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REPRESENTATIONS OF SEMIGROUPS OF IDEMPOTENTS

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0. INTRODUCTION

In this paper we consider the question of which semigroups of idempotents (bands) have faithful representations over fields. Most of the literature on representations of semigroups (c.f. [1], ch. 5, or [2]) undertakes to describe the representations of a given algebraic semigroup, without identifying which representations, if any, are faithful. Our goal is somewhat different, namely to describe the algebraic structure of a class of matrix semigroups. We first prove that a necessary condition for a semigroup of idempotents to have a faithful representation is that thesemi group possess only finitely many two-sided ideals. It follows that the semigroup will have a composition series having completely (0−) simple factors. It is an easy matter to construct faithful representations of 0-simple semigroups of idempotents, but this does not mean that faithful representations can be found for all semigroups of idempotents with only finitely many two-sided ideals. The reason for this is that the actions of a quotient semigroup on a simple ideal need not be representable by the action of a semigroup of matrices on a finite dimensional vector space. After giving some examples to illustrate what can go wrong, we prove that whenever the actions can be so represented then we can construct a faithful representation of the whole semigroup.

Throughout, F is a field, F_n the ring of $n \times n$ matrices over F, ${}^nF$ the space of n-dimensional column vectors over F, and F^n the space of n-dimensional row vectors.

1. IDEAL STRUCTURE AND COMPOSITION SERIES IN SEMIGROUPS OF IDEMPOTENT MATRICES; SIMPLE SEMIGROUPS OF IDEMPOTENTS

In order to show that a semigroup of idempotent matrices over F has only finitely many two-sided ideals we need two preliminary propositions, both of which are elementary.

Proposition 1. *If $H \in F_n$ is an upper triangular idempotent matrix, then the rank of H is equal to the number of ones on the diagonal of H.*

Proof. Suppose $H = (h_{ij})$, and let $e_1, \ldots, e_n$ denote the standard basis. Then $He_1 = e_1 h_{11}$, and $\langle He_1 \rangle$ has basis $\{He_j \mid h_{jj} = 1, j \leqq 1\}$. Suppose $V_i = \langle He_1, \ldots \ldots, He_i \rangle$ $(1 \leqq i < n)$ has as basis $\{He_j \mid h_{jj} = 1, j \leqq i\}$. We write $He_{i+1} = = e_{i+1} h_{i+1\, i+1} + v$, where $v \in \langle e_1, \ldots, e_i \rangle$. If $h_{i+1\, i+1} = 1$, V_{i+1} clearly has basis $\{He_j \mid h_{jj} = 1, j \leqq i + 1\}$, as $He_{i+1} \notin \langle He_1, \ldots, He_i \rangle$. Otherwise, $h_{i+1\, i+1} = 0$, and $He_{i+1} = H^2 e_{i+1} = Hv \in V_i$, and V_{i+1} has basis $\{He_j \mid h_{jj} = 1, j \leqq i + 1\}$ by assumption. By induction we conclude that the rank of V_n, which is just the rank of H, is equal to the number of ones on the diagonal of H. ■

Proposition 2. *If $G, H \in F_n$ are idempotents of the same rank and $G = HGH$ then $G = H$.*

Proof. We change the basis of ${}^nF$ to get H in the form

$$H = \begin{Bmatrix} I_r & 0 \\ 0 & 0 \end{Bmatrix}.$$

Then G has the form

$$G = \begin{Bmatrix} A & B \\ C & D \end{Bmatrix},$$

but since $G = HGH$, B, C, and D are all zero matrices. Then

$$G = \begin{Bmatrix} A & 0 \\ 0 & 0 \end{Bmatrix}$$

can be diagonalized by a change of basis which leaves H unchanged, and because G and H have the same rank we see that $A = I$, and so $G = H$. ■

We are now ready to prove

Theorem 3. *A semigroup of idempotent matrices over a field has only finitely many two-sided ideals.*

