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REPRESENTATIONS OF THE l^1-ALGEBRA OF AN INVERSE SEMIGROUP(1)

BY

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ABSTRACT. In this paper the star representations on Hilbert space of the l^1-algebra of an inverse semigroup are studied. It is shown that the set of all irreducible star representations form a separating family for the l^1-algebra. Then specific examples of star representations are constructed, and some theory of star representations is developed for the l^1-algebra of a number of the most important examples of inverse semigroups.

Introduction. Let S be a semigroup (as defined in [2, p.1]). If $a, b \in S$, we write ab for the semigroup product of a with b. Let $l^1(S)$ be the set of all complex-valued functions f on S such that

$$\|f\|_1 = \sum_{a\in S} |f(a)| < \infty.$$

If $f, g \in l^1(S)$, then the convolution product $f * g$ is given by the definition

$$(f * g)(c) = \sum_{a,b \text{ with } ab=c} f(a)g(b), \qquad c \in S.$$

With convolution multiplication and norm $\| \cdot \|_1$, $l^1(S)$ is a Banach algebra. If $a \in S$, we identify $\dot{a}$ with the function which takes the value 1 at a and is 0 everywhere else. In this way S is embedded in $l^1(S)$. Having made this identification, when $f \in l^1(S)$ we have

$$f = \sum_{a\in S} f(a)a.$$

A map $a \longrightarrow a^*$ of S into S is called an involution on S if

$$(ab)^* = b^*a^* \quad \text{all } a, b \in S, \text{ and}$$

$$(a^*)^* = a \qquad \text{all } a \in S.$$

If S has an involution $*$, then $l^1(S)$ has an involution $*$ defined by the rule

$$f^* = \sum_{a\in S} f(a)^*a^*, \qquad f \in l^1(S),$$

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where here the complex conjugate of a complex number λ is denoted λ^*. In the familiar case when S is a group, the natural involution $*$ on S is $g^* = g^{-1}$, $g \in S$. The algebra $l^1(S)$ is the usual convolution group algebra of S.

For a general semigroup S the algebra $l^1(S)$ was first studied by E. Hewitt and H. Zuckerman [10]–[12] and by W. D. Munn [16]. In their fundamental paper [12], Hewitt and Zuckerman consider $l^1(S)$ for S an abelian semigroup. In this case $l^1(S)$ is a commutative Banach algebra, and the main problem is to find conditions on S that insure the existence of a separating family of multiplicative linear functionals on $l^1(S)$. Hewitt and Zuckerman completely solve this problem, proving in [12, Theorems 3.5 and 5.8] that there exists a separating family of multiplicative linear functionals on $l^1(S)$ if and only if the collection of semicharacters of S separates the points of S if and only if S has the property that whenever $x^2 = y^2 = xy$, $x, y \in S$, then $x = y$. In terms of representation theory, the set of nonzero multiplicative linear functionals on $l^1(S)$ (or equivalently the set of semicharacters of S) is in a natural one-to-one correspondence with the set of one dimensional (irreducible) representations of $l^1(S)$. Thus, when S is abelian, S has the property that whenever $x^2 = y^2 = xy$, $x, y, \in S$, then $x = y$ if and only if the set of irreducible representations of $l^1(S)$ form a separating family, i.e. $l^1(S)$ is Jacobson semisimple.

For a general semigroup S there is an extensive theory concerning the representations of S by finite matrices over a field due to A. H. Clifford, W. D. Munn, G. B. Preston, Hewitt and Zuckerman, and others; see [2, Chapter 5]. At least for certain types of finite (nonabelian) semigroups S, the irreducible representations of $l^1(S)$ can be determined, and it can be shown that there exists a separating family of irreducible representations of $l^1(S)$ [13], [2, Chapter 5]. Of course, when S is a group it is a standard fact that $l^1(S)$ has a separating family of irreducible representations (in fact the set of irreducible star representations of $l^1(S)$ on Hilbert space form a separating family).

However, if the semigroup S is neither abelian, nor finite, nor a group, little is known about the properties of the Banach algebra $l^1(S)$, about the infinite dimensional representations of $l^1(S)$ as bounded operators on a Banach space, about the set of irreducible representations of $l^1(S)$, or whether $l^1(S)$ is Jacobson semisimple. In a recent paper [1], B. Barnes and J. Duncan made some progress on these questions when S is the free semigroup with a finite or countably infinite number of generators (and also, in some cases where the generators satisfied reasonable relations). The semigroups considered in [1] all have natural involutions. Barnes and Duncan determine irreducible star representations of the star algebra $l^1(S)$ on Hilbert space, and prove that these form a separating family for $l^1(S)$. In particular, $l^1(S)$ is Jacobson semisimple in this case.

In this paper we consider the representation theory of S, or what is equivalent, the representation theory of $l^1(S)$ where S is an inverse semigroup. In this case, if $a \in S$, then by definition there exists a unique element $b \in S$ such that the following two equalities hold $aba = a$, $bab = b$; see [2, p. 27]. When a and b satisfy these equalities we write $a^* = b$ (in the usual notation $a^{-1} = b$). Then $a \longrightarrow a^*$ is an involution on S, and lifting this involution on S to an involution $*$ on $l^1(S)$, we have that $l^1(S)$ is a Banach star algebra. A star representation of S on a Hilbert space H is a semigroup homomorphism π: $S \longrightarrow B(H)$ with the property that $\pi(a^*) = \pi(a)^*$ for $a \in S$. Here $B(H)$ denotes the set of all bounded operators on H. Since $aa^*a = a$, we have that a^*a is an idempotent in S. Then $\pi(a^*a) = \pi(a)^*\pi(a)$ is a self adjoint idempotent in $B(H)$. This means that $\pi(a)$ is a partial isometry on H for all $a \in A$ [8, pp. 62–63]. Thus a star representation of S is a representation of S as a semigroup of partial isometries on some Hilbert space. Groups are special examples of inverse semigroups, and this notion of star representation is a natural extension of the idea of representing groups as groups of unitary operators on a Hilbert space. Every star representation π of S lifts to a star representation π of $l^1(S)$ by letting

$$\pi(f) = \sum_{a \in S} f(a)\pi(a), \qquad f \in l^1(S).$$

In the other direction, the restriction of a star representation of $l^1(S)$ to S is a representation of S by a semigroup of partial isometries on the representation space.

The first result we prove is that when S is an inverse semigroup then the set of the irreducible star representations of $l^1(S)$ is a separating family (§2). Most of the rest of the paper is devoted to the determination of irreducible star representations of $l^1(S)$ for some of the most interesting examples of inverse semigroups: certain semigroups of partial transformations on a set, completely 0-simple semigroups, and the bicyclic semigroup; see §1, Examples (1.1)–(1.4).

In §3 a general theory is developed concerning star representations of $l^1(S)$ determined by idempotents e in S with eSe a finite set. These results are applied in §4 to the symmetric inverse semigroup on a set X (see Example 1.1). More examples of irreducible star representations of the l^1-algebra of this semigroup are given in §5. In §6 a detailed theory of representations of completely 0-simple inverse semigroups (Example 1.3) is presented. The last section, §7, is concerned with the representations of the l^1-algebra of the bicyclic semigroup (Example 1.4).

The class of inverse semigroups contains many different types of examples. We necessarily deal with only a few types here, but even for these few we have found the representation theory to be rich, varied, and very interesting.

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1. **Inverse semigroups: basic facts, examples, notation.** In this section we make a brief review of some of the basic facts and terminology concerning inverse semigroups. Usually we use the same terminology and notation as [2] and [3], but there are some differences. Also, we give a very brief discussion of the examples of inverse semigroups that will most concern us here. In the last part of this section we establish some general notation.

A semigroup S is an inverse semigroup if for any $a \in S$, there exists a unique element $a^* \in S$ such that

$$aa^*a = a \quad \text{and} \quad a^*aa^* = a^*.$$

Then the map $a \longrightarrow a^*$ is an involution on S. Throughout the remainder of this paper S will always denote an inverse semigroup. The idempotents of S play a crucial role in the algebraic theory of S, and in some parts of the representation theory of $l^1(S)$. Let E_S denote the set of all idempotents of S. It is immediate that $a^*a \in E_S$ for all $a \in S$. Thus, E_S is never empty. An important fact that we use repeatedly is that $ef = fe$ whenever $e, f \in E_S$ [2, Theorem 1.17].

There are several useful relations on the set E_S. First we define an equivalence relation on E_S as follows:

DEFINITION. If $e, f \in E_S$, then $e \sim f$ if there exists an element $a \in S$ such that $e = a^*a$ and $f = aa^*$.

We check the transitivity of the relation $\sim$. Assume that $e, f, g \in E_S$, $e \sim f$, and $f \sim g$. Then by definition there exist $a, c \in S$ such that $e = a^*a$, $f = aa^*$, $f = c^*c$, and $g = cc^*$. Then $e = a^*aa^*a = a^*fa = (ca)^*ca$, and $g = cc^*cc^* = cfc^* = (ca)(ca)^*$. Thus, $e \sim g$. It is not difficult to show that $e \sim f$ if and only if $e\mathcal{D}f$ (see [2, pp. 47–48] and [3, p. 102]), but we make no use of this fact.

A second relation on E_S is the usual partial ordering of idempotents. If $e, f \in E_S$, then $e \leqslant f$ if $ef = fe = e$ [2, pp. 23–24]. The relations $\geqslant, <, >$, are defined as usual in terms of $\leqslant$.

An element $\theta \in S$ is called the zero element of S if $a\theta = \theta a = \theta$ for all $a \in S$. We reserve the notation θ for the zero of S whenever S has a zero.

Let I be an ideal of S. We denote the Rees quotient semigroup of S modulo I as S/I [2, p. 17]. The elements of S/I are the elements $\{a\}, a \in S\backslash I$ (set difference), and the element I. The semigroup S acts on S/I in a natural way: if $a \in S$, then

$$a\{b\} = \{ab\} \quad \text{if } ab \notin I,$$

$$a\{b\} = I \qquad \text{if } ab \in I,\ b \notin I,$$

$$aI = I.$$

These equations define how a acts on the left on S/I. The action of a on the right on S/I is defined in a similar manner.

The concept of primitive idempotent plays a role of central importance in the algebraic theory of inverse semigroups. An idempotent $e \in E_S$, $e \neq \theta$, is primitive if whenever $f \in E_S$ and $f \leqslant e$, then $f = e$, or in the case S has a zero, either $f = e$ or $f = \theta$. We use a slightly more general concept in §3 to relate the representation theory of $l^1(S)$ to the representation theory of the l^1-algebra of certain groups. We say an idempotent e is primitive modulo an ideal I of S if $\{e\}$ is a primitive idempotent of S/I. When e is primitive modulo I, then it is not difficult to verify that the set $\{eSe\}$ in S/I is a group with zero [2, p. 5]. We let

$$G_e = \{\{eae\}: \{eae\} \in S/I,\ eae \notin I\}.$$

When S has a zero, then we naturally identify $S/\{\theta\}$ with S. In this case primitive idempotents modulo $\{\theta\}$ are identified with primitive idempotents of S.

In the case when S has a zero we make certain technical changes in the terminology used in the introduction. First, $l^1(S)$ denotes the set of complex-valued functions f on S such that $\Sigma_{a \in S} |f(a)| < \infty$ and $f(\theta) = 0$. Second, if π is a representation of S, we always assume that $\pi(\theta) = 0$.

Now we turn to a brief description of the examples with which we are chiefly concerned in this paper.

EXAMPLE 1.1. $\mathcal{I}_X$, the symmetric inverse semigroup on a set X. Let X be a nonempty set. The elements of $\mathcal{I}_X$ are the one-to-one maps b defined on a domain D_b in X with values in X. The set of values of b we denote by R_b. We also assume that the empty map θ is in $\mathcal{I}_X$. By convention $\theta b = b\theta = \theta$ for all $b \in \mathcal{I}_X$. If $b, c \in \mathcal{I}_X$, $b \neq \theta$, $c \neq \theta$, define $D_{bc} = \{x \in X: x \in D_c \text{ and } c(x) \in D_b\}$. If D_{bc} is empty, then let $bc = \theta$. Otherwise, bc is the usual composition of the maps b and c defined on D_{bc}. With this multiplication $\mathcal{I}_X$ is a semigroup. If $b \in \mathcal{I}_X$, $b \neq \theta$, let b^* be the map with domain R_b defined by $b^*(x) = y$ if and only if $b(y) = x$. It is not difficult to verify that b^* is the unique element in $\mathcal{I}_X$ satisfying the equations $bb^*b = b$ and $b^*bb^* = b^*$ [2, p. 29]. Thus $\mathcal{I}_X$ is an inverse semigroup. The idempotent maps in $\mathcal{I}_X$ are the maps e such that $e(x) = x$ for all $x \in D_e$.

The inverse semigroup $\mathcal{I}_X$ is universal in the sense that if S is any inverse semigroup, then S is isomorphic with an inverse subsemigroup of $\mathcal{I}_S$ [2, Theorem 1.20].

EXAMPLE 1.2. $\mathcal{F}_X$, the semigroup of finite one-to-one maps on X. Let X

be a nonempty set. We denote by $\mathcal{F}_X$ the inverse subsemigroup of $\mathcal{I}_X$ consisting of those maps $b \in \mathcal{I}_X$ such that R_b is finite and the empty map θ. In §§4 and 5 we shall mainly deal with representations of $l^1(S)$ where S is some inverse subsemigroup of $\mathcal{I}_X$ with $\mathcal{F}_X \subset S \subset \mathcal{I}_X$ or where $S \subset \mathcal{F}_X$.

$\mathcal{F}_X$ is an ideal in $\mathcal{I}_X$. For each $k > 0$, let $\mathcal{F}_k$ be the set of all maps $b \in \mathcal{I}_X$ such that R_b contains at most k elements, and the empty map θ. Also let $\mathcal{F}_0 = \{\theta\}$. Each of the sets $\mathcal{F}_k$, $k \geq 0$, are inverse subsemigroups of $\mathcal{I}_X$, and also ideals of $\mathcal{I}_X$.

If e is any idempotent map in $\mathcal{F}_{k+1}$, it is not difficult to verify that e is primitive modulo $\mathcal{F}_k$, and that G_e is the symmetric group on $k + 1$ elements.

