type.

This completes the proof.

3.2. AN INDUCTION FORMULA

Let $\mu = (\mu_1, \ldots, \mu_s)$ be a double partition of n and let χ^μ denote the character of the representation π^μ of $\mathfrak{A}^k(B_n)$. Set

$$(3.2.1) \qquad C^\mu = (f^\mu)^{-1} \sum_{w \in W(B_n)} \nu(a_w)^{-1} \chi^\mu(a_w) \chi^\mu(\hat{a}_w)$$

We show that the inductive construction of the representations π^μ yields an inductive formula for C^μ, which will provide the means to determine the generic degree associated with χ^μ .

Let $(W(B_n), R)$ be as in Section 2.2 . Take $J_{n-1} \subset R$ to be the subset $J_{n-1} = \{ w_1, \ldots, w_{n-1} \}$ and let $W_{J_{n-1}} = \langle J_{n-1} \rangle$. Then $W_{J_{n-1}} \cong W(B_{n-1})$. Let $M^\mu(a_w)$ denote the matrix of $\pi^\mu(a_w)$. From the proof of Theorem (2.2.7) we have the decomposition

$$(3.2.2) \qquad M^\mu(a_w) = M^{(\mu_s -)}(a_w) \dotplus \ldots \dotplus M^{(\mu_1 -)}(a_w), \qquad w \in W_{J_{n-1}}$$

where the sum is taken over those $(\mu_t -)$ which are non-zero. Let g_t and f_t denote the position, in the arrangement according to the last letter sequence, of the first and last tableaux of shape (μ) respectively, which upon deletion of $\underline{n}$ yield standard tableaux of shape $(\mu_t -)$. Then for $a_t, b_r, \ g_t \le a_t \le f_t, \ g_r \le b_r \le f_r$, (3.2.2) implies

$$(3.2.3) \qquad M^\mu_{a_t, b_r}(a_w) = 0 \quad \text{for } t \neq r , \ w \in W_{J_{n-1}} .$$

and

(3.2.4) $$M^{\mu}_{a_t,b_t}(a_w) = M^{(\mu_t-)}_{a_t,b_t}(a_w) , \quad w \in W_{J_{n-1}} .$$

Since $|W(B_n)| = 2^n n!$, $|W(B_n) : W_{J_{n-1}}| = 2n$. From ([2], p.37), there exists a set $\{ x_k \mid k = 1, \ldots, 2n \}$ of coset representation of the left $W_{J_{n-1}}$-cosets of $W(B_n)$ such that $\ell(x_k w) = \ell(x_k) + \ell(w)$, $w \in W_{J_{n-1}}$, $k = 1, \ldots, 2n$. The $\{x_k\}$ are the unique elements of minimal length in the left cosets $x_k W_{J_{n-1}}$. We will determine these elements explicitly for our choice of R and J_{n-1} in the next section.

DEFINITION (3.2.5) *With the notation as above, let*

$$E^{(\mu_t-)} = \sum_{k=1}^{2n} \sum_{j=g_t}^{f_t} \nu(a_{x_k})^{-1} M^{\mu}_{1,j}(a_{x_k}) M^{\mu}_{j,1}(\hat{a}_{x_k}) .$$

We now prove

THEOREM (3.2.6) $$C^{\mu} = C^{(\mu_t-)} E^{(\mu_t-)} .$$

PROOF : Choose p such that $g_t \leq p \leq f_t$. Let $\{ x_k : k = 1, \ldots, 2n \}$ be the set of left $W_{J_{n-1}}$-coset representatives of minimal length as above. By (3.1.4) and (3.1.5),

$$C^{\mu} = (M^{\mu}_{1,p}, M^{\mu}_{p,1}) = \sum_{w \in W(B_n)} \nu(a_w)^{-1} M^{\mu}_{1,p}(a_w) M^{\mu}_{p,1}(\hat{a}_w)$$

$$= \sum_{k=1}^{2n} \sum_{w \in W_{J_{n-1}}} \nu(a_{x_k})^{-1} \nu(a_w)^{-1} M^{\mu}_{1,p}(a_{x_k} a_w) M^{\mu}_{p,1}(\hat{a}_w \hat{a}_{x_k})$$

$$= \sum_{k=1}^{2n} \sum_{w \in W_{J_{n-1}}} \nu(a_{x_k})^{-1} \nu(a_w)^{-1} \left(\sum_{i=1}^{f} M^{\mu}_{1,i}(a_{x_k}) M^{\mu}_{i,p}(a_w) \right) \times \left(\sum_{j=1}^{f} M^{\mu}_{p,j}(\hat{a}_w) M^{\mu}_{j,1}(\hat{a}_{x_k}) \right)$$

$$(3.2.7) \qquad = \sum_{k=1}^{2n} \sum_{i,j=1}^{f} \nu(a_{x_k})^{-1} M^{\mu}_{1,i}(a_{x_k}) M^{\mu}_{j,1}(\hat{a}_{x_k}) \times \left(\sum_{w \in W_{J_{n-1}}} \nu(a_w)^{-1} M^{\mu}_{i,p}(a_w) M^{\mu}_{p,j}(\hat{a}_w) \right)$$

The second step follows from the fact that by the choice of the $\{x_k\}$, $\ell(x_k w) = \ell(x_k) + \ell(w)$. By (3.2.3)

$$\sum_{w \in W_{J_{n-1}}} \nu(a_w) M^{\mu}_{i,p}(a_w) M^{\mu}_{p,j}(\hat{a}_w) = 0$$

for either i or j not lying between g_t and f_t , while by (3.2.4) and (3.1.4)

$$\sum_{w \in W_{J_{n-1}}} \nu(a_w)^{-1} M^{\mu}_{i,p}(a_w) M^{\mu}_{p,j}(\hat{a}_w) = (M^{(\mu_t -)}_{i,p} , M^{(\mu_t -)}_{p,j}) = \delta_{ij} C^{(\mu_t -)}$$

for $g_t \leq i,j \leq f_t$, as $\pi^{(\mu_t -)}$ is an absolutely irreducible representation of $\mathfrak{A}^K(B_{n-1})$ by theorem (2.2.14). Combining the above formulae with (3.2.7) gives

$$C^{\mu} = C^{(\mu_t -)} \sum_{k=1}^{2n} \sum_{j=g_t}^{f_t} \nu(a_{x_k})^{-1} M^{\mu}_{1,j}(a_{x_k}) M^{\mu}_{j,1}(\hat{a}_{x_k}) = C^{(\mu_t -)} E^{(\mu_t -)}$$

which is the required result.

While the above theorem provides the induction step for a variety of factorizations of C^{μ}, no such factorization lends itself to an explicit formula for C^{μ} without tedious calculation. To obtain a formula for C^{μ} by induction using Theorem (3.2.6) requires the evaluation of $E^{(\mu_t -)}$ for some row t. The most convenient choice is the first allowable row from which the last square can be deteted. Let c denote the index of this row. Then in the Young diagram of shape $(\mu) = (\alpha, \beta)$, the row c is a row of the Young diagram of shape (α) if $(\alpha) \neq (0)$. Moreover, the standard tableaux of shape (μ) which upon deletion of $\underline{n}$ yield standard tableaux of shape $(\mu_c -)$ occur last in the ordering according to the last letter sequence. We defer the evaluation of $E^{(\mu_c -)}$ until the next section. We close this section with a deduction which will prove valuable in the calculations to come.

PROPOSITION (3.2.8) *Let* $(W(B_n), R)$ *be as before and let* w *be any element of* $W(B_n)$ *which can be expressed as a product of distinct generators chosen from* R *in increasing order, i.e.* $w = w_{i_1} \cdots w_{i_k}$, $w_{i_j} \in R$, $j = 1, \ldots, k$ *and* $i_1 < i_2 < \ldots < i_k$. *For the matrix representation* M^{μ} *of* $\mathfrak{A}^K(B_n)$, $\mu = (\alpha, \beta)$ *a double partition of* n,

$$(3.2.9) \qquad M^{\mu}_{c,d}(a_w) = \sum_{s,t,\ldots,u} M^{\mu}_{c,s}(a_{i_1}) M^{\mu}_{s,t}(a_{i_2}) \cdots M^{\mu}_{u,d}(a_{i_k})$$

and there is at most one non-zero term in the above summation. In particular $M^{\mu}_{c,d}(a_w) \neq 0$ *if and only if there exists a* $\bar{w} \in W(B_n)$, $\bar{w} = w_{j_1} \cdots w_{j_s}$, *where* $j_1 < \ldots < j_s$ *and* $\{j_1, \ldots, j_s\} \subset \{i_1, \ldots, i_k\}$, *or* $\bar{w} = 1$, *such that* $\bar{w}^{-1} T^{\mu}_c = T^{\mu}_d$.

PROOF : As the $\{w_{i_j}\}$ $j = 1, \ldots, k$ are distinct, $a_w = a_{i_1} \ldots a_{i_k}$ and (3.2.9) is just the definition of matrix multiplication. Furthermore the second statement in the proposition follows immediately from the first since by definition (2.2.6) and theorem (2.2.7), a matrix entry $M^{\mu}_{i,j}(a_k) \neq 0$, for $k = 2, \ldots, n$, if and only if $i = j$ or $w_k T^{\mu}_i = T^{\mu}_j$ while $M^{\mu}_{i,j}(a_1) \neq 0$ if and only if $i = j$. Thus a product of matrix entries of the form

$$M^{\mu}_{c,c_1}(a_{i_1}) M^{\mu}_{c_1,c_2}(a_{i_2}) \ldots M^{\mu}_{c_{k-1},d}(a_{i_k}), \quad i_1 < i_2 < \ldots < i_k$$

is non-zero if and only if there exists a $w \in W$, $w = w_{j_1} \ldots w_{j_s}$ with $j_1 < \ldots < j_s$, $\{j_1, \ldots, j_s\} \subset \{i_1, \ldots, i_k\}$ or $w = 1$ such that $w^{-1} T^{\mu}_c = T^{\mu}_d$. Hence to complete the proof of the proposition we must show there is at most one non-zero term in the summation (3.2.9). We do this by induction on k. It is certainly true for $k = 1$. Assume true for $k - 1$. By the rules of matrix multiplication,

$$(3.2.10) \qquad M^{\mu}_{c,d}(a_w) = \sum_s M^{\mu}_{c,s}(a_{w'}) M^{\mu}_{s,d}(a_{i_k}), \qquad w' = w_{i_1} \ldots w_{i_{k-1}} .$$

By the induction hypothesis it is enough to show at most one term in (3.2.10) is non-zero. Consider the position of the letters $\underline{i_k}$ and $\underline{i_{k-1}}$ in T_d^{μ}.