Proof. Let $\mathcal{S} \subseteq F_n$ be a semigroup of idempotent matrices. By the corollary following theorem 4.2 of [3], we can take $\mathcal{S}$ to be triangular. We claim that for $E \in \mathcal{S}$, $\mathcal{S}E\mathcal{S} = \{H \in \mathcal{S} \mid h_{ii} = 1 \Rightarrow e_{ii} = 1\}$. First, let $H \in \mathcal{S}E\mathcal{S}$, so $H = AEB$, $A, B \in \mathcal{S}$. We see that $h_{ii} = a_{ii} e_{ii} b_{ii}$, so if $h_{ii} = 1$ then $e_{ii} = 1$. Now suppose $e_{ii} = 1$ whenever $h_{ii} = 1$. Then $G = HEH$ will have the property that $g_{ii} = 1 \Leftrightarrow h_{ii} = 1$, so by the first proposition G and H have the same rank. Clearly $G = HGH$ as H is idempotent, so by the second proposition, $G = H \in \mathcal{S}E\mathcal{S}$, and the claim is established. It follows from the claim that there are as many principal ideals as there are different arrangements of 0's and 1's on the diagonals of elements of $\mathcal{S}$, and this number is at most 2^n. Since any ideal of $\mathcal{S}$ is the union of the principal ideals it contains, and there are only finitely many principal ideals, there are only finitely many two-sided ideals of $\mathcal{S}$. ■

It follows from theorem 3 that a semigroup of idempotent matrices has a composition series. The factors in any composition series are $(0-)$ simple semigroups of idempotents, so by [1], thms. 2.55 and 2.48, these factors are completely $(0-)$ simple semigroups, i.e., they have minimal right and left ideals.

As is well known (c.f., [1], p. 94), a completely simple semigroup of idempotents is given up to isomorphism by $\{(r, l) \mid r \in R, l \in L\}$, where R is the set of minimal right ideals and L is the set of minimal left ideals, and multiplication is defined by $(r, l)(r', l') = (r, l')$. If we take any one-one correspondences $r \to A_r$, $l \to B_l$, where the A's and B's are rectangular matrices over a field F, then we get a faithful representation of our semigroup

$$(r, l) \to \begin{Bmatrix} 0 & 0 & A_r & 0 \\ 0 & I & 0 & B_l \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}.$$

Thus any $(0-)$ simple semigroup of idempotents has a faithful representation over some field. (It can be shown, however, that not all representations will be equivalent to one of the ones just constructed).

2. PIECING TOGETHER

We have seen how to construct representations of $0-$ simple semigroups of idempotents, and we have seen that these are the building blocks (in the sense of theorem 1.3 and the remarks following it) of all semigroups of idempotents which have faithful representations over some field. However, it does not follow that all semigroups of idempotents having only finitely many two-sided ideals have faithful representations. Other, less easily described, restrictions are imposed on the semigroup by the way the semigroup as a whole acts on an ideal.

To consider the simplest case, suppose $\mathcal{S} \subseteq F_n$ is a semigroup of idempotents containing only two ideals, $\mathcal{S}$ and $\mathcal{I}$. There are two actions of $\mathcal{S} - \mathcal{I}$ $(=\mathcal{S}/\mathcal{I} - 0)$ on $\mathcal{I}$, one on the right and one on the left, namely $(E)\,S = ES$ and $S(E) = SE$ $(E \in \mathcal{I}, S \in \mathcal{S} - \mathcal{I})$. Because $\mathcal{S} \subseteq F_n$ each action of the simple semigroup $\mathcal{S} - \mathcal{I}$ on $\mathcal{I}$ can be represented as an action of a semigroup of matrices on elements of a finite dimensional vector space. This fact further limits the semigroups of idempotents with faithful representations, as we see from the following proposition and examples.