EXAMPLE 1.3. Completely 0-simple inverse semigroups. A semigroup S with zero is completely 0-simple if the only ideals of S are $\{\theta\}$ and S, and S contains a primitive idempotent e [2, §2.7]. It is easy to show that $\mathcal{F}_{k+1}/\mathcal{F}_k$ is completely 0-simple for $k \geq 0$ [3, p. 223].

We give an abstract example which is in fact typical of the genre. Let S be an inverse semigroup with zero. Assume that J is a subset of E_S with the properties

(i) $\theta \in J$, $e \in J$ for some $e \neq \theta$,

(ii) if $e, f \in J$, $e \neq f$, then $ef = \theta$, and

(iii) if $e, f \in J$, $e \neq \theta$, $f \neq \theta$, then $e \sim f$.

Then let

$$S_J = \{a \in S: a^*a \in J \text{ and } aa^* \in J\}.$$

The nonzero idempotents in S_J are obviously primitive. Let I be an ideal of S_J such that $a \in I$, $a \neq \theta$. Let b be any element of S_J. There exists $c \in S_J$ such that $b^*b = c^*c$ and $cc^* = a^*a$. Then

$$b = bb^*bb^*b = bc^*cc^*c = bc^*a^*ac \in I.$$

This proves that $\{\theta\}$ and S_J are the only ideals of S_J.

EXAMPLE 1.4. $\mathcal{C}$, the bicyclic semigroup [2, pp. 43–44]. Let $\mathcal{C}$ be the semigroup consisting of an identity 1 and all the words in two letters p and q subject to the single relation $qp = 1$. Specifically, $\mathcal{C} = \{p^mq^n, m \geq 0, n \geq 0\}$. The product of p^mq^n and p^jq^k is the word $p^mq^np^jq^k$ simplified by the relation $qp = 1$. It is easy to verify that $p^* = q$, and more generally, $(p^mq^n)^* = p^nq^m$. $\mathcal{C}$ is the most important specific example in the class of bisimple inverse semigroups. It is also the member of this class which has the simplest structure.

Some miscellaneous notation: The scalar field involved is always the field of complex numbers, **C**. If $\lambda \in \mathbf{C}$, then λ^* denotes the complex conjugate of λ.

Since we deal only with star representations of $l^1(S)$ on Hilbert space, we take "representation" to mean automatically "star representation". Let π be a representation of a star algebra on a Hilbert space K. We often use the pair (π, K) to designate the star homomorphism π together with the representation space K. If J is a π-invariant subspace of K, then $\pi | J$ denotes the restriction of the representation π to the subspace J. A representation is a subrepresentation of (π, K) if it is of the form $(\pi | J, J)$ where J is some π-invariant subspace of K. If two representations (π_1, K_1) and (π_2, K_2) are unitarily equivalent, we use the notation $\pi_1 \approx \pi_2$.

If H is a Hilbert space, we use the notation (φ, ψ) for the inner product of φ and $\psi \in H$ unless a different notation is explicitly introduced. A pre-inner product on a vector space is a form which satisfies all of the axioms of an inner-product except that it may be degenerate.

If X is a set, then $|X|$ denotes the cardinality of X. If T and S are subsets of X, then $T \backslash S = \{x \in T: x \notin S\}$. If X has a topology and T is a subset of X, then $\mathrm{cl}(T)$ is the closure of T in X.

2. **The existence of a separating family of irreducible representations for $l^1(S)$.** In order to prove that a Banach star algebra A has a separating family of irreducible representations on Hilbert space, it suffices to prove that A has a faithful representation on some Hilbert space [17, Theorem (4.6.7)]. We show that $l^1(S)$ has a faithful representation via the construction of the left regular representation of $l^1(S)$ on $l^2(S)$ which we now define.

The space $l^2(S)$ is the usual Hilbert space of all complex-valued functions f defined on S such that $\Sigma_{a \in S} |f(a)|^2 < \infty$, and with the additional convention that $f(\theta) = 0$ if S has a zero. We let $\{\varphi(a): a \in S, a \neq \theta\}$ denote the standard orthonormal basis for $l^2(S)$.

If $a, b \in S, b \neq \theta$, define

$$\pi(a)\varphi(b) = \begin{cases} \varphi(ab) & \text{if } a^*ab = b, \\ 0 & \text{if } a^*ab \neq b. \end{cases}$$

If $f \in l^1(S), f = \Sigma \lambda_k a_k$, and $g \in l^2(S), g = \Sigma \mu_j \varphi(b_j)$, we define

$$\pi(f)g = \sum_{k,j} \lambda_k \mu_j \pi(a_k)\varphi(b_j).$$

PROPOSITION 2.1. *The map $f \longrightarrow \pi(f)$ for $f \in l^1(S)$ is a representation of $l^1(S)$ on $l^2(S)$.*

PROOF. Assume that $a, b, c \in S$. First we verify that

$$\{a^*abc = bc \text{ and } b^*bc = c \iff \{b^*a^*abc = c. \tag{1}$$

If the left-hand side of (1) holds, then $b^*(a^*abc) = b^*bc = c$. Conversely, if $b^*a^*abc = c$, then

$$b^*bc = (b^*b)b^*a^*abc = c,$$

$$bc = bb^*a^*abc = a^*a(bb^*b)c = a^*abc.$$

This establishes (1).

Now suppose that $a, b, c \in S$, $c \neq \theta$. Then from the definition of π we have $\pi(a)\pi(b)\varphi(c) = \varphi(abc) \Longleftrightarrow$ the left-hand side of (1) holds $\Longleftrightarrow$ the right-hand side of (1) holds $\Longleftrightarrow \pi(ab)\varphi(c) = \varphi(abc)$. Thus, $\pi(ab)\varphi(c) = \pi(a)\pi(b)\varphi(c)$, so that π defines a homomorphism of $l^1(S)$ into the algebra of bounded linear operators on $l^2(S)$.

Next we show that when $a, b, c \in S$, then

$$\text{(2)} \qquad \{a^*ab = b, \text{ and } ab = c \Longleftrightarrow \{aa^*c = c, \text{ and } a^*c = b.$$

Suppose the left-hand side of (2) holds. Then $a^*ab = a^*c$ and $b = a^*ab = a^*c$. Also, $aa^*c = ab = c$. Therefore the right-hand side of (2) holds. The reverse argument is the same.

Now to prove that π is a representation it suffices to check that if $a, b, c \in S\backslash\{\theta\}$, then

$$(\pi(a)\varphi(b), \varphi(c)) = (\varphi(b), \pi(a^*)\varphi(c))$$

where $(\cdot, \cdot)$ is the inner product on $l^2(S)$. Thus, it is enough to show that $(\pi(a)\varphi(b), \varphi(c)) = 1$ if and only if $(\varphi(b), \pi(a^*)\varphi(c)) = 1$. We have

$$\begin{aligned}(\pi(a)\varphi(b), \varphi(c)) = 1 &\Longleftrightarrow \pi(a)\varphi(b) = \varphi(c) \Longleftrightarrow a^*ab = b \text{ and } ab = c \\ &\Longleftrightarrow \text{(by (2)) } aa^*c = c \text{ and } a^*c = b \Longleftrightarrow \pi(a^*)\varphi(c) = \varphi(b) \\ &\Longleftrightarrow (\varphi(b), \pi(a^*)\varphi(c)) = 1.\end{aligned}$$

This completes the proof.

Let X be a nonempty set. Until further notice we assume that $S = I_X$ [Example 1.1]. Let π be the left regular representation of $l^1(S)$ on $l^2(S)$ as described above. Our immediate aim is to show that in this case π is faithful. Suppose on the contrary that there exists $f \in l^1(S)$ such that $f \neq 0$ but $\pi(f) = 0$. We write $f = \Sigma\lambda_k a_k$ where $\lambda_k \neq 0$ for all k, and $a_k \neq a_j$ if $k \neq j$. If $f(a) \neq 0$, let

$$W(a) = \{b \in S: f(b) \neq 0, b^*b \geqslant a^*a, \text{ and } ba^*a = a\}.$$

Note that $a \in W(a)$. Also let

$$V = \{b \in S: f(b) \neq 0 \text{ and } b^*b \geqslant a^*a\}.$$

Then we have

$$0 = \pi(f)\varphi(a^*a) = \sum_{b\in V} f(b)\varphi(ba^*a)$$
$$= \left(\sum_{b\in W(a)} f(b)\right)\varphi(a) + \sum_{b\in V\setminus W(a)} f(b)\varphi(ba^*a).$$

It follows from this equality that

$$\sum_{b\in W(a)} f(b) = 0. \tag{3}$$

If e and f are two idempotent maps in S, let $e \vee f$ be the idempotent map with domain the union of the domains of e and f. Set $W_1 = W(a_1)$. Note that if $b \in W(a)$ and $b \neq a$, then $W(b) \subset W(a)$ and $a \notin W(b)$. Let $b_1, b_2, \ldots, b_m$ be any collection of elements in W_1 with $b_k \neq a_1$, $1 \leqslant k \leqslant m$. Next we show that

$$\sum\left\{f(b): b \in \bigcap_{k=1}^{m} W(b_k)\right\} = 0. \tag{4}$$

To prove (4), let $e_k = b_k^*b_k$ for all k, and let $h = e_1 \vee e_2 \vee \cdots \vee e_m$. Set $Z = \bigcap_{k=1}^m W(b_k)$. Suppose that $f(a) \neq 0$, $b \in Z$, $a^*a \geqslant h$, and $ah = bh$. Then $a^*a \geqslant e_k$ and $ae_k = be_k = b_k$ for $1 \leqslant k \leqslant m$. Therefore $a \in Z$. Then

$$0 = \pi(f)\varphi(h) = \sum\{f(a)\varphi(ah): a^*a \geqslant h\}$$
$$= \sum_{b\in Z} f(b)\varphi(bh) + \sum\{f(a)\varphi(ah): a^*a \geqslant h, a \notin Z\}.$$

It follows that $\Sigma_{b\in Z} f(b)\varphi(bh) = 0$ which implies (4).

Again, let $b_1, \ldots, b_m$ be a collection of elements in W_1 such that $b_k \neq a_1$, $1 \leqslant k \leqslant m$. Next we prove that

$$\sum\left\{f(b): b \in \bigcup_{k=1}^{m} W(b_k)\right\} = 0. \tag{5}$$

The idea of the proof of (5) is to show that $\bigcup_{k=1}^m W(b_k)$ can be written as a disjoint union of sets A each of which has the property that $\Sigma_{b\in A} f(b) = 0$. We proceed to define the sets A involved. Let $M = \{1, 2, \ldots, m\}$. In the context of this proof K and J will always denote subsets of M (including possibly the empty set φ). Also, $|K|$ will denote the number of elements in K. For each J, let $B_J = \bigcap_{k\in M\setminus J} W(b_k)$. By (4) we have $\Sigma_{b\in B_J} W(b_k) = 0$ for any J. When $J = \varphi$, let $A_J = B_J = \bigcap_{k=1}^m W(b_k)$. Then $\Sigma_{b\in A_J} f(b) = 0$. Now for each set J with $|J| = 1$, let $A_J = B_J\setminus A_\varphi$. Note that the collection $\{A_J: |J| \leqslant 1\}$ is disjoint and that for each J, $|J| \leqslant 1$, we have $\Sigma_{b\in A_J} f(b) = 0$ (the elementary principle we use here, and continue to use in the course of the proof, is that if B and C are subsets of S such that $\Sigma_{b\in B} f(b) = 0$ and $\Sigma_{b\in C\cap B} f(b) = 0$, and if $A = B\backslash C$, then $\Sigma_{b\in A} f(b) = 0$). For each J with $|J| = 2$, let $A_J = B_J\backslash(\bigcup_{|K|<2} A_K)$. Again the collection $\{A_J: |J| \leqslant 2\}$ is disjoint, and for $|J| \leqslant 2$, $\Sigma_{b\in A_J} f(b) = 0$. The proof continues in this fashion. We outline the nth step where $n < m$. For each J with $|J| = n$, let $A_J = B_J\backslash(\bigcup_{|K|<n} A_K)$. Then $\{A_J: |J| \leqslant n\}$ is a disjoint collection and for each J, $|J| \leqslant n$, $\Sigma_{b\in A_J} f(b) = 0$. It remains to note that

$$\bigcup_{k=1}^{m} W(b_k) = \bigcup_{|K|<m} A_K.$$

This completes the proof of (5).

Now we are in a position to prove that π is faithful in the case $S = \mathcal{I}_X$.

PROPOSITION 2.2. *When $S = \mathcal{I}_X$, then the left regular representation π is faithful.*

PROOF. We assume the results and notation above. Now $\lambda_1 \neq 0$. Therefore by (3)

$$\sum \{f(b): b \in W_1,\ b \neq a_1\} = -\lambda_1 \neq 0.$$

Choose a collection of distinct elements $b_1, \ldots, b_m$ in $W_1\backslash\{a_1\}$ such that

$$\sum\{|f(b)|: b \in W_1\backslash\{b_1, \ldots, b_m, a_1\}\} < |\lambda_1|. \tag{6}$$

Let $U = \bigcup_{k=1}^{m} W(b_k)$. Then

$$\begin{aligned} |\lambda_1| &= \left|\sum\{f(b): b \in W_1\backslash\{a_1\}\}\right| \\ &\leqslant \left|\sum\{f(b): b \in U\}\right| + \sum\{|f(b)|: b \in W_1\backslash(U \cup \{a_1\})\}. \end{aligned}$$

By (5) we have $\Sigma\{f(b): b \in U\} = 0$, so that the inequality above contradicts (6).

THEOREM 2.3. *Let S be an inverse semigroup. Then there exists a faithful representation of $l^1(S)$ on a Hilbert space. In particular, the set of irreducible representations of $l^1(S)$ on Hilbert space is a separating family.*

PROOF. By [2, Theorem 1.20] S can be embedded as an inverse subsemigroup of $S' = \mathcal{I}_S$. It follows that there exists a star monomorphism γ of $l^1(S)$ into $l^1(S')$. Let π be the left regular representation of $l^1(S')$ on $l^2(S')$. By Proposition 2.2 π is faithful. Then $\pi \circ \gamma$ is a faithful representation of $l^1(S)$ on $l^2(S')$. It follows immediately from [17, Theorem (4.6.7)] that the irreducible representations of $l^1(S)$ form a separating family.

By Theorem 2.3 and [17, Theorem (4.1.19)] we have the next result.