(i) If they belong either to the same row or column of the tableaux of shape (α) or the tableaux of shape (β) of T^{μ}_d, $M^{\mu}_{s,d}(a_{i_k}) \neq 0$ if and only if $s = d$. Thus

$$M^{\mu}_{c,d}(a_w) = M^{\mu}_{c,d}(a_{w'})M^{\mu}_{d,d}(a_{i_k})$$

and the proposition is proved for this case.

(ii) If the letters $\underline{i_k}$ and $\underline{i_{k-1}}$ do not belong to the same row or column of either the tableaux T^{α} or the tableaux T^{β} of $T^{\mu}_d = (T^{\alpha}, T^{\beta})$, set $T^{\mu}_e = w_{i_k}T^{\mu}_d$. Then $e \neq d$, $M^{\mu}_{s,d}(a_{i_k}) \neq 0$ if and only if $s = d$ or e and

$$M^{\mu}_{c,d}(a_w) = M^{\mu}_{c,d}(a_{w'})M^{\mu}_{d,d}(a_{i_k}) + M^{\mu}_{c,e}(a_{w'})M^{\mu}_{e,d}(a_{i_k}) .$$

Suppose $M^{\mu}_{c,d}(a_{w'}) \neq 0$. By the inductive hypothesis and the first part of the proposition, there exists a $\overline{w} \in W$ expressible as a product of distinct generators chosen from the set $\{w_{i_1}, \ldots, w_{i_{k-1}}\}$ such that $\overline{w}T^{\mu}_c = T^{\mu}_d$. As $i_1 < \ldots < i_k$, the letter $\underline{i_k}$ is left fixed under the action of $\overline{w}$ on T^{μ}_c, i.e. $\underline{i_k}$ occupies the same position in the standard tableaux T^{μ}_c and T^{μ}_d. Therefore by the choice of d, the letter $\underline{i_k}$ occupies different positions in the standard tableaux T^{μ}_c and T^{μ}_e. But then there cannot exist a $\overline{w} \in W$ expressible as a product of generators taken from $\{w_{i_1}, \ldots, w_{i_{k-1}}\}$ such that $\overline{w}T^{\mu}_c = T^{\mu}_e$, i.e. $M^{\mu}_{c,e}(a_{w'}) = 0$. Thus

$$M^{\mu}_{c,d}(a_{i_1}\ldots a_{i_k}) = M^{\mu}_{c,s}(a_{i_1}\ldots a_{i_{k-1}})M^{\mu}_{s,d}(a_{i_k})$$

where $s = e$ or d, depending on the position of the letter $\underline{i_k}$ in T^{μ}_d. This completes the proof.

The above proposition is a distinctive property of the shape of the matrices $M^{\mu}(a_i)$ in that it clearly depends only on the position of the zero entries. Furthermore if $w \varepsilon W(B_n)$ is as in the statement of the proposition, the proposition is clearly valid for w^{-1} as well. Finally, for the diagonal entries $M^{\mu}_{c,c}(a_w)$, the proposition is valid for any w expressible as a product of distinct generators, not necessarily on increasing (or decreasing) product (see [13], pp 43-44).

3.3. THE EVALUATION OF $E^{(\mu_c-)}$

Our aim in this section is an expression for $E^{(\mu_c-)}$ in terms of the polynomials $\Delta(m, y)$ and $\Delta(m, -1)$. Throughout, $\mu = (\alpha, \beta)$ will denote a double partition of n, with the partition (α) having s_α parts $(\alpha) = (\alpha_1, \ldots, \alpha_{s_\alpha})$ and the partition (β) having s_β parts $(\beta) = (\beta_1, \ldots, \beta_{s_\beta})$. As before (μ_c-) denotes the non-zero double partition of $n-1$ obtained from (μ) by deletion of a square from the end of row c of the Young diagram $D(\mu)$, i.e. the first allowable row.

We first determine explicitly the elements $\{x_k\}$ of minimal length in the left $W_{J_{n-1}}$-cosets of $(W(B_n), R)$, R and J_{n-1} defined as in the previous section. We introduce some more notation. For any set of consecutive integers $1 \leq k, k+1, \ldots, \ell \leq n$, set $w(k, \ell) = w_k w_{k+1} \cdots w_\ell$ so that $w(k, k) = w_k$. For ease of notation we also define $w(k+1, k) = 1$. Furthermore, set $w(k) = w(2, k)^{-1} w_1 w(2, k)$, $k = 1, \ldots, n$. Thus $w(1) = w_1$. $w(k)$ is the k^{th} sign change $-(k)$ of $W(B_n)$.

LEMMA (3.3.1) *The set*

$$S = \{1\} \quad \{w(k, n) : k = 2, \ldots, n\} \quad \{w(k)w(k+1, n) : k = 1, \ldots, n\}$$

is the unique set of elements of minimal length in the left $W_{J_{n-1}}$*-cosets of* $W(B_n)$.

PROOF : $W(B_n)$ acts on a fixed orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of $\mathbb{R}^n$ as all permutations and sign changes. For the given choice of R, the set of fundamental roots of $W(B_n)$ is

$$\{\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n-1}\}$$

and the set of positive roots is

$$\{\varepsilon_i, \varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \mid 1 \leq i, j \leq n, \ j > i\}$$

see ([2]). For an element $x = w(k, n)$, x^{-1} is the $n - k + 2$ cycle $(n \ \ n-1 \ \ldots \ k-1)$, working from right to left. Hence x^{-1} sends the positive roots $\varepsilon_j - \varepsilon_{k-1}$, $j > k - 1$, to the negative roots $\varepsilon_{j-1} - \varepsilon_n$, $j = k, \ldots, n$. By the choice of J_{n-1}, these roots remain negative under the action of $W_{J_{n-1}} \cong W(B_{n-1})$. Thus $\ell(w'x^{-1}) \geq \ell(x^{-1})$ for all $w' \in W_{J_{n-1}}$, as $\ell(w)$ equals the number of positive roots sent to negative roots under w (see [2] or [14], appendix). Therefore x is of minimal length in the left $W_{J_{n-1}}$-coset $xW_{J_{n-1}}$. A similar argument shows $\ell(w'x^{-1}) \geq \ell(x^{-1})$ for $x = w(k)w(k+1, n)$, $k = 1, \ldots, n$ and for all $w' \in W_{J_{n-1}}$. As $\ell(x) \neq \ell(x')$ for any $x, x' \in S$, the elements of S must belong to distinct cosets. As $|W(B_n) : W_{J_{n-1}}| = 2n$ and $|S| = 2n$, S must be a set of coset representatives for the left $W_{J_{n-1}}$-cosets of $W(B_n)$. This completes the proof.

Let $a_{w(2,\,k)} = a(2, k)$ and $a_{w(k)} = a(k)$ in $\mathfrak{A}(B_n)$. As the expressions for $w(2, k)$ and $w(k)$ as a product of generators from R is reduced

$$\begin{aligned} a(2, k) &= a_2 \ldots a_k \\ a(k) &= (a_k \ldots a_2) a_1 (a_2 \ldots a_k) . \end{aligned} \tag{3.3.2}$$

In order to state the next proposition concerning the matrices $M^{\mu}(a(k))$ corresponding to the k^{th} sign change of $W(B_n)$ we need some notations. Let $\rho_{i,p}$, $i = 2, \ldots, n$ denote the axial distance from i-1 to i in the standard tableaux T_p^{μ} . Let

$$\rho_p(i) = \sum_{j=2}^{i} (\rho_{j,p} + 1), \quad i = 2, \ldots, n$$

and define $\rho_p(1) = 0$ for all indices $p = 1, \ldots, f^{\mu}$.

PROPOSITION (3.3.3) *The matrix* $M^{\mu}(a(k))$, $k = 1, \ldots, n$ *is a diagonal matrix with the* p,p^{th} *entry equal to* $zx^{\rho_p(k)}$, *where* $z = y$ *if the letter* k *appears in the tableaux* T^{α} *of* $T_p^{\mu} = (T^{\alpha}, T^{\beta})$ *and* $z = -1$ *if* k *appears in the tableaux* T^{β} *of* T_p^{μ} .

PROOF : The proof is by induction on k . For $k = 1$ the statement of the proposition is just the definition of $M^{\mu}(a_1)$. Now assume $M^{\mu}(a(k-1))$ is diagonal. By (3.3.2)

$$M^{\mu}(a(k)) = M^{\mu}(a_k)M^{\mu}(a(k-1))M^{\mu}(a_k) .$$

Write V_{μ}^{K} as the direct sum $V_{\mu}^{K} = \oplus V_{p,q}$ of a_k invariant subspaces, where $V_{p,q}$ has basis $\{t_p, t_q\}$ if $(k-1, k)T_p = T_q$, $p < q$, and $V_{p,p}$ has basis $\{t_p\}$ if the letters k-1 and k appear either in the same row or column of the same tableaux. With this ordering of the basis, $M^{\mu}(a(k))$ has the corresponding block form $M^{\mu}(a(k)) = \dot{+} M_{p,q}^{\mu}(a(k))$.
Thus the usual case by case argument on the configuration of the letter k-1 and k will suffice.

1. $\underline{k-1}$ and $\underline{k}$ in the same row or column of the same of T_p .

