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REPRESENTATIONS OF THE NORMALIZERS OF MAXIMAL TORI OF SIMPLE LIE GROUPS

By

Jun-ichi MATSUZAWA* and Makoto TAKAHASHI

Abstract. We study the branching rule for the restriction from a complex simple Lie group G to the normalizer of a maximal torus of G. We show that the problem is reduced to the determination of the Weyl group module structures induced on the zero weight spaces of representations of semisimple Lie groups. The concrete formulas are obtained for $SL(n, \mathbf{C})$ in terms of generalized q-binomial coefficients and Schur functions.

1. Introduction

The zero weight space of a representation V of a complex semisimple Lie group G naturally affords a representation of the Weyl group W of G, because W is by definition the quotient group N/T, where T is a maximal torus of G and $N = N_G(T)$ is the normalizer of T in G. We call this representation of W the zero weight representation for V, which gives interesting relationship between the representations of G and those of W. In particular, the W-module structure of the zero weight space has deep-rooted connection to various problems in the representation theory of Lie groups such as the analysis of the plethysm and generalized exponents, etc. ([2], [9], [13]). A natural extension of this problem is to study the restriction of the whole representation of G to N. This is what we pursue in this paper. We have a similar problem if we replace G by its compact real form, which is really an equivalent problem (see §9).

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This paper is organized as follows. In §§2–5 we discuss the parametrization of irreducible representations of N. In §§6–10 we study the branching rules from G to N.

In §2, we recall Clifford's theory on the representations of an extension group E of a group H by a finite group F over the field $\mathbf{C}$ of complex numbers ([3]), which is a purely algebraic version of the method of little groups or Mackey machine ([19], [11]). For an irreducible representation χ of H, define E_χ to be the subgroup leaving the equivalence class of χ invariant and take an irreducible representation τ of E_χ such that $\tau|_H$ is a multiple of χ. Then the representation of E induced from τ is irreducible. Every irreducible representation of E is given in this way. Furthermore it is shown in ([3]) that τ is the tensor product of two irreducible projective representations τ_1 and τ_2 of E_χ, where τ_1 has the same degree as χ and satisfies $\tau_1(gh) = \tau_1(g)\chi(h)$ $(g \in G, h \in H)$, and τ_2 is the pullback of an irreducible projective representation of E_χ/H.

If χ can be extended to an ordinary representation (by which we mean a linear representation) of E_χ, then these two projective representations can be replaced by ordinary representations. If E is a semidirect product of H and F, then this condition holds for all χ and therefore all irreducible representations of E can be obtained from irreducible representations of H and subgroups of F. In §3 we study the case where H is an abelian group and give a sufficient condition for all irreducible representations χ to be extended to ordinary representations of E_χ. Under this condition, the equivalence classes of irreducible representations of E are parametrized by the *conjugacy classes*, under the action of E, of the pairs (χ, ϕ), where χ is an irreducible character of H and ϕ is an irreducible character of the factor group E_χ/H. The characters of E are given in §4.

These results hold for abstract groups and finite-dimensional representations over algebraically closed fields. Also they are valid for complex simple Lie groups and their holomorphic finite-dimensional representations, or compact simple Lie groups and their continuous finite-dimensional representations in complex vector spaces. In §5 we study the case where H is a maximal torus T of a connected complex semisimple Lie group G and E is its normalizer N in G, which is an extension of T by the Weyl group W. Although N is not a semidirect product of T and W in general, the equivalence classes of irreducible holomorphic representations of N can be parametrized by the conjugacy classes of (χ, ϕ), where χ is a holomorphic character of T and ϕ is an irreducible representation of the parabolic subgroup N_χ/T of W.

We show this in two ways. One method is to choose good representatives in N of the elements of W and apply the results of §2. The other is to find a semidirect product group containing N and to apply the results of § 3. The former is canonical, but sometimes the latter is convenient for concrete calculation.

In § 6 we discuss the structure of the N-module $V{\downarrow}_N^G$ obtained from a G-module V by restriction from G to N, by applying the results of the preceding sections for a complex simple Lie group G and the normalizer N of a maximal torus T of G. We show that the problem can be reduced to the determination of the structure of "zero weight representations" for $V{\downarrow}_{L_P'}^G$, where L_P' varies over the derived groups of the Levi parts of parabolic subgroups of G. The same results can also be formulated starting with the compact real forms of G (see § 9).

In the last four sections we study in detail the case where $G = SL(n, \mathbf{C})$ by using Young diagrams and Schur functions. We calculate the multiplicity of an irreducible representation of N in the restriction of an irreducible representation of G to N in two ways.

In § 7 and § 8 we apply the results of § 6 and determine the zero weight representations for $V{\downarrow}_{L_P'}^G$. The multiplicities can be written in terms of Littlewood-Richardson's coefficient, characters of parabolic subgroups of W and generalized q-binomial coefficients.

In § 9 we adopt the compact group formulation and consider the unitary group $U(n)$ and the normalizer N' of a maximal torus of $U(n)$. Since an element of N' is a product of a permutation matrix and a diagonal matrix, the restriction of an irreducible character of $U(n)$ to a connected component of N' can be regarded as a function on T. This enables us to calculate the multiplicities combinatorially in terms of the Schur functions and Weyl groups. The multiplicity formula for $Sp(n)$ on this line is obtained by the second author ([18]).

In the last section we give a series of examples of irreducible modules V for $GL(n, \mathbf{C})$ and their weights μ such that $\bigoplus_{\nu \in W_\mu} V_\nu$ is irreducible as an N-module, where V_ν denotes the ν-weight space of V and $W_\mu = N_\mu / T$. For the case where $\mu = 0$, every irreducible representation of the symmetric group $\mathfrak{S}_n$ can be obtained as the zero weight representation of a suitable irreducible representation of $SL(n, \mathbf{C})$. ([8], [9], see also [2]).

Naruse ([14]) and Nishiyama ([15]) have obtained the results in the case where $G = GL(n, \mathbf{C})$ and N is replaced by the symmetric group of degree n, which are related to the plethysms and the representations of party algebras.

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2. Representations of Finite Extensions of Groups

Throughout this paper, all representations are finite dimensional defined over the field $\mathbf{C}$ of complex numbers.

We begin with recalling Clifford's theory on the relationship of representations of a group and those of its normal subgroup ([3], see also [7] § 11). We include proofs for completeness as well as to confirm some delicate points (see Remark 2.5 in particular).

The results in this section holds for abstract groups and finite-dimensional representations over algebraically closed fields. Also they are valid for complex (resp. compact) simple Lie groups and their holomorphic (resp. continuous) finite-dimensional representations in complex vector spaces which we will study in the subsequent sections (see Remark 2.7). In each of thse cases, all homomorphisms and mappings are morphisms in the relevant category.

Let E be an extension of a group H by a finite group F, i.e. there is an exact sequence of groups:

$$1 \to H \to E \to F \to 1. \tag{2.1}$$

A section $\sigma : F \to E$ is a morphism in the relevant category not necessarily preserving group multiplications. Since F is finite, there always exists a section.

The group E acts on the representations of H by

$$g \cdot \chi(h) = \chi(g^{-1}hg),$$

where χ is a representation of H, $g \in E$, $h \in H$. The representation $g \cdot \chi$ is called a conjugate representation of χ and denoted by χ^g. Let E_χ be the subgroup of E leaving the equivalence class of χ invariant:

$$E_\chi = \{g \in E \mid \chi^g \text{ is equivalent to } \chi\}.$$

We denote by $\mathrm{Irr}(G)$ a complete set of representatives of the equivalence classes of irreducible representations of a group G and by $\mathrm{Irr}(G, \alpha)$ a complete set of representatives of the equivalence classes of irreducible projective representations of G with factor set $\alpha : G \times G \to \mathbf{C}^\times$.

For $\chi \in \mathrm{Irr}(H)$ let

$$\mathrm{Irr}(E_\chi)_\chi = \{\tau \in \mathrm{Irr}(E_\chi) \mid \tau|_H \text{ is a multiple of } \chi\} \tag{2.2}$$

Theorem 2.1 (Clifford). *Let χ be an irreducible representation of H and τ an irreducible representation of E_χ such that the restriction $\tau|_H$ of τ to H is a multiple of χ. Let $\theta_{\chi,\tau}$ be the induced representation of E afforded by*

$$V_{\chi,\tau} = \bigoplus_{g \in E/E_\chi} gU_{\chi,\tau}, \tag{2.3}$$

where $U_{\chi,\tau}$ affords τ and the summation is taken over a complete set of coset representatives of E/E_χ. Then $\theta_{\chi,\tau}$ is irreducible and every irreducible representation of E is given in this way. Two irreducible representations $\theta_{\chi,\tau}$ and $\theta_{\chi',\tau'}$ are equivalent if and only if the pair (χ',τ') is equivalent to a conjugate of (χ,τ): there exists an element g' of E such that χ' is equivalent to $\chi^{g'}$ and τ' is equivalent to the representation $\tau \circ \varphi_{g'}^{-1}$, where $\varphi_{g'}$ is the isomorphism of E_χ to $E_{\chi'}$ defined by $g \mapsto g'gg'^{-1}$.

An irreducible representation τ of E_χ decomposes as the tensor product of two irreducible projective representations of E_χ: one has the same degree as χ, the other is the pullback $\hat{\sigma}$ of an irreducible projective representation σ of E_χ/H. The two projective representations are given as follows. Since χ^g is equivalent to χ for $g \in E_\chi$, there exists an invertible matrix $\tau_1(g)$ of the same degree as that of χ such that

$$\chi^g(h) = \tau_1(g)^{-1}\chi(h)\tau_1(g) \tag{2.4}$$

for all $h \in H$. Since

$$\tau(h)\tau(g) = \tau(g)\tau(g^{-1}hg)$$

for $g \in E_\chi$, $h \in H$, if we take the matrix representation such that

$$\tau(h) = \begin{pmatrix} \chi(h) & & \\ & \ddots & \\ & & \chi(h) \end{pmatrix}, \quad \text{and put } \tau(g) = \begin{pmatrix} T_{11}(g) & \cdots & T_{1k}(g) \\ \vdots & & \vdots \\ T_{k1}(g) & \cdots & T_{kk}(g) \end{pmatrix},$$

then we have

$$\chi(h)T_{ij}(g) = T_{ij}(g)\chi(g^{-1}hg),$$

and hence, together with (2.4), we have

$$\chi(h)T_{ij}(g)\tau_1(g)^{-1} = T_{ij}(g)\tau_1(g)^{-1}\chi(h).$$

By Schur's lemma, for fixed g the matrix $T_{ij}(g)\tau_1(g)^{-1}$ is a scalar matrix:

$$T_{ij}(g) = c_{ij}(g)\tau_1(g), \quad c_{ij}(g) \in \mathbf{C}.$$

Hence we have a matrix $\tau_2(g)$ of degree k whose (i, j)-entry is $c_{ij}(g)$ and obtain

$$\tau(g) = \tau_1(g) \otimes \tau_2(g). \tag{2.5}$$

Consider two elements $g, g' \in E_\chi$. Since $\tau_1(gg')$ and $\tau_1(g)\tau_1(g')$ both intertwine χ, they only differ by a scalar factor by Schur's lemma:

$$\tau_1(g)\tau_1(g') = \alpha(g, g')\tau_1(gg'), \quad \alpha(g, g') \in \mathbf{C}^\times.$$

Moreover, since

$$\tau_1(gg') \otimes \tau_2(gg') = \tau(gg') = \tau(g)\tau(g') = \tau_1(g)\tau_1(g') \otimes \tau_2(g)\tau_2(g'),$$

we have

$$\tau_2(g)\tau_2(g') = \alpha(g, g')^{-1}\tau_2(gg').$$

Hence τ_1 and τ_2 are both projective representations of E_χ, whose factor sets are inverse to each other, and we have $\tau = \tau_1 \otimes \tau_2$.

Since $T_{ij}(h) = 0$ for $i \neq j$ and $T_{ii}(h) = \chi(h)$, we can assume that $\tau_1(h) = \chi(h)$ and that $\tau_2(h)$ is the identity for $h \in H$. Furthermore we can take τ_1 in such a way that the factor set α depends only on the coset E_χ/H.

Let $\{s_f \mid f \in F\}$ be a complete set of coset representatives of E/H with $\pi(s_f) = f$ and $s_1 = 1$, where $\pi : E \to F$ is the projection. Let $F_\chi = \pi(E_\chi)$. Let $\tau_1(s_1)$ be the identity matrix and fix $\tau_1(s_f)$ $(f \in F_\chi, f \neq 1)$ in any way. Define

$$\tau_1(s_f h) = \tau_1(s_f)\chi(h), \quad \tau_1(h) = \chi(h), \quad f \in F_\chi,\ h \in H.$$

Then we have

$$\tau_1(gh) = \tau_1(g)\chi(h), \quad g \in E_\chi,\ h \in H, \tag{2.6}$$

and

$$\begin{aligned}
\tau_1(gh)\tau_1(g'h') &= \tau_1(g)\chi(h)\tau_1(g')\chi(h') \\
&= \tau_1(g)\tau_1(g')\chi(g'^{-1}hg')\chi(h') \\
&= \alpha(g, g')\tau_1(gg')\chi(g'^{-1}hg'h') \\
&= \alpha(g, g')\tau_1(ghg'h'),
\end{aligned}$$

hence we have

$$\alpha(gh, g'h') = \alpha(g, g').$$

This shows that the factor set α is associated to the factor set of the coset E_χ/H. Therefore, by (2.5), we have $\tau_2(gh) = \tau_2(g)$ $(g \in E_\chi, h \in H)$ provided that τ_1

satisfies (2.6). Hence τ_2 defines a projective representation of E_χ/H, whose factor set is inverse to the factor set of E_χ/H given by α as above.