COROLLARY 2.4. *If S is an inverse semigroup, then $l^1(S)$ is Jacobson semisimple.*

COROLLARY 2.5. *Let S be an inverse semigroup, and let I be an ideal of S. Then $l^1(S/I)$ is star isomorphic to $l^1(S)/l^1(I)$, and thus $l^1(S)/l^1(I)$ is Jacobson semisimple.*

PROOF. By definition $S/I = \{\{a\}, I: a \in S\backslash I\}$. Since S is an inverse semigroup, then S/I is an inverse semigroup. Define $\varphi: l^1(S) \to l^1(S/I)$ by

$$\varphi\left(\sum \lambda_k a_k\right) = \sum_{a_k \notin I} \lambda_k\{a_k\}.$$

Then φ is a star homomorphism of $l^1(S)$ onto $l^1(S/I)$, and the kernel of φ is exactly $l^1(I)$. This proves that $l^1(S/I)$ is star isomorphic to $l^1(S)/l^1(I)$. It follows from Corollary 2.4 that $l^1(S)/l^1(I)$ is Jacobson semisimple.

REMARK. Let π be the left regular representation of $l^1(S)$ on $l^2(S)$. For the special case where $S = \mathcal{I}_X$, we proved that π was faithful [Proposition 2.2]. It would be interesting to know whether π is always faithful. The proof of Proposition 2.2 establishes that π is faithful when S is an inverse semigroup with the following property:

If $e, f \in E_S$, then there is an element g $(= e \vee f)$ in E_S such that $g \geq e$, $g \geq f$, and whenever $h \in E_S$ with $h \geq e$, $h \geq f$, then $h \geq g$.

3. Representations of $l^1(S)$ determined by finite idempotents of S. In this section we consider representations of $l^1(S)$ which are determined by the idempotents in S which have the property that eSe is a finite set. We call idempotents of S with this property finite idempotents. The results of this section apply to inverse subsemigroups S of $\mathcal{I}_X$, since in this case, every idempotent in S which is also in $\mathcal{F}_X$ is finite.

Assume that $e \in E_S$ is finite. An important fact concerning e is that e is primitive modulo some ideal of S. We prove this below. Let

$$K_e = \{f \in E_S\colon \exists g \in E_S \text{ with } f \sim g < e\}.$$

Then define

$$I_e = \{a \in S\colon a^*a \in K_e\}.$$

PROPOSITION 3.1. *Let $e \in E_S$ be finite, and let I_e be as above. Then I_e is an ideal of S, and e is primitive modulo I_e.*

PROOF. Assume that $a \in I_e$ and $b \in S$. There exists $g \in E_S$ such that $a^*a \sim g < e$. Then there exists $c \in S$ such that $a^*a = c^*c$ and $g = cc^*$. We have

$$b^*a^*ab = b^*c^*cb \sim cbb^*c^* \leq g < e.$$

Thus $b^*a^*ab \in K_e$, so that $ab \in I_e$. Therefore I_e is a right ideal of S. But also if $a \in I_e$, then $aa^* \sim a^*a \in K_e$, so that $a^* \in I_e$. It follows that I_e is an ideal of S.

Now suppose that $f \in E_S$ and $f < e$. Then $f \in I_e$. This proves that e is primitive modulo I_e.

Let $e \in E_S$ be finite. As we proved in the previous proposition, e is primitive modulo I_e. In this case the group G_e $(= \{\{eae\}: eae \in S \backslash I_e\})$ is a finite group. Later in this section we use the representation theory of the finite dimensional group algebra $l^1(G_e)$ to give information concerning the representations of $l^1(S)$. A crucial role in this presentation is played by the representation theory of Banach star algebras with minimal left ideals. We now very briefly review parts of this theory, and then proceed to apply it to the case at hand.

Let A be a Banach algebra with proper involution * (i.e. if $f \in A$ and $f^*f = 0$, then $f = 0$). If L is a minimal left ideal of A, then there exists a unique selfadjoint (abbreviated in the future as s.a.) idempotent $h \in A$ such that $L = Ah$ [17, Lemma (4.10.1)]. Furthermore, h is a minimal idempotent of A [17, p. 45] which means in this case that $hAh = \{\lambda h: \lambda \in \mathbf{C}\}$. Following [17, p. 261], we define a conjugate linear form $\langle \cdot, \cdot \rangle$ on $Ah \times Ah$ by the rule

$$\langle fh, gh \rangle h = hg^*fh, \qquad f, g \in A.$$

If $\langle fh, fh \rangle = 0$ for some $f \in A$, then $hf^*fh = 0$, so that $fh = 0$. Thus $\langle \cdot, \cdot \rangle$ is nondegenerate. Also, this form is positive definite. Then as in [17, p. 261] we define for $f \in A$ an operator $\pi_h(f)$ acting on Ah by

$$\pi_h(f)gh = fgh, \qquad g \in A.$$

The map $f \longrightarrow \pi_h(f)$ is a star representation on the inner product space $(Ah, \langle \cdot, \cdot \rangle)$. This representation extends to a representation of A on the completion H_h of the inner product space [17, Theorem (4.10.3)]. We denote this extended representation again by π_h. The representation (π_h, H_h) is irreducible. This can be verified as follows. Let K be a closed π_h-invariant subspace of H_h. As usual, we consider Ah to be a dense subspace of H_h. If $bh \in Ah$, then $\pi_h(h)bh = \lambda h$ for some scalar λ. Since Ah is dense in H_h, for every $x \in H_h$ there exists a scalar λ such that $\pi_h(h)x = \lambda h$. It follows that either $\pi_h(h)K = \{0\}$ or $h \in K$, and either $\pi_h(h)K^\perp = \{0\}$ or $h \in K^\perp$. But $\pi_h(h)$ is not the zero operator, so either $h \in K$ or $h \in K^\perp$. In the former case $Ah \subset K$, so that $K = H_h$, while in the latter case, $Ah \subset K^\perp$, so that $K = \{0\}$.

Now let A be, as before, a Banach algebra with proper involution *. Let J be a closed star ideal of A such that the natural involution on A/J is also proper. Denote by Q_J the natural quotient map Q_J: $A \longrightarrow A/J$. If f is a s.a. minimal idempotent of A/J, then f determines the representation π_f of A/J on H_f. Then the map $a \longrightarrow \pi_f(Q_J(a))$, $a \in A$, extends π_f to an irreducible representation of A on H_f. We denote this extension by $\pi_f \circ Q_J$.

THEOREM 3.2. *Let A and J be as in the discussion above. Let (π, H) be a representation of A with $J \subset \ker(\pi)$. Let h be a s.a. minimal idempotent of A/J and choose h' s.a. such that $Q_J(h') = h$.*

If $x_0 \in H$ and $K = \mathrm{cl}\{\pi(Ah')x_0\} \neq \{0\}$, then $(\pi|K) \approx \pi_h \circ Q_J$.

PROOF. Let $(\cdot\,,\cdot)$ denote the inner product on H. Let $y_0 = \pi(h')x_0$. We may assume that $(y_0, y_0) = 1$. If $a, b \in A$ and $Q_J(a - b)h = 0$, then $(a - b)h' \in J$, so that $\pi(ah') = \pi(bh')$. Define U: $(A/J)h \longrightarrow K$ by $U(Q_J(a)h) = \pi(ah')y_0$. By the previous argument we have that U is well defined. Also U maps onto a dense subspace of K.

If $a \in A$,

$$Q_J(h'a^*ah' - \langle Q_J(a)h, Q_J(a)h\rangle h') = h(Q_J(a)^*Q_J(a))h - \langle Q_J(a)h, Q_J(a)h\rangle h = 0.$$

Therefore

$$\pi(h'a^*ah') = \langle Q_J(a)h, Q_J(a)h\rangle \pi(h').$$

Note also that $\pi(h')y_0 = y_0$. If $a \in A$, then

$$(U(Q_J(a)h), U(Q_J(a)h)) = (\pi(ah')y_0, \pi(ah')y_0) = (\pi(h'a^*ah')y_0, y_0) = \langle Q_J(a)h, Q_J(a)h\rangle(\pi(h')y_0, y_0) = \langle Q_J(a)h, Q_J(a)h\rangle.$$

Thus U extends to a unitary transformation of H_h onto K. If $a, b \in A$, we have

$$\pi(b)U(Q_J(a)h) = \pi(bah')y_0 = UQ_J(b)Q_J(a)h = U(\pi_h \circ Q_J)(b)(Q_J(a)h).$$

Therefore, $(\pi|K) \approx \pi_h \circ Q_J$.

COROLLARY 3.3. *Let A, J, and h be as above. Let (π, H) be an irreducible representation of A. Then $\pi \approx \pi_h \circ Q_J$ if and only if* $\ker(\pi) = \ker(\pi_h \circ Q_J)$.

PROOF. Assume that $\ker(\pi) = \ker(\pi_h \circ Q_J)$. Choose h' s.a. in A such that $Q_J(h') = h$. Since $(\pi_h \circ Q_J)(h') = \pi_h(h) \neq 0$, we have $\pi(h') \neq 0$. The result now follows from Theorem 3.2.

For the remainder of this section we assume that e is a finite idempotent of S and that I is an ideal of S such that e is primitive modulo I. As mentioned in §1, S acts on S/I in a natural way. Thus, if $a, b \in S$ and $c \in S/I$, then the product acb makes sense as an element of S/I.

Let $A = l^1(S)$ and $J = l^1(I)$. Then J is a closed star ideal of A and A/J is isomorphic to $l^1(S/I)$ [Corollary 2.5]. By Theorem 2.3 both A and A/J have a faithful representation on Hilbert space. In particular both A and A/J have proper involution. As before, we use the notation Q_J for the natural quotient map of A onto A/J.

We use the notation B for the algebra $l^1(G_e) = l^1(e(S/I)e) = e(A/J)e$. B is finite dimensional and Jacobson semisimple. This means that all of the Wedderburn theory for such algebras is available to us; see [5, pp. 163–190]. In particular, B contains minimal left ideals each of which is of the form Bh, h a s.a. minimal idempotent of B; B is the direct sum of its minimal ideals; and every irreducible representation of B is equivalent to the left regular representation of B on some minimal left ideal. We use these facts and other results from the Wedderburn theory freely in what follows.

Let h be a s.a. minimal idempotent of B. Since $he = eh = h$, we have

$$h(A/J)h = h(e(A/J)e)h = \{\lambda h \colon \lambda \in \mathbf{C}\}.$$

Thus, h is a s.a. minimal idempotent of A/J. Then we can construct the irreducible star representation $(\pi_h \circ Q_J, H_h)$ as indicated previously. The algebra B is the direct sum of minimal ideals M_k, $1 \leqslant k \leqslant n$, where $M_k M_j = \{0\}$ if $k \neq j$. Then there are exactly n inequivalent irreducible representations of B. These representations can be determined by minimal idempotents of B (which are then also minimal idempotents of A/J). Two minimal idempotents determine equivalent representations if and only if they belong to the same minimal ideal of B. If h and f are s.a. minimal idempotents of B contained in different minimal ideals of B, then $hBf = \{0\}$. By definition $B = e(A/J)e$. Thus, $h(A/J)f = h(e(A/J)e)f = hBf = \{0\}$. It follows that $h \in \ker(\pi_f)$, and similarly, $f \in \ker(\pi_h)$. Therefore $\pi_f \circ Q_J$ and $\pi_h \circ Q_J$ must be inequivalent representations of A. We summarize the previous discussion in the next result.

PROPOSITION 3.4. *Let e and J be as above. If $\{h_1, \ldots, h_n\}$ is a collection of s.a. minimal idempotents of $l^1(G_e)$ which determines a complete set of inequivalent irreducible representations of $l^1(G_e)$, then $\{\pi_{h_k} \circ Q_J, 1 \leqslant k \leqslant n\}$ is a collection of inequivalent irreducible representations of $l^1(S)$.*

A s.a. minimal idempotent h of $l^1(G_e)$ determines the irreducible representation $\pi_h \circ Q_J$ of $l^1(S)$. The problem of finding the minimal idempotents (or equivalently, the irreducible representations) of a group algebra such as $l^1(G_e)$ can be a very difficult problem. However, the irreducible representations of $l^1(S)$ determined by s.a. minimal idempotents of $l^1(G_e)$ are each contained in a representation of $l^1(S)$ which is very simply described, the left regular representation of $l^1(S)$ on $l^2((S/I)e)$. We define this representation next.

The space $l^2((S/I)e)$ is the usual Hilbert space of complex-valued functions f on the set $(S/I)e$ such that $\Sigma\{|f(a)|^2 \colon a \in (S/I)e\} < \infty$ with the additional convention that $f(I) = 0$. We denote the standard orthonormal basis of this

Hilbert space by $\{\varphi(a): a \in (S/I)e, a \neq I\}$. If $b \in S$ and $a \in (S/I)e$, $a \neq I$, then define

$$\pi(b)\varphi(a) = \begin{cases} \varphi(ba) & \text{if } b^*ba = a, \\ 0 & \text{if } b^*ba \neq a. \end{cases}$$

Then if $f \in l^1(S)$, $f = \Sigma\lambda_k b_k$, and $\psi = \Sigma\mu_j\varphi(a_j) \in l^2((S/I)e)$, let

$$\pi(f)\psi = \sum_{k,j} \lambda_k\mu_j\pi(b_k)\,\varphi(a_j).$$

Just as in the proof of Proposition 2.1, we have that $f \longrightarrow \pi(f)$ is a representation of $l^1(S)$ on $l^2((S/I)e)$. For convenience of notation we let $H = l^2((S/I)e)$.

PROPOSITION 3.5. *If h is a s.a. minimal idempotent of $l^1(G_e)$, then $\pi_h \circ Q_J$ is equivalent to some subrepresentation of π.*

PROOF. There exist distinct elements $a_k \in S$, $1 \leqslant k \leqslant n$, and nonzero scalars λ_k such that $Q_J(h') = h$ where $h' = \Sigma_{k=1}^n \lambda_k a_k$ and $a_k^* a_k = e$ for all k. Then $\pi(a_k)\varphi(e) = \varphi(a_k)$, and $\varphi(a_k) \neq \varphi(a_j)$ if $k \neq j$. Thus,

$$\pi(h')\varphi(e) = \sum_{k=1}^{n} \lambda_k\pi(a_k)\varphi(e) = \sum_{k=1}^{n} \lambda_k\varphi(a_k) \neq 0.$$

Then the result follows from Theorem 3.2.