If $\underline{k-1}$ and $\underline{k}$ are in the same row, $\rho_{k,p} = 1$. Therefore $\rho_p(k) = \rho_p(k-1) + 2$. As $M^{\mu}_{p,p}(a_k) = x$ by theorem (2.2.7)

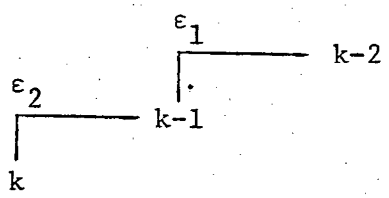
$$M^{\mu}_{p,p}(a(k)) = x\left(zx^{\rho_p(k-1)}\right)x = zx^{\rho_p(k-1)+2} = zx^{\rho_p(k)} .$$

If $\underline{k-1}$ and $\underline{k}$ are in the same column, $\rho_{k,p} = -1$. Therefore $\rho_p(k) = \rho_p(k-1)$. As $M^{\mu}_{p,p}(a_k) = -1$ by theorem (2.2.7)

$$M^{\mu}_{p,p}(a(k)) = (-1)\left(zx^{\rho_p(k-1)}\right)(-1) = zx^{\rho_p(k)} .$$

2. $\underline{k-1}$ and $\underline{k}$ in distinct rows and columns of T^{μ}_p .

Set $T^{\mu}_q = (k-1, k)T^{\mu}_p$ and take $p < q$. The deletion of all letters $\geq k-1$ in T^{μ}_p and T^{μ}_q yield the same tableaux of $k-2$ letters. Therefore $\rho_p(k-2) = \rho_q(k-2)$. Let



denote the configuration of the letters $\underline{k-2}$, $\underline{k-1}$ and $\underline{k}$ in T^{μ}_p , with ε_1, ε_2 the respective axial distances. With this notation we have,

$$\rho_p(k-1) = \rho_p(k-2) + \varepsilon_1 + 1$$

$$\rho_p(k) = \rho_p(k-2) + \varepsilon_1 + \varepsilon_2 + 2$$

$$\rho_q(k-1) = \rho_q(k-2) + \varepsilon_1 + \varepsilon_2 + 1$$

$$\rho_q(k) = \rho_q(k-2) + \varepsilon_2 + 2 .$$

Therefore if <u>k-1</u> and <u>k</u> belong to the same tableaux of T^{μ}_{p} , $M^{\mu}_{p,q}(a_k) = M(\varepsilon_2, -1)$ and $M^{\mu}_{p,q}(a(k-1)) = zx^{\rho_p(k-2)+\varepsilon_1+1} D(1, x^{\varepsilon_2})$ by theorem (2.2.7) and the induction hypothesis. Direct computation verifies the relation

$$M(\varepsilon_2, -1)D(1, x^{\varepsilon_2})M(\varepsilon_2, -1) = xD(x^{\varepsilon_2}, 1) .$$

Therefore

$$M^{\mu}_{p,q}(a(k)) = zx^{\rho_p(k-2)+\varepsilon_1+2} D(x^{\varepsilon_2}, 1)$$

$$= D(zx^{\rho_p(k)}, zx^{\rho_q(k)})$$

If <u>k-1</u> and <u>k</u> belong to distinct tableaux of T^{μ}_{p} , $M^{\mu}_{p,q}(a_k) = M(-\varepsilon_2, y)$ by theorem (2.2.7) . As we have taken $p < q$ with respect to the last letter sequence, <u>k-1</u> appears in the tableaux corresponding to (α) of T^{μ}_{p} . Thus by the induction hypothesis

$$M^{\mu}_{p,q}(a(k-1)) = x^{\rho_p(k-2)+\varepsilon_1+1} D(y, -x^{\varepsilon_2}) .$$

Direct computation verifies the relation

$$M(-\varepsilon_2, y)D(y, -x^{\varepsilon_2})M(-\varepsilon_2, y) = xD(-x^{\varepsilon_2}, y) .$$

Therefore

$$M^{\mu}_{p,q}(a(k)) = x^{\rho_p(k-2)+\varepsilon_1+2} D(-x^{\varepsilon_2}, y) = D(-x^{\rho_p(k)}, yx^{\rho_q(k)})$$

This completes the proof of the proposition.

We use this proposition to affect a reduction of $E^{(\mu_c -)}$.

Pairing the $W_{J_{n-1}}$-coset representations $a(k)a(k+1, n)$ and $a(k+1, n)$ of $W(B_n)$ (lemma (3.3.1)) we have

$$M^{\mu}_{1,i}(a(k+1, n)) + M^{\mu}_{1,i}(a(k)a(k+1, n))$$

(3.3.4) $$= \left(1 + M^{\mu}_{1,1}(a(k))\right) M^{\mu}_{1,i}(a(k+1, n)) , \quad i = 1, \ldots, f ,$$

as $M^{\mu}(a(k))$ is diagonal by proposition (3.3.3)

Furthermore for letters $\underline{k-1}$ and $\underline{k}$ both appearing in the same row of the canonical tableaux T^{μ}_1 of shape (μ) (see section 1.1), $w_k = (k-1, k)$, $k = 1, \ldots, n$, is a row permutation of the canonical tableaux. As any row permutation w can be written as a product of such transpositions

(3.3.5) $$M^{\mu}_{1,i}(a_w) = M^{\mu}_{i,1}(a_w) = 0 , \quad i \neq 1 ,$$

and

(3.3.6) $$M^{\mu}_{1,1}(a_w) = M^{\mu}_{1,1}(\hat{a}_w) = x$$

by theorem (2.2.7), w a row permutation of T^{μ}_1. Let r_i, $i = 1, \ldots, s$, $s = s_{\alpha} + s_{\beta}$, denote the last letter in the i^{th} row of the canonical tableaux T^{μ}_1, i.e.

$$r_i = \sum_{j=1}^{i} \alpha_j \quad \text{if} \quad i \leq s_{\alpha} ,$$

$$r_i = \sum_{j=1}^{s_{\alpha}} \alpha_j + \sum_{j=1}^{k} \beta_j \quad \text{if} \quad i = s_{\alpha} + k .$$

Set

$$(3.3.7) \qquad R_i = \sum_{k=r_{i-1}+1}^{r_i} \left[1 + \nu(a(k))^{-1} M^{\mu}_{1,1}(a(k))^2\right] \nu(a(k+1,\ r_i))^{-1} \times M^{\mu}_{1,1}(a(k+1,\ r_i))^2 .$$

Combining (3.3.5) and (3.3.6) with the definitions of R_i and $E^{(\mu_c -)}$ and using the explicit coset representations given by lemma (3.4.1) we have

$$(3.3.8) \qquad E^{(\mu_c -)} = \sum_{i=1}^{s} R_i \nu(a(r_i+1,\ n))^{-1} \sum_{j=g_c}^{f^{\mu}} M^{\mu}_{1,j}(a(r_i+1,\ n)) \times M^{\mu}_{j,1}(\widehat{a(r_i+1,\ n)})$$

Set $\Delta(m,\ y^{-1}) = 1+x^m y^{-1}$. Then

PROPOSITION (3.3.9). *For* $1 \le i \le s_\alpha$ *denote* R_i *by* R_{α_i} . *For* $s_\alpha + 1 \le i \le s_\alpha + s_\beta$, *denote* R_i *by* R_{β_j} *where* $i = s_\alpha + j$. *Then*

$$R_{\alpha_i} = \Delta(\alpha_i - 2i+1,\ y)\Delta(\alpha_i,\ -1) ,$$

$$R_{\beta_i} = \Delta(\beta_i - 2i+1,\ y^{-1})\Delta(\beta_i,\ -1) .$$

PROOF : First let $1 \le i \le s_\alpha$. The axial distance from $\underline{r_i}$ to $\underline{r_i+1}$ is $-\alpha_i$ while for $\underline{k}$ and $\underline{k+1}$ in the same row of T^{μ}_1 , the axial distance from $\underline{k}$ to $\underline{k+1}$ is 1 . Therefore

$$\rho_1(r_{i-1}+1) = \sum_{k=2}^{r_{i-1}+1} (\rho_{k,1} + 1) = \sum_{j=1}^{i-1} (2(\alpha_j - 1) + (1-\alpha_j)) = \sum_{j=1}^{i-1} (\alpha_j - 1).$$

Also $\nu(a(r_{i-1}+1+k)) = yx^{2(m+k)}$ where $m = \sum_{j=1}^{i-1} \alpha_j$ and $k = 0, \ldots, \alpha_i$.
Therefore

$$\nu(a(r_{i-1}+1+k))^{-1} M^{\mu}_{1,1}(a(r_{i-1}+1+k))^2$$

$$= [yx^{2(m+k)}]^{-1} y^2 x^{2[m-(i-1)+2k]} = yx^{-2(i-1)+2k}$$

for $k = 0, \ldots, \alpha_i$.