We next show α is essentially determined by χ and the extension (2.1). We explain this in terms of non-abelian cohomology ([4], [5], [6], [16]), which is defined as follows.

For groups G and A, let $\kappa : G \to \mathrm{Aut}(A)/\mathrm{Int}(A)$ be a homomorphism of abstract groups (even if G and A are Lie groups), where $\mathrm{Aut}(A)$ (resp. $\mathrm{Int}(A)$) is the group of automorphisms (resp. inner automorphisms) of A. Let $Z^2(G, A, \kappa)$ denote the set of all pairs (γ, δ) of mappings

$$\gamma : G \times A \to A((g, a) \mapsto \gamma(g)(a)), \quad \delta : G \times G \to A,$$

such that

$$\gamma(g) \bmod \mathrm{Int}(A) = \kappa(g), \quad g \in G, \tag{2.7}$$

$$\gamma(g')(\gamma(g)(a)) = \delta(g, g')^{-1}(\gamma(gg')(a))\delta(g, g'), \tag{2.8}$$

$$\delta(g, g'g'')\delta(g', g'') = \delta(gg', g'')\gamma(g'')(\delta(g, g')), \tag{2.9}$$

for $a \in A,\ g, g', g'' \in G$.

An equivalence relation is defined as follows: (γ, δ) is equivalent to (γ', δ') if there exists a mapping $h : G \to A$ such that

$$\gamma'(g)(a) = h(g)^{-1}(\gamma(g)(a))h(g), \tag{2.10}$$

$$\delta'(g, g') = h(gg')^{-1}\delta(g, g')(\gamma(g')(h(g)))h(g'), \tag{2.11}$$

for $a \in A,\ g, g' \in G$. The cohomology set $H^2(G, A, \kappa)$ is the set of equivalence classes in $Z^2(G, A, \kappa)$, which is the set of equivalence classes of extensions of A by G,

$$1 \to A \to E \to G \to 1,$$

such that there exists a section $s : G \to E$ and κ is the image in $\mathrm{Aut}(A)/\mathrm{Int}(A)$ of the automorphism of A given by $a \mapsto s(g)^{-1}as(g)$.

In our case, let $G = F_\chi$, $A = GL(V_\chi)$ and $\kappa = 1$ (sending F_χ to the identity), where $\chi : H \to GL(V_\chi)$ is the irreducible representation. Let β be the normalized factor set of the extension (2.1) associated with the representatives $\{s_f \mid f \in F\}$:

$$s_f s_{f'} = s_{ff'}\beta(f, f'), \quad f, f' \in F, \beta(f, f') \in H,$$
$$\beta(1, f') = \beta(f, 1) = 1.$$

LEMMA 2.2. *Let* $\gamma : F_\chi \to \mathrm{Aut}(GL(V_\chi))$ *be the mapping defined by* $\gamma(f)(X) = \tau_1(s_f)^{-1} X \tau_1(s_f)$. *Then*

(i) $(1, \alpha^{-1} I_{V_\chi}), (\gamma, \chi \circ \beta) \in Z^2(F_\chi, GL(V_\chi), 1)$, *where* $I_{V_\chi} \in GL(V_\chi)$ *is the identity transformation and* $(\alpha^{-1} I_{V_\chi})(f, f') = \alpha^{-1}(f, f') I_{V_\chi}$.

(ii) $(1, \alpha^{-1} I_{V_\chi})$ *is equivalent to* $(\gamma, \chi \circ \beta)$.

PROOF. (i) Since α^{-1} satisfies the cocycle condition, $(1, \alpha^{-1} I_{V_\chi})$ satisfies (2.7), (2.8) and (2.9). Since $\gamma(f) \in \mathrm{Int}(GL(V_\chi))$, (2.7) is satisfied for γ. For $f, f' \in F_\chi$ and $X \in GL(V_\chi)$, since

$$\begin{aligned} \tau_1(s_f)\tau_1(s_{f'}) &= \alpha(f, f')\tau_1(s_f s_{f'}) \\ &= \alpha(f, f')\tau_1(s_{ff'}\beta(f, f')) \\ &= \alpha(f, f')\tau_1(s_{ff'})\chi(\beta(f, f')) \quad (by\ \ (2.6)) \end{aligned} \tag{2.12}$$

we have

$$\begin{aligned} \gamma(f')(\gamma(f)(X)) &= (\tau_1(s_f)\tau_1(s_{f'}))^{-1} X (\tau_1(s_f)\tau_1(s_{f'})) \\ &= \chi(\beta(f, f'))^{-1}(\tau_1(s_{ff'})^{-1} X \tau_1(s_{ff'}))\chi(\beta(f, f')) \\ &= ((\chi \circ \beta)(f, f'))^{-1}(\gamma(ff')(X))(\chi \circ \beta)(f, f'). \end{aligned}$$

Since β satisfies the cocycle condition

$$\beta(f, f'f'')\beta(f', f'') = \beta(ff', f'')s_{f''}^{-1}\beta(f, f')s_{f''},$$

we have

$$\begin{aligned} (\chi \circ \beta)(f, f'f'')(\chi \circ \beta)(f', f'') &= \chi(\beta(f, f'f''))\chi(\beta(f', f'')) \\ &= \chi(\beta(f, f'f'')\beta(f', f'')) \\ &= \chi(\beta(ff', f'')s_{f''}^{-1}\beta(f, f')s_{f''}) \\ &\overset{(2.4)}{=} \chi(\beta(ff', f''))\tau_1(s_{f''})^{-1}\chi(\beta(f, f'))\tau_1(s_{f''}) \\ &= (\chi \circ \beta)(ff', f'')\gamma(f'')(\chi \circ \beta(f, f')). \end{aligned}$$

(ii) Let $h : F_\chi \to GL(V_\chi)$ be the mapping defined by $h(f) = \tau_1(s_f)$. Then, for $f \in F_\chi$, $X \in GL(V_\chi)$, we have

$$\begin{aligned} \gamma(f)(X) &= \tau_1(s_f)^{-1} X \tau_1(s_f) \\ &= h(f)^{-1} X h(f). \end{aligned}$$

It follows from (2.12), we have

$$\chi(\beta(f,f')) = \tau_1(s_{ff'})^{-1}\alpha(f,f')^{-1}\tau_1(s_f)\tau_1(s_{f'}), \quad f,f' \in F_\chi.$$

Thus (2.10) and (2.11) are satisfied. $\square$

It follows from Lemma 2.2 that two cocycles $(1,\alpha^{-1}I_{V_\chi}), (\gamma, \chi \circ \beta) \in Z^2(F_\chi, GL(V_\chi), 1)$ ($\gamma : F_\chi \to \mathrm{Aut}(GL(V_\chi))$ is defined by $\gamma(f)(X) = \tau_1(s_f)^{-1}X\tau_1(s_f)$) define the same cohomology class in $H^2(F_\chi, GL(V_\chi), 1)$. If κ is given by a homomorphism $G \to \mathrm{Aut}(A)$, then $H^2(G, A, \kappa)$ is canonically identified with $H^2(G, Z(A), \kappa)$, where $Z(A)$ is the center of A ([16] 1.17). If A is abelian, $H^2(G, A, \kappa)$ coincides with ordinary cohomology $H^2(G, A)$, where G acts on A via κ. Thus we have

$$H^2(F_\chi, GL(V_\chi), 1) = H^2(F_\chi, \mathbf{C}^\times),$$

since $Z(GL(V_\chi)) \simeq \mathbf{C}^\times$.

The discussion above shows that the factor set α is determined by the irreducible representation χ of H and the factor set of the extension (2.1).

The results are summarized as follows:

Theorem 2.3 (Clifford).

(i) *Let χ, τ be as in the Theorem* 2.1. *Then*

$$\tau = \tau_1 \otimes \tau_2$$

where τ_1 and τ_2 are irreducible projective representations of E_χ such that the degree of τ_1 is same as the degree of χ and $\tau_1(gh) = \tau_1(g)\chi(h)$, $\tau_2(gh) = \tau_2(g)$ for all $g \in E_\chi$ and $h \in H$, so that τ_2 can be viewed as a projective representation of the factor group $F_\chi = E_\chi/H$. The representations τ_1 and τ_2 can be taken to be ordinary representations of E_χ if there exists an ordinary representation τ_1 such that $\tau_1(h) = \chi(h)$ for all $h \in H$.

(ii) *Conversely, for a given $\chi \in \mathrm{Irr}(H)$, there exists an irreducible projective representation τ_1 of E_χ such that $\tau_1(gh) = \tau_1(g)\chi(h)$ $(g \in E_\chi, h \in H)$ and $\tau_1|_H = \chi$. The factor set α of τ_1 is determined by the cohomology class of a* 2*-cocycle $\bar{\alpha} \in Z^2(F_\chi, \mathbf{C}^\times)$, which is given by χ and the extension* (2.1). *Let τ_2 be the pull back of an irreducible projective representation of $F_\chi = E_\chi/H$ with factor set $\bar{\alpha}^{-1}$. Then $\tau = \tau_1 \otimes \tau_2$ is an irreducible ordinary representation of E_χ such that $\tau|_H$ is a multiple of χ. Furthermore if τ_1 is fixed, then there is a one-to-one correspondence between $\mathrm{Irr}(E_\chi)_\chi$ and $\mathrm{Irr}(F_\chi, \bar{\alpha}^{-1})$ by the relation that τ is equivalent to $\tau_1 \otimes \tau_2$.*

Remark 2.4. If E is a semidirect product of H by F, then F is regarded as a subgroup of E. We can write an element of E uniquely in the form fh, $f \in F$, $h \in H$. If furthermore H is an abelian group A, then any linear character χ can be extended to a linear character τ_1 of E_χ by putting

$$\tau_1(fa) = \chi(a), \quad f \in F_\chi, a \in A, \tag{2.13}$$

since $(fa)(f'a') = (ff')(a^{f'}a')$ and $\chi(a^{f'}) = \chi^{f'}(a) = \chi(a)$ if $f' \in F_\chi$. Hence, by Theorem 2.3 (i), the irreducible representations τ of E_χ restricting to multiples of χ on A are given, up to equivalence, by the representations $\tau_1 \otimes \hat{\sigma}$, where $\hat{\sigma} = \sigma \circ \pi|_{E_\chi}$, $\sigma \in \mathrm{Irr}(F_\chi)$. We will denote $\tau_1 \otimes \hat{\sigma}$ by $\chi \rtimes \sigma$ in the sequel. We have

$$\mathrm{Irr}(E) = \{\theta_{\chi, \chi \rtimes \sigma} \mid \chi \in \mathrm{Hom}(A, \mathbf{C}^\times)/F, \sigma \in \mathrm{Irr}(F_\chi)\},$$

where $\chi \in \mathrm{Hom}(A, \mathbf{C}^\times)/F$ means, by abuse of notation, that χ varies over a complete set of representatives of the F-orbits of linear characters of A.

Remark 2.5. The condition (2.6) is worth special attention. We cannot replace τ_1 by another projective representation τ_1' of E_χ equivalent to τ_1 if τ_1' doesn't satisfy (2.6).

For example, if χ is one-dimensional, e.g. if H is abelian, the projective representation τ_1 of E_χ is also one-dimensional, and hence equivalent to any one-dimensional linear representation τ_1', e.g., the trivial representation. However, this does not mean that χ can be extended to a linear representation of E_χ, unless $\tau_1'|_H = \chi$ honestly holds. Accordingly, the condition that H is abelian does not assure that τ_2 can be taken to be linear representations.

In fact, let $E = Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ be the quaternion group and H its center $Z = \langle a^2 \rangle$, which is the cyclic group Z_2 of order 2. If χ is the trivial representation $\mathbf{1}$ of Z, then $E_\chi = Q_8$ and we can take τ_1 to be the trivial representation of Q_8. Since the factor group Q_8/Z is isomorphic to $Z_2 \oplus Z_2$, $\mathrm{Irr}(Q_8/Z)$ consists of 4 one-dimensional representations, which, together with τ_1, define 4 one-dimensional representations of Q_8 by Theorem 2.3 (ii). If χ is non-trivial representation $-\mathbf{1}$ of Z, then $E_\chi = Q_8$. Since $\mathrm{Irr}(Q_8)$ consists of 4 one-dimensional representations obtained as above and a two-dimensional faithful representation, there is no one-dimensional representation of $E_\chi = Q_8$ such that its restriction to Z is $-\mathbf{1}$.

In § 5, we will study the case where H is a maximal torus T of a complex simple Lie group G and E is its normalizer N in G. Any character χ of T can be extended to N_χ (Theorem 5.1). This follows not only from the commutativeness of T but also from special conditions for the representatives in N of the elements of the Weyl groups.

Remark 2.6. Theorem 2.1 and 2.3 show that the irreducible representations of E can be parametrized by the equivalence classes of pairs $(\chi, \bar{\tau}_2)$ with $\chi \in \mathrm{Irr}(H)$ and $\bar{\tau}_2 \in \mathrm{Irr}(F_\chi, \alpha^{-1})$, where $\alpha \in H^2(F_\chi, \mathbf{C}^\times)$ is determined from the non-abelian cohomology class $\beta \in H^2(F_\chi, H, \kappa)$ attached to the extension (2.1) by the condition

$$(\lambda_{V_\chi})_*(\alpha)^{-1} = \chi_*(\beta).$$

Here $\chi : H \to GL(V_\chi)$ and $\lambda_{V_\chi} : \mathbf{C}^\times \to GL(V_\chi)$ is the isomorphism from $\mathbf{C}^\times$ to the center of $GL(V_\chi)$.