Proposition 1. *Suppose $\mathcal{S} \subseteq F_n$ is a semigroup of matrices, and $\mathcal{I}$ is an ideal of $\mathcal{S}$.*

(a) *Whenever we have elements $A_i \in \mathcal{S}$, $S_i, T_i \in \mathcal{I}$ $(1 \leqq i \leqq m)$ with $A_iS_j = A_iT_j$ $(j > i)$, $A_iS_i \neq A_iT_i$, then $m \leqq n^2$.*

(b) *Whenever there exist elements $A_i, B_i \in \mathcal{S}$, $S_i \in \mathcal{I}$ $(1 \leqq i \leqq m)$ with $A_iS_j = B_iS_j$ $(j > i)$, $A_iS_i \neq B_iS_i$, then $m \leqq n^2$.*

Proof. (a) Since $\mathcal{S} \subseteq F_n$, $\langle\{T_i - S_i\}\rangle$ is of dimension at most n^2 over F (letting the scalars appear on the right, with $(M)\,a = M\,.\,(a\,.\,I)$). If we have sequences of length greater than n^2 then there exists an i with $T_i - S_i = \sum_{j>i} (T_j - S_j)\, a_j$. Then

$0 \neq A_i(T_i - S_i) = \sum_{j>i} A_i(T_j - S_j)\, a_j = 0$, a contradiction. Therefore all such sequences have length $m \leqq n^2$.

(b) This part is proved similarly. ■

Comparable results hold when we consider $\mathscr{S} - \mathscr{I}$ acting on $\mathscr{I}$ on the right.

To illustrate the necessity of these conditions and their independence, we present the following examples.

Example 1, *of a semigroup $\mathscr{S}$ of idempotents containing an ideal $\mathscr{I}$ with $\mathscr{I}$ simple and $\mathscr{S}/\mathscr{I}$ 0-simple, but for which neither* (a) *nor* (b) *holds.*

Let $\mathscr{X}$ be a subset of End ${}^{\omega}\mathbb{R}$ and $\mathscr{Y}$ a subset of ${}^{\omega}\mathbb{R}$. Take the underlying set of $\mathscr{S}$ to be $\mathscr{X} \cup \{(v, 0) \mid v \in {}^{\omega}\mathbb{R}\} \cup \{(0, w) \mid w \in \mathscr{Y}\}$, and set $\mathscr{I} = \mathscr{S} - \mathscr{X}$. Let multiplication in $\mathscr{I}$ be left trivial – i.e., $(v_1, v_2)(w_1, w_2) = (v_1, v_2)$ – and let multiplication in $\mathscr{S} - \mathscr{I}$ be right trivial $(A \,.\, B = B)$. Define $B(v, w) = (v + Bw, 0)$, $(v, w) B = (v, w)$. It is not difficult to check that $\mathscr{S}$ is a semigroup of idempotents, $\mathscr{I}$ is an ideal of $\mathscr{S}$ which is simple, and $\mathscr{S}/\mathscr{I}$ is 0-simple. If we take $\mathscr{X}$ to be End ${}^{\omega}\mathbb{R}$ and $\mathscr{Y}$ to be ${}^{\omega}\mathbb{R}$, this example violates (a) and (b). Specifically, if $\{e_i \mid i \in \mathbb{N}\}$ is a basis for ${}^{\omega}\mathbb{R}$, the sequences which violate (a) and (b) can be taken as follows:

(a) Take $T_i = (0, e_i)$, $S_i = (0, 0)$, $A_i : e_j \to \begin{cases} e_j & j \leqq i\,, \\ 0 & j > i\,. \end{cases}$

(b) Take $S_i = (0, e_i)$, $A_i : e_j \to \begin{cases} e_j & j > i\,, \\ 0 & j \leqq i\,, \end{cases}$ $B_i : e_j \to e_j\,.$

For this example, though, the right action analogues of (a) and (b) both hold: if $S_jA_i = T_jA_i$ for some A_i then $S_j = T_j$ and $S_jA_k = T_jA_k$ for all k; and of course $S_iA_j = S_iB_j$ for all A_j and B_j. Thus it does not suffice to assume just that one action satisfies finiteness conditions like (a) and (b).

It is also the case that neither (a) nor (b) implies the other, as the following two examples show.

Example 2, *of a semigroup of idempotents containing only two ideals which satisfies* (b) *but not* (a).