Furthermore $\nu(a(r_{i-1}+1+k, r_i)) = x^{\alpha_i - k}$ and

$$M^{\mu}_{1,1}(a(r_{i-1}+1+k, r_i))^2 = x^{2(r_i - r_{i-1} - k)} = x^{2(\alpha_i - k)}$$

for $k = 1, \ldots, \alpha_i$ by (3.3.6), as we have defined $a(m+1, m) = 1$. Thus (3.3.7) implies

$$R_{\alpha_i} = \sum_{k=1}^{\alpha_i} (1 + yx^{-2(i-1)+2(k-1)})x^{\alpha_i - k}$$

$$= \sum_{k=1}^{\alpha_i} (x^{\alpha_i - k} + yx^{\alpha_i + k - 2i})$$

$$= (1 + yx^{\alpha_i - 2i+1})(1 + \ldots + x^{\alpha_i - 1})$$

$$= \Delta(\alpha_i - 2i+1, y)\Delta(\alpha_i, -1) .$$

We now turn to the second part of the proposition. Observe first that the axial distance from $\underline{r_{s_\alpha}}$ to $\underline{r_{s_\alpha}+1}$ in the canonical tableaux T^{μ}_1 is by definition (1.1.6) the axial distance from the last square in the diagram (α) to the first square in the first row of (α). Therefore the path traversed from the letter $\underline{1}$ to the letter $\underline{r_{s_\alpha}+1}$ is a closed path in terms of axial distance. As a result $\sum_{j=2}^{r_{s_\alpha}+1} \rho_{j,1} = 0$ and $\rho_1(r_{s_\alpha}+1) = \sum_{j=2}^{r_{s_\alpha}+1} \rho_{j,1} + 1 = |\alpha|$. For $i > s_\alpha$ set $\ell = i - s_\alpha$. Then

$$\rho_1(r_{i-1}+1) = |\alpha| + \sum_{k=r_{s_\alpha}+2}^{r_{i-1}+1} \rho_{k,1} + 1$$

$$= |\alpha| + \sum_{j=1}^{\ell-1} \beta_j - 1 .$$

Also $\nu(a(r_{i-1}+1+k)) = -x^{2(m+k)}$ where $m = |\alpha| + \sum_{j=1}^{\ell-1} \beta_j$ and $k = 0, \ldots, \beta_\ell$. Therefore

$$\nu(a(r_{i-1}+1+k))M^\mu_{1,1}(a(r_{i-1}+1+k)) = [yx^{2(m+k)}]^{-1}x^{2[m-(\ell-1)+2k]}$$

$$= y^{-1}x^{-2(\ell-1)+2k} .$$

The argument used in the first part of the proposition can now be used to give

$$R_{\beta_i} = \Delta(\beta_i-2i+1, y^{-1})\Delta(\beta_i, -1) .$$

This completes the proof.

We now start evaluation of the sum

$$\sum_{j=g_c}^{f^\mu} M^\mu_{1,j}(a(r_i+1, n))M^\mu_{j,1}(\widehat{a(r_i+1, n)})$$

in the expression for $E^{(\mu_c-)}$ given by (3.3.8).

PROPOSITION (3.3.10) *Let* c *be as above. Then*

(i) $M^{\mu}_{i,j}(a(k, n)) = 0$ *for* $k > r_c+1$ *and for all* $j \geq g_c$.

(ii) $M^{\mu}_{i,j}(a(r_c+1, n)) = 0$ *for all* $j > g_c$ *while*

$$M^{\mu}_{1,g_c}(a(r_c+1, n)) = \prod_{i=1}^{n-r_c} M^{\mu}_{e_{i-1},e_i}(a_{r_c+i})$$

where $T^{\mu}_{e_o} = T^{\mu}_1$, *the canonical tableaux of shape* (μ), *and*

$T^{\mu}_{e_i} = w_{r_c+i}T^{\mu}_{e_{i-1}}$, $i \geq 1$.

(iii) $M^{\mu}_{i,j}(a(k, n)) = \left(\sum_{i=1}^{f^{\mu}} M_{1,i}(a(k, r_c))\right)M^{\mu}_{1,g_c}(a(r_c+1, n))$ *for*

$j \geq f_1$ *and* $k \leq r_c$.

(iv) $\sum_{j=1}^{f^{\mu}} M^{\mu}_{1,j}(a(r_k+1, r_c)) = \prod_{i=k}^{c-1}\left(\sum_{j=1}^{f^{\mu}} M^{\mu}_{1,j}(a(r_i+1, r_{i+1}))\right)$ *for* $k < r_c$.

PROOF : (i) and (ii). For $j \geq g_c$, the letter $\underline{n}$ occupies the last square in row c by definition of (μ_c-). Suppose $M^{\mu}_{1,j}(a(k, n)) \neq 0$, $j \geq g_c$. By proposition (3.2.8) there exists a $w \in W$ such that $wT^{\mu}_1 = T^{\mu}_j$, w expressible as a product of distinct generators chosen from $\{w_k, \ldots, w_n\}$. It follows that w fixes all letters $\leq k-1$ in T^{μ}_1 . Hence if $k > r_c+1$, the letter $\underline{r_c}$ occupies the same position in the tableaux T^{μ}_1 and T^{μ}_j , i.e. the last square in row c . This is impossible. Therefore $M^{\mu}_{1,j}(a(k, n)) = 0$ for $k > r_c+1$ which proves (i).

On the other hand, if $k = r_c+1$, $wT_1^\mu = T_j^\mu$ implies $w = w'w_{r_c+1}$, since the letter $\underline{r_c}$ must be moved under the action of w. Therefore r_c+1 is in the last square of row c in $w_{r_c+1}T_1^\mu = T_{e_1}^\mu$. Similarly, as w_{r_c+1} does not occur in the expression of w' as a product of distinguished generators, $w' = w''w_{r_c+2}$ because the letter r_c+1 must be moved under the action of w'. Therefore r_c+2 is in the last square of row c in $w_{r_c+2}T_{e_1}^\mu = T_{e_2}^\mu$. Continuation of this argument allows us to conclude $j = g_c$ and

$$M_{1,g_c}^\mu(a_{r_c+1}\cdots a_n) = M_{1,e_1}^\mu(a_{r_c+1})M_{e_1,e_2}^\mu(a_{r_c+2})\cdots M_{e_{n-k},g_c}^\mu(a_n)$$

where $w_{r_c+i}T_{e_{i-1}}^\mu = T_{e_i}^\mu$. This proves (ii).

(iii) and (iv). Let T_e^μ denote a standard tableaux obtained from T_1^μ by any rearrangement of the letters $\underline{1}, \ldots, \underline{r_j}$, for $j \leq c$. This amounts to a rearrangement of the above letters among the first j rows of (μ). Because the first j rows, $(j \leq c)$ are of equal length, any such arrangment of $\underline{1}, \ldots, \underline{r_j}$ in the first j rows must have $\underline{r_j}$ in the last box of row j. Thus the position of the letters $\underline{r_j}, \ldots, \underline{n}$ is the same in wT_e^μ and wT_1^μ, for any w whose reduced expression as a product of elements from R is made up entirely of the generators w_i, $i = r_j+1, \ldots, n$. It follows that

$$M_{e,i}^\mu(a(r_j+1,\ k)) \;=\; M_{1,i}^\mu(a(r_j+1,\ k))$$

for $j \leq c$ and $k \geq r_j+1$.

Therefore

$$M^{\mu}_{1,i}(a(r_j+1, r_c)) = \sum_{s,t,\ldots,u} M^{\mu}_{1,s}(a(r_j+1, r_{j+1})) \times$$

$$M^{\mu}_{s,t}(a(r_{j+1}+1, r_{j+2}))\ldots M^{\mu}_{u,i}(a(r_{c-1}+1, r_c))$$

$$= \sum_{s,t,\ldots} M^{\mu}_{1,s}(a(r_j+1, r_{j+1})) \times$$

$$M^{\mu}_{1,t}(a(r_{j+1}+1, r_{j+2}))\ldots M^{\mu}_{1,i}(a(r_{c-1}+1, r_c))$$

$$= \prod_{i=j}^{c-1} \sum_{k=1}^{f^{\mu}} M^{\mu}_{1,k}(a(r_i+1, r_{i+1}))$$

by the above argument and the ordering of the last letter sequence. For the same reason

$$M^{\mu}_{1,i}(a(k, n)) = \sum_{j=1}^{f^{\mu}} M^{\mu}_{1,j}(a(k, r_c))M^{\mu}_{j,i}(a(r_c+1, n))$$

$$= \left(\sum_{j=1}^{f^{\mu}} M^{\mu}_{1,j}(a(k, r_c))\right)M^{\mu}_{1,g_c}(a(r_c+1, n)) .$$

This proves (iii) and (iv).

Using this proposition and proposition (3.2.8) it is straight forward from (3.3.8) that $E^{(\mu_c-)}$ can be written as

$$E^{(\mu_c-)} = \sum_{i=1}^{c} R_i \nu(a(r_i+1, n))^{-1} \sum_{j=g_c}^{f^{\mu}} M^{\mu}_{1,j}(a(r_i+1, n))M^{\mu}_{j,1}(\widehat{a(r_i+1, n)})$$

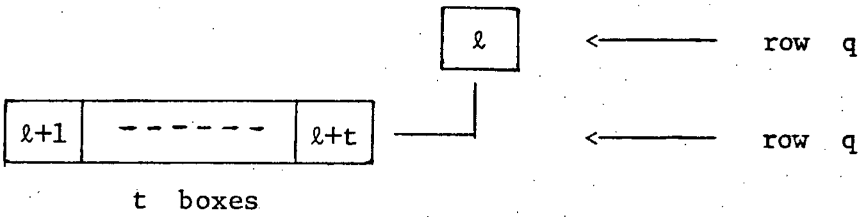
$$= D_1D_2$$

where

$$D_1 = \sum_{i=1}^{c} R_i \nu(a(r_i+1,\ r_c))^{-1} \prod_{j=1}^{c-1} \left\{ \sum_{k=1}^{f^\mu} M^\mu_{1,k}(a(r_j+1,\ r_{j+1})) \times M^\mu_{k,1}(\widehat{a(r_j+1,\ r_{j+1})}) \right\},$$

$$D_2 = \nu(a(r_c+1,\ n))^{-1} M^\mu_{1,g_c}(a(r_c+1,\ n)) M^\mu_{g_c,1}(\widehat{a(r_c+1,\ n)}) .$$