Remark 2.7. If E and H are complex (resp. compact) Lie groups, then this parametrization nicely "restricts" to that of the irreducible holomorphic (resp. continuous) representations of E by those pairs $(\chi, \bar{\tau}_2)$ in which χ is holomorphic (resp. continuous). This can be verified by checking that each operation used in the proof of Theorem 2.1 and 2.3 "preserves" the kind of the representations considered. We observe this in the paragraphs below for complex Lie groups and holomorphic representations. The other case is utterly parallel.

Under our assumption, the groups $E \supset E_\chi \supset H$ form a chain of open subgroups. A representation of such a group is holomorphic if and only if its restriction to an open subgroup is holomorphic. Also, subrepresentations, direct sums and conjugates of holomorphic representations are holomorphic. (By a conjugate we mean a conjugate of a representation of H or E_χ by an element g of E, which is a representation of H or $E_{g\cdot\chi}$ respectively.)

Now if χ and τ are holomorphic in Theorem 2.1, then the induced representation $\theta_{\chi,\tau}$ is also holomorphic since $\theta_{\chi,\tau}|_H$ is a direct sum of conjugates of χ. In the other direction, if $\theta \in \mathrm{Irr}(E)$, then χ and τ such that $\theta \sim \theta_{\chi,\tau}$ are obtained as subrepresentations of $\theta|_H$ and $\theta|_{E_\chi}$. Hence if θ is holomorphic, then χ and τ are holomorphic. Note also that the equivalence of (χ, τ) as defined in Theorem 2.1 preserved holomorphicity.

Moreover if χ is holomorphic, then any $\tau \in \mathrm{Irr}(E_\chi)_\chi$ is holomorphic since its restriction to H is a multiple of χ by definition. Hence, in this case, Theorem 2.3 readily gives a parametrization of the irreducible holomorphic representations of E_χ restricting to multiples of χ on H by the irreducible projective representations of E_χ/H with cocycle $\bar{\alpha}^{-1}$. Note that, in this case, any projective representation τ_1 of E_χ which extends χ and satisfies (2.6) is a holomorphic map from E_χ to

$GL(V_\chi)$ since it is so on each of the cosets of H, which form a (disjoint) open covering of E_χ; and that the pullback τ_2 of a projective representation of E_χ/H is also a holomorphic map since it is constant on each coset of H.

3. Relation to Semidirect Products

We will study the representations of normalizer N of a maximal torus of $SL(n, \mathbf{C})$ in the last four sections of this paper. The representations are closely related to those of the normalizer $\tilde{N}$ of a maximal torus of $GL(n, \mathbf{C})$. The group $\tilde{N}$ is the semidirect product of the subgroup consisting of the diagonal matrices by the symmetric group of degree n and contains N. With this example in mind, we consider, more generally, the case where H is an abelian group and study the relationship between the representations of a semidirect product group and its subgroups.

Let us keep the notation of §2.

ASSUMPTION 3.1. Let $\tilde{E}$ be the semidirect product of an abelian group $\tilde{A}$ by a finite group F:

$$1 \to \tilde{A} \to \tilde{E} \xrightarrow{\tilde{\pi}} F \to 1$$

and E a subgroup of $\tilde{E}$ satisfying $\tilde{\pi}(E) = F$. Putting $\pi = \tilde{\pi}|_E$ and $A = \ker \pi$, we have an extension

$$1 \to A \to E \xrightarrow{\pi} F \to 1,$$

which is not split in general.

For $a \in A$, $f \in F$, the element $s_f^{-1} a s_f$ is independent of the choice of a representative s_f with $\pi(s_f) = f$, because A is abelian. Thus we denote it by a^f.

Since E is a subgroup of $\tilde{E}$, an element of E is written in two ways: $s_f a$ in E and fa' in $\tilde{E}$ with $a, a' \in A$, $f \in F$. These are related as follows. For each $f \in F$, there exists a unique element $\tilde{\epsilon}_f \in \tilde{A}$ which satisfies

$$s_f = f\tilde{\epsilon}_f, \tag{3.1}$$

thus we have

$$a' = \tilde{\epsilon}_f a.$$

LEMMA 3.2. *The factor set β is given by*

$$\beta(f, f') = \frac{\tilde{\epsilon}_f^{f'} \tilde{\epsilon}_{f'}}{\tilde{\epsilon}_{ff'}}.$$

PROOF. By (3.1), we have

$$\begin{aligned} s_f s_{f'} &= s_{ff'}\beta(f,f') \\ &= ff'\tilde{\epsilon}_{ff'}\beta(f,f'). \end{aligned}$$

We also have

$$\begin{aligned} s_f s_{f'} &= f\tilde{\epsilon}_f f'\tilde{\epsilon}_{f'} \\ &= ff'\tilde{\epsilon}_f^{f'}\tilde{\epsilon}_{f'}. \end{aligned}$$

Thus we have

$$\beta(f,f') = \frac{\tilde{\epsilon}_f^{f'}\tilde{\epsilon}_{f'}}{\tilde{\epsilon}_{ff'}}. \qquad \square$$

The action of E on the character group X of A induces an action of F on X. For a character χ of A, let F_χ be the stabilizer subgroup of χ in F.

By Remark 2.4, the irreducible representations of $\tilde{E}$ are given, up to equivalence, by the representations $\theta_{\tilde{\chi},\tilde{\chi}\rtimes\sigma}$ where $\tilde{\chi} \in \mathrm{Hom}(\tilde{\chi},\mathbf{C}^\times)/F$ and $\sigma \in \mathrm{Irr}(F_\chi)$.

THEOREM 3.3. *Let χ be a linear character of A, and suppose that there exists a linear character $\tilde{\chi}$ of $\tilde{A}$ such that $\tilde{\chi}|_A = \chi$ and $F_{\tilde{\chi}} = F_\chi$.*

(i) *Every irreducible representation $\theta_{\tilde{\chi},\tilde{\chi}\rtimes\sigma}$ of $\tilde{E}$ remains irreducible upon restriction to E.*

(ii) *For an irreducible representation $\theta_{\chi,\tau}$ of E, the projective representations τ_1 and τ_2 of Theorem 2.3 can be taken to be ordinary representations in the following way. Let $\tilde{\tau}_1$ be a linear character of $\tilde{E}_\chi$ defined as in (2.13) and $\tau_1 = \tilde{\tau}_1|_{E_\chi}$. Then τ_1 is a linear character of E_χ with $\tau_1(a) = \chi(a)$ for all $a \in A$. Hence τ_2 is an ordinary representation, which defines the irreducible representation σ of $F_\chi = F_{\tilde{\chi}}$. Put $\tilde{\tau}_2 = \hat{\sigma}$ and $\tilde{\tau} = \tilde{\tau}_1 \otimes \tilde{\tau}_2$, where $\hat{\sigma} = \sigma \circ \tilde{\pi}|_{\tilde{E}_{\tilde{\chi}}}$. Then*

$$\theta_{\tilde{\chi},\tilde{\chi}\rtimes\sigma}|_E = \theta_{\chi,\tau}. \tag{3.2}$$

(iii) *Suppose that such $\tilde{\chi}$ exists for every linear character χ of A. Fix the choice of such a linear character $\tilde{\chi}$ for every χ. Then the representations $\theta_{\chi,\tau} = \theta_{\tilde{\chi},\tilde{\chi}\rtimes\sigma|_E}$, parametrized by the pairs (χ,σ) in the following way, form a complete set of representatives of the equivalence classes of irreducible representations of E: χ varies over a complete set of representatives of* *the F-orbits of linear characters of A, and $\tilde{\tau} = \tilde{\tau}_1 \otimes \hat{\sigma}$, where $\tilde{\tau}_1$ is determined from $\tilde{\chi}$ as in* (2.13) *and σ varies over* $\mathrm{Irr}(F_\chi)$, *a complete set of representatives of the equivalence classes of irreducible representations of $F_{\tilde{\chi}} = F_\chi$.*

PROOF. (i) Since

$$\tilde{\tau}(f\tilde{a}) = \tilde{\chi}(a\tilde{\epsilon}_f^{-1})\tilde{\tau}(s_f), \quad f\tilde{a} \in \tilde{E}_\chi,$$

a $\tilde{\tau}(E_\chi)$-invariant subspace is also $\tilde{\tau}(\tilde{E}_\chi)$-invariant. Hence $\tau = \tilde{\tau}|_{E_\chi}$ is irreducible and $\theta_{\tilde{\chi},\tilde{\tau}}|_E = \theta_{\chi,\tau}$.

(ii) We have $\tau_1|_A = \chi$, since $\tilde{\tau}_1|_{\tilde{A}} = \tilde{\chi}$, $A = E_\chi \cap \tilde{A}$ and $\chi = \tilde{\chi}|_A$. Since τ_1 is a linear character of E_χ, τ_2 is also an ordinary representation by (2.5). By construction, we have $\tilde{\tau}_1|_{E_\chi} = \tau_1$ and $\tilde{\tau}_2|_{E_\chi} = \tau_2$, so that $\tilde{\tau}|_{E_\chi} = \tau$. Hence

$$\theta_{\tilde{\chi},\tilde{\tau}}|_E = \theta_{\chi,\tau}.$$

(iii) Now suppose that the assumption at the beginning of this Theorem holds for every linear character χ of A. Let $R(A,E)$ be a complete set of representatives of conjugacy classes of linear characters of A under the action of E.

It follows from Theorem 2.1 that every irreducible representation of E is equivalent to some $\theta_{\chi,\tau}$, where $\chi \in R(A,E)$ and $\tau \in \mathrm{Irr}(E_\chi)_\chi$. By (ii), we have $\tau = \tau_1 \otimes (\sigma \circ \pi|_{E_\chi})$. Here τ_1 is given by $\tilde{\chi}$ as in (ii) and σ is an irreducible representation of F_χ. Let σ' be an element of $\mathrm{Irr}(F_\chi)$ which is equivalent to σ. Then $\theta_{\chi,\tau}$ is equivalent to $\theta_{\chi,\tau_1 \otimes (\sigma' \circ \pi|_{E_\chi})}$, since τ is equivalent to $\tau_1 \otimes (\sigma' \circ \pi|_{E_\chi})$. By Theorem 2.1 again, $\theta_{\chi,\tau_1 \otimes (\sigma \circ \pi|_{E_\chi})}$ and $\theta_{\chi',\tau_1' \otimes (\sigma' \circ \pi|_{E_{\chi'}})}$ $(\chi,\chi' \in R(A,E), \sigma \in \mathrm{Irr}(F_\chi)$, $\sigma' \in \mathrm{Irr}(F_{\chi'}))$ are equivalent if and only if $\chi = \chi'$ and $\sigma = \sigma'$.

Conversely, let χ be an element of $R(A,E)$ and $\sigma \in \mathrm{Irr}(F_\chi)$. Then we obtain an irreducible representation $\theta_{\tilde{\chi},\tilde{\tau}}$ with $\tilde{\tau} = \tilde{\chi} \rtimes \sigma$ as in Remark 2.4. By restricting $\theta_{\tilde{\chi},\tilde{\tau}}$ to E, we have an irreducible representation $\theta_{\chi,\tau}$. □

For $\tilde{\chi}$ we define a function $e_{\tilde{\chi}}$ on F by

$$e_{\tilde{\chi}}(f) = \tilde{\chi}(\tilde{\epsilon}_f). \tag{3.3}$$

COROLLARY 3.4. *For $s_f a \in E_\chi$,*

$$\tilde{\tau}(s_f a) = e_{\tilde{\chi}}(f)\chi(a)\sigma(f), \tag{3.4}$$

where $\sigma \in \mathrm{Irr}(F_\chi)$ is defined as in Theorem 3.3 (ii).

Proof. For $f\tilde{a} \in \tilde{E}_{\tilde{\chi}}$ we have

$$\begin{aligned}\tilde{\tau}(f\tilde{a}) &= \tilde{\tau}_1(f\tilde{a}) \otimes \tilde{\tau}_2(f\tilde{a}) \\ &= \tilde{\chi}(\tilde{a})\sigma(f).\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{\tau}(s_f a) &= \tilde{\tau}(fa\tilde{\epsilon}_f) \\ &= \tilde{\tau}(fa)\tilde{\tau}(\tilde{\epsilon}_f) \\ &= \tilde{\chi}(a)\sigma(f)\tilde{\chi}(\tilde{\epsilon}_f) \\ &= e_{\tilde{\chi}}(f)\chi(a)\sigma(f),\end{aligned}$$

since $\tilde{\chi}(a) = \chi(a)$ for $a \in A$. □

Example 3.5. (1) Let $\tilde{T}$ (resp. T) be the group of all diagonal matrices in $\tilde{G} = GL(n, \mathbf{C})$ (resp. $G = SL(n, \mathbf{C})$) and $\tilde{N}$ (resp. N) its normalizer in $\tilde{G}$ (resp. G). The extensions

$$1 \to T \to N \to W \to 1$$

and

$$1 \to \tilde{T} \to \tilde{N} \to W \to 1$$

satisfy the conditions in Assumption 3.1, where W is the Weyl group of $\tilde{G}$ with respect to $\tilde{T}$ (resp. of G with respect to T) isomorphic to $\mathfrak{S}_n$.