PROPOSITION 3.6. *The representation (π, H) is a finite orthogonal direct sum of representations of the form $(\pi_f \circ Q_J, H_f)$ where f is chosen from the set of s.a. minimal idempotents of $l^1(G_e)$.*

PROOF. Let M_k, $1 \leqslant k \leqslant n$, be the set of all the distinct minimal ideals of $B = l^1(G_e)$. Each of the algebras M_k has an identity u_k, and we have $e = u_1 + \cdots + u_n$ and $u_k u_j = 0$ if $k \neq j$. For each k choose $u_k' \in A$, u_k' s.a., such that $Q_J(u_k') = u_k$. For each i, let

$$K_i = \mathrm{cl}\{\pi(gu_i')\varphi(e): g \in A\}.$$

Suppose $i \neq j$. Then $u_i(A/J)u_j = \{0\}$, so if $f, g \in A$, $(\pi(fu_j')\varphi(e), \pi(gu_i')\varphi(e)) = 0$. Thus, $K_i \perp K_j$. Furthermore, since $\varphi(e)$ is a cyclic vector for π, H is the orthogonal sum of the π-invariant subspaces K_i.

Now fix i. Choose h a s.a. minimal idempotent of B with $h \in M_i$. Choose h' s.a. in A such that $Q_J(h') = h$. $\pi(h')$ is a s.a. projection, and as in the proof of Proposition 3.5, $\pi(h') \neq 0$. Note that $\pi(h')K_j = \{0\}$ if $i \neq j$. Thus the range of $\pi(h')$ is contained in K_i. Let $\{x_1, \ldots, x_m\}$ be an orthonormal basis for the range of $\pi(h')$. For each j, $1 \leqslant j \leqslant m$, let

$$L_j = \mathrm{cl}\{\pi(A)x_j\}.$$

If $j \neq k$, then $x_j \perp x_k$, and

$$(\pi(f)x_j, \pi(g)x_i) = (\pi(fh')x_j, \pi(gh')x_i) = (\pi(h'g^*fh')x_j, x_k)$$
$$= \lambda(x_j, x_k) \quad \text{for some scalar } \lambda,$$
$$= 0.$$

Set $L = L_1 + \cdots + L_m$. We show that L is dense in K_i, and it will follow that $K_i = L$. Since $u_i \in M_i = BhB$, we can choose $f, g \in A$ such that $u_i' = fh'g$ modulo J. Then if $k \in A$, we have

$$\pi(ku_i')\varphi(e) = \pi(kf)\pi(h'g)\varphi(e).$$

Now $\pi(h'g)\varphi(e) \in \text{span}\{x_1, \ldots, x_m\}$, so that $\pi(ku_i')\varphi(e) \in L_1 + \cdots + L_m = L$. Therefore K_i is the orthogonal sum of $L_1, L_2, \ldots, L_m$. By Theorem 3.2, each of the representations $(\pi|L_j)$ is equivalent to $\pi_h \circ Q_J$. This proves the result.

Proposition 3.7. *If K is a closed π-invariant subspace of H and $(\pi|K)$ is irreducible, then there exists a s.a. minimal idempotent h of $l^1(G_e)$ such that $(\pi|K) \approx \pi_h \circ Q_J$.*

Proof. Assume that $x \in K$, $x \neq 0$. As shown in Proposition 3.6, H is the orthogonal direct sum of π-invariant subspaces J_i, $1 \leqslant i \leqslant p$, with the property that each representation $(\pi|J_i)$ is equivalent to some representation of the form $\pi_f \circ Q_J$, f a s.a. minimal idempotent of B. Let $x = x_1 + \cdots + x_p$ where $x_k \in J_k$, $1 \leqslant k \leqslant p$. Suppose $x_i \neq 0$, and let h be a s.a. minimal idempotent of B such that $(\pi|J_i) \approx \pi_h \circ Q_J$. Choose h' s.a. in A such that $Q_J(h') = h$. Then $\pi(h'A)x_i \neq \{0\}$, so that $\pi(h'A)x \neq \{0\}$. Since $\pi(A)x \subset K$, we have $\pi(h')K \neq \{0\}$. Then the result follows from Theorem 3.2.

Now assume that f is also in E_S and that f is primitive modulo I. Let γ be the left regular representation of $l^1(S)$ on $K = l^2((S/I)f)$.

Proposition 3.8. *If $e \sim f$, then (π, H) is equivalent to (γ, K). In particular every irreducible subrepresentation of (γ, K) is equivalent to some irreducible subrepresentation of (π, H).*

Proof. We denote the elements of S/I as $\{a\}$, $a \in S\backslash I$, and I. There exists $b \in S$ such that $e = b^*b$ and $f = bb^*$. If $ae \in S\backslash I$, then $ab^*f \in S\backslash I$, and we have $\{a\}eb^* = \{ab^*bb^*\} = \{ab^*\}f$. Therefore $(S/I)eb^* \subset (S/I)f$.

Define U: $H \longrightarrow K$ as follows. If $\psi = \Sigma\lambda_k\varphi(a_ke)$, let $U\psi = \Sigma\lambda_k\varphi(a_keb^*)$. Note that if $a_keb^*, a_jeb^* \in S\backslash I$ and $a_keb^* = a_jeb^*$, then $a_ke, a_je \in S\backslash I$ and $a_ke = a_je$. This implies that U is an isometry. Since U maps H onto K, U is unitary. Let ψ be as above and assume $a \in S$. Let $M = \{k: a^*aa_ke = a_ke\}$. Note that $a^*aa_ke = a_ke$ if and only if $a^*aa_keb^* = a_keb^*$. Then

$$\gamma(a)U\psi = \sum_{k\in M} \lambda_k \varphi(aa_k eb^*) = U\Big(\sum_{k\in M} \lambda_k \varphi(aa_k e)\Big) = U\pi(a)\psi.$$

Therefore $\gamma \approx \pi$.

Define $F = \{a \in S: a^*a \text{ is finite}\}$. It is easy to verify that F is an ideal of S. In fact F is the smallest ideal of S that contains every finite idempotent in E_S.

PROPOSITION 3.9. *Assume that (γ, K) is a representation of $l^1(S)$ such that $l^1(F) \not\subset \ker(\gamma)$. Then there exists $e \in E_S$, e finite, and a s.a. minimal idempotent of $l^1(S)/J$, where $J = l^1(I_e)$, such that $\pi_h \circ Q_J$ is equivalent to some subrepresentation of γ.*

PROOF. Since $l^1(F) \not\subset \ker(\gamma)$, there exists $e \in E_S$, e finite, such that $\gamma(e) \neq 0$. Furthermore, we may assume that e is minimal with respect to the partial order $\leqslant$ in the set $\{f \in E_S: f \text{ is finite and } \gamma(f) \neq 0\}$. With this assumption we now verify that $\gamma(a) = 0$ for all $a \in I_e$. For suppose $a \in I_e$. Then by definition $a^*a \sim g < e$ for some $g \in E_S$. There exists $c \in S$ such that $a^*a = c^*c$ and $g = cc^*$. By the choice of e we have $\gamma(g) = 0$, so that $\gamma(c)\gamma(c^*) = 0$. It follows that $\gamma(c) = 0$, and thus, $\gamma(a) = 0$. Therefore $\gamma = 0$ on I_e.

Let $J = l^1(I_e)$. The algebra $l^1(G_e)$ is a finite sum of minimal left ideals. Since $\gamma(e) \neq 0$, there must exist a minimal left ideal L of $l^1(G_e)$ such that $Q_J^{-1}(L) \not\subset \ker(\gamma)$. The left ideal L is generated by a s.a. minimal idempotent h of $l^1(G_e)$. Then h is also a s.a. minimal idempotent of $l^1(S)/J$. Choose a s.a. element $h' \in l^1(S)$ such that $Q_J(h') = h$. Then $Q_J^{-1}(L) \subset Ah' + J$. If $\gamma(h') = 0$, then since γ is 0 on J, we have $Ah' + J \subset \ker(\gamma)$. This is impossible by the choice of L. Thus $\gamma(h') \neq 0$, and the result follows from Theorem 3.2.

Now we prove a structure theorem for representations.

THEOREM 3.10. *Let (π, H) be a representation of $l^1(S)$. Then there exist π-invariant subspaces of H, H_1 and H_2, with $H_2 = H_1^{\perp}$, such that*

(1) $l^1(F) \subset \ker(\pi | H_2)$,

(2) *if K is a nonzero π-invariant subspace of H_1, then $(\pi | K)$ contains an irreducible subrepresentation equivalent to a representation of the form $(\pi_h \circ Q_J, H_h)$, and*

(3) *if $H_1 \neq \{0\}$, then $(\pi | H_1)$ is the orthogonal direct sum of irreducible representations each of which is equivalent to some representation of the form $(\pi_h \circ Q_J, H_h)$.*

PROOF. First let

$$H_1 = \operatorname{span}\{\pi(e)\varphi: e \in E_S,\ e \text{ finite},\ \varphi \in H\},$$

and

$$H_2 = \{\psi \in H\colon \pi(e)\psi = 0 \text{ for all } e \in E_S,\ e \text{ finite}\}.$$

Then

$$\begin{aligned}\psi \in H_1^\perp &\Longleftrightarrow (\pi(e)\varphi, \psi) = 0,\ e \in E_S,\ e \text{ finite},\ \varphi \in H \\ &\Longleftrightarrow (\varphi, \pi(e)\psi) = 0,\ e \in E_S,\ e \text{ finite},\ \varphi \in H \\ &\Longleftrightarrow \psi \in H_2.\end{aligned}$$

Thus, $H_2 = H_1^\perp$. Suppose $\pi(e)\varphi \in H_1$ where $e \in E_S$, e finite, $\varphi \in H$. If $a \in F$, then there exists a finite idempotent $f \in S$ such that $fae = ae$. Therefore

$$\pi(a)(\pi(e)\varphi) = \pi(f)(\pi(ae)\varphi) \in H_1.$$

It follows that H_1 and $H_2 = H_1^\perp$ are π-invariant.

This establishes the basic decomposition of π into an orthogonal direct sum of $(\pi|H_1)$ and $(\pi|H_2)$. Now we verify properties (1)–(3) of these two subrepresentations. By the definition of H_2 we have that $(\pi|H_2)(e) = 0$ for every finite idempotent e. It follows that $F \subset \ker(\pi|H_2)$, so that $l^1(F) \subset \ker(\pi|H_2)$. This proves (1).

Let K be as in the statement of (2). If $l^1(F) \subset \ker(\pi|K)$, then $\pi(e)\psi = 0$ whenever $e \in E_S$, e finite, and $\psi \in K$. Then if $e \in E_S$ is finite, $\psi \in K$, and $\varphi \in H$, we have $(\psi, \pi(e)\varphi) = (\pi(e)\psi, \varphi) = 0$. This implies that $K \subset H_1^\perp$, so that $K = \{0\}$, a contradiction. Thus, $l^1(F) \not\subset \ker(\pi|K)$. Then (2) follows from Proposition 3.9.

To prove (3), assume that $H_1 \neq \{0\}$. By (2) it follows that $(\pi|H_1)$ has an irreducible subrepresentation equivalent to a representation of the form $(\pi_h \circ Q_J, H_h)$. Then a Zorn's Lemma argument shows that there exists a maximal (with respect to inclusion) π-invariant subspace $M \subset H_1$ such that $(\pi|M)$ is an orthogonal direct sum of the sort described in (3). Let $K = M^\perp \cap H_1$. By the maximal property of J, $(\pi|K)$ contains no irreducible subrepresentation equivalent to a representation of the form $(\pi_h \circ Q_J, H_h)$. Then by (2), $K = \{0\}$. Thus $H_1 = M$ which proves (3).

4. **Applications to the representation theory of $\mathcal{I}_X$.** Let X be an infinite set. We adopt the notation of Examples 1.1 and 1.2. The aim of this section is to apply the results of §3 to the case where the semigroup S is $\mathcal{I}_X$.

First we consider some basic facts concerning $\mathcal{I}_X$, all of which are easily verified. The finite idempotents of $\mathcal{I}_X$ are exactly the idempotent maps in $\mathcal{F}_X$, and the ideal $F = \{a \in \mathcal{I}_X\colon a^*a \text{ is finite}\}$ is the ideal $\mathcal{F}_X$. Let E be the set of idempotent maps in $\mathcal{I}_X$. If $e, f \in E$, then $e \sim f$ if and only if $|D_e| = |D_f|$. If $e \in E$ and e is finite, then the ideal I_e defined just prior to Proposition 3.1 is $\mathcal{F}_{n-1}$ where $n = |D_e|$. Then e is primitive modulo $\mathcal{F}_{n-1}$ [Proposition 3.1], and the group G_e is the symmetric group on n elements. Thus, for each positive integer n, the group algebra of the symmetric group on n elements can be used to determine irreducible representations of $l^1(\mathcal{I}_X)$ as in §3. A technique is available for explicitly constructing minimal left ideals of the group algebra of the symmetric group; see [5, pp. 190–198]. Therefore specific examples of irreducible representations of the form $(\pi_h \circ Q_J, H_h)$ can be constructed. What we do next is consider a collection of representations of $l^1(\mathcal{I}_X)$ of this form.

Fix a positive integer n, and let e be any idempotent map in $\mathcal{I}_X$ with $|D_e| = n$. Let

$$h = \frac{1}{n!}\left(\sum_{g \in G_e} g\right).$$

It is easy to check that h is a s.a. minimal idempotent of $l^1(G_e)$, and hence of $l^1(\mathcal{I}_X)/l^1(\mathcal{F}_{n-1})$. For convenience let $A = l^1(\mathcal{I}_X)$ and $J = l^1(\mathcal{F}_{n-1})$. We have that $(\pi_h \circ Q_J, H_h)$ is an irreducible representation of A. In this case it is very easy to write down an elementary equivalent form of this representation. Let P_n be the collection of all subsets $T \subset X$ with $|T| = n$. Let $\{\varphi(T): T \in P_n\}$ be the standard orthonormal basis of $l^2(P_n)$. If $b \in \mathcal{I}_X$ and $T \in P_n$, define

$$\gamma_n(b)\varphi(T) = \begin{cases} \varphi(b(T)) & \text{if } T \subset D_b, \\ 0 & \text{if } T \not\subset D_b \end{cases}$$

(here $b(T)$ denotes the set of values the map b takes on T). If $f = \Sigma\lambda_k b_k \in A$ and $\psi = \Sigma\mu_j\varphi(T_j) \in l^2(P_n)$, then we extend γ_n by the usual rule

$$\gamma_n(f)\psi = \sum_{k,j} \lambda_k\mu_j\gamma_n(b_k)\,\varphi(T_j).$$

It is easy to verify that $f \longrightarrow \gamma_n(f)$ is a representation of A on the Hilbert space $l^2(P_n)$. Also, it is interesting to note that $\gamma_n \not\approx \gamma_m$ if $m \neq n$.