Consider a part of the Young diagram of shape (α, β) consisting of the last box in the p^{th} row and the entire q^{th} row, $p < q$. Let $T^\mu_{e_o}$ be a standard tableaux with $t+1$ letters $\underline{\ell}$, $\underline{\ell+1}$, ..., $\underline{\ell+t}$ distributed in



increasing order in this part. Set $T^\mu_{e_i} = w_{\ell+i} \cdots w_{\ell+1} T^\mu_{e_o}$, $i = 1, \ldots, t$. Then $w_{\ell+i}$ is a row permutation of $T^\mu_{e_j}$ for $j \neq i-1$ or i, as the letters $\underline{\ell+i-1}$ and $\underline{\ell+i}$ have either not been moved from row q or have been returned to row q. Hence

$$M^\mu_{e_j,k}(a_{\ell+i}) = M^\mu_{k,e_j}(a_{\ell+i}) = 0$$

for $k \neq e_j$ and $j \neq i-1$ or i, by theorem (2.2.4). Therefore

$$\sum_{j=1}^{f^\mu} M^\mu_{e_o,j}(a(\ell+1,\ \ell+t)) M^\mu_{j,e_o}(\widehat{a(\ell+1,\ \ell+t)})$$

$$= \prod_{i=1}^{t} [M^\mu_{e_o,e_o}(a_{\ell+i})]^2$$

$$+ \sum_{k=1}^{t-1} \prod_{j=1}^{k} M^{\mu}_{e_{j-1},e_j}(a_{\ell+j}) M^{\mu}_{e_j,e_{j-1}}(a_{\ell+j}) \prod_{j=k+1}^{t} [M^{\mu}_{e_k,e_k}(a_{\ell+j})]^2$$

$$(3.3.12) \qquad + \prod_{i=1}^{t} M^{\mu}_{e_{i-1},e_i}(a_{\ell+i}) M^{\mu}_{e_i,e_{i-1}}(a_{\ell+i}) ,$$

by proposition (3.2.8). Label these three terms A, B, and C respectively. As the entries of $M^{\mu}(a_i)$ depend only on the position of the letters $\underline{i-1}$ and $\underline{i}$ in a standard tableaux, the above computation is independent of the letter $\underline{\ell}$ and depends only on the rows p and q. Hence set

$$(3.3.13) \qquad \begin{aligned} F_{p,q} &= \nu(a(\ell+1, \ell+t))^{-1}(A + B) = x^{-t}(A + B) , \\ f_{p,q} &= \nu(a(\ell+1, \ell+t))^{-1} C = x^{-t} C . \end{aligned}$$

We now rewrite (3.3.11) as follows .

Let d_j be such that $w(r_{c+1}, r_j)^{-1} T_1 = T_{d_j}$. Then $d_j = e_{r_j - r_c}$ where the e_i's are defined as in (ii) of proposition (3.3.10). The proof of (3.3.10(ii)) shows the letters $\underline{r_j}, \underline{r_j+1}, \ldots, \underline{r_{j+1}}$ occur in the last square of row c and in the $j+1$ st row of T_{d_j} in increasing order, i.e. in a configuration as described above. Hence (3.3.10(ii)) and proposition (3.2.8) show

$$\nu(a(r_c+1, n))^{-1} M^{\mu}_{1,g_c}(a(r_c+1, n)) M^{\mu}_{g_c,1}(\widehat{a(r_c+1, n)})$$

$$= \prod_{j=c}^{s-1} \nu(a(r_j+1, r_{j+1}))^{-1} \prod_{i=r_j-r_c+1}^{r_{j+1}-r_c} M^{\mu}_{e_{i-1},e_i}(a_{r_c+i}) M^{\mu}_{e_i,e_{i-1}}(a_{r_c+i})$$

$$(3.3.14) \qquad = \prod_{j=c+1}^{s} f_{c,j} \quad .$$

Similarly, for $j < c$ the letter $\underline{r_j}$, $\underline{r_j+1}$, ..., $\underline{r_{j+1}}$ appear in the last square of row j and in the $j+1$ st row of T_1^μ. Noting that $f_{p,q} = 0$ for rows p and q of equal length and both belonging to the same tableaux of T_1^μ the computation (3.3.12) shows

$$\nu(a(r_j+1,\ r_{j+1}))^{-1} \sum_{i=1}^{f^\mu} M^\mu_{1,i}(a(r_j+1,\ r_{j+1}))M_{i,1}(\widehat{a(r_j+1,\ r_{j+1})})$$

$$(3.3.15) \qquad = F_{j,j+1}$$

for $j < c$. Substition of (3.3.14) and (3.3.15) into (3.3.11) now gives

$$(3.3.16) \qquad E^{(\mu_c-)} = \left(\sum_{i=1}^{c-1} R_i \prod_{j=i}^{c-1} F_{j,j+1} + R_c\right) \prod_{i=c+1}^{s} f_{c,i} .$$

PROPOSITION (3.3.17) *Let* m *be the axial distance from the first square in the* q^{th} *row to the last square in the* p^{th} *row and assume row* p *is a row of the tableaux* T^α *and row* q *is a row of the tableaux* T^β *of* $T_{e_o} = (T^\alpha, T^\beta)$. *Let* t *be the length of the* q^{th} *row. Then*

$$\text{(i)} \qquad f_{p,q} = \frac{\Delta(m+1,\ y)\Delta(m-t,\ y)}{\Delta(m,\ y)\Delta(m-t+1,\ y)} ,$$

$$\text{(ii)} \qquad F_{p,q} = \frac{(x-1)^2\Delta(t,\ -1)}{x\Delta(m,\ y)\Delta(m-t+1,\ y)} .$$

PROOF : We use the notations of (3.3.13).

(i) Because the letter $\underline{\ell}$ appears in the last box in row p and $\underline{\ell+1}$ in the first box of row q, the axial distance from $\underline{\ell+1}$ to $\underline{\ell}$ in $T^\mu_{e_o}$ is m and the axial distance from $\underline{\ell+i}$ to $\ell+i-1$ in $T^\mu_{e_{i-1}}$ is $m-i+1$, $i = 1, \ldots, t$. Therefore

$$M^{\mu}_{e_{i-1},e_i}(a_{\ell+i})M^{\mu}_{e_i,e_{i-1}}(a_{\ell+i}) = \frac{x\Delta(m-i,\ y)\Delta(m-i+2,\ y)}{[\Delta(m-i+1,\ y)]^2}$$

by theorem (2.2.4) . Thus (3.3.11) implies

$$C = \prod_{i=1}^{t} \frac{x\Delta(m-i,\ y)\Delta(m-i+2,\ y)}{[\Delta(m-i+1,\ y)]^2}$$

$$= \frac{x^t\Delta(m+1,\ y)\Delta(m-t,\ y)}{\Delta(m,\ y)\Delta(m-t+1,\ y)}$$

As $f_{p,q} = x^{-t}c$, we have the required result.

(ii) We have

$$M^{\mu}_{e_{i-1},e_{i-1}}(a_{\ell+i}) = \frac{x-1}{\Delta(m-i+1,\ y)}, \qquad i = 1, \ldots, t ,$$

by theorem (2.2.7) and as $w_{\ell+i}$ is a row permutation of $T^{\mu}_{e_j}$ for $i > j+1$, $M^{\mu}_{e_j,e_j}(a_{\ell+i}) = x$ for $i > j+1$, by theorem (2.2.4) . Hence (3.3.11) implies

$$A = \frac{x^{2(t-1)}(x-1)^2}{[\Delta(m,\ y)]^2} .$$

Furthermore, using (i) and (3.3.11)

$$B = \sum_{k=1}^{t-1}\left(\prod_{j=1}^{k} \frac{x\Delta(m-i,\ y)\Delta(m-i+2,\ y)}{[\Delta(m-i+1,\ y)]^2}\right)\frac{(x-1)^2x^{2(t-1-k)}}{[\Delta(m-k,\ y)]^2}$$

$$= \sum_{k=1}^{t-1} \frac{x^k\Delta(m+1,\ y)\Delta(m-k,\ y)}{\Delta(m,\ y)\Delta(m-k+1,\ y)} \cdot \frac{(x-1)^2x^{2(t-1-k)}}{[\Delta(m-k,\ y)]^2}$$

$$= \left(\frac{x^{t-1}(x-1)^2\Delta(m+1,\ y)}{\Delta(m,\ y)}\right)\sum_{k=1}^{t-1} \frac{x^{t-1-k}}{\Delta(m-k+1,\ y)\Delta(m-k,\ y)}$$

An easy induction argument, using the fact that

$$x\Delta(k, -1)\Delta(m-k-1, y) + \Delta(m, y) = \Delta(k+1, -1)\Delta(m-k, y)$$

shows

$$\sum_{k=1}^{t-1} \frac{x^{t-1-k}}{\Delta(m-k+1, y)\Delta(m-k, y)} = \frac{\Delta(t-1, -1)}{\Delta(m, y)\Delta(m-t+1, y)} .$$

Therefore

$$B = \frac{x^{t-1}(x-1)^2\Delta(t-1, -1)\Delta(m+1, y)}{[\Delta(m, y)]^2\Delta(m-t+1, y)} .$$

Finally, the above computations give

$$A + B = \frac{x^{t-1}(x-1)^2}{[\Delta(m, y)]^2} \; x^{t-1} + \frac{\Delta(t-1, -1)\Delta(m+1, y)}{\Delta(m-t+1, y)}$$

$$= \frac{x^{t-1}(x-1)^2}{[\Delta(m, y)]^2} \; \frac{\Delta(m, y)\Delta(t, -1)}{\Delta(m-t+1, y)}$$

$$= \frac{x^{t-1}(x-1)^2\Delta(t, -1)}{\Delta(m, y)\Delta(m-t+1, y)}$$

As $F_{p,q} = x^{-t}[A + B]$, we have the required result.

COROLLARY (3.3.18) *Let* p *and* q *be rows belonging to the same tableaux of* $T^{\mu}_{e_o}$. *With the same notations as in proposition* (3.3.17) *we have*

(i) $$f_{p,q} = \frac{\Delta(m+1, -1)\Delta(m-t, -1)}{\Delta(m, -1)\Delta(m-t+1, -1)}$$

(ii) $$F_{p,q} = \frac{\Delta(t, -1)}{x\Delta(m, -1)\Delta(m-t+1, -1)}$$

If, furthermore, $p = q-1$ *and the rows* q *and* $q-1$ *have the same length, then* $F_{q-1,q} = x^{-1}$.