Any character χ of T is the restriction of a character $\tilde{\chi}$ of $\tilde{T}$. For a sequence of integers $\boldsymbol{m} = (m_1, \dots, m_n)$, let $\tilde{\chi}_{\boldsymbol{m}} : \mathrm{diag}(\tilde{t}_1, \dots, \tilde{t}_n) \mapsto \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$ be a character of $\tilde{T}$. Then two characters $\tilde{\chi} = \tilde{\chi}_{\boldsymbol{m}}$ and $\tilde{\chi}' = \tilde{\chi}_{\boldsymbol{m}'}$ of $\tilde{T}$ restrict to the same character of T if and only if $\boldsymbol{m}' = \boldsymbol{m} + (k, \dots, k)$ or equivalently $\tilde{\chi}' = \tilde{\chi} \otimes (\det)^k$. Therefore if $\tilde{\chi}|_T = \chi$, then $W_{\tilde{\chi}} = W_\chi$ and the assumption of Theorem 3.3 is satisfied.

For $n = 2$,

$$\tilde{T} = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \middle| t_i \in \mathbf{C}^\times \right\}, \quad \tilde{N} = \tilde{T} \cup \left\{ \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix} \middle| t_i \in \mathbf{C}^\times \right\}$$

$$T = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} \middle| t_1 \in \mathbf{C}^\times \right\}, \quad N = T \cup \left\{ \begin{pmatrix} 0 & t_1 \\ -t_1^{-1} & 0 \end{pmatrix} \middle| t_1 \in \mathbf{C}^\times \right\}.$$

Let

$$\tilde{t} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \quad \widetilde{t'} = \begin{pmatrix} 0 & t_1 \\ t_2 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, \quad t' = \begin{pmatrix} 0 & t_1 \\ -t_1^{-1} & 0 \end{pmatrix},$$

$$t' = tg, \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For an integer m let $\chi(t) = t_1^m$ and $\tilde{\chi}(\tilde{t}) = t_1^{m_1} t_2^{m_2}$ with $m = m_1 - m_2$, $m_i \in \mathbf{Z}$. Then $\tilde{\chi}|_T = \chi$.

(i) If $m \neq 0$, then $N_\chi = T$ and W_χ is trivial. Let $\tau = \chi$. Then

$$V_{\chi,\tau} = U_{\chi,\tau} \oplus gU_{\chi,\tau}.$$

Take a non-zero vector $e_1 \in U_{\chi,\tau}$, and put $e_2 = ge_1 \in gU_{\chi,\tau}$. The matrix representation of $\theta_{\chi,\tau}$ with respect to the basis $\{e_1, e_2\}$ has the form

$$\theta_{\chi,\tau}(t) = \begin{pmatrix} t_1^m & 0 \\ 0 & t_1^{-m} \end{pmatrix}, \quad \theta_{\chi,\tau}(t') = \begin{pmatrix} 0 & (-t_1)^m \\ t_1^{-m} & 0 \end{pmatrix}.$$

(ii) If $m = 0$, then $N_\chi = N$ and $W_\chi = W \simeq \mathfrak{S}_2$. Let $\tilde{\chi}(\tilde{t}) = (t_1 t_2)^k$ and $\tilde{\tau}_1(\tilde{t}) = \tilde{\tau}_1(\widetilde{t'}) = (t_1 t_2)^k$. Define $\tau_1 = \tilde{\tau}_1|_N$, then $\tau_1(t) = 1$ and

$$\tau_1(t') = \tilde{\tau}_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix}\right) = \tilde{\chi}\left(\begin{pmatrix} -t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix}\right) = (-1)^k.$$

(a) For the trivial representation of $W_\chi \simeq \mathfrak{S}_2$, we have $\tau_2 = 1$ and

$$\theta_{\chi,\tau} = \tau = \tau_1 \otimes \tau_2 = \tau_1 \otimes 1.$$

Hence we have

$$\theta_{\chi,\tau}(t) = 1, \quad \theta_{\chi,\tau}(t') = (-1)^k.$$

(b) For the sign representation of $W_\chi \simeq \mathfrak{S}_2$, we have $\tau_2(t) = 1$, $\tau_2(t') = -1$. In this case we have

$$\theta_{\chi,\tau}(t) = 1, \quad \theta_{\chi,\tau}(t') = \tau_1(t')\tau_2(t') = (-1)^{k+1}.$$

We next consider the representations of $\tilde{N}$.

(iii) If $m_1 \neq m_2$, then $\tilde{N}_{\tilde{\chi}} = \tilde{T}$. Let $\tilde{\tau} = \tilde{\chi}$. Then

$$V_{\tilde{\chi},\tilde{\tau}} = U_{\tilde{\chi},\tilde{\tau}} \oplus gU_{\tilde{\chi},\tilde{\tau}}.$$

Take a non-zero element $\tilde{e}_1 \in U_{\tilde{\chi},\tilde{\tau}}$, and put $\tilde{e}_2 = g\tilde{e}_1$. Then we have the matrix representation with respect to the basis $\{e_1, e_2\}$

$$\theta_{\tilde{\chi},\tilde{\tau}}(\tilde{t}) = \begin{pmatrix} t_1^{m_1} t_2^{m_2} & 0 \\ 0 & t_1^{m_2} t_2^{m_1} \end{pmatrix}, \quad \theta_{\tilde{\chi},\tilde{\tau}}(\widetilde{t'}) = \begin{pmatrix} 0 & (-t_1)^{m_1} t_2^{m_2} \\ t_1^{m_2}(-t_2)^{m_1} & 0 \end{pmatrix}.$$

The restriction $\theta_{\tilde{\chi},\tilde{\tau}}|_N$ is the representation in (i).

(iv) If $m_1 = m_2 = k$, then $\tilde{N}_{\tilde{\chi}} = \tilde{N}$ and $W_{\tilde{\chi}} = W \simeq \mathfrak{S}_2$. Then $\tilde{\tau}_1(\tilde{t}) = \tilde{\chi}(\tilde{t}) = (t_1 t_2)^k$ and

$$\tilde{\tau}_1(\widetilde{t'}) = \tilde{\tau}_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) = \tilde{\chi}\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix} = (t_1 t_2)^k.$$

(a) For the trivial representation of $W_{\tilde{\chi}} \simeq \mathfrak{S}_2$, we have $\tilde{\tau}_2 = 1$ and $\theta_{\tilde{\chi},\tilde{\tau}} = \tilde{\tau}_1 \otimes \tilde{\tau}_2$. Hence

$$\theta_{\tilde{\chi},\tilde{\tau}}(\tilde{t}) = \theta_{\tilde{\chi},\tilde{\tau}}(\widetilde{t'}) = (t_1 t_2)^k.$$

The restriction $\theta_{\tilde{\chi},\tilde{\tau}}|_N$ is the representation in (ii)(a).

(b) For the sign representation of $W_{\tilde{\chi}} \simeq \mathfrak{S}_2$, we have $\theta_{\tilde{\chi},\tilde{\tau}}(\tilde{t}) = (t_1 t_2)^k$ and

$$\begin{aligned} \theta_{\tilde{\chi},\tilde{\tau}}(\widetilde{t'}) &= \theta_{\tilde{\chi},\tilde{\tau}}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) \\ &= \tilde{\tau}_1\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) \otimes \tilde{\tau}_2\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}\right) \\ &= -(t_1 t_2)^k \end{aligned}$$

Hence $\theta_{\tilde{\chi},\tilde{\tau}}|_N$ is the representation in (ii)(b).

(2) We may realize the symplectic group $Sp(2n, \mathbf{C})$ as the set of matrices X of $GL(2n, \mathbf{C})$ satisfying ${}^tXJX = J$, where J is the matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let T be the group of all diagonal matrices in $Sp(2n, \mathbf{C})$ and N the normalizer of T in $Sp(2n, \mathbf{C})$. The Weyl group $W = N/T$, faithfully represented as a group of automorphisms of T acting by conjugation, is generated by the permutations of the indices $\{1, 2, \ldots, n\}$ of the elements $t = \mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$ (which permute $t_1, \ldots, t_n$ and $t_1^{-1}, \ldots, t_n^{-1}$ in a parallel manner) and the transposition that exchange t_n and t_n^{-1}. This naturally gives an embedding $\iota : W \hookrightarrow \mathfrak{S}_{2n}$ as the permutations that commute with $(1\ 2n)(2\ 2n-1) \cdots (n\ n+1)$.

Let $\tilde{T}$ be the group of all diagonal matrices in $GL(2n, \mathbf{C})$. The group W can be realized as a subgroup of $GL(2n, \mathbf{C})$ consisting of the permutation matrices representing the elements of $\iota(W) \subset \mathfrak{S}_{2n}$. Let $\tilde{N}$ be the subgroup of $GL(2n, \mathbf{C})$ generated by $\tilde{T}$ and W, which is isomorphic to the semidirect product of $\tilde{T}$ by W. Then N is a subgroup of $\tilde{N}$, and $(W, T, N, \tilde{T}, \tilde{N})$ satisfies Assumption 3.1. For a character χ of T given by $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$, define a character $\tilde{\chi}$ of $\tilde{T}$ by $\tilde{\chi}(\tilde{t}) = \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$. Then we have $\tilde{\chi}|_T = \chi$ and $W_{\tilde{\chi}} = W_\chi$.

(3) The orthogonal group $SO(m, \mathbf{C})$ is realized as the set of matrices X of $SL(m, \mathbf{C})$ satisfying ${}^tXJX = J$, where J is the matrix

$$J = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } m = 2n+1, \qquad J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad \text{for } m = 2n.$$

Let T be the group of all diagonal matrices in $SO(m, \mathbf{C})$ and N the normalizer of T in $SO(m, \mathbf{C})$. If $m = 2n+1$, the Weyl group $W = N/T$, as in (2), is generated by the permutations of the indices $\{1, 2, \dots, n\}$ of the elements $t = \mathrm{diag}(t_1, \dots, t_n, t_1^{-1}, \dots t_n^{-1}, 1)$ of T (which permute $t_1, \dots, t_n$ and $t_1^{-1}, \dots, t_n^{-1}$ in a parallel manner) and the transposition that exchange t_n and t_n^{-1}. If $m = 2n$, the Weyl group is generated by the permutations of the indices $\{1, 2, \dots, n\}$ of the elements $t = \mathrm{diag}(t_1, \dots, t_n, t_1^{-1}, \dots t_n^{-1})$ of T (which permute $t_1, \dots, t_n$ and $t_1^{-1}, \dots, t_n^{-1}$ in a parallel manner) and products of even numbers of transpositions that exchange t_i and t_i^{-1}. This naturally gives an embedding $\iota : W \to \mathfrak{S}_{2n}$ as the permutations that commute with $(1\ 2n)(2\ 2n-1) \cdots (n\ n+1)$.

Let $\tilde{T}$ be the group of all diagonal matrices in $GL(m, \mathbf{C})$. The group W can be realized as a subgroup of $GL(m, \mathbf{C})$ consisting of the permutation matrices representing the elements of $\iota(W) \subset \mathfrak{S}_{2n}$. Let $\tilde{N}$ be the subgroup of $GL(m, \mathbf{C})$ generated by $\tilde{T}$ and W, which is isomorphic to the semidirect product of $\tilde{T}$ by W. Then N is a subgroup of $\tilde{N}$, and $(W, T, N, \tilde{T}, \tilde{N})$ satisfies Assumption 3.1. For a character χ of T given by $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$, define a character $\tilde{\chi}$ of $\tilde{T}$ by $\tilde{\chi}(\tilde{t}) = \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$. Then we have $\tilde{\chi}|_T = \chi$ and $W_{\tilde{\chi}} = W_\chi$.

4. Formula for Irreducible Characters of Finite Extensions of Abelian Groups

In this section, we calculate the character $\psi_{\chi,\tau}$ of $\theta_{\chi,\tau}$ for the case where E is an extension of an abelian group A by a finite group F:

$$1 \to A \to E \xrightarrow{\pi} F \to 1.$$

The notation is as in the preceding sections. Let τ be an irreducible representation of E_χ whose restriction to A is a multiple of χ. Let $\varphi_{\chi,\tau}$ be the character of τ. By (2.3), we have

$$\psi_{\chi,\tau}(g) = \sum_{\substack{g' \in E/E_\chi \\ g'^{-1}gg' \in E_\chi}} \varphi_{\chi,\tau}(g'^{-1}gg'),$$

where the summation is taken over a complete set of coset representatives of E/E_χ. Since $E/E_\chi = F/F_\chi$,

$$\psi_{\chi,\tau}(g) = \sum_{\substack{f \in F/F_\chi \\ f^{-1}\pi(g)f \in F_\chi}} \varphi_{\chi,\tau}(\tilde{f}^{-1}g\tilde{f}),$$

where the summation is taken over a complete set of coset representatives of F/F_χ and $\tilde{f}$ is an element of $\pi^{-1}(f)$.

For $g \in E$, put

$$I(g) = \{f \in F \mid f^{-1}\pi(g)f \in F_\chi\}.$$

Then $I(g)F_\chi = I(g)$ and

$$\psi_{\chi,\tau}(g) = \frac{1}{|F_\chi|}\sum_{f \in I(g)} \varphi_{\chi,\tau}(\tilde{f}^{-1}g\tilde{f}), \tag{4.1}$$

where $\tilde{f}$ is an element of $\pi^{-1}(f)$.