PROPOSITION 4.1. *Let h be as above. Then $\gamma_n \approx \pi_h \circ Q_J$.*

PROOF. Let G denote the symmetric group of permutations on $\{1, 2, \ldots, n\}$. Write $D_e = \{x_1, \ldots, x_n\}$. For each $\sigma \in G$, let a_σ be the map in $\mathcal{I}_X$ with domain D_e defined by

$$a_\sigma(x_k) = x_{\sigma(k)}, \qquad 1 \leqslant k \leqslant n.$$

Then $h' = (1/n!)\,\Sigma_{\sigma\in G}\, a_\sigma \in l^1(\mathcal{I}_X)$ and $Q_J(h') = h$. Note that $\gamma_n(a_\sigma)\varphi(D_e) = \varphi(a_\sigma(D_e)) = \varphi(D_e)$ for all $\sigma \in G$. Thus $\gamma_n(h')\varphi(D_e) = \varphi(D_e)$. Let $\lambda_1, \ldots, \lambda_m$ be scalars, and $T_1, \ldots, T_m$ be in P_n. Choose $b_1, \ldots, b_m \in \mathcal{I}_X$ such that b_k has domain D_e and range T_k, $1 \leqslant k \leqslant m$. Then

$$\gamma_n((\lambda_1 b_1 + \cdots + \lambda_m b_m)h')\varphi(D_e) = \lambda_1\varphi(T_1) + \cdots + \lambda_m\varphi(T_m).$$

This proves that

$$l^2(P_n) = \mathrm{cl}\{\gamma_n(Ah')\varphi(D_e)\}.$$

Then a direct application of Theorem 3.2 completes the proof that $\gamma_n \approx \pi_h \circ Q_J$.

As we have just shown, the representation $(\pi_h \circ Q_J, H_h)$ for certain h has an elementary equivalent form in terms of a natural representation of $l^1(\mathcal{I}_X)$ on l^2 of a certain collection of subsets of X. It may be true that all the irreducible representations γ of $l^1(\mathcal{I}_X)$ with $l^1(\mathcal{F}_X) \not\subset \ker(\gamma)$ have some equivalent form of this type where the subsets of X involved are allowed certain orderings. However, we have not been able to prove such a result in general.

Let e be a finite idempotent map in $\mathcal{I}_X$, and assume that $n = |D_e|$. It was shown in §3 that the left regular representation of $l^1(\mathcal{I}_X)$ on $l^2((\mathcal{I}_X/\mathcal{F}_{n-1})e)$ contains all the irreducible representations of $l^1(\mathcal{I}_X)$ determined by s.a. minimal idempotents of $l^1(G_e)$. Now we construct an elementary equivalent form of this representation. This construction can be done for each positive integer n. Let X_n be the set of all ordered n-tuples of distinct elements in X, i.e.

$$X_n = \{(y_1, \ldots, y_n): y_k \in X, y_k \neq y_j \text{ if } k \neq j\}.$$

Form $l^2(X_n)$ with the standard orthonormal basis of this space denoted by $\{\varphi(y_1, \ldots, y_n): (y_1, \ldots, y_n) \in X_n\}$. If $b \in \mathcal{I}_X$ and $(y_1, \ldots, y_n) \in X_n$, let

$$\pi_n(b)\varphi(y_1, \ldots, y_n) = \begin{cases} \varphi(b(y_1), \ldots, b(y_n)) & \text{if } y_k \in D_b,\ 1 \leqslant k \leqslant n, \\ 0 \quad \text{if some } y_k \notin D_b. \end{cases}$$

Then π_n extends in the usual way to a representation of $l^1(\mathcal{I}_X)$ on $l^2(X_n)$.

THEOREM 4.2. *Let e be any idempotent map in $\mathcal{I}_X$ with $n = |D_e|$. Then*

(1) *$(\pi_n, l^2(X_n))$ is equivalent to the left regular representation of $l^1(\mathcal{I}_X)$ on $l^2((\mathcal{I}_X/\mathcal{F}_{n-1})e)$.*

(2) *$(\pi_n, l^2(X_n))$ is a finite direct sum of the irreducible representations determined by s.a. minimal idempotents of $l^1(G_e)$.*

(3) *If (π, K) is an irreducible representation of $l^1(\mathcal{F}_X) \not\subset \ker(\pi)$, then (π, K) is equivalent to some subrepresentation of $(\pi_n, l^2(X_n))$ for some n.*

PROOF. Let $\{x_1, \ldots, x_n\}$ be the elements of D_e. The elements of $(\mathcal{I}_X/\mathcal{F}_{n-1})e$ are $\{ae\}$ where $ae \in \mathcal{I}_X \setminus \mathcal{F}_{n-1}$ and $\mathcal{F}_{n-1}$. For convenience let $Q_n = (\mathcal{I}_X/\mathcal{F}_{n-1})e$. Denote by $\{\varphi(ae): ae \in \mathcal{I}_X \setminus \mathcal{F}_{n-1}\}$ the standard orthonormal basis for $l^2(Q_n)$. If $ae \in \mathcal{I}_X \setminus \mathcal{F}_{n-1}$, define

$$U\varphi(ae) = \varphi(a(x_1), \ldots, a(x_n)).$$

Then U maps the basis of $l^2(Q_n)$ onto the basis of $l^2(X_n)$. Note that when ae,

$be \in I_X \setminus F_{n-1}$, then $ae = be$ if and only if $(a(x_1), \ldots, a(x_n)) = (b(x_1), \ldots, b(x_n))$. Thus the mapping U is one-to-one on the basis of $l^2(Q_n)$. It follows that the extension of U to all of $l^2(Q_n)$ onto $l^2(X_n)$ given by

$$U\Big(\sum \lambda_k \varphi(a_k e)\Big) = \sum \lambda_k \varphi(a_k(x_1), \ldots, a_k(x_n))$$

is a unitary operator. The fact that U intertwines the left regular representation of $l^1(I_X)$ on $l^2(Q_n)$ with the representation $(\pi_n, l^2(X_n))$ is easily verified. Therefore these two representations are equivalent. This proves (1). Then (2) follows immediately from Proposition 3.6, and (3) follows from Proposition 3.9 and Proposition 3.5.

Now we turn to some results that concern the dimension of the representations of $l^1(I_X)$. If (π, H) is a representation of a star algebra, then the dimension of (π, H) is $\dim(H)$ (= the cardinality of any orthonormal basis of H). Let $(\pi_n, l^2(X_n))$ be the representations constructed in the previous paragraph $n \geqslant 1$.

PROPOSITION 4.3. *Let K be a nonzero π_n-invariant subspace of $l^2(X_n)$. Then* $\dim(K) = |X|$.

PROOF. Choose $\varphi \in K$, $\varphi \neq 0$. Then φ has the form $\varphi = \Sigma \lambda_k \varphi(T_k)$ where each scalar $\lambda_k \neq 0$ and each $T_k \in X_n$. Assume $T_1 = (y_1, \ldots, y_n)$, $y_j \in X$. Let Λ be an index set with $|\Lambda| = |X|$. Choose a collection of mutually disjoint subsets of X, $\{W_\lambda : \lambda \in \Lambda\}$, with the properties that each W_λ contains exactly n elements. For each $\lambda \in \Lambda$, choose $a_\lambda \in I_X$ with domain $\{y_1, \ldots, y_n\}$ and range W_λ. Then $\pi_n(a_\lambda)\varphi \neq 0$ for all $\lambda \in \Lambda$, and the collection $\{\pi_n(a_\lambda)\varphi : \lambda \in \Lambda\}$ is a mutually orthogonal subset of K. Thus, $\dim(K) \geqslant |\Lambda| = |X|$. The reverse inequality is obvious since $\dim(l^2(X_n)) = |X|$.

COROLLARY 4.4. *If (π, H) is irreducible representation of $l^1(I_X)$ and $l^1(F_X) \not\subset \ker(\pi)$, then* $\dim(H) = |X|$.

PROOF. By Theorem 4.2(3) such a representation (π, H) is equivalent to a subrepresentation of $(\pi_n, l^2(X_n))$ for some n. Then Proposition 4.3 implies the result.

In two of the next results we assume that X is countably infinite. It is very likely that these results generalize with no restriction on the cardinality of X (except that X be infinite), but the tools to prove the general case do not seem to be readily available in the literature.

THEOREM 4.5. *Assume that X is countably infinite. If (π, H) is a nonzero representation of $l^1(I_X)$ with $l^1(F_X) \subset \ker(\pi)$, then* $\dim(H) > |X|$.

PROOF. Let E be the collection of idempotent maps in I_X. Let $d = |X|$. By hypothesis we have $e \in \ker(\pi)$ whenever $e \in E$ and R_e is finite. Also, if f and

g are in E and $|R_f| = |R_g| = d$, then $f \sim g$. Suppose that $f = c^*c$ and $g = cc^*$, $c \in \mathcal{I}_X$. If $\pi(f) = 0$, then $\pi(c)^*\pi(c) = 0$, so that $\pi(c) = 0$. Therefore $\pi(g) = 0$. This proves that if any $f \in E$ with $|R_f| = d$ is in $\ker(\pi)$, then every $g \in E$ is in $\ker(\pi)$. Thus π would be the zero representation, a contradiction.

We have that $e \in E$ is in $\ker(\pi)$ if and only if R_e is finite. Now by [15, Lemma 2] there exists a subset Γ of E with the properties

(i) $e \in \Gamma \Rightarrow |R_e| = d$,

(ii) $e, f \in \Gamma$, $e \neq f \Rightarrow R_{ef}$ is finite,

(iii) $|\Gamma| > d$.

Thus by (i) $\pi(e) \neq 0$ for all $e \in \Gamma$, while by (ii), $\pi(ef) = 0$ whenever $e, f \in \Gamma$, $e \neq f$. Therefore $\{\pi(e): e \in \Gamma\}$ is a mutually orthogonal set of nonzero projections on H. Then $\dim(H) \geqslant |\Gamma| > d$.

THEOREM 4.6. *The basic structure theorem for representations* [*Theorem* 3.10] *holds with* $S = \mathcal{I}_X$ *and* $F = \mathcal{F}_X$.

COROLLARY 4.7. *If X is countably infinite and (π, H) is a separable representation of $l^1(\mathcal{I}_X)$ (i.e. $\dim(H) = |X|$), then (π, H) is the orthogonal direct sum of irreducible representations each of which is equivalent to some representation of the form $(\pi_h \circ Q_J, H_h)$.*

PROOF. By Theorem 3.10, π is the direct sum of two subrepresentations $(\pi|H_1)$ and $(\pi|H_2)$. Furthermore, $l^1(\mathcal{F}_X) \subset \ker(\pi|H_2)$. Therefore by Theorem 4.5, $H_2 = \{0\}$. Then the result follows from Theorem 3.10(3).

5. **Some Calkin-type irreducible representations.** Let X be an infinite set. We assume throughout this section that S is an inverse subsemigroup of $\mathcal{I}_X$ such that S contains every idempotent map in $\mathcal{I}_X$. The aim of this section is to construct a collection of irreducible representations of $l^1(S)$ each of which annihilates every finite idempotent of S.

First we represent $l^1(S)$ on $l^2(X)$. Let $B = \{\varphi(x): x \in X\}$ be the standard orthonormal basis of $l^2(X)$. If $a \in S$, define

$$\pi(a)\varphi(x) = \begin{cases} \varphi(a(x)) & \text{if } x \in D_a, \\ 0 & \text{if } x \notin D_a. \end{cases}$$

Then π extends in the usual fashion to a representation of $l^1(S)$ on $l^2(X)$. For convenience we use the notation $H = l^2(X)$. Let $\mathcal{A}$ be the uniformly closed algebra of operators on H generated by $\pi(l^1(S))$. Let $\mathcal{D}$ be the algebra of all operators T on H such that every element of the basis B is an eigenvector of T. Let Y be a nonempty subset of X. If e is the idempotent map in S with $D_e = Y$, then $\pi(e)$ is the projection on the subspace of H spanned by $\{\varphi(y): y \in Y\}$. Then it is easy to verify that $\mathcal{D}$ is the uniformly closed algebra of operators on

H generated by $\{\pi(e)\colon e \in E_S\}$. Thus, $\mathcal{D} \subset \mathcal{A}$. If $T \in \mathcal{D}$, let $f_T(x) = (T\varphi(x), \varphi(x))$, $x \in X$. The map $T \longrightarrow f_T$ is an isometric isomorphism of $\mathcal{D}$ onto $l^\infty(X)$, the algebra of all bounded functions on X with the sup norm. We identify $\mathcal{D}$ with $l^\infty(X)$ in what follows.

Now we proceed to construct a collection $\{\pi_\alpha\}$ of irreducible representations of $\mathcal{A}$. Then $\{\pi_\alpha \circ \pi\}$ is a collection of irreducible representations of $l^1(S)$. Denote by $\{h_x\}$ a set of vectors in H indexed by the set of all $x \in X$. Let W be the set of all such sets $\{h_x\}$ which have the property that given any $\epsilon > 0$ and any $g \in H$, then

$$(1) \qquad \{x \in X\colon |(h_x, g)| \geq \epsilon\} \text{ is finite.}$$

W is a vector space with the obvious definitions of scalar multiplication and vector addition, e.g. $\{h_x\} + \{g_x\} = \{h_x + g_x\}$. In the particular case when X is countably infinite, W can be identified with the set of all sequences in H which converge weakly to zero. If $T \in \mathcal{A}$, let T act on W by the definition $T\{h_x\} = \{Th_x\}$.