PROOF : If the rows p and q belong to the same tableaux of $T^{\mu}_{e_o}$, the matrices $M(k, y)$ are replaced by the matrices $M(k, -1)$ in proposition (3.3.17) by theorem (2.2.7) . The matrices $M(k, -1)$ are obtained from $M(k, y)$ by setting $y = -1$, whence the first statement. For the second statement we have $m = t$ and $\Delta(1, -1) = 1$.

3.4. GENERIC DEGREES

Recall from (1.1) that $(\alpha)' = (\alpha_1', \ldots, \alpha_{s_\alpha'}')$ denotes the partition conjugate to the partition $(\alpha) = (\alpha_1, \ldots, \alpha_{s_\alpha})$

DEFINITION (3.4.1) *Let* (α, β) *be a double partition with corresponding ordered pair of Young diagrams* $(D(\alpha), D(\beta))$. *For the* (i,j)*-square of* $D(\alpha)$, *set*

$$h^\alpha_{i,j} = (\alpha_i - j) + (\alpha_j' - i) + 1 ,$$

$$g^\alpha_{i,j} = (\alpha_i - j) + (\beta_j' - i) + 1 ,$$

where $\beta_j' = 0$ *for* $j > s_{\beta'}$. *For the* (i,j)*-square of* $D(\beta)$, *set*

$$h^\beta_{i,j} = (\beta_i - j) + (\beta_j' - i) + 1 ,$$

$$g^\beta_{i,j} = (\beta_i - j) + (\alpha_j' - i) + 1 ,$$

where $\alpha_j' = 0$ *for* $j > s_{\alpha'}$. *We call* $h^\alpha_{i,j}$ *(resp.* $h^\beta_{i,j}$*) the hook length of the* (i,j)*-square of* $D(\alpha)$ *(resp.* $D(\beta)$*). We call* $g^\alpha_{i,j}$ *(resp.* $g^\beta_{i,j}$*) the split hook length of the* (i,j)*-square of* $D(\alpha)$ *(resp.* $D(\beta)$*).*

From (1.1.1), $h^\alpha_{i,j}$ $(h^\beta_{i,j})$ is the length of the (i,j)-hook of $D(\alpha)$ $(D(\beta))$. As the last square in the i^{th} row of $D(\alpha)$ has coordinates (i, α_i) while the last square in the j^{th} column of $D(\alpha)$ has coordinates (α_j', j), (1.1.5) shows $h^\alpha_{i,j}$ equals the axial distance from the (α_j',j)-square to the (i,α_i)-square plus one in $D(\alpha)$. $h^\beta_{i,j}$ has the same interpretation for the diagram $D(\beta)$. The split hook lengths have a corresponding interpretation. Namely, $g^{\alpha}_{i,j}$ is the axial distance from the square in the j^{th} column of $D(\beta)$, the (β'_j, j)-square, to the last square in the i^{th} row of $D(\alpha)$, the (i,α_i)-square, plus one. Similarly $g^{\beta}_{i,j}$ equals the axial distance from the end of the j^{th} column of $D(\alpha)$ to the end of the i^{th} row of $D(\beta)$, plus one.

PROPOSITION (3.4.2) (i) *For the double partition* $(\alpha, (0))$

$$E^{((\alpha_c-),\ (0))} = \Delta(\alpha_c - c,\ y)H^{\alpha} \quad ,$$

where

$$H^{\alpha} = \prod_{j=1}^{c-1} \frac{\Delta(h_{j,\alpha_c},\ -1)}{x\Delta(h_{j,\alpha_c}-1,\ -1)} \prod_{j=1}^{\alpha_c-1} \frac{\Delta(h^{\alpha}_{c,j},\ -1)}{\Delta(h^{\alpha}_{c,j}-1,\ -1)} \ .$$

Furthermore

$$\Delta(\alpha_c - c,\ y) = \Delta(g^{\alpha}_{c,\alpha_c},\ y) \prod_{j=1}^{\alpha_c-1} \frac{\Delta(g^{\alpha}_{c,j},\ y)}{\Delta(g^{\alpha}_{c,j}-1,\ y)} \ .$$

(ii) *For the double partition* $((0), \beta)$

$$E^{((0),\ (\beta_c-))} = \Delta(\beta_c - c,\ y^{-1})H^{\beta} \quad ,$$

where H^{β} *is defined as in* (i) *the partition* (β) *replacing the partition* (α). *Furthermore*

$$\Delta(\beta_c - c,\ y^{-1}) = (yx^{1-c})^{-1}\Delta(-g^{\beta}_{c,\beta_c},\ y) \prod_{j=1}^{\beta_c-1} \frac{\Delta(-g^{\beta}_{c,j},\ y)}{x\Delta(-g^{\beta}_{c,j}+1,\ y)} \ .$$

(iii) *For the double partition* (α, β), $(\alpha) \neq (0)$, $(\beta) \neq (0)$, *let row* d *be the row of* D(β) *such that the* α_c*th column of* D(β) *ends in row* d *if* $\alpha_c \leq \beta_1$. *If* $\beta_1 < \alpha_c$, *set* $d = 0$. *Then*

$$E^{((\alpha_c-),\ (\beta))} = E^{((\alpha_c-),\ (0))} G$$

where

$$G = \frac{\Delta(g^{\alpha}_{c,\alpha_c},\ y)}{\Delta(\alpha_c-c,\ y)} \prod_{j=1}^{\alpha_c-1} \frac{\Delta(g^{\alpha}_{c,j},\ y)}{\Delta(g^{\alpha}_{c,j}-1,\ y)} \prod_{j=1}^{d} \frac{\Delta(-g^{\beta}_{j,\alpha_c},\ y)}{\Delta(-g^{\beta}_{j,\alpha_c}+1,\ y)}$$

It is understood that the last product in the definition of G *is taken to be equal to* 1 *if* $d = 0$.

PROOF : We will show the expression for $E^{(\mu_c-)}$ given by (3.3.16) has the desired form for each of the cases mentioned. Direct computations are all that is required.

(i) For the double partition (α, (0)), the row c is a row of D(α) and $\alpha_i = \alpha_c$ for $i = 1, \ldots, c$. Then by Proposition (3.3.9) and Corollary (3.3.18)

$$\sum_{i=1}^{c-1} R_i \prod_{i=1}^{c-1} F_{i,i+1} + R_c$$

$$= \Delta(\alpha_c,\ -1) \sum_{i=1}^{c} \Delta(\alpha_c-2i+1,\ y)x^{i-c}$$

$$(3.4.\ ) \qquad = x^{1-c}\Delta(\alpha_c,\ -1)\Delta(c,\ -1)\Delta(\alpha_c-c,\ y) \quad .$$

Let m_i denote the axial distance from the first square in row i to the last square in row c, $i > c$. Then $m_{i+1} = m_i+1$ and using Corollary (3.3.18)

$$\prod_{i=c+1}^{s_\alpha} f_{c,i} = \prod_{i=c+1}^{s_\alpha} \frac{\Delta(m_i+1, -1)\Delta(m_i-\alpha_i, -1)}{\Delta(m_i, -1)\Delta(m_i-\alpha_i+1, -1)}$$

$$= \frac{\Delta(m_{c+1}-\alpha_{c+1}, -1)}{\Delta(m_{c+1}, -1)} \prod_{i=c+1}^{s_\alpha} \frac{\Delta(m_{i+1}-\alpha_{i+1}, -1)}{\Delta(m_{i+1}-\alpha_i, -1)} .$$

Set $\alpha_j = 0$ for $j > s_\alpha$ and rewrite the above as

$$\prod_{i=c+1}^{s_\alpha} f_{c,i} = \frac{\Delta(m_{c+1}-\alpha_{c+1}, -1)}{\Delta(m_{c+1}, -1)} \prod_{i=c+1}^{s_\alpha} \prod_{k=\alpha_{i+1}+1}^{\alpha_i} \frac{\Delta(m_{i+1}-k+1, -1)}{\Delta(m_{i+1}-k, -1)}$$

where the second product is taken equal to 1 for all i such that $\alpha_{i+1} = \alpha_i$. If $\alpha_{i+1} < \alpha_i$, $\alpha_i-\alpha_{i+1}$ equals the number of columns of $D(\alpha)$ which end in row i while if $\alpha_{i+1} = \alpha_i$, no column ends in row i. Then for $i \geq c$ such that $\alpha_{i+1} < \alpha_i$ we have by (3.4.1) and (1.1.5)

$$m_{i+1} - k + 1 = \alpha_c + (i-c) - k + 1 = h^\alpha_{c,k} , \quad \alpha_{i+1}+1 \leq k \leq \alpha_i .$$

Thus

$$\prod_{k=\alpha_{i+1}+1}^{\alpha_i} \frac{\Delta(m_{i+1}-k+1, -1)}{\Delta(m_{i+1}-k, -1)} = \prod_{k=\alpha_{i+1}+1}^{\alpha_i} \frac{\Delta(h^\alpha_{c,k}, -1)}{\Delta(h^\alpha_{c,k}-1, -1)} .$$

Furthermore, by the choice of the row c, $m_{c+1} = \alpha_c > \alpha_{c+1}$ and $h^\alpha_{c,\alpha_c} = 1$, so that

$$\frac{\Delta(m_{c+1}-\alpha_{c+1}, -1)}{\Delta(m_{c+1}, -1)} = \frac{1}{\Delta(\alpha_c, -1)} \prod_{k=\alpha_{c+1}+1}^{\alpha_c-1} \frac{\Delta(h^\alpha_{c,k}, -1)}{\Delta(h^\alpha_{c,k}-1, -1)} .$$

The above computations show

$$(3.4.4) \qquad \prod_{i=c+1}^{s_\alpha} f_{c,i} = \frac{1}{\Delta(\alpha_c, -1)} \prod_{i=1}^{\alpha_c -1} \frac{\Delta(h^\alpha_{c,i}, -1)}{\Delta(h^\alpha_{c,i}-1, -1)} .$$

Finally, $c = h^\alpha_{1,\alpha_c}$, so that

$$(3.4.5) \qquad x^{1-c}\Delta(c, -1) = x^{1-c}\Delta(h^\alpha_{1,\alpha_c}, -1) = \prod_{j=1}^{c-1} \frac{\Delta(h^\alpha_{j,\alpha_c}, -1)}{x\Delta(h^\alpha_{j,\alpha_c}-1, -1)}$$

(3.4.3) - (3.4.5) combine to give the desired expression for $E^{((\alpha_c-), (0))}$. As the second partition is the empty partition (0), $\alpha_c - c = g_{c,1}$ and $g^\alpha_{c,j} = g^\alpha_{c,j+1} + 1$, $j = 1, \ldots, c-1$, so that the expression for $\Delta(\alpha_c - c, y)$ given in the statement of (i) is just a telescopic product.