Let K_f be the conjugacy class of $f \in F$ in F. If $k_0 \in F$ and $k_0^{-1}\pi(g)k_0 = f_0 \in K_{\pi(g)} \cap F_\chi$, then, for $k \in F$, $k^{-1}\pi(g)k = f_0$ if and only if kk_0^{-1} is an element of the centralizer $Z_F(\pi(g))$ of $\pi(g)$ in F.

Let $\Phi_f : F \to F$ be the mapping given by $k \mapsto k^{-1}fk$. Note that $\operatorname{Im} \Phi_{\pi(g)} = K_{\pi(g)}$. We have

$$I(g) = \Phi_{\pi(g)}^{-1}(F_\chi) = \bigsqcup_{f_0 \in K_{\pi(g)} \cap F_\chi} \Phi_{\pi(g)}^{-1}(f_0) \quad \text{(disjoint union)} \tag{4.2}$$

and, for each $f_0 \in K_{\pi(g)} \cap F_\chi$, we have

$$\Phi_{\pi(g)}^{-1}(f_0) = Z_F(\pi(g))k_0 \quad \text{if we pick any } k_0 \in \Phi_{\pi(g)}^{-1}(f_0). \tag{4.3}$$

Now we have the following.

PROPOSITION 4.1. *Let* $g \in E$.

(i) *For each* $f_0 \in F_\chi \cap K_{\pi(g)}$, *fix an element* $k_0 \in F$ *with* $f_0 = k_0^{-1}\pi(g)k_0$, *then*

$$\psi_{\chi,\tau}(g) = \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_{\pi(g)}} \sum_{x \in Z_F(\pi(g))k_0} \varphi_{\chi,\tau}(\tilde{x}^{-1} g \tilde{x}),$$

where $\varphi_{\chi,\tau}$ *is the character of* τ *and* $\tilde{x}$ *is an element of* $\pi^{-1}(x)$.

(ii) *Suppose* E *is a semidirect product of* A *by* F *and* τ *is given by* $\chi \rtimes \sigma$ (*see Remark* 2.4). *Let* $g = fa \in E$ *with* $f \in F$ *and* $a \in A$. *Then we have*

$$\psi_{\chi,\tau}(fa) = \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \phi_\sigma(f_0) \left(\sum_{x \in \Phi_f^{-1}(f_0)} \chi(x^{-1}ax) \right),$$

where ϕ_σ *is the character of* σ.

PROOF. (i) This follows from (4.1), (4.2) and (4.3).

(ii) By (4.1), (4.2), we have

$$\psi_{\chi,\tau}(fa) = \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \sum_{x \in \Phi_f^{-1}(f_0)} \varphi_{\chi,\tau}(x^{-1}(fa)x).$$

Since $x^{-1}(fa)x = (x^{-1}fx)(x^{-1}ax) = f^x a^x$, we have

$$\varphi_{\chi,\tau}(x^{-1}(fa)x) = \chi(a^x)\phi_\sigma(f^x)$$

and hence

$$\begin{aligned} \psi_{\chi,\tau}(fa) &= \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \sum_{x \in \Phi_f^{-1}(f_0)} \chi(a^x)\phi_\sigma(f^x) \\ &= \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \sum_{x \in \Phi_f^{-1}(f_0)} \chi(a^x)\phi_\sigma(f_0) \\ &= \frac{1}{|F_\chi|} \sum_{f_0 \in F_\chi \cap K_f} \phi_\sigma(f_0) \left(\sum_{x \in \Phi_f^{-1}(f_0)} \chi(a^x) \right) \end{aligned}$$

□

EXAMPLE 4.2. Let T be the group of all diagonal matrices in $GL(n, \mathbf{C})$ and N the normalizer of T in $GL(n, \mathbf{C})$. Then $W = N/T$ is isomorphic to the symmetric group $\mathfrak{S}_n$ of degree n and N is a semidirect product of T by $\mathfrak{S}_n$:

$$1 \to T \to N \to W = \mathfrak{S}_n \to 1$$

(i) For $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$, let $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$, where $m_1, \ldots, m_n$ are distinct integers. Then W_χ is trivial and the trivial representation is the only irreducible representation of W_χ. Hence $\tau = \chi \rtimes \mathbf{1}$ (see Remark 2.4). Since $W_\chi \cap K_w = \varnothing$ for any non-trivial element w of W. Hence $\psi_{\chi,\tau}(wt) = 0$ for $w \in W$, $w \neq 1$, $t \in T$. For $w = 1$, we have $W_\chi \cap K_1 = \{1\}$ and

$$\Phi_1^{-1}(1) = W = \mathfrak{S}_n.$$

Hence we have

$$\psi_{\chi,\tau}(wt) = \sum_{w \in \mathfrak{S}_n} \chi(t^w)$$
$$= \sum_{w \in \mathfrak{S}_n} t_{w(1)}^{m_1} \cdots t_{w(n)}^{m_n}.$$

(ii) Let $\chi(t) = (t_1 \cdots t_n)^m$. Then $W_\chi = \mathfrak{S}_n$ and $W_\chi \cap K_w = K_w$ for all $w \in \mathfrak{S}_n$. Hence we have

$$\psi_{\chi,\tau}(wt) = \frac{1}{n!} \sum_{u \in K_w} \phi_\sigma(u) \sum_{v \in \Phi_w^{-1}(u)} \chi(t^v).$$

5. Case of the Normalizers of Maximal Tori

Let G be a connected complex simple Lie group. We apply the results of preceding sections to the case where A is a maximal torus T of G, E the normalizer N of T in G and F the Weyl group W of G:

$$1 \to T \to N \xrightarrow{\pi} W \to 1.$$

In the following sections we write μ for a weight of a representation of G with respect to T and regard μ as a linear character of T. Thus, for the sake of brevity, we use the letter μ, instead of χ which is used for an irreducible representation of H or A in the previous sections, for a linear character of T when H or A is a maximal torus T of G.

When G is the classical group $SL(n,\mathbf{C})$, $Sp(2n,\mathbf{C})$ or $SO(m,\mathbf{C})$, if we take $\tilde{T}$ and $\tilde{N}$ as in Example 3.5, then $(W, T, N, \tilde{T}, \tilde{N})$ satisfies Assumption 3.1 and the assumption of Theorem 3.3 (iii). Thus we can apply the results of § 3. In general we can extend a linear character μ to a linear character τ_1 of N_μ by choosing good representatives in N of the elements of W and apply the results of § 2.

We first recall some results about the representatives in N of the elements of W ([17] 9.3).

Let Φ be the root system of G with respect to T and $(,)$ the inner product induced on the character group of T or the weight lattice by the Killing form. Let B be a Borel subgroup containing T, $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the corresponding basis of Φ and Φ^+ the set of positive roots. For each $\alpha \in \Phi$, let u_α be an isomorphism of $\mathbf{C}$ into G such that

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad t \in T,\ x \in \mathbf{C}. \tag{5.1}$$

We can choose u_α and $u_{-\alpha}$ for each $\alpha \in \Phi^+$ in such a way that

$$n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$$

is an element of N and represents the reflection $s_\alpha \in W$ associated to α. Then n_α^2 lies in T. We denote it by t_α, which is given by $\alpha^\vee(-1)$, where $\alpha^\vee$ is the coroot with $\langle \alpha, \alpha^\vee \rangle = 2$.

The normalizer N is generated by $T \cup \{n_{\alpha_1}, \dots, n_{\alpha_l}\}$ and determined by the relations:

$$tt' = t't, \quad t, t' \in T \tag{5.2}$$

$$n_\alpha^2 = \alpha^\vee(-1), \quad \alpha \in \Pi \tag{5.3}$$

$$n_\alpha t n_\alpha^{-1} = s_\alpha(t), \quad t \in T,\ \alpha \in \Pi \tag{5.4}$$

$$\underbrace{n_{\alpha_i}n_{\alpha_j}n_{\alpha_i}\cdots}_{m_{ij}} = \underbrace{n_{\alpha_j}n_{\alpha_i}n_{\alpha_j}\cdots}_{m_{ij}} \tag{5.5}$$

where m_{ij} is the order of $s_{\alpha_i}s_{\alpha_j}$.

For $w \in W$, let $w = s_{\gamma_1} \cdots s_{\gamma_k}$ be a reduced expression of w with $\gamma_i \in \Pi$. Then the element $n_{\gamma_1} \cdots n_{\gamma_k}$ of N is independent of the choice of the reduced expression of w. Denote this by n_w. Hence we have a set of representatives $\{n_w \mid w \in W\}$ in N of the elements of W provided that we fix $\{u_\alpha \mid \alpha \in \Phi\}$ and a basis Π of Φ.

THEOREM 5.1. (i) *Let μ be a linear character of T and N_μ the stabilizer of μ in N. Let τ_1 be the mapping of N_μ to $\mathbf{C}$ defined by $\tau_1(n_w t) = \mu(t)$, $w \in W_\mu$, $t \in T$. Then τ_1 is a linear character of N_μ satisfying $\tau_1|_T = \mu$.*

Let Λ^+ denote the set of the linear characters of T which are dominant with respect to Π. If $\mu \in \Lambda^+$ and $\sigma \in \mathrm{Irr}(W_\mu)$, then we write $\tau(\mu, \sigma)$ for the representation $\tau_1 \otimes \tau_2$ of N_μ, where τ_1 is defined from μ as in (i) *and $\tau_2 = \hat{\sigma} = \sigma \circ (\pi|_{N_\mu})$, i.e.*

$$\tau(\mu, \sigma)(n_w t) = \mu(t)\sigma(w).$$

(ii) *The representations* $\tau(\mu,\sigma)$ *of* N_μ *just defined above, where* σ *varies over* $\mathrm{Irr}(W_\mu)$, *form a complete set of representatives of the equivalence classes of irreducible representations of* N_μ *whose restrictions to* T *are multiples of* μ. *Moreover, the representations* $\theta_{\mu,\tau(\mu,\sigma)}$ *of* N *with* $\mu \in \Lambda^+$ *and* $\sigma \in \mathrm{Irr}(W_\mu)$ *form a complete set of representatives of the equivalence classes of irreducible representations of* N.

If $\tau \in \mathrm{Irr}(N_\mu)_\mu$ (*see* §2), *then there exists a unique element* $\sigma \in \mathrm{Irr}(W_\mu)$ *such that* $\tau = \tau(\mu,\sigma)$ *in the parametrization currently discussed* (*using the representatives* $\{n_w \mid w \in W\}$).

Proof. (i) Since μ is dominant, W_μ is a standard parabolic subgroup. Thus N_μ is determined by the relations (5.2)~(5.5) for $\Pi_\mu = \{\alpha \in \Pi \mid (\alpha,\mu) = 0\}$. Thus we have to check

$$\tau_1(n_\alpha)^2 = \tau_1(\alpha^\vee(-1)), \quad \alpha \in \Pi_\mu$$

$$\tau_1(n_\alpha)\tau_1(t)\tau_1(n_\alpha)^{-1} = \tau_1(s_\alpha(t)), \quad t \in T,\ \alpha \in \Pi_\mu$$

$$\underbrace{\tau_1(n_{\alpha_i})\tau_1(n_{\alpha_j})\tau_1(n_{\alpha_i})\cdots}_{m_{ij}} = \underbrace{\tau_1(n_{\alpha_j})\tau_1(n_{\alpha_i})\tau_1(n_{\alpha_j})\cdots}_{m_{ij}}, \quad \alpha_i, \alpha_j \in \Pi_\mu.$$

The last equation is obvious since $\tau_1(n_\alpha) = 1$ for $\alpha \in \Pi_\mu$. Since $\langle \alpha^\vee, \mu\rangle = 0$, we have

$$\tau_1(\alpha^\vee(-1)) = \mu(\alpha^\vee(-1)) = (-1)^{\langle \alpha^\vee,\mu\rangle} = 1,$$

which is equal to $\tau_1(n_\alpha)^2$. Since $\tau_1(n_\alpha) = 1$, we have

$$\tau_1(n_\alpha)\tau_1(t)\tau_1(n_\alpha)^{-1} = \tau_1(t) = \mu(t).$$

On the other hand, we have

$$\tau_1(s_\alpha(t)) = \mu(s_\alpha(t)) = \mu(t),$$

since $s_\alpha \in W_\chi$.

(ii) By (i) every character μ of T can be extended to a character τ_1 of N_μ. Since every W-orbit of linear characters of T contains a unique element of Λ^+, the statement follows from Theorem 2.1 and 2.3. □

The proof of Theorem 5.1 (i) is based on the special properties of the representatives $\{n_w \mid w \in W\}$. The next proposition gives a necessary and sufficient condition for the existence of τ_1 for a finite extension of a complex algebraic torus:

PROPOSITION 5.2. *Let E be an extension of a complex algebraic torus A by a finite group F*:

$$1 \to A \to E \xrightarrow{\pi} F \to 1.$$

For a linear character χ of A, there exists a linear character τ_1 of E_χ such that $\tau_1|_A = \chi$ if and only if there is a complete set $\{e_f\}_{f \in F_\chi}$ of coset representatives of E_χ/A such that $e_f e_{f'} e_{ff'}^{-1} \in \ker \chi$ for any $f, f' \in F_\chi$.