Fix α a pure state (equivalently, a multiplicative linear functional) on $l^\infty(X)$. Using α we define a pre-inner product on W as follows: If $\{h_x\}, \{g_x\} \in W$, then the function $f(x) = (h_x, g_x)$, $x \in X$, is in $l^\infty(X)$. Then define

$$\langle\{h_x\}, \{g_x\}\rangle = \alpha(f).$$

It is not difficult to verify that $\langle\cdot\,,\cdot\rangle$ is a pre-inner product on W. Let K' be the inner product space obtained by factoring W by the linear subspace of all vectors $\{h_x\} \in W$ such that $\langle\{h_x\}, \{h_x\}\rangle = 0$. We denote the natural quotient projection of W onto K' by Q. Let K be the Hilbert space completion of K'. We denote the inner product on K by $\langle h, g\rangle$, $h, g \in K$. Let $T \in \mathcal{A}$. If $\{h_x\} \in W$ and $\langle\{h_x\}, \{h_x\}\rangle = 0$, then

$$0 \leq \langle\{Th_x\}, \{Th_x\}\rangle = \alpha((Th_x, Th_x)) \leq \alpha(\|T\|^2 \|h_x\|^2) = \|T\|^2\alpha(\|h_x\|^2) = 0.$$

Thus if $k \in W$ and $Q(k) = 0$, then $Q(Tk) = 0$. This implies that the following definition makes sense. If $\{h_x\} \in W$, let $\gamma'(T)(Q(\{h_x\})) = Q(\{Th_x\})$. Then $T \longrightarrow \gamma'(T)$ is a representation of $\mathcal{A}$ on K'. This representation extends uniquely to a representation γ of $\mathcal{A}$ on K. Note that the element $\{\varphi(x)\} \in W$. Denote this element by φ. Then $Q(\varphi)$ is a nonzero vector in K. Let H_α be the closed subspace of K generated by $\{\gamma(T)Q(\varphi)\colon T \in \mathcal{A}\}$. Finally, let π_α be the restriction of the representation γ to H_α. In what follows we derive some of the properties of the representations $(\pi_\alpha \circ \pi, H_\alpha)$. We start by establishing the irreducibility of these representations.

THEOREM 5.1. *The representation* (π_α, H_α) *is an irreducible representation* *of* $\mathcal{A}$, *and therefore* $(\pi_\alpha \circ \pi, H_\alpha)$ *is an irreducible representation of* $l^1(S)$.

PROOF. Define a positive functional $\widetilde{\alpha}$ on $\mathcal{A}$ by

$$\widetilde{\alpha}(T) = \langle TQ(\varphi), Q(\varphi)\rangle = \alpha((T\varphi(x), \varphi(x))), \qquad T \in \mathcal{A}.$$

If $T \in \mathcal{D}$, we have $\widetilde{\alpha}(T) = \alpha(f_T)$ where as before $f_T(x) = (T\varphi(x), \varphi(x))$. We identify T and f_T, so that $\widetilde{\alpha}$ coincides with α on $\mathcal{D}$, i.e. $\widetilde{\alpha}$ is an extension of α to $\mathcal{A}$. Now we show using a result of R. Kadison and I. Singer that $\widetilde{\alpha}$ is the unique positive extension of α to $\mathcal{A}$. If $a \in S$, $\varphi(x) \in B$, then by definition $\pi(a)\varphi(x)$ is either 0 or the element $\varphi(a(x)) \in B$. Therefore by [14, Theorem 3] and the remarks following the proof of Theorem 3, all the states on $\mathcal{A}$ that coincide with α on $\mathcal{D}$ must coincide with $\widetilde{\alpha}$ on $\{\pi(a): a \in S\}$. (*Note*. The result quoted [14, Theorem 3] is proved only in the case where H is separable. However, the proof can be extended to work in Hilbert spaces of arbitrary dimension.) Then since $\{\pi(a): a \in S\}$ generates the algebra $\mathcal{A}$, $\widetilde{\alpha}$ must be the unique state on $\mathcal{A}$ that coincides with α on $\mathcal{D}$ [14, Remark 6, p. 396]. It now follows from [6, Lemma (2.10.1)] that $\widetilde{\alpha}$ is a pure state of $\mathcal{A}$, so that the representation of $\mathcal{A}$ determined by $\widetilde{\alpha}$ is irreducible. Finally, by [17, Lemma (4.5.8)] this representation is equivalent to (π_α, H_α), and this completes the proof.

As before, let $F = \{a \in S: a^*a \text{ is finite}\}$. There is a simple condition on α which insures that $l^1(F) \subset \ker(\pi_\alpha \circ \pi)$. We verify this condition next. We need one bit of notation. Let $c_0(X)$ be the set of all complex-valued functions f on X such that given any $\epsilon > 0$, the set $\{x \in X: |f(x)| \geqslant \epsilon\}$ is finite. Then $c_0(X)$ is an ideal in $l^\infty(X)$.

PROPOSITION 5.2. *If* $\alpha(f) = 0$ *for all* $f \in c_0(X)$, *then* $l^1(F) \subset \ker(\pi_\alpha \circ \pi)$.

PROOF. It is enough to show that if e is any finite idempotent map in $\mathcal{I}_X$, then $\pi_\alpha(\pi(e)) = 0$. To prove this it suffices to show that whenever $\{h_x\} \in W$, then $\langle \pi(e)\{h_x\}, \pi(e)\{h_x\}\rangle = 0$. Fix $\{h_x\} \in W$, and let $g(x) = (\pi(e)h_x, \pi(e)h_x)$, $x \in X$. We verify that $g \in c_0(X)$, so that $\langle \pi(e)h_x, \pi(e)h_x\rangle = \alpha(g(x)) = 0$. Note that the range of $\pi(e)$ is $\operatorname{span}\{\varphi(x): x \in D_e\}$. For each $x \in X$, h_x has the Hilbert space expansion in terms of the basis B,

$$h_x = \sum_{y \in X} (h_x, \varphi(y))\varphi(y).$$

Then

$$\pi(e)h_x = \sum_{y \in D_e} (h_x, \varphi(y))\varphi(y).$$

Now $x \longrightarrow (h_x, \varphi(y))$ is in $c_0(X)$ by (1). Therefore

$$g(x) = \|\pi(e)h_x\|^2 = \sum_{y \in D_e} |(h_x, \varphi(y))|^2 \in c_0(X).$$

This completes the proof.

We note the next result without proof. It can be established by arguments similar to those in [18, pp. 524–526].

PROPOSITION 5.3. *There are at least* $\exp(\exp(|X|))$ *inequivalent irreducible representations of* $l^1(S)$ *in the collection* $\{(\pi_\alpha \circ \pi, H_\alpha)$: α *a pure state of* $l^\infty(X)\}$ *with the property that* $l^1(F) \subset \ker(\pi_\alpha \circ \pi)$.

6. **Representation of completely 0-simple semigroups.** Unless explicitly stated otherwise, S will denote a completely 0-simple semigroup throughout this section (see Example 1.3). Every idempotent in S is primitive [2, Exercise 5, p. 83]. A simple fact we need in what follows is

$$\text{if } e, f \in E_S,\ e \neq f, \text{ then } ef = \theta \tag{1}$$

(*Proof.* $ef \leqslant f$ and $ef \leqslant e$; since e and f are primitive and $e \neq f$, then $ef = \theta$).

Assume that $e \in E_S$. Then e is primitive, so that $G_e = eSe\backslash\{\theta\}$ is a group. The aim of this section is to show how the cyclic representations of $l^1(S)$ can be induced from the cyclic representations of $l^1(G_e)$. (*Note.* If (π, K) is a representation of $l^1(G_e)$, then we automatically assume that $\pi(e)$ is the identity operator on K.) The technique involved is similar to (and is motivated by) the one used to induce representations of a group from those of a subgroup. In our case, the fact that S is completely 0-simple is crucial. This is clear from the proof of the following important lemma.

LEMMA 6.1. *Assume that* $e \in E_S$, $e \neq \theta$. *Let* $A = l^1(S)$, *and assume that* (π, K) *is a representation of* eAe. *If* $f_k \in A$, $\varphi_k \in K$, $1 \leqslant k \leqslant n$, *then*

$$\sum_{p=1,k=1}^{n} (\pi(ef_p^*f_ke)\varphi_k, \varphi_p) \geqslant 0.$$

PROOF. First we prove that when $a, b \in S$, then

$$eb^*ae \neq \theta \iff a^*a = b^*b = e, \quad \text{and} \quad aa^* = bb^*. \tag{2}$$

Assume that the right-hand side of (2) holds. Then $ae = a$, $eb^* = b^*$, and so $b^*a = eb^*ae$. If $b^*a = \theta$, then $a = bb^*a = \theta$, a contradiction. Therefore $eb^*ae \neq \theta$. Conversely, assume that $eb^*ae \neq \theta$. Then $ae \neq \theta$, so that $ea^*ae = e(a^*a) \neq \theta$. It follows from (1) that $a^*a = e$. Similarly, $b^*b = e$. Therefore $b^*a = eb^*ae \neq \theta$. Then $bb^*aa^* \neq \theta$, so that $bb^* = aa^*$. This proves (2).

Now assume that $f_ke = \Sigma\lambda_{jk}a_{jk}$, $1 \leqslant k \leqslant n$, where each $a_{jk} \in Se\backslash\{\theta\}$, $a_{jk} \neq a_{ik}$ if $i \neq j$, and each $\lambda_{jk} \neq 0$. Let $\{e_m\}$ be the collection of distinct idempotents in the set $\{a_{jk}a_{jk}^*\}$, subscripted so that $e_m \neq e_p$ if $m \neq p$. Let

$$K_m = \{(j, k): a_{jk}a_{jk}^* = e_m\}.$$

Then $f_k e = \Sigma_m(\Sigma_{(j,k)\in K_m} \lambda_{jk} a_{jk})$ and by (2) we have

$$(3) \qquad ef_p^* f_k e = \sum_m \left(\sum_{(j,p),(i,k)\in K_m} \lambda_{jp}^* \lambda_{ik} a_{jp}^* a_{ik} \right).$$

For each m choose some $b_m \in \{a_{ik}: (i, k) \in K_m\}$. Define

$$h_{mk} = \sum_{(j,k)\in K_m} \lambda_{jk} b_m^* a_{jk}.$$

Note that $h_{mk} \in eAe$. Also, if $(j, p), (i, k) \in K_m$, then $a_{jp}^* b_m b_m^* a_{ik} = a_{jp}^* e_m a_{ik} = a_{jp}^* a_{ik}$. Thus,

$$h_{mp}^* h_{mk} = \sum_{(j,p),(i,k)\in K_m} \lambda_{jp}^* \lambda_{ik} a_{jp}^* a_{ik}.$$

From this equality and (3) we have

$$ef_p^* f_k e = \sum_m h_{mp}^* h_{mk}.$$

Therefore,

$$\sum_{p=1,k=1}^{n} (\pi(ef_p^* f_k e)\varphi_k, \varphi_p) = \sum_m \left(\sum_{p=1,k=1}^{n} (\pi(h_{mp}^* h_{mk})\varphi_k, \varphi_p) \right)$$
$$= \sum_m \left(\sum_{p=1,k=1}^{n} (\pi(h_{mk})\varphi_k, \pi(h_{mp})\varphi_p) \right)$$
$$= \sum_m \left(\sum_{k=1}^{n} \pi(h_{mk})\varphi_k, \sum_{p=1}^{n} \pi(h_{mp})\varphi_p \right) \geq 0.$$

Now fix $e \in E_S$, and assume that (π, K) is a cyclic representation of $l^1(G_e) = eAe$. We use the representation (π, K) to induce a representation $\widetilde{\pi}$ of A on some Hilbert space that contains K in such a way that $(\widetilde{\pi}|K, K) \approx (\pi, K)$. The construction of the induced representation involves the formation of the tensor product of modules over an algebra. We use the notation and terminology of [5, §12]. We shall assume that the reader is familiar with this portion of [5] rather than reproducing a summary of it here. Although [5, §12] deals with the tensor product of modules over a ring, the process generalizes to the case where the modules are also vector spaces and the rings involved are algebras. In this case, the resulting tensor product is a vector space and the action of the algebra on this vector space is a linear operation.

The space K is a left eAe-module where $f \in eAe$ acting on $\varphi \in K$ is given by $\pi(f)\varphi$. Also A is in the obvious way a left and right eAe-module. Thus we can form the A-module $A \otimes_{eAe} K$; see [5, p. 66]. We simplify this notation to $A \otimes_e K$. If $f \in A$ and $\gamma \in A \otimes_e K$, we denote by $\widetilde{\pi}(f)\gamma$ the module product of f with γ. Our first tasks are to introduce a pre-inner product on $A \otimes_e K$ and to verify that $f \longrightarrow \widetilde{\pi}(f)$ is a representation of A on this pre-inner product space.