Let $C_i \in$ End ${}^{\omega}\mathbb{R}$ be defined by

$$C_i(e_j) = \begin{cases} e_j i & j \leqq i\,, \\ e_{i+1} i & j > i\,. \end{cases}$$

Then in example 1, take $\mathscr{X}$ to be $\{C_i \mid i \in \mathbb{N}\}$, $\mathscr{Y} = \{e_k \mid k \in \mathbb{N}\}$.

not (a): Take $A_i = C_i$, $S_i = (0, e_{i+1})$, $T_i = (0, e_i)$.

(b): We show that we cannot have $A_iS_j = B_iS_j$ $(j > i)$, $A_iS_i \neq B_iS_i$ for $1 \leqq i, j \leqq 2$. Suppose not, and let $A_1 = C_i$, $B_1 = C_j$, $S_2 = (v, w)$, where $w \in \{0\} \cup \mathscr{Y}$. Then $(v + C_iw, 0) = C_i(v, w) = C_j(v, w) = (v + C_jw, 0)$. Therefore $C_iw = C_jw$, so either $w = 0$ or $i = j$. If $w = 0$, $S_2 = (v, 0)$, and for any A_2, B_2, $A_2S_2 = (v, 0) = B_2S_2$, a contradiction. If $i = j$, $A_1 = B_1$, so $A_1S_1 = B_1S_1$, a contradiction. Therefore (b) holds with $n = 1$.

Example 3, *of a semigroup of idempotents with just two ideals and which satisfies* (a) *and not* (b).

Let the underlying set of $\mathcal{S}$ be End ${}^{\omega}\mathbb{R} \cup \{(v, w) \mid v, w \in {}^{\omega}\mathbb{R}\}$, and set $\mathcal{I} = \{(v, w)\}$. Define multiplication in $\mathcal{I}$, $\mathcal{S} - \mathcal{I}$ to be left trivial, and define $A(v, w) = (Aw, w)$, $(v, w) A = (v, w)$. It is easily verified that $\mathcal{S}$ is a semigroup of idempotents with only two ideals, $\mathcal{S}$ and $\mathcal{I}$.

(a): Suppose $A_1 S_2 = A_1 T_2$, and write $S_2 = (v, w)$, $T_2 = (v', w')$. Then $A_1 S_2 = (A_1 w, w) = A_1 T_2 = (A_1 w', w')$. Thus $w = w'$, so for any choice of A_2, $A_2 S_2 = A_2 T_2$, and we see that (a) holds for $n = 1$.

not (b): Again let $\{e_i \mid i \in \mathbb{N}\}$ be a basis of ${}^{\omega}\mathbb{R}$, and define $A_i \in$ End ${}^{\omega}\mathbb{R}$ by

$$A_i(e_j) = \begin{cases} e_j & j < i, \\ 0 & j \geq i. \end{cases}$$

Using these A_i's, $B_i = 0$ for all $i \in \mathbb{N}$, and $S_i = (0, e_i)$ we see that (b) does not hold.

The reason why the semigroups in the above examples do not have faithful representations over any field is essentially that the left action of $\mathcal{S} - \mathcal{I}$ on $\mathcal{I}$ cannot be represented as a semigroup of linear transformations acting on vectors in a finite dimensional vector space. It turns out that if both the left and right actions of $\mathcal{S} - \mathcal{I}$ on $\mathcal{I}$ have faithful representations then we can get a representation of the whole semigroup.

Theorem 2. *Let $\mathcal{S}$ be a semigroup of idempotents with an ideal $\mathcal{I}$ which is simple, and suppose $\mathcal{S}/\mathcal{I}$ has no zerodivisors. Suppose there exist faithful representations of $\mathcal{S} - \mathcal{I}$ into F_n and F_m ($S \to M_S$ and $S \to N_S$ respectively) and set monomorphisms $\mathcal{I} \to {}^nF$ and $\mathcal{I} \to F^m$ ($T \to v_T$ and $T \to w_T$ respectively) such that $M_S v_T = v_{ST}$, $w_T N_S = w_{TS}$. Then $\mathcal{S}$ has a faithful representation in F_{n+m+2}.*

Proof. Recall that a simple idempotent semigroup, can be described as $R \times L$, where R is the set of minimal right ideals of $\mathcal{I}$ and L the set of minimal left ideals, and multiplication is defined by $(r, l)(r', l') = (r, l')$.