PROOF. The result is obvious if χ is trivial. Suppose χ is not trivial and hence $\operatorname{Im} \chi = \mathbf{C}^\times$. Denote by $\Gamma(F_\chi)_\chi$ the set of complete sets $\{e_f\}_{f \in F_\chi}$ of coset representatives of E_χ/A satisfying the condition $e_f e_{f'} e_{ff'}^{-1} \in \ker \chi$, $f, f' \in F_\chi$ and by $\operatorname{Hom}(E_\chi, \mathbf{C}^\times)_\chi$ the set of linear characters τ_1 of E_χ with $\tau_1|_A = \chi$. For $\{e_f\}_{f \in F_\chi} \in \Gamma(F_\chi)_\chi$, let τ_1 be the mapping of E_χ to $\mathbf{C}^\times$ defined by $\tau_1(e_f a) = \chi(a)$, $a \in A$. Since

$$\tau_1(e_f a)\tau_1(e_{f'} a')\tau_1(e_{ff'} aa')^{-1} = \chi(a)\chi(a')\chi(aa')^{-1} = 1,$$

we have $\tau_1 \in \operatorname{Hom}(E_\chi, \mathbf{C}^\times)_\chi$. This induces a mapping $\Phi : \Gamma(F_\chi)_\chi \to \operatorname{Hom}(E_\chi, \mathbf{C}^\times)_\chi$.

Conversely given $\tau_1 \in \operatorname{Hom}(E_\chi, \mathbf{C}^\times)_\chi$, we have $\pi^{-1}(f) \cap \ker \tau_1 \neq \emptyset$, since $\tau_1(\pi^{-1}(f)) = \mathbf{C}^\times$ for every $f \in F_\chi$. Taking an element $e_f \in \pi^{-1}(f) \cap \ker \tau_1$ for every $f \in F_\chi$, we have $\{e_f\}_{f \in F_\chi} \in \Gamma(F_\chi)_\chi$, since $e_f e_{f'} e_{ff'}^{-1} \in A$ and $\chi(e_f e_{f'} e_{ff'}^{-1}) = \tau_1(e_f e_{f'} e_{ff'}^{-1}) = 1$. Hence Φ is surjective. □

REMARK 5.3. The set of representatives $\{n_w \mid w \in W\}$ satisfies the condition in Proposition 5.2.

REMARK 5.4. (i) We may choose $\{u_\alpha \mid \alpha \in \Phi\}$ in such a way that every element n_α $(\alpha \in \Phi)$ lies in a compact real form of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ the Cartan subalgebra corresponding to T. For the root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$, let $\{H_\alpha, X_\alpha \mid H_\alpha \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Phi\}$ be a Chevalley basis. Then

$$\mathfrak{g}_c = \sum_{\alpha \in \Pi} \mathbf{R}(\sqrt{-1}H_\alpha) + \sum_{\alpha \in \Phi^+} \mathbf{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Phi^+} \mathbf{R}\sqrt{-1}(X_\alpha + X_{-\alpha})$$

is a compact real form. Let G_c be the compact real form of G corresponding to $\mathfrak{g}_c$, $T_c = T \cap G_c$ and N_c its normalizer in G_c. Then $W = N/T = N_c/T_c$.

Let us take

$$u_\alpha(x) = \exp(xX_\alpha).$$

Then (5.1)~(5.5) are satisfied ([17] 8.1, 9.3).

Let

$$X' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a Chevally basis of $\mathfrak{sl}(2, \mathbf{C})$ and $u'_\pm : \mathbf{C} \to SL(2, \mathbf{C})$ the mapping defined by

$$u'_+(x) = \exp(xX') = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad u'_-(x) = \exp(xY') = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Then we have

$$n' = u'_+(1)u'_-(-1)u'_+(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is an element of the compact real form $SU(2)$ of $SL(2, \mathbf{C})$.

For $\alpha \in \Phi^+$, there exists an isomorphism, sending H', X', Y' to H_α, X_α, $X_{-\alpha}$ respectively, of $\mathfrak{sl}(2, \mathbf{C})$ to the subalgebra of $\mathfrak{g}$ generated by H_α, X_α, $X_{-\alpha}$. This induces a homomorphism φ of $SL(2, \mathbf{C})$ to G such that $\varphi(SU(2)) \subset G_c$. Since

$$\begin{aligned} \varphi(n') &= \varphi(\exp(X')\exp(-Y')\exp(X')) \\ &= \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha) \\ &= n_\alpha, \end{aligned}$$

we have $n_\alpha \in G_c$. Hence the same statement of Theorem 5.1 is hold for a compact real form of G.

(ii) By (5.1) we have

$$tu_\alpha(x)t^{-1}u_\alpha(x)^{-1} = u_\alpha((\alpha(t) - 1)x).$$

Hence the image of u_α is in the commutator subgroup of G ([17] 7.3).

6. Branching from G to N: Reduction to Zero Weight Representations

The notations are as in §5. In this section we consider the restriction of a holomorphic representation of G to N.

Let ρ be an irreducible representation of G afforded by V. Let V_μ be the weight space of T on V corresponding to a dominant weight μ. Let N_μ and W_μ be the stabilizer subgroups of μ in N and W respectively.

Put $\Phi_\mu = \{\alpha \in \Phi \mid (\alpha, \mu) = 0\}$. Let P be the parabolic subgroup of G corresponding to $\Pi_\mu = \Pi \cap \Phi_\mu$ and G_P its Levi subgroup, which is reductive. The commutator subgroup G_P' of G_P is a semisimple Lie subgroup whose root system can be identified with Φ_μ.

Let

$$\rho|_{G_P} = \sum_{\nu \in \mathrm{Irr}(G_P)} c_\nu^\rho \nu$$

be the decomposition of the restriction of ρ to G_P into irreducible representations of G_P.

LEMMA 6.1. *If the weight μ is a weight of an irreducible representation ν in $\rho|_{G_P}$, then μ is the only weight of ν that W_μ fixes.*

PROOF. Suppose W_μ fixes a weight μ' of ν. Let ν be afforded by V^ν. By the irreducibility of ν, V^ν is spanned by a weight vector of weight μ under the action of the Lie algebra $\mathfrak{g}_P'$ of G_P'. Then μ' must be in the form

$$\mu + \sum_{\alpha \in \Pi_\mu} m_\alpha \alpha.$$

Thus W_μ fixes $\sum_{\alpha \in \Pi_\mu} m_\alpha \alpha (\in \mathbf{Z}\Phi_\mu)$, which must be 0. Thus we have $\mu = \mu'$. $\square$

Put

$$\rho_\mu = \sum_{\nu \in M_\mu} c_\nu^\rho \nu, \tag{6.1}$$

where M_μ is the set of irreducible representations of G_P appearing in $\rho|_{G_P}$ of which μ is a weight.

PROPOSITION 6.2. *The weight space V_μ of weight μ in ρ is the zero weight space of ρ_μ as a G_P'-module.*

PROOF. Since the zero weights of G_P'-module are those that are fixed by the action of the Weyl group W_μ of G_P', V_μ is the zero weight space of ρ_μ by Lemma 6.1. $\square$

Since the Weyl group acts on the zero weight space, V_μ can be regarded as a W_μ-module. Let

$$\nu_0 = \sum_{\sigma \in \mathrm{Irr}(W_\mu)} d^0_{\nu,\sigma}\sigma \tag{6.2}$$

be the decomposition of the zero weight space of $\nu|_{G'_P}$ into the irreducible representations of W_μ. Then $(\rho_\mu)_0$ is decomposed as a W_μ-module:

$$\begin{aligned}(\rho_\mu)_0 &= \sum_{\nu \in M_\mu} c^\rho_\nu \nu_0 \\ &= \sum_{\substack{\nu \in M_\mu \\ \sigma \in \mathrm{Irr}(W_\mu)}} c^\rho_\nu d^0_{\nu,\sigma}\sigma. \end{aligned} \tag{6.3}$$

If we take the complete set of representatives

$$\{\theta_{\mu,\tau(\mu,\sigma)} \mid \mu \in \Lambda^+, \sigma \in \mathrm{Irr}(W_\mu)\}$$

of the equivalence classes of irreducible representations of N, as given in Theorem 5.1 using $\{n_w \mid w \in W\}$, then we have the decomposition of the restriction of ρ to N.

Theorem 6.3. *Let G be a connected complex simple Lie group and T, N, W as in §5. If ρ is an irreducible representation of G, then the restriction of ρ to N decomposes as*

$$\rho|_N = \sum_{\mu \in \Lambda^+_\rho} \sum_{\sigma \in \mathrm{Irr}(W_\mu)} d_{\mu,\tau(\mu,\sigma)}\theta_{\mu,\tau(\mu,\sigma)},$$

where Λ^+_ρ is the set of dominant weights of ρ and

$$d_{\mu,\tau(\mu,\sigma)} = \sum_{\nu \in M_\mu} c^\rho_\nu d^0_{\nu,\sigma}$$

with M_μ, c^ρ_ν defined in (6.1) *and $d^0_{\nu,\sigma}$ in* (6.2).

Proof. Let U be a $(\rho_\mu)_0(W_\mu)$-invariant subspace of V_μ affording the irreducible representation $\sigma \in \mathrm{Irr}(W_\mu)$. Note that N_μ is generated by T and the elements n_w with $w \in W_\mu$ and since $n_w \in G'_P$ for all $w \in W_\mu$ by Remark 5.4 (ii), the action of $\rho(n_w)$ on V_μ is the same as that of $(\rho_\mu)_0(w)$. It follows that U is also invariant under $\rho(N_\mu)$ and affords the representation of $\tau(\mu,\sigma) \in \mathrm{Irr}(N_\mu)_\mu$ by Theorem 5.1 (i). Therefore we have

$$V_\mu \simeq \bigoplus_{\sigma \in \mathrm{Irr}(W_\mu)} U_{\mu,\tau(\mu,\sigma)}^{\oplus d_{\mu,\tau(\mu,\sigma)}}$$

as an N_μ-module with the numbers $d_{\mu,\tau(\mu,\sigma)}$ defined as in the statement.

Then by comparing the construction of the induced representation (2.3) and

$$\rho(N)V_\mu = \bigoplus_{g \in N/N_\mu} \rho(g)V_\mu,$$

we have

$$\rho(N)V_\mu \simeq \bigoplus_{\sigma \in \mathrm{Irr}(W_\mu)} V_{\mu,\tau(\mu,\sigma)}^{\oplus d_{\mu,\tau(\mu,\sigma)}}.$$

The results follows. □

There are two extremal cases.

EXAMPLE 6.4. (i) Let μ be the highest weight of ρ and $\rho' = d\rho$ the differential of ρ. Let $\alpha \in \Phi^+$ be a root such that $(\mu,\alpha) = 0$ and $\mathfrak{a}$ the subalgebra of $\mathfrak{g}$ generated by X_α, H_α, $X_{-\alpha}$. (see Remark 5.4(i)). By the representation theory of $sl(2,\mathbf{C})$, we have $\dim \rho'(\mathfrak{a})v_\mu = (\mu,\alpha) + 1 = 1$ for a highest weight vector v_μ and hence $\rho'(X_\alpha)v_\mu = \rho'(X_{-\alpha})v_\mu = 0$. Since $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$, we have $\rho(n_\alpha)v_\mu = v_\mu$. It follows from Theorem 5.1 that $\rho(N)V_\mu$ affords $\theta_{\mu,\tau(\mu,1)}$.

(ii) If $N_\mu = T$, e.g. μ is regular, then W_μ is trivial, and hence we have

$$\rho(N_\mu)V_\mu = V_\mu = \mathbf{C}_\mu^{\oplus \dim V_\mu},$$

where $\mathbf{C}_\mu$ is an irreducible T-module with weight μ. Thus $\rho(N)V_\mu$ affords

$$\theta_{\mu,\tau(\mu,1)}^{\oplus \dim V_\mu}.$$

We next consider the case where N is a subgroup of some semidirect product group $\tilde{N}$ of a complex torus $\tilde{T}$ by W such that $T \subset \tilde{T}$ and the assumption of Theorem 3.3(iii) is satisfied: for any irreducible character μ of T, there exists an irreducible character $\tilde{\mu}$ of $\tilde{T}$ satisfying $\tilde{\mu}|_T = \mu$ and $W_{\tilde{\mu}} = W_\mu$.

Having chosen such $\tilde{\mu}$ for each μ, it follows from Theorem 3.3(iii) that a complete set of representatives of the equivalence classes of irreducible representations of N is given by $\{\theta_{\tilde{\mu},\tilde{\mu}\rtimes\sigma}|_N \mid \mu \in \Lambda^+, \sigma \in \mathrm{Irr}(W_\mu)\}$.

Let $\mu \in \Lambda^+$ and $\sigma' \in \mathrm{Irr}(W_\mu)$. Then it follows from Theorem 3.3 that there is a unique $\sigma \in \mathrm{Irr}(W_{\tilde{\mu}})$ (recall our assumption that $W_{\tilde{\mu}} = W_\mu$) such that the representation $\tilde{\mu} \rtimes \sigma$ of $\tilde{N}_{\tilde{\mu}}$ restricts to the representation $\tau(\mu, \sigma')$ of N_μ, so that $\theta_{\tilde{\mu},\tilde{\mu}\rtimes\sigma}|_N = \theta_{\mu,\tau(\mu,\sigma')}$. We want to compare this σ with σ'.