(6.2) The construction of the pre-inner product on $A \otimes_e K$. For the present fix $f \in A$ and $\psi \in K$. Define a map $W'\colon A \times K \to \mathbf{C}$ by

$$W'(g, \varphi) = (\pi(ef^*ge)\varphi, \psi), \qquad g \in A, \varphi \in K.$$

If $h, g \in A$, $\varphi \in K$, we have

$$W'(gehe, \varphi) = (\pi(ef^*gehe)\varphi, \psi) = (\pi(ef^*ge)\pi(ehe)\varphi, \psi) = W'(g, \pi(ehe)\varphi).$$

Thus, W' is a balanced map. It follows from [5, Theorem (12.3)] that there is a homomorphism (depending on f and ψ) $W(f, \psi)\colon A \otimes_e K \to \mathbf{C}$ such that

$$W(f, \psi)g \otimes \varphi = W'(g, \varphi) = (\pi(ef^*ge)\varphi, \psi). \tag{4}$$

If $f \in A$, $\psi \in K$, and $\gamma \in A \otimes_e K$, define $I'_\gamma\colon A \times K \to \mathbf{C}$ by

$$I'_\gamma(f, \psi) = W(f, \psi)\gamma.$$

If $h \in A$ and f, ψ, γ are as above, then

$$I'_\gamma(fehe, \psi) = W(fehe, \psi)\gamma = W(f, \pi(ehe)\psi)\gamma = I'_\gamma(f, \pi(ehe)\psi).$$

Thus, I'_γ is a balanced map, so that by [5, Theorem (12.3)] there exists a homomorphism $I_\gamma\colon A \otimes_e K \to \mathbf{C}$ such that

$$I_\gamma(f \otimes \psi) = I'_\gamma(f, \psi) = W(f, \psi)\gamma. \tag{5}$$

If $\gamma, \beta \in A \otimes_e K$, define

$$\langle \gamma, \beta \rangle = I_\gamma(\beta).$$

By the construction it follows that $\langle \cdot\,, \cdot \rangle$ is linear in the first variable and conjugate linear in the second. Also, using (4) and (5) we have for $f, g \in A$, $\varphi, \psi \in K$,

$$\langle g \otimes \varphi, f \otimes \psi \rangle = I_{g \otimes \varphi}(f \otimes \psi) = W(f, \psi)g \otimes \varphi = (\pi(ef^*ge)\varphi, \psi). \tag{6}$$

Then

$$\langle g \otimes \varphi, f \otimes \psi \rangle = (\pi(ef^*ge)\varphi, \psi) = (\pi(eg^*fe)\psi, \varphi)^* = \langle f \otimes \psi, g \otimes \varphi \rangle^*.$$

Therefore $\langle \gamma, \tau \rangle = \langle \tau, \gamma \rangle^*$ whenever $\gamma, \tau \in A \otimes_e K$. Suppose that $\gamma = \sum_{k=1}^n f_k \otimes \varphi_k \in A \otimes_e K$. Then

$$\begin{aligned}\langle \gamma, \gamma \rangle &= \sum_{p=1,k=1}^{n} \langle f_k \otimes \varphi_k, f_p \otimes \varphi_p \rangle \\ &= \sum_{p=1,k=1}^{n} (\pi(ef_p^*f_ke)\varphi_k, \varphi_p) \quad \text{by (6)} \\ &\geqslant 0 \quad \text{by Lemma 6.1.}\end{aligned}$$

To summarize, we have shown

(6.3) The form $\langle \cdot\,, \cdot \rangle$ is a pre-inner product on $A \otimes_e K$ with the property that when $g, f \in A$ and $\varphi, \psi \in K$, then

$$\langle g \otimes \varphi, f \otimes \psi \rangle = (\pi(ef^*ge)\varphi, \psi).$$

(6.4) The construction of the induced representation. Recall that we denote the result of the induced module operation of $f \in A$ on $\gamma \in A \otimes_e K$ by $\widetilde{\pi}(f)\gamma$. In particular, it follows from [5, p. 66] that if $f, g \in A$ and $\varphi \in K$, then $\widetilde{\pi}(f)g \otimes \varphi = fg \otimes \varphi$. First we verify that if $\gamma, \tau \in A \otimes_e K$ and $f \in A$, then $\langle \widetilde{\pi}(f)\gamma, \tau \rangle = \langle \gamma, \widetilde{\pi}(f^*)\tau \rangle$. It is enough to check this equality in the case where $\gamma = g \otimes \varphi$ and $\tau = h \otimes \psi$. Then

$$\begin{aligned}\langle \widetilde{\pi}(f)g \otimes \varphi, h \otimes \psi \rangle = \langle fg \otimes \varphi, h \otimes \psi \rangle &= (\pi(eh^*fge)\varphi, \psi) \quad \text{by (6.3)}\\ &= (\varphi, \pi(eg^*f^*he)\psi) = \langle f^*h \otimes \psi, g \otimes \varphi \rangle^* \quad \text{by (6.3)}\\ &= \langle g \otimes \varphi, \widetilde{\pi}(f^*)h \otimes \psi \rangle.\end{aligned}$$

Thus $f \longrightarrow \widetilde{\pi}(f)$ is a star representation of A on the pre-inner product space $A \otimes_e K$.

Before proving that $\widetilde{\pi}$ is a bounded operator, we prove a necessary lemma.

LEMMA 6.5. *If $\varphi \in K$ is a cyclic vector for π, then $e \otimes \varphi$ is a cyclic vector for $\widetilde{\pi}$ on $A \otimes_e K$.*

PROOF. If $f \otimes \psi$ is considered a function of $f \in A$ and $\psi \in K$ with values in the pre-inner product space $A \otimes_e K$, then using (6) it is not difficult to verify that this function is continuous in both variables separately. Let $\gamma = h_1 \otimes \psi_1 + \cdots + h_n \otimes \psi_n$ be given. Since φ is a cyclic vector for π, we can choose $g_k \in eAe$ such that $\|\psi_k - \pi(g_k)\varphi\|$ is as small as we wish for each k. Thus, since $\psi \longrightarrow f \otimes \psi$ is continuous, we can arrange by an appropriate choice of $g_k \in eAe$ that $\tau = h_1 \otimes \pi(g_1)\varphi + \cdots + h_n \otimes \pi(g_n)\varphi$ is as close to γ as we wish in the topology on $A \otimes_e K$ determined by the pre-inner product. But

$$\tau = (h_1g_1 + \cdots + h_ng_n) \otimes \varphi = \widetilde{\pi}(h_1g_1 + \cdots + h_ng_n)e \otimes \varphi.$$

This proves the lemma.

Let $\varphi \in K$ be a cyclic vector for π. Define F on A by

$$F(f) = \langle \widetilde{\pi}(f)e \otimes \varphi, e \otimes \varphi \rangle, \qquad f \in A.$$

F is a positive functional on A. Let $\|\gamma\| = \langle \gamma, \gamma \rangle^{1/2}$ for $\gamma \in A \otimes_e K$. Note that if $g, f \in A$, we have

$$F(g^*f^*fg) = \|\widetilde{\pi}(f)g \otimes \varphi\|^2, \quad \text{and} \quad F(g^*g) = \|g \otimes \varphi\|^2.$$

Then by [17, Theorem (4.5.2)] F is admissible on A so that there exists $M > 0$ with

$$F(g^*f^*fg) \leqslant MF(g^*g) \quad \text{for all } g \in A.$$

This inequality and Lemma 6.5 imply that $\tilde{\pi}(f), f \in A$, is a bounded operator on $A \otimes {}_eK$.

Now let Z be the set of all γ in $A \otimes {}_eK$ such that $\|\gamma\| = 0$. Then $(A \otimes {}_eK)/Z$ is an inner product space. Let p_Z be the natural quotient map of $A \otimes {}_eK$ onto this inner product space. For the present fix $\gamma \in Z$. We deal with the case that A has no identity (otherwise, $A = eAe$). Let A_1 be the algebra A with an identity 1 adjoined. Define α on A_1 by

$$\alpha(\lambda 1 + f) = \langle \lambda\gamma + \tilde{\pi}(f)\gamma, \gamma\rangle$$

where $f \in A$, $\lambda \in \mathbf{C}$. Then α is a positive functional on A_1. By the usual general Cauchy-Schwarz inequality for α we have for $f \in A$

$$\|\tilde{\pi}(f)\gamma\|^2 = \alpha(f^*f) \leqslant \alpha(1)^{1/2}\alpha((f^*f)^2)^{1/2}.$$

Since $\alpha(1) = \|\gamma\|^2 = 0$, we have $\|\tilde{\pi}(f)\gamma\| = 0$. Therefore Z is invariant under $\tilde{\pi}$. Thus, if $f \in A$, $\tilde{\pi}(f)$ determines a bounded linear operator on $(A \otimes {}_eK)/Z$ by the rule $\tilde{\pi}(f)p_Z(\gamma) = p_Z(\tilde{\pi}(f)\gamma)$, $\gamma \in A \otimes {}_eK$. Let $\widetilde{K}$ be the Hilbert space completion of $(A \otimes {}_eK)/Z$. Then if $f \in A$, $\tilde{\pi}(f)$ extends to a bounded operator on $\widetilde{K}$ which we also denote by $\tilde{\pi}$. Furthermore, it follows from the construction that $f \longrightarrow \tilde{\pi}(f)$ is a representation of A on $\widetilde{K}$.

In what follows we follow the usual practice of considering $p_Z(A \otimes {}_eK)$ as a subspace of its completion $\widetilde{K}$.

(6.6) Verification that (π, K) is equivalent to $\tilde{\pi}|_{eAe}$ restricted to some closed subspace of $\widetilde{K}$. Let K_0 be the subspace $\{p_Z(e \otimes \varphi): \varphi \in K\}$ of $\widetilde{K}$. Define $U_0: K_0 \longrightarrow K$ and $W_0: K \longrightarrow K_0$ by

$$U_0(p_Z(e \otimes \varphi)) = \varphi, \quad W_0(\varphi) = p_Z(e \otimes \varphi), \qquad \varphi \in K.$$

(*Note*. It is easy to verify that U_0 is well defined.) Then for all $\varphi \in K$, $U_0W_0(\varphi) = \varphi$, and $W_0U_0(p_Z(e \otimes \varphi)) = p_Z(e \otimes \varphi)$. Also, if $\varphi, \psi \in K$, then

$$\begin{aligned}(U_0p_Z(e \otimes \varphi), \psi) &= (\varphi, \psi)\\ &= \langle p_Z(e \otimes \varphi), p_Z(e \otimes \psi)\rangle \quad \text{by (6)}\\ &= \langle p_Z(e \otimes \varphi), W_0(\psi)\rangle.\end{aligned}$$

Thus, $W_0 = U_0^*$. It follows W_0 maps K isometrically onto K_0. Therefore K_0 is a closed subspace of $\widetilde{K}$. The argument above implies that U_0 is a unitary operator from K_0 onto K.

If $f \in eAe$, then

$$U_0\widetilde{\pi}(f)p_Z(e \otimes \varphi) = U_0 p_Z(e \otimes \pi(f)\varphi) = \pi(f)\varphi = \pi(f)U_0 p_Z(e \otimes \varphi).$$

Let π_0 denote the representation $\widetilde{\pi}|_{eAe}$ restricted to K_0. We have shown that $(\pi_0, K_0) \approx (\pi, K)$.

We call the representation $(\widetilde{\pi}, \widetilde{K})$ constructed in (6.2) and (6.4) the representation of A induced by (π, K).

The next several results develop some of the properties of induced representations.

PROPOSITION 6.7. *Let (π_1, K_1) and (π_2, K_2) be two cyclic representations of eAe. If the corresponding induced representations $(\widetilde{\pi}_1, \widetilde{K}_1)$ and $(\widetilde{\pi}_2, \widetilde{K}_2)$ are equivalent, then $\pi_1 \approx \pi_2$.*

PROOF. Let $U: \widetilde{K}_1 \longrightarrow \widetilde{K}_2$ be a unitary operator that intertwines $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$. For $j = 1, 2$, let $Z_j = \{\gamma \in A \otimes {}_eK_j: \|\gamma\| = 0\}$, and let p_j be the natural projection of $A \otimes {}_eK_j$ onto $(A \otimes {}_eK_j)/Z_j$. Again for $j = 1, 2$, let $H_j = \{p_j(e \otimes \varphi): \varphi \in K_j\}$, and let γ_j be $\widetilde{\pi}_j|_{eAe}$ restricted to H_j. By (6.6) we have that $\gamma_1 \approx \pi_1$ and $\gamma_2 \approx \pi_2$.

Now we prove that $\gamma_1 \approx \gamma_2$. First we show that U maps H_1 into H_2. If $f \in A$, $\psi \in K_2$, we have

$$\widetilde{\pi}_2(e)p_2(f \otimes \psi) = p_2(efe \otimes \psi) = p_2(e \otimes \pi_2(efe)\psi). \tag{7}$$

Then for $\varphi \in K_1$

$$\begin{aligned} Up_1(e \otimes \varphi) &= U\widetilde{\pi}_1(e)p_1(e \otimes \varphi) \\ &= \widetilde{\pi}_2(e)Up_1(e \otimes \varphi) \in H_2 \quad \text{(by (7))}. \end{aligned}$$

A similar argument shows that U^* maps H_2 into H_1. It follows that U maps H_1 isometrically onto H_2. Also, since U intertwines $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$, $U\gamma_1 = \gamma_2 U$ on H_1. This completes the proof that $\pi_1 \approx \pi_2$.

THEOREM 6.8. *Assume that π is an irreducible representation of eAe on a Hilbert space K. Then $(\widetilde{\pi}, \widetilde{K})$ is irreducible.*

PROOF. Fix $\varphi \in K$, $\|\varphi\| = 1$. By Lemma 6.5, $p_Z(e \otimes \varphi)$ is a cyclic vector for $\widetilde{\pi}$ on $\widetilde{K}$ (here Z is as in (6.4)). Let

$$\alpha(f) = (\pi(f)\varphi, \varphi), \qquad f \in eAe,$$

and

$$\widetilde{\alpha}(f) = \langle \widetilde{\pi}(f)p_Z(e \otimes \varphi), p_Z(e \otimes \varphi)\rangle, \qquad f \in A.$$

Then if $f \in A$

$$\begin{aligned} \tilde{\alpha}(efe) &= \langle \tilde{\pi}(efe)p_Z(e \otimes \varphi), p_Z(e \otimes \varphi)\rangle \\ &= (\pi(efe)\varphi, \varphi) \quad \text{by (6.3)} \\ &= \alpha(efe). \end{aligned} \tag{8}$$

Suppose $\tilde{\alpha} = \frac{1}{2}(\beta + \gamma)$ where β and γ are states on A (here δ is a state means that δ is a positive functional on A with $\delta(g^*) = \delta(g)^*$, and $|\delta(g)|^2 \leq \delta(g^*g)$, $g \in A$). Since β and γ are states of A, we have $\beta(e)^2 \leq \beta(e)$ and $\gamma(e)^2 \leq \gamma(e)$. Thus, $\beta(e) \leq 1$ and $\gamma(e) \leq 1$. Now

$$1 = \tilde{\alpha}(e) = \tfrac{1}{2}(\beta(e) + \gamma(e)) \leq 1.$$

Therefore $\beta(e) = \gamma(e) = 1$. Since α determines the irreducible representation (π, K), we have by [9, Theorem 21.34)] that

$$\alpha(efe) = \beta(efe) = \gamma(efe), \qquad f \in A. \tag{9}$$

If δ is a state and $\delta(e) = 1$, then it follows from the general Cauchy-Schwarz inequality for δ that $\delta(g(1-e)) = \delta((1-e)g) = 0$ for any $g \in A$. Thus, in this case $\delta(g) = \delta(ege)$ for all $g \in A$. Applying this fact to β and γ, we have that

$$\beta(f) = \gamma(f) = \gamma(efe), \qquad f \in A.$$

Then (8) and (9) imply that $\tilde{\alpha} = \beta = \gamma$. If follows from [9, Theorem (21.34)] that $(\tilde{\pi}, \tilde{K})$ is irreducible.