(ii) The proof of (ii) is entirely similar to (i) and will be omitted. Simply replace (α) with (β) and note that by Proposition (3.3.9) y must be replaced with y^{-1} in (3.4.3). Also $\beta_c - c = g^\beta_{c,1}$ so that

$$\Delta(\beta_c - c, y^{-1}) = (yx^m)^{-1}\Delta(m, y) , \qquad m = -g^\beta_{c,1} .$$

(iii) In this case the row c is a row of $D(\alpha)$. Thus employing (3.3.16)

$$\begin{aligned} E^{((\alpha_c-), (\beta))} &= \left(\sum_{i=1}^{c-1} R_i \prod_{j=1}^{i} F_{j,j+1} + R_c \right) \prod_{i=c+1}^{s_\alpha} f_{c,i} \prod_{i=s_\alpha+1}^{s_\alpha+s_\beta} f_{c,i} \\ &= E^{((\alpha_c-), (0))} \prod_{i=s_\alpha+1}^{s_\beta} f_{c,i} . \end{aligned}$$

It suffices to show

$$(3.4.6) \qquad \prod_{i=s_\alpha+1}^{s_\beta} f_{c,i} = G .$$

Let m_i, $i = 1, \ldots, s$, denote the axial distance from the first box in the i^{th} row of $D(\beta)$ to the last box in row c of $D(\alpha)$. Then $m_{i+1} = m_i + 1$ and as in (i)

$$\prod_{i=s_\alpha+1}^{s_\alpha+s_\beta} f_{c,i} = \frac{\Delta(m_{s_\beta}+1, y)}{\Delta(m, y)} \prod_{i=1}^{s_\beta} \frac{\Delta(m_i-\beta_i, y)}{\Delta(m_{i+1}-\beta_i, y)} .$$

Let $\beta_j = 0$ for $j > s_\beta$ and let d be as in the statement of the theorem. Rewrite the above as

$$(3.4.7)\quad \left(\frac{\Delta(m_{d+1}-d_c+1, y)}{\Delta(m, y)} \prod_{i=1}^{d} \frac{\Delta(m_i-\beta_i, y)}{\Delta(m_{i+1}-\beta_i, y)} \right) \times$$

$$\left(\frac{\Delta(m_{d+1}-\beta_{d+1}, y)}{\Delta(m_{d+1}-\alpha_c+1, y)} \prod_{i=d+1}^{s_\beta} \frac{\Delta(m_{i+1}-\beta_{i+1}, y)}{\Delta(m_{i+1}-\beta_i, y)} \right)$$

Label the two bracketed expressions in (3.4.7) A and B respectively.

By (3.4.1) and (1.1.5),

$$m_{d+1} - \alpha_c + 1 = (\alpha_c-\alpha_c) + (d-c) + 1 = g^{\alpha}_{c,\alpha_c}$$

and

$$m_i - \beta_i = \alpha_c + (i-c) - 1 - \beta_i$$

$$= -[(\beta_i-\alpha_c) + (c-i) + 1] = -g^{\beta}_{i,\alpha_c} .$$

Therefore

$$A = \frac{\Delta(g^{\alpha}_{c,\alpha_c}, y)}{\Delta(\alpha_c-c, y)} \prod_{i=1}^{d} \frac{\Delta(-g^{\beta}_{i,\alpha_c}, y)}{\Delta(-g^{\beta}_{i,\alpha_c}+1, y)} .$$

As $\beta_{d+1} < \alpha_c$, and as for $\beta_{i+1} \leq k \leq \beta_i - 1$,

$$m_{i+1} - k = \alpha_c + (i-c) - k = g^{\alpha}_{c,k+1} ,$$

computations identical to those employed in (i) show

$$B = \prod_{i=1}^{\alpha_c - 1} \frac{\Delta(g^{\alpha}_{c,i}, y)}{\Delta(g^{\alpha}_{c,i} - 1, y)} .$$

The above expressions for A and B and (3.4.7) show (3.4.6) as required. This completes the proof of (iii) and the proposition.

We can now obtain an elegant formula for

$$C^{\mu} = (\deg \chi^{\mu})^{-1} \sum_{w \in W(B_n)} \nu(a_w)^{-1} \chi^{\mu}(a_w) \chi^{\mu}(\hat{a}_w) .$$

For the double partition $(\mu) = (\alpha, \beta)$ set

(3.4.8)

$$H^{\alpha}_{i,j} = x^{1-\overline{\alpha}_j} \Delta(h^{\alpha}_{i,j}, -1) ,$$

$$H^{\beta}_{i,j} = x^{1-\overline{\beta}_j} \Delta(h^{\beta}_{i,j}, -1) ,$$

$$G^{\alpha}_{i,j} = \Delta(g^{\alpha}_{i,j}, y) ,$$

$$G^{\beta}_{i,j} = (yx^m)^{-1} \Delta(m, y) , \quad m = - g^{\beta}_{i,j} .$$

THEOREM (3.4.9) *For the irreducible character* χ^{μ} *of* $\mathfrak{A}^{K}(B_n)$, $(\mu) = (\alpha, \beta)$ *a double partition of* $n \geq 2$,

$$C^{\mu} = \prod_{(i,j)\varepsilon(\alpha)} H^{\alpha}_{i,j} G^{\alpha}_{i,j} \prod_{(i,j)\varepsilon(\beta)} H^{\beta}_{i,j} G^{\beta}_{i,j} .$$

PROOF : By induction on $n \geq 2$. The statement of the theorem readily be checked for the representation of $\mathfrak{A}^{K}(B_2)$ given by Theorem (2.2.7). Let $n > 2$. The Young diagram $D(\mu_c-)$ is obtained from the Young diagram $D(\mu)$ by deleting the last square from row c . Thus the hook lengths of the squares of $D(\mu_c-)$ differ from the hook lengths of the squares of $D(\mu)$ only for squares in the c^{th} row and $\overline{\alpha}_c$ or $\overline{\beta}_c$ column depending on whether the row c is a row of the diagram $D(\alpha)$ or the diagram $D(\beta)$ of $D(\mu) = (D(\alpha), D(\beta))$. Indeed, the hook lengths of these squares of $D(\mu_c-)$ are one less than the corresponding squares of $D(\mu)$. Similarly the split hook lengths of the squares of $D(\mu_c-)$ differ from the split hook lengths of the squares of $D(\mu)$ only in the c^{th} row and, if c is a row of $D(\alpha)$ and $\overline{\beta}_c \neq 0$, in the $\overline{\beta}_c$ column of $D(\beta)$. Again the split hook lengths of these squares of $D(\mu_c-)$ are one less than the corresponding squares of $D(\mu)$. Thus, from the computations of Proposition (3.4.2) we have that $E^{(\mu_c-)}$ is of the form

$$E^{(\mu_c-)} = G_{c,\lambda_c} \prod_{(i,j)} \frac{G^{\mu}_{i,j}}{G^{(\mu_c-)}_{i,j}} \prod_{(s,t)} \frac{H^{\mu}_{s,t}}{H^{(\mu_c-)}_{s,t}}$$

where $\lambda = \alpha$ or β, depending on whether the row c is a row of $D(\alpha)$ or $D(\beta)$, and where (i,j) and (s,t) run over the appropriate squares of

$D(\mu_c -)$, mentioned above, for which the hook lengths and split hook lengths differ from those of $D(\mu)$. As $C^{\mu} = C^{(\mu_c -)} E^{(\mu_c -)}$ by Theorem (3.2.6), this completes the induction and the proof of the theorem.

We can now give an explicit expression for the generic degree d_{χ} (3.1.3), of the irreducible characters χ of $\mathfrak{A}^K(B_n)$. Let $P_{B_n}(x, y)$ be the *Poincare polynomial* of $\mathfrak{A}^K(B_n)$,

(3.4.10) $$P_{B_n}(x, y) = \sum_{w \varepsilon W(B_n)} \nu(a_w) .$$

From ([10]),

$$P_{B_n}(x, y) = \sum_{i=0}^{n-1} (1 + x^i y)(1 + \ldots + x^i) .$$

COROLLARY (3.4.11) *For the irreducible character* χ^{μ} *of* $\mathfrak{A}^K(B_n)$, $(\mu) = (\alpha, \beta)$,

$$d_{\chi^{\mu}} = P_{B_n}(x, y) \Big/ \prod_{(i,j)\varepsilon(\alpha)} H^{\alpha}_{i,j} G^{\alpha}_{i,j} \prod_{(i,j)\varepsilon(\beta)} H^{\beta}_{i,j} G^{\beta}_{i,j} .$$

PROOF : This follows from the definition of $d_{\chi^{\mu}}$ (3.1.3), (3.4.10) and Theorem (3.4.9).