Recall that $\tilde{\mu} \rtimes \sigma = \tilde{\tau}_1 \otimes \hat{\sigma}$ where $\tilde{\tau}_1(w\tilde{t}) = \tilde{\mu}(\tilde{t})$ $(w \in W_{\tilde{\mu}} = W_\mu, \tilde{t} \in \tilde{T})$ and $\hat{\sigma} = \sigma \circ (\tilde{\pi}|_{\tilde{N}_{\tilde{\mu}}})$ (Remark 2.4), and that $\tau(\mu, \sigma') = \tau_1 \otimes \widehat{\sigma'}$ where $\tau_1(n_w t) = \mu(t)$ $(w \in W_\mu, t \in T)$ and $\widehat{\sigma'} = \sigma' \circ (\pi|_{N_\mu})$. Since $\tilde{\mu} \rtimes \sigma|_{N_\mu} = \tau(\mu, \sigma')$, namely $(\widetilde{\tau_1}|_{N_\mu}) \otimes \hat{\sigma} = \tau_1 \otimes \widehat{\sigma'}$, the difference between σ and σ' is determined by the difference between $\widetilde{\tau_1}|_{N_\mu}$ and τ_1. We have

$$\widetilde{\tau_1}(n_w t)\tau_1(n_w t)^{-1} = \widetilde{\tau_1}(w\tilde{\epsilon}_w t)\tau_1(n_w t)^{-1} = \tilde{\mu}(\tilde{\epsilon}_w t)\mu(t)^{-1} = \tilde{\mu}(\tilde{\epsilon}_w),$$

which we have defined to be $e_{\tilde{\mu}}(w)$ ((3.3)). Note that the function $e_{\tilde{\mu}}|_{W_\mu}$ is a linear character since, by putting $t = 1$ we have

$$e_{\tilde{\mu}}(w) = \tilde{\tau}_1(n_w)\tau_1(n_w)^{-1} = \tilde{\tau}_1(n_w),$$

and if $w, w' \in W_\mu$ then

$$n_w n_{w'} n_{ww'}^{-1} \in \ker \tau_1 \cap T = \ker \mu \subset \ker \tilde{\mu} \subset \ker \tilde{\tau}_1.$$

Hence we have

$$\tilde{\tau}_1(n_w t) = e_{\tilde{\mu}}(w)\tau_1(n_w t), \quad w \in W_\mu,\ t \in T,$$

namely

$$\tilde{\tau}_1|_{N_\mu} = \widehat{e_{\tilde{\mu}}|_{W_\mu}} \otimes \tau_1.$$

By what we remarked just above, this shows that $\sigma' = \sigma \otimes (e_{\tilde{\mu}}|_{W_\mu})$.

Summarizing the above, we have the following:

Theorem 6.5. *Let G be a connected complex simple Lie group and T, N, W as in* §5. *Let $\tilde{N}$ be a semidirect product of a complex torus $\tilde{T}$ by W such that $\tilde{T}$ contains T and Assumption* 3.1 *is satisfied. Suppose that the assumption of Theorem* 3.3(iii) *is satisfied for $\tilde{N}$, $\tilde{T}$, N, T, W and fix the choice of an irreducible character $\tilde{\mu}$ of $\tilde{T}$ for every irreducible character μ of T with $\tilde{\mu}|_T = \mu$ and $W_{\tilde{\mu}} = W_\mu$. If $\theta_{\tilde{\mu},\tilde{\mu}\rtimes\sigma}|_N = \theta_{\mu,\tau(\mu,\sigma')}$, then*

$$\sigma' = \sigma \otimes (e_{\tilde{\mu}}|_{W_\mu}).$$

If ρ is an irreducible representation of G, then the restriction of ρ to N decomposes as

$$\rho|_N = \sum_{\mu \in \Lambda_\rho^+} \sum_{\sigma \in \mathrm{Irr}(W_\mu)} d_{\mu,\tau(\mu,\sigma)} \theta_{\mu,\tau(\mu,\sigma)},$$

where Λ_ρ^+ is the set of dominant weights of ρ and

$$d_{\mu,\tilde{\mu}\rtimes\sigma|_{N_\mu}} = d_{\mu,\tau(\mu,\sigma\otimes(e_{\tilde{\mu}}|_{W_\mu}))} = \sum_{\nu \in M_\mu} c_\nu^\rho d^0_{\nu,\sigma\otimes(e_{\tilde{\mu}}|_{W_\mu})}$$

with M_μ, c_ν^ρ defined in (6.1) *and $d^0_{\nu,\sigma}$ in* (6.2).

EXAMPLE 6.6. Let $G = SL(n, \mathbf{C})$. We consider the situation as in Example 3.5. For $w \in \mathfrak{S}_n$, let $n_w \in N$ be a representative as in §5 whose non-zero entries are ± 1:

$$n_w = \tilde{n}_w \tilde{t}_w,$$

where

$$\tilde{t}_w = \mathrm{diag}(\epsilon_1, \dots, \epsilon_n), \quad \epsilon_i = (-1)^{\#\{j \mid 1 \le j \le n,\, i<j,\, w(i)>w(j)\}}$$

and $\tilde{n}_w$ is the permutation matrix such that

$$\tilde{n}_w^{-1} \mathrm{diag}(t_1, \dots, t_n) \tilde{n}_w = \mathrm{diag}(t_{w(1)}, \dots, t_{w(n)}).$$

Then we have (see (3.1))

$$\tilde{\epsilon}_w = \tilde{t}_w.$$

Furthermore since

$$\begin{aligned} \det(n_w) &= \det(\tilde{n}_w) \det(\tilde{t}_w) \\ &= \mathrm{sgn}(w) \det(\tilde{t}_w) \end{aligned}$$

and $\det(n_w) = 1$, we have

$$\det(\tilde{t}_w) = \mathrm{sgn}(w).$$

For a character μ of T,

$$\mu(t) = \prod_{i=1}^n t_i^{m_i},$$

we have chosen $\tilde{\mu}$ by setting

$$\tilde{\mu}(\tilde{t}) = \left(\prod_{i=1}^n \tilde{t}_i^{m_i} \right) \det(\tilde{t})^k$$

for some integer k. In particular if μ is trivial, then $G_P' = G$, $N_\mu = N$, $W_\mu = \mathfrak{S}_n$ and $\tilde{\mu} = \det(\tilde{t})^k$. Then we have

$$e_{\tilde{\mu}}(w) = \tilde{\mu}(\tilde{t}_w) = \det(\tilde{t}_w)^k = \operatorname{sgn}(w)^k.$$

Hence the character $e_{\tilde{\mu}}$ is $\operatorname{sgn}^{\otimes k}$ (see [2] Lemma 1.2).

7. The Case of $SL(n, \mathbf{C})$

In this section, we consider a special class of simple Lie groups, $SL(n, \mathbf{C})$. We regard $G = SL(n, \mathbf{C})$ as a subgroup of $\tilde{G} = GL(n, \mathbf{C})$.

Let $\tilde{T}$ and T be the groups of all diagonal matrices in $\tilde{G}$ and G respectively. The Weyl groups W of $\tilde{G}$ and G are both isomorphic to the symmetric group $\mathfrak{S}_n$ of degree n. The normalizer $\tilde{N}$ (resp. N) of $\tilde{T}$ (resp. T) is the group of matrices of $\tilde{G}$ (resp. G) that have precisely one non-zero entry in each row and column. Then $\tilde{N}$ is the semidirect product of $\tilde{T}$ by W.

Take the subgroup of all upper triangular matrices as a Borel subgroup of $\tilde{G}$ and G. The equivalence classes of irreducible representations of $\tilde{G}$ are parametrized by Young diagrams with at most n rows in such a way that every Young diagram $(m_1, \dots, m_n)$ corresponds to the representation with highest weight $\tilde{\rho}$ given by $\tilde{\rho}(\tilde{t}) = \tilde{t}_1^{m_1} \cdots \tilde{t}_n^{m_n}$ for $\tilde{t} = \operatorname{diag}(\tilde{t}_1, \dots, \tilde{t}_n)$.

Every irreducible representation of G is obtained by restricting an irreducible representation of $\tilde{G}$. Let det be the linear representation of $\tilde{G}$ defined by taking the determinant of matrices. For an irreducible representation $\tilde{\rho}$ of $\tilde{G}$ corresponding to a Young diagram $\tilde{\rho} = (\tilde{\rho}_1, \dots, \tilde{\rho}_n)$ (we use a same symbol for a representation and the corresponding Young diagram) and an integer k, the irreducible representation $(\det)^{\otimes k} \otimes \tilde{\rho}$ corresponds to the Young diagram $\tilde{\rho}[k] = (\tilde{\rho}_1 + k, \dots, \tilde{\rho}_n + k)$. The restrictions of the representations $\tilde{\rho}[k]$ to G define the same irreducible representation of G for any $k \in \mathbf{Z}$. Thus the irreducible representations of G are parametrized by Young diagram with at most $n-1$ rows.

Convention 7.1. In the sequel, unless otherwise stated, we use the notation $\tilde{\rho}$, for a given irreducible representation ρ of G, to denote the uniquely determined Young diagram with at most $n-1$ rows such that $\tilde{\rho}|_G = \rho$.

Let $\tilde{\rho}$ be an irreducible representation of $\tilde{G}$ corresponding to a Young diagram $\tilde{\rho}$ with at most $n-1$ rows. Let

$$\tilde{\mu} = a_1\omega_{i_1} + \cdots + a_{r-1}\omega_{i_{r-1}} + a_r\omega_n$$

$$(a_1, \dots, a_{r-1} > 0, a_r \geq 0, 1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n-1),$$

be a dominant weight of $\tilde{\rho}$ such that $|\tilde{\mu}| = |\tilde{\rho}|$, where ω_i is a weight corresponding to Young diagram (1^i) and $|\lambda|$ denotes the sum of the parts λ_i of a Young diagram $\lambda = (\lambda_1, \dots, \lambda_n)$. The stabilizer $W_{\tilde{\mu}}$ of $\tilde{\mu}$ is a direct product of symmetric groups:

$$W_{\tilde{\mu}} = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}, \tag{7.1}$$

where $n_1 = i_1, n_2 = i_2 - i_1, \dots, n_r = n - i_{r-1}$. Let $\tilde{P}$ be the corresponding parabolic subgroup of $\tilde{G}$ containing the Borel subgroup. The Levi subgroup $\tilde{G}_{\tilde{P}}$ of $\tilde{P}$ is a product of $GL(n_i, \mathbf{C})$:

$$\tilde{G}_{\tilde{P}} = GL(n_1, \mathbf{C}) \times \cdots \times GL(n_r, \mathbf{C}). \tag{7.2}$$

Let

$$\tilde{\rho}|_{\tilde{G}_{\tilde{P}}} = \sum_{\boldsymbol{v}} c_{\boldsymbol{v}}^{\tilde{\rho}} \boldsymbol{v} \tag{7.3}$$

be the decomposition of the restriction of $\tilde{\rho}$ to $\tilde{G}_{\tilde{P}}$ into irreducible representations $\boldsymbol{v}$ of $\tilde{G}_{\tilde{P}}$. Every irreducible representation $\boldsymbol{v}$ of $\tilde{G}_{\tilde{P}}$ is given by a set of Young diagrams v^i, $1 \leq i \leq r$, with at most n_i rows, which correspond to irreducible representations of $GL(n_i, \mathbf{C})$. The summation is taken over v^i's with $\sum_{i=1}^r |v^i| = |\tilde{\rho}|$. The coefficient $c_{\boldsymbol{v}}^{\tilde{\rho}}$ is given by the Littlewood-Richardson rule:

$$c_{\boldsymbol{v}}^{\tilde{\rho}} = \sum_{(\kappa^1, \dots, \kappa^{r-2})} c_{v^1 \kappa^1}^{\tilde{\rho}} c_{v^2 \kappa^2}^{\kappa^1} \cdots c_{v^{r-1} v^r}^{\kappa^{r-2}}, \tag{7.4}$$

where $c_{\mu, v}^{\rho}$ is the Littlewood-Richardson coefficient. We call this a multiple Littlewood-Richardson coefficient.

By (7.1), $\tilde{T}$ acts on the weight space of $\tilde{\mu} = a_1 \omega_{i_1} + \cdots + a_{r-1} \omega_{i_{r-1}} + a_r \omega_n$ as the multiplication by

$$(\tilde{t}_1 \cdots \tilde{t}_{n_1})^{f_1} \cdots (\tilde{t}_{n-n_r+1} \cdots \tilde{t}_n)^{f_r}, \quad \mathrm{diag}(\tilde{t}_1, \dots, \tilde{t}_n) \in \tilde{T}, \tag{7.5}$$

where $f_i = a_i + \cdots + a_r$, $1 \leq i \leq r$. That is, the multiplicities of f_i is equal to n_i in the Young diagram $\tilde{\mu}$.

Thus, by the definition of $\tilde{\rho}_{\tilde{\mu}}$ (see (6.1)), we have

$$\tilde{\rho}_{\tilde{\mu}} = \sum_{\substack{\boldsymbol{v} = (v^1, \dots, v^r) \\ |v^i| = f_i n_i}} c_{\boldsymbol{v}}^{\tilde{\rho}} \boldsymbol{v}.$$

Let $\rho = \tilde{\rho}|_G$ and μ be the dominant weight of ρ with $\tilde{\mu}|_T = \mu$. Note that the correspondence of the weights $\tilde{\mu}$ of $\tilde{\rho}$ to the weights μ of ρ is one-to-one, since the difference between the highest weight of $\tilde{\rho}$ (resp. ρ) and a weight $\tilde{\mu}$ of $\tilde{\rho}$ (resp. a weight μ of ρ) is a linear combination of roots and $|\tilde{\mu}| = |\tilde{\rho}|$. Then $G_P = \tilde{G}_{\tilde{P}} \cap G$ and

$$G'_P = SL(n_1, \mathbf{C}) \times \cdots \times SL(n_r, \mathbf{C}).$$

Now we have the main result in this section.