PROPOSITION 6.9. *Let (π', H) be an irreducible representation of A. Assume that $\pi'(e) \neq 0$ where $e \in E_S$. Define a representation (π, K) of eAe in the natural way where $K = \pi'(e)H$. Then $\tilde{\pi} \approx \pi'$.*

PROOF. Choose $\varphi \in K$ with $\|\varphi\| = 1$. By Lemma 6.5 $p_Z(e \otimes \varphi)$ is cyclic for $\tilde{\pi}$ on $\tilde{K}$. For $f \in A$ let

$$\alpha'(f) = (\pi'(f)\varphi, \varphi) \quad \text{and} \quad \tilde{\alpha}(f) = \langle \tilde{\pi}(f)p_Z(e \otimes \varphi), p_Z(e \otimes \varphi)\rangle.$$

By [17, Lemma (4.5.8)] it is enough to show that $\alpha' = \tilde{\alpha}$. If $f \in A$, then

$$\begin{aligned} \tilde{\alpha}(f) &= \langle \tilde{\pi}(f)p_Z(e \otimes \varphi), p_Z(e \otimes \varphi)\rangle \\ &= (\pi(efe)\varphi, \varphi) \quad \text{by (6.3)} \\ &= (\pi'(efe)\varphi, \varphi) = (\pi'(f)\varphi, \varphi) = \alpha'(f). \end{aligned}$$

For some inverse semigroups S, the irreducible representations of $l^1(S)$ can be determined from the irreducible representations of the l^1-algebras of a collection of completely 0-simple factors of S. This is true when S has a composition series.

DEFINITION. Let S be a semigroup with zero. A composition series for S is an increasing family of ideals of S, $\{I_\alpha\}$, $0 \leqslant \alpha \leqslant \gamma$, where γ is a fixed ordinal and the indices are all ordinals α such that $0 \leqslant \alpha \leqslant \gamma$, having the properties that

(i) $I_0 = \{\theta\}$ and $I_\gamma = S$, and

(ii) if α is a limit ordinal, $\alpha \leqslant \gamma$, then $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$.

If whenever α is a nonlimit ordinal, $\alpha \leqslant \gamma$, we have $I_{\alpha+1}/I_\alpha$ is completely 0-simple, then we say the composition series has all completely 0-simple factors.

The semigroup $\mathcal{F}_X$ has composition series $\{\mathcal{F}_n\}$, $0 \leqslant n \leqslant \omega$, ω the first limit ordinal, where $\mathcal{F}_\omega = \mathcal{F}_X$, and $\mathcal{F}_n$ for $0 \leqslant n < \omega$ has the usual definition. The factors of this composition series are completely 0-simple. More generally, we have the following result.

PROPOSITION 6.10. *Assume that S has a zero. Assume that every nonempty subset of E_S has a minimal element with respect to the partial order $\leqslant$. Then S has a composition series with completely 0-simple factors.*

PROOF. Let $I_0 = \{\theta\}$. Assume that an increasing family of ideals, $\{I_\alpha\}$, $0 \leqslant \alpha < \beta$, has been chosen with $I_\alpha/I_{\alpha-1}$ completely 0-simple for all nonlimit ordinals $1 < \alpha < \beta$. If β is a limit ordinal, let $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$. Now assume that β is a nonlimit ordinal. Choose e a minimal element of $\{f \in E_S: f \notin I_{\beta-1}\}$. Let

$$I_\beta = \{a \in S: a^*a \sim e\} \cup I_{\beta-1}.$$

If $a \in I_\beta$, then $a^* \in I_\beta$. Next we prove that I_β is a right ideal, hence an ideal. Suppose $b \in S$ and $a \in I_\beta \backslash I_{\beta-1}$. Then there exists $c \in S$ such that $a^*a = c^*c$ and $e = cc^*$. Then $b^*a^*ab = b^*c^*cb \sim cbb^*c^*$, and also, $(cbb^*c^*)e = (cbb^*c^*)cc^* = cbb^*c^*$. Thus, $cbb^*c^* \leqslant e$, so that by the choice of e either $cbb^*c^* = e$ or $cbb^*c^* \in I_{\beta-1}$. In the first case $b^*a^*ab \sim e$, so that $ab \in I_\beta$. In the second case,

$$\begin{aligned} ac^*(cbb^*c^*)cb \in I_{\beta-1} &\Rightarrow (ac^*c)(bb^*b) \in I_{\beta-1} \\ &\Rightarrow ab = (aa^*a)(bb^*b) \in I_{\beta-1} \subset I_\beta. \end{aligned}$$

Note that by the choice of e, we have $e \in I_\beta$ and e is primitive modulo $I_{\beta-1}$. Now we verify that $I_\beta/I_{\beta-1}$ is 0-simple. Let I be an ideal of I_β with $I_{\beta-1} \subset I$. Suppose that $b \in I \backslash I_{\beta-1}$. We prove that $I = I_\beta$. We have $b^*b \sim e$. Assume $a \in I_\beta \backslash I_{\beta-1}$. Then $a^*a \sim e \sim b^*b$, so there exists $c \in S$ with $b^*b = c^*c$ and $cc^* = a^*a$. Therefore

$$c^*c \in I \Rightarrow c = cc^*c \in I \Rightarrow a^*a = cc^* \in I \Rightarrow a \in I.$$

This proves that $I = I_\beta$. We have shown that $I_\beta/I_{\beta-1}$ is completely 0-simple. Therefore by transfinite induction we can construct a composition series for S with completely 0-simple factors.

Let A be a star algebra and let J_2 and J_1 be star ideals of A with $J_1 \subset J_2$. Now we describe a procedure for lifting a cyclic representation of the quotient algebra J_2/J_1 to a representation of A. Denote the residue class of J_2/J_1 that contains $g \in J_2$ by $g + J_1$. The space J_2/J_1 is a module over A where $f \in A$ acts on $g + J_1$, $g \in J_2$, by the rule $f(g + J_1) = fg + J_1$. Let (π, H) be a cyclic representation of J_2/J_1 with φ a cyclic vector for π. Let

$$K = \{\pi(g + J_1)\varphi: g \in J_2\}.$$

If $f \in A$, define $\overline{\pi}(f)$ on $\psi = \pi(g + J_1)\varphi \in K$ by

$$\overline{\pi}(f)\psi = \pi(fg + J_1)\varphi.$$

Since K is dense in H, $\overline{\pi}$ extends uniquely to a cyclic representation $\overline{\pi}$ of A on H. Clearly, if π is irreducible, then the extension $\overline{\pi}$ is irreducible. If (π, H) is a cyclic representation of J_2/J_1, we use the notation $(\overline{\pi}, H)$ for the representation of A constructed above for some choice of cyclic vector φ.

THEOREM 6.11. *Let $\{I_\alpha\}$, $0 \leqslant \alpha \leqslant \gamma$, be a composition series for S with completely 0-simply factors. Let $J_\alpha = l^1(I_\alpha)$ for $0 \leqslant \alpha \leqslant \gamma$.*

(1) *If $\alpha \geqslant 1$ is a nonlimit ordinal and (π, H) is a cyclic representation of $J_\alpha/J_{\alpha-1}$, then (π, H) extends to a cyclic representation $(\overline{\pi}, H)$ of $l^1(S)$.*

(2) *If (τ, K) is an irreducible representation of $l^1(S)$, then there exists a nonlimit ordinal $\alpha \geqslant 1$ and an irreducible representation (π, K) of $J_\alpha/J_{\alpha-1}$ such that the extension $(\overline{\pi}, K) = (\tau, K)$.*

PROOF. (1) follows immediately by the preceding construction. We prove (2). Let (τ, K) be an irreducible representation of $l^1(S)$. Let α be the smallest ordinal with $\tau(J_\alpha) \neq \{0\}$. Then $\tau(J_\beta) = \{0\}$ for all $\beta < \alpha$. If α were a limit ordinal, then

$$J_\alpha = \operatorname{cl}\left\{\bigcup J_\beta: \beta < \alpha\right\}.$$

Thus in this case, $\tau(J_\alpha) = \{0\}$, a contradiction. Therefore α is a nonlimit ordinal and $\tau(J_{\alpha-1}) = \{0\}$. Define a representation π of $J_\alpha/J_{\alpha-1}$ on K by

$$\pi(f + J_{\alpha-1}) = \tau(f), \quad f \in J_\alpha.$$

Then (π, K) is an irreducible representation of $J_\alpha/J_{\alpha-1}$. Let φ be a cyclic vector for π. Form the extension $(\overline{\pi}, K)$ as indicated previously. Then if $f \in l^1(S)$ and $g \in J_\alpha$,

$$\overline{\pi}(f)\pi(g + J_{\alpha-1})\varphi = \pi(fg + J_{\alpha-1})\varphi = \tau(fg)\varphi = \tau(f)\tau(g)\varphi = \tau(f)\pi(g + J_{\alpha-1})\varphi.$$

Thus $\overline{\pi} = \pi$ on K.

7. **Representations of the bicyclic semigroup.** The simplest example of a simple inverse semigroup that contains no primitive idempotent is the bicyclic semigroup. As in Example 1.4, we use the notation $\mathcal{C}$ for this semigroup, and let p, q denote the generators of $\mathcal{C}$ with relation $qp = 1$ (recall that $p^* = q$). As we shall see, by applying well-known results from operator theory concerning the structure of an isometry on a Hilbert space, it is possible to describe explicitly the irreducible representations of $l^1(\mathcal{C})$.

Let π be a representation of $l^1(\mathcal{C})$ on a Hilbert space H. We assume here and throughout this section that $\pi(1) = I$, the identity operator on H. Let $V = \pi(p)$. Then $\pi(q) = \pi(p^*) = V^*$, and $V^*V = \pi(qp) = \pi(1) = I$. Therefore V is an isometry on H. Conversely, given a Hilbert space H and an isometry V on H, let $\pi(p) = V$, $\pi(q) = V^*$, and $\pi(1) = I$. Since $\mathcal{C}$ is generated by p, q and 1, there is a unique representation π of $l^1(\mathcal{C})$ on H satisfying these equations. Thus:

(7.1) Every representation of $l^1(\mathcal{C})$ is completely determined as above by a Hilbert space H and an isometry V on H.

Important particular examples of isometries are the unilateral shifts. Let α be a cardinal, and let K be a Hilbert space of dimension α. Let H_α be the Hilbert space direct sum of a countably infinite number of copies of K, and let S_α be the unilateral shift on H_α; see [7, p. 15] where the notation $l^2_+(K)$ and U_+ is used in place of H_α and S_α. We denote by π_α the representation of $l^1(\mathcal{C})$ on H_α determined by the choice $\pi_\alpha(p) = S_\alpha$. If ψ is a vector in H_α, then ψ is determined by its coordinates $\{\psi_n\}$ where each $\psi_n \in K$ and $\Sigma \|\psi_n\|^2 < \infty$. Fix $\varphi \in K$, $\varphi \neq 0$. For each $j \geq 1$, let φ^j be the vector in H_α with coordinates

$$(\varphi^j)_n = \begin{cases} \varphi & \text{if } j = n, \\ 0 & \text{if } j \neq n. \end{cases}$$

Let J be the closed linear span of $\{\varphi^j : j \geq 1\}$ in H_α. Since

$$S_\alpha(\varphi^j) = \varphi^{j+1}, \quad j \geq 1,$$

and

$$S_\alpha^*(\varphi^j) = \varphi^{j-1}, \quad j > 1, \quad S_\alpha^*(\varphi^1) = 0,$$

we have J is invariant under S_α and S_α^*. If $\alpha > 1$, we can choose $\psi \in K$ such that $\psi \neq 0$ and $\psi \perp \varphi$. Then the vector in H_α with first coordinate ψ and all other coordinates 0 is orthogonal to J. Therefore when $\alpha > 1$, J is a proper closed π_α-invariant subspace of H_α. On the other hand, for the case $\alpha = 1$, we have that (π_1, H_1) is irreducible by [4, Corollary 1.2]. Thus, the following result holds.

(7.2) The representation (π_α, H_α) of $l^1(\mathcal{C})$ is irreducible if and only if $\alpha = 1$.

Now let π be a representation of $l^1(C)$ on a Hilbert space H, and let $V = \pi(p)$. Since V is an isometry, it follows from [7, pp. 15–16] that there exists a subspace M of H that reduces V and has the property that $V|M$ is unitary and $V|M^\perp$ is equivalent to a unilateral shift. Therefore $\pi = (\pi|M) \oplus \pi_\alpha$ where $(\pi|M)(p)$ is unitary on M and α is the multiplicity of $V|M^\perp$. Thus:

(7.3) A representation of $l^1(C)$ is a direct sum of a representation of $l^1(C)$ determined by a unitary operator and π_α for some α.

Now we can easily identify all the irreducible representations of $l^1(C)$. First note that a representation of $l^1(C)$ determined by a unitary operator on a Hilbert space H is irreducible if and only if H is one dimensional. Also, to identify the one-dimensional representations of $l^1(C)$ it is sufficient to identify the semicharacters of C. For each $\lambda \in \mathbf{C}$, $|\lambda| = 1$, let $\varphi_\lambda(p) = \lambda$, $\varphi_\lambda(q) = \lambda^*$, and $\varphi_\lambda(1) = 1$. Then φ_λ extends to a semicharacter on C (which we again denote by φ_λ). In fact it is easy to see that $\{\varphi_\lambda: |\lambda| = 1\}$ is the set of all semicharacters of C.

PROPOSITION 7.4. *The set of all irreducible representations of $l^1(C)$ consists of the one-dimensional representations determined by the semicharacters $\{\varphi_\lambda: |\lambda| = 1\}$ and the representation (π_1, H_1).*

PROOF. The result follows immediately from (7.2), (7.3), and the remarks above concerning semicharacters of C.

ADDED IN PROOF. (1) Part of the proof of Proposition 3.1 has been omitted. It should be verified that $e \notin I_e$. This can be proved as follows. Suppose on the contrary that $e \in I_e$. Then there exists $g \in E_S$ such that $e \sim g < e$. Choose $a \in S$ such that $e = a^*a$ and $g = aa^*$. For $h \in E_S$, define the set $M_h = \{k \in E_S: k \leqslant h\}$. Observe that since $g < e$, the finite sets M_g and M_e have different cardinalities. But $k \longrightarrow aka^*$ is a one-to-one map of M_e onto M_g (the inverse of this map is $k \longrightarrow a^*ka$). This contradiction proves that $e \notin I_e$.

(2) We have recently submitted a paper concerned with some of the topics treated here. It contains an easier proof of Theorem 2.3, and a complete solution to the problem mentioned in the remarks immediately following the proof of Proposition 4.1.

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