The generic degrees of the representation of $\mathfrak{A}^K(A_n)$ and $\mathfrak{A}^K(D_n)$ are readily obtained from Theorem (3.4.9) as well. In particular

COROLLARY (3.4.12) *Let* $\phi : D = Q[x, y] \longrightarrow Q(x)$ *be the homomorphism defined by* $\phi(y) = 0$. *Let* (α) *be a partition of* n *and let* χ^{α} *and*

$\chi^{(\alpha,\ (0))}$ *be the irreducible characters of* $\mathfrak{A}^K(A_{n-1})$ *and* $\mathfrak{A}^K(B_n)$ *corresponding to* (α) *and* $(\alpha, (0))$. *Let* $\phi^* : D_P \longrightarrow Q(x)$ *be the extension of* ϕ *to the ring of fractions,* $P = \ker \phi$. *Then* $d_{\chi^{(\alpha,\ (0))}} \in D_P$ *and* $\phi^*(d_{\chi^{(\alpha,\ (0))}}) = d_{\chi^\alpha}$.

PROOF : For $w \in W(B_n)$, define $\ell_1(w)$ to be the number of times w_1 occurs in a reduced expression for w in R, and set $\ell_2(w) = \ell(w) - \ell_1(w)$. Then

$$\nu(a_w) = y^{\ell_1(w)} x^{\ell_2(w)}, \qquad a_w \in \mathfrak{A}(B_n).$$

We first show

$$(3.4.13) \qquad \chi^{(\alpha,\ (0))}(a_w) = y^{\ell_1(w)} \chi^{(\alpha,\ (0))}(\bar{a}_w) \in D_P, \qquad w \in W(B_n).$$

Let $M(a)$ denote the matrix of $\pi^{(\alpha,\ (0))}(a)$ with respect to the basis $\{t_1, \ldots, t_f\}$ of $V^K_{(\alpha,\ (0))}$, $a \in \mathfrak{A}^K(B_n)$. By (2.2.6), $M(a_1) = yI$ and thus commutes with $M(a_i)$, $i = 2, \ldots, n$. Hence for $w \in W(B_n)$

$$M(a_w) = (M(a_1))^{\ell_1(w)} M(\bar{a}_w)$$

where

$$\bar{a}_w = \sum_{g \in W(A_{n-1})} c_g^w a_g, \qquad c_g^w \in Q[x].$$

From (2.2.6), $\chi^{(\alpha,\ (0))}(\bar{a}_w) \in Q(x) \subset D_P$, D_P considered as a subring of $K = Q(x, y)$. Thus we have shown (3.4.13). The rest of the proof is now clear. As $\ell_1(w) = \ell_1(w^{-1})$, we have

$$\phi^*(C^{(\alpha,\ (0))}) = \sum_{w\in W(A_{n-1})} \nu(a_w)^{-1}\chi^\alpha(a_w)\chi^\alpha(\hat{a}_w) = C^\alpha .$$

Similarly

$$\phi\Big(\sum_{w\in W(B_n)} \nu(a_w)\Big) = \sum_{w\in W(A_{n-1})} \nu(a_w)$$

and the statement of the corollary follows.

Thus by the above corollary and Theorem (3.4.9) we have, for (α) a partition of n ,

(3.4.14) $$d_{\chi^\alpha} = P_{A_{n-1}}(x) \Big/ \prod_{i,j} H^\alpha_{i,j}$$

where

$$P_{A_{n-1}}(x) = \prod_{i=0}^{n} (1 + \ldots + x^i).$$

COROLLARY (3.4.15) *Let* $\phi : D = Q[x, y] \longrightarrow Q(x)$ *be the homomorphism defined by* $\phi(y) = 1$. *Let* $(\mu) = (\alpha, \beta)$ *be a double partition of* n *with* $(\alpha) \neq (\beta)$ *and let* χ^μ *and* ψ^μ *be the irreducible characters of* $\mathfrak{A}^K(B_n)$ *and* $\mathfrak{A}^K(D_n)$ *corresponding to* (μ) . *Then* $\phi^*(d_{\chi^\mu}) = d_{\psi^\mu}$, *where* $\phi^* : D_P \longrightarrow Q(x)$ *is the extension of* ϕ *to the ring of fractions* D_P , $P = \ker \phi$

PROOF : From Theorem (2.3.9), ψ^μ is the restriction to $\mathfrak{A}^K(D_n)$ of the irreducible character χ^μ_ϕ of $\mathfrak{A}_{\phi,K}(B_n)$, $K = Q(x)$, where $\chi^\mu_\phi(a_w) = \phi^*(\chi^\mu(a_w))$. Thus by the definition of generic degree, to prove the corollary it is sufficient to prove

$$(3.4.16) \qquad \sum_{w\in W(B_n)} \nu(a_{w\phi})^{-1}\chi^\mu_\phi(a_{w\phi})\chi^\mu_\phi(\hat{a}_{w\phi}) = 2\sum_{w\in W(D_n)} \nu(a_{w\phi})^{-1}\chi^\mu_\phi(a_{w\phi})\chi^\mu_\phi(\hat{a}_{w\phi})$$

for (μ) as in the statement of the corollary. For any $w \in W(B_n)$ and corresponding basis element $a_{w\phi} \in \mathfrak{A}_{\phi,Q(x)}(B_n)$, we have $a_{w\phi}a_{w_1\phi} = a_{ww_1\phi}$, as $(a_{w_1\phi})^2 = 1$. Thus, using the orthogonality relations (3.1.4) and the coset decomposition $W(B_n) = W(D_n) \sqcup W(D_n)w_1$, we have

$$\phi^*(C\ ) = \sum_{w\in W(B_n)} \nu(a_{w\phi})\chi^\mu_\phi(a_{w\phi})\chi^\mu_\phi(\hat{a}_{w\phi})$$

$$= \sum_{w\in W(B_n)} \sum_{i=1}^{f^\mu} \nu(a_{w\phi})^{-1}M^\mu_{ii}(a_{w\phi})M^\mu_{ii}(\hat{a}_{w\phi})$$

$$(3.4.17) \qquad = \sum_{w\in W(D_n)} \sum_{i=1}^{f^\mu} \nu(a_{w\phi})^{-1}\Big(M^\mu_{ii}(a_{w\phi})M^\mu_{ii}(a_{w\phi}) + M^\mu_{ii}(a_{w\phi}a_{w_1\phi})M^\mu(a_{w_1\phi}\hat{a}_{w\phi})\Big)$$

where $M^\mu(a_{w\phi})$ is the matrix of $\pi^\mu_\phi(a_w)$ with respect to the last letter sequence arrangement of the basis $\{t_1, \ldots, t_f\}$ of V^K_μ. Then $M^\mu(a_{w_1\phi})$ is a diagonal matrix with entries ± 1 by (2.3.9) so

$$M^\mu_{ii}(a_{w\phi}a_{w_1\phi})M^\mu_{ii}(a_{w_1\phi}\hat{a}_{w\phi}) = M^\mu_{ii}(a_{w\phi})M^\mu_{ii}(\hat{a}_{w\phi}) .$$

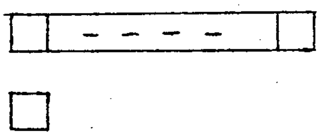
Then (3.4.17) becomes

$$\phi^*(C\ ) = 2\sum_{w\in W(D_n)} \sum_{i=1}^{f^\mu} \nu(a_{w\phi})^{-1}M^\mu_{ii}(a_{w\phi})M^\mu_{ii}(\hat{a}_{w\phi})$$

$$= 2\sum_{w\in W(D_n)} \nu(a_{w\phi})^{-1}\chi^\mu_\phi(a_{w\phi})\chi^\mu_\phi(\hat{a}_{w\phi})$$

as the restriction of π_ϕ^μ to $\mathfrak{A}^K(D_n)$ is an absolutely irreducible representation of $\mathfrak{A}^K(D_n)$. This proves (3.4.16) and completes the proof.

We conclude with an example. We calculate the generic degree of the *reflection representation* of the generic algebra of classical type, a computation also given in ([6]) (as a polynomial in one variable). The double partition $(\mu) = ((n-1), (1))$ yields the *reflection representation* of $\mathfrak{A}^K(B_n)$.



n-1 squares

(D(n-1), D(1))

We have

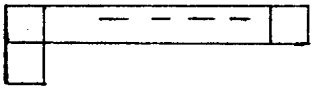
$$c^{((n-1),\ (1))} = y^{-1}(1 + x^{n-1}y)\prod_{i=-1}^{n-3}(1+ x^i y)\prod_{i=1}^{n-2}(1 + \dots + x^i) .$$

Thus

$$d_{\chi^\mu} = P_{B_n}(x, y)\Big/ c^\mu = \frac{y(1 + \dots + x^{n-1})(1 + x^{n-2}y)}{(1 + x^{-1}y)}$$

Setting $y = 1$ in the above we have by (3.4.15) the generic degree of the reflection representation of $\mathfrak{A}^K(D_n)$.

The partition $(\alpha) = (n-1, 1)$ of n yields the reflection representation of $\mathfrak{A}^K(A_{n-1})$.

 n-1 squares

D(n-1, 1)

The generic degree of the representation of $\mathfrak{A}^K(B_n)$ corresponding to the double partition $(\alpha, (0)) = ((n-1, 1), (0))$ is

$$d_{\chi(\alpha, (0))} = \frac{x(1 + x^{n-1}y)(1 + \ldots + x^{n-2})}{(1 + x^{-1}y)} .$$

Setting $y = 0$ in the above we have by (3.4.12) the generic degree of the reflection representation of $\mathfrak{A}^K(A_{n-1})$.

Finally we remark that from (3.1.7) the above corollaries give the degrees of the irreducible constituents of 1_B^G for G a finite group with BN-pair with Coxeter system of classical type by substitution of the index parameters in the formula for the generic degree. In particular, these computations apply to the families of Chevalley groups $A_\ell(q)$, $B_\ell(q)$, $D_\ell(q)$, $A^1_{2\ell}(q^2)$, $A^1_{2\ell-1}(q^2)$, $D^1_\ell(q^2)$. The degrees of these characters for the families of type $A_\ell(q)$ were already known (see [16]).

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