Let j be a fixed principal left ideal of $\mathcal{I}$, and let r be a fixed principal right ideal of $\mathcal{I}$. Then j contains just one element from each minimal right ideal of $\mathcal{I}$ and r contains just one element from each minimal left ideal. For each minimal right ideal a of $\mathcal{I}$ let $v_a \in {}^nF$ be the image of the element in $a \cap j$, and for each minimal left ideal b let $w_b \in F^m$ be the image of the element of $b \cap r$. Define a map

$$\mathcal{I} \to F_{m+n+2} \quad \text{by} \quad T \to \begin{pmatrix} 0 & 0 & v_a & 0 \\ 0 & 1 & 0 & w_b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = P_T ,$$

where $T\mathcal{I}$ is the minimal right ideal a and $\mathcal{I}T$ is the minimal left ideal b.

For S, $T \in \mathscr{I}$, the left ideal determined by T is the same as the left ideal determined by ST, and the right ideal determined by S is the same as the right ideal determined by ST. Hence, if we have

$$P_T = \begin{pmatrix} 0 & 0 & v_a & 0 \\ 0 & 1 & 0 & w_b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_S = \begin{pmatrix} 0 & 0 & v_c & 0 \\ 0 & 1 & 0 & w_d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

$$P_{ST} = \begin{pmatrix} 0 & 0 & v_c & 0 \\ 0 & 1 & 0 & w_b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = P_S P_T\,,$$

and our mapping is a faithful representation of $\mathscr{I}$.

For $S \in \mathscr{S} - \mathscr{I}$, let

$$P_S = \begin{pmatrix} M_S & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & N_S \end{pmatrix}.$$

This clearly gives a faithful representation of $\mathscr{S} - \mathscr{I}$, and it remains to show that the actions of $\mathscr{S} - \mathscr{I}$ on $\mathscr{I}$ are as they should be.

Let $S \in \mathscr{S} - \mathscr{I}$, $T \in \mathscr{I}$, and consider the product

$$P_S P_T = \begin{pmatrix} M_S & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & N_S \end{pmatrix} \begin{pmatrix} 0 & 0 & v_a & 0 \\ 0 & 1 & 0 & w_b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & M_S v_a & 0 \\ 0 & 1 & 0 & w_b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = M\,.$$

To show that $M = P_{ST}$ we must verify that

(1) w_b corresponds to the element of $\mathscr{I}ST \cap r$; and

(2) $M_S v_a$ corresponds to the element of $ST\mathscr{I} \cap j$.

(1) Clearly, $\mathscr{I}ST \subseteq \mathscr{I}T$, and since $\mathscr{I}T$ is minimal, $\mathscr{I}ST = \mathscr{I}T$. But w_b was chosen to correspond to the element of $\mathscr{I}T \cap r$, so w_b corresponds to the element of $\mathscr{I}ST \cap r$.

(2) Let T' denote the element of $T\mathscr{I} \cap j$, so that $v_a = v_{T'}$. By the way our representation acts, $M_S v_{T'} = v_{ST'}$. By definition, $v_{ST'}$ corresponds to the element of $j \cap$ $\cap\, ST'\mathscr{I}$. But since $T' \in T\mathscr{I}$, and since all principal right ideals of $\mathscr{I}$ are minimal, $ST'\mathscr{I} = ST\mathscr{I}$. Thus $M_S v_a = v_{ST'}$ corresponds to the element of $ST\mathscr{I} \cap j$, as desired.

The proof that the right action is preserved is similar, and is omitted, and we have the desired faithful representation of $\mathscr{S}$. ■

References

[1] *Clifford, A. H.,* and *G. B. Preston:* The Algebraic Theory of Semigroups I, A.M.S., Providence, R.I., 1961.

[2] *McAlister, Donald B.:* Representations of Semigroups by Linear Transformations I, II, Semigroup Forum *2* (1971), 189—263, 283—320.

[3] *Sizer, Walter S.:* Triangularizing Semigroups of Matrices over a Skew Field, Linear Algebra and Its Applications *16* (1977), 177—187.

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