Theorem 7.2. *Let ρ be an irreducible representation of $SL(n, \mathbf{C})$ and $\tilde{\rho}$ as in Convention* 7.1. *For a weight μ of ρ, let $\tilde{\mu}$ be the weight of $\tilde{\rho}$ such that $|\tilde{\mu}| = |\tilde{\rho}|$ and $\tilde{\mu}|_T = \mu$. Then the restriction of ρ to N decomposes as*

$$\rho|_N = \sum_{\mu \in \Lambda_\rho^+} \sum_{\tau \in \mathrm{Irr}(N_\mu)} d_{\mu,\tau} \theta_{\mu,\tau},$$

where

$$d_{\mu, \tilde{\mu} \rtimes \sigma|_{N_\mu}} = d_{\mu, \tau(\mu, \sigma')} = \sum_{\substack{\boldsymbol{\nu} = (\nu^1, \dots, \nu^r) \\ |\nu^i| = f_i n_i}} c_{\boldsymbol{\nu}}^{\tilde{\rho}} d_{\boldsymbol{\nu}, \sigma'}^0,$$

$$\sigma' = \sigma \otimes (\mathrm{sgn}^{\otimes f_1} \times \cdots \times \mathrm{sgn}^{\otimes f_r})$$

with $(f_1, \dots, f_r)$ *defined in* (7.5) *for* $\tilde{\mu}$.

Proof. By (7.5) and Example 6.6, we have

$$e_{\tilde{\mu}}(w) = \mathrm{sgn}(w_1)^{f_1} \cdots \mathrm{sgn}(w_r)^{f_r},$$

for an element $w = w_1 \cdots w_r \in W_{\tilde{\mu}}$ $(w_i \in \mathfrak{S}_{n_i})$. Thus we have

$$\sigma \otimes (e_{\tilde{\mu}}|_{W_\mu}) = \sigma \otimes (\mathrm{sgn}^{\otimes f_1} \times \cdots \times \mathrm{sgn}^{\otimes f_r}).$$

The result follows from Theorem 6.5. □

8. Explicit Formula for the Branching from $SL(n, \mathbf{C})$ to N (I)

We give a multiplicity formula of an irreducible representation of N in a given irreducible representation of G for the case $G = SL(n, \mathbf{C})$. The notation is as in section 7.

Let $\boldsymbol{\nu} = (\nu^1, \dots, \nu^r)$, $|\nu^i| = f_i n_i$, be an irreducible representation of $\tilde{G}_{\tilde{P}}$. Let $\zeta_{(\nu^i)_0}$ be the character of the representation of $\mathfrak{S}_{n_i}$ induced on the zero weight space of $\nu^i|_{SL(n_i, \mathbf{C})}$. The character value of $\zeta_{(\nu^i)_0}$ is calculated as follows. Let w_i be an element of $\mathfrak{S}_{n_i}$ with cyclic factorization $w_i = w_{i1} \cdots w_{ik_i}$. Let l_{ij} be the order of w_{ij}, then for every i, the set $\{l_{ij}\}$ form a uniquely determined partition π_i of n_i.

Let

$$H_i = GL(l_{i1}, \mathbf{C}) \times \cdots \times GL(l_{ik_i}, \mathbf{C})$$

and

$$v^i|_{H_i} = \sum_{\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^{k_i})} c_{\boldsymbol{\lambda}}^{v^i} \boldsymbol{\lambda}, \tag{8.1}$$

where $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^{k_i})$ is a sequence of Young diagrams λ^j with at most l_{ij} rows, which corresponds to an irreducible representation of $GL(l_{ij}, \mathbf{C})$. Then we have ([2] §4)

$$\zeta_{(v^i)_0}(w_i) = \sum_{\substack{\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^{k_i}) \\ |\lambda^j|=l_{ij}f_i}} c_{\boldsymbol{\lambda}}^{v^i} \operatorname{sgn}(w_i)^{f_i k_i} \eta_{(\lambda^1)_0}(w_{i1}) \cdots \eta_{(\lambda^{k_i})_0}(w_{ik_i}), \tag{8.2}$$

where $\eta_{(\lambda^j)_0}$ is the character of the representation of $\mathfrak{S}_{l_{ij}}$ induced on the zero weight space of representation $\lambda^j|_{SL(l_{ij}, \mathbf{C})}$.

Let x_{ij} be the permutation matrix of $GL(l_{ij}, \mathbf{C})$ corresponding to the cycle w_{ij}. Since w_{ij} is an l_{ij}-cycle, w_{ij} is a Coxeter element of the Weyl group $\mathfrak{S}_{l_{ij}}$ of $SL(l_{ij}, \mathbf{C})$. Hence the trace of $\lambda^j(x_{ij})$ is equal to $\operatorname{sgn}(w_{ij})^{f_i} \eta_{(\lambda^j)_0}(w_{ij})$. In [9], it is shown that the trace of $\lambda^j(x_{ij}) = 0$ or ± 1.

The trace of $\lambda^j(x_{ij})$ is also calculated by the generalized q-binomial coefficients as follows. For an integer m and a Young diagram $\gamma = (\gamma_1, \dots, \gamma_s)$, the generalized q-binomial coefficients in the indeterminate q is defined to be

$$\begin{bmatrix} m \\ \gamma' \end{bmatrix} = \prod_{x \in \gamma} \frac{1 - q^{m-c(x)}}{1 - q^{h(x)}},$$

where $\gamma' = (\gamma'_1, \dots, \gamma'_t)$ is the transpose of γ, $c(x) = j - i$ is the content and $h(x) = \gamma_i + \gamma'_j - i - j + 1$ is the hook length for each $x = (i, j) \in \gamma$ (here we regard γ as a matrix. See e.g. [10] Chap. I §1, Ex. 3, §3, Ex. 1). It is shown that the generalized q-binomial coefficient is a polynomial in q and

$$s_\gamma(1, q, q^2, \dots, q^{m-1}) = q^{n(\gamma)} \begin{bmatrix} m \\ \gamma' \end{bmatrix},$$

where s_γ is the Schur function of m variables corresponding to γ and $n(\gamma) = \sum_{i=1}^{s} (i-1)\gamma_i$.

On the other hand, s_γ is the restriction to T of the irreducible character of $GL(m, \mathbf{C})$ corresponding to γ and the permutation matrix corresponding to a cycle of order m is conjugate to the diagonal matrix $\operatorname{diag}(1, \omega, \omega^2, \dots, \omega^{m-1})$, $\omega = e^{2\pi\sqrt{-1}/m}$. Thus we have

$$\operatorname{sgn}(w_{ij})^{f_i} \eta_{(\lambda^j)_0}(w_{ij}) = q^{n(\lambda^j)} \begin{bmatrix} l_{ij} \\ \lambda^{j\prime} \end{bmatrix} \Bigg|_{q=\exp(2\pi\sqrt{-1}/l_{ij})}. \tag{8.3}$$

Theorem 8.1. *For an irreducible representation ρ of G and an irreducible representation $\theta_{\mu,\tau}$ of N, the multiplicity $[\rho|_N : \theta_{\mu,\tau}]$ is*

$$\frac{1}{|W_\mu|} \sum_{\substack{\boldsymbol{v}=(v^1,\dots,v^r) \\ |v^i|=f_i n_i}} c_{\boldsymbol{v}}^{\tilde{\rho}} \sum_{\substack{w=w_1\cdots w_r \in W_\mu \\ w_i \in \mathfrak{S}_{n_i}}} \phi_{\sigma'}(w) \prod_{i=1}^{r} \left(\sum_{\substack{\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^{k_i}) \\ |\lambda^j|=l_{ij} f_i}} c_{\boldsymbol{\lambda}}^{v^i} \prod_{j=1}^{k_i} b(\lambda^j, l_{ij}) \right),$$

where $\tilde{\rho}$ is determined from ρ as in Convention 7.1, *$W_\mu = \mathfrak{S}_{n_1} \times \cdots \mathfrak{S}_{n_r}$ is the stabilizer of μ in W, $c_{\boldsymbol{v}}^{\tilde{\rho}}$ and $c_{\boldsymbol{\lambda}}^{v^i}$ are multiple Littlewood-Richardson coefficients* ((7.4), (8.1)), *$\phi_{\sigma'}$ is the character of irreducible representation*

$$\sigma' = (\operatorname{sgn}^{\otimes f_1} \times \cdots \times \operatorname{sgn}^{\otimes f_r}) \otimes \sigma$$

of W_μ, the integer f_i is given in (7.5), *σ is defined from τ as in Theorem* 5.1, *l_{ij} is the order of cycle of w_{ij}, and $b(\lambda^j, l_{ij})$ is the integer given by* (8.3).

Proof. Let $\zeta_{(\boldsymbol{v})_0}$ be the character of W_μ-module $\boldsymbol{v}_0$. By Theorem 7.2 we have

$$\begin{aligned}
[\rho|_N : \theta_{\mu,\tau}] &= \sum_{\substack{\boldsymbol{v}=(v^1,\dots,v^r) \\ |v^i|=f_i n_i}} c_{\boldsymbol{v}}^{\tilde{\rho}} [(\boldsymbol{v})_0 : \sigma'] \\
&= \sum_{\substack{\boldsymbol{v}=(v^1,\dots,v^r) \\ |v^i|=f_i n_i}} c_{\boldsymbol{v}}^{\tilde{\rho}} (\zeta_{(\boldsymbol{v})_0}, \phi_{\sigma'}) \\
&= \sum_{\substack{\boldsymbol{v}=(v^1,\dots,v^r) \\ |v^i|=f_i n_i}} c_{\boldsymbol{v}}^{\tilde{\rho}} \frac{1}{|W_\mu|} \sum_{w \in W_\mu} \zeta_{(\boldsymbol{v})_0}(w) \phi_{\sigma'}(w) \\
&= \sum_{\substack{\boldsymbol{v}=(v^1,\dots,v^r) \\ |v^i|=f_i n_i}} c_{\boldsymbol{v}}^{\tilde{\rho}} \frac{1}{|W_\mu|} \sum_{\substack{w=w_1\cdots w_r \in W_\mu \\ w_i \in \mathfrak{S}_{n_i}}} \phi_{\sigma'}(w) \prod_{i=1}^{r} \zeta_{(v^i)_0}(w_i).
\end{aligned}$$

By (8.2) and (8.3), we have the result. $\square$

9. Explicit Formula for the Branching from $SL(n,\mathbf{C})$ to N (II)

Let G_c be a compact real form of a complex semisimple Lie group G, T_c a maximal torus of G_c and N_c its normalizer in G_c. Let T be the maximal torus of G containing T_c and N its normalizer in G. Then $T_c = T \cap G_c$ and the Weyl group $W = N/T$ is isomorphic to N_c/T_c. Every finite dimensional irreducible continuous representation of G_c (resp. T_c) is obtained from a finite dimensional irreducible holomorphic representation of G (resp. T) by restriction. This gives a one-to-one correspondence between these representations of G_c and G (resp. T_c and T). Note that our representatives n_w of the Weyl group in N actually lie in G_c (see Remark 5.4 (i)). By the construction of irreducible representations of N and N_c, we have a one-to-one correspondence between the equivalence classes of irreducible representations of N and those of N_c by restriction.

In this section we give an explicit formula for the branching from $SL(n,\mathbf{C})$ to N in a way different from that of the previous section.

Let $\tilde{G}$, $\tilde{N}$, $\tilde{T}$ be as in §7. Let $\tilde{N}_c = \tilde{N} \cap U(n)$ and $\tilde{T}_c = \tilde{T} \cap U(n)$. $\tilde{N}_c$ is a semidirect product of $\tilde{T}_c$ by $\mathfrak{S}_n$. Then we have

$$\mathrm{Irr}(\tilde{N}_c) = \{\theta_{\tilde{\mu},\tilde{\mu}\rtimes\sigma} \mid \tilde{\mu} \in \tilde{\Lambda}^+, \sigma \in \mathrm{Irr}(W_{\tilde{\mu}})\},$$

where $\tilde{\Lambda}^+$ is the set of all characters of $\tilde{T}_c$ which are dominant with respect to Π. Every irreducible representation of $U(n)$ is afforded by an irreducible module of $GL(n,\mathbf{C})$ and every irreducible representation of $\tilde{N}_c$ is also afforded by an irreducible module of $\tilde{N}$ by taking the restriction. Hence the multiplicities $[\tilde{\rho}|_{\tilde{N}} : \theta_{\tilde{\mu},\tilde{\tau}}]$ are equal to those for $U(n)$ and $\tilde{N}_c$.

Since we work entirely within $U(n)$ in this section, we replace $\tilde{T}_c$, $\tilde{N}_c$, $\tilde{\mu}$, $\tilde{\tau}$ by T, N, μ, τ to avoid complicated notation.

Let dn be the Haar measure on N normalized by $\int_N dn = 1$. Let $\psi_{\mu,\tau}$ be the character of $\theta_{\mu,\tau}$ and S_ρ the character of irreducible representation ρ of $U(n)$, whose restriction to T is Schur function s_ρ. Then we have

$$[\rho|_N : \theta_{\mu,\tau}] = \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} \int_T \psi_{\mu,\tau}(n_w t)\overline{S_\rho(n_w t)}\, dt, \tag{9.1}$$

where $n_w t \in N$ and dt is the normalized Haar measure on T.

By Proposition 4.1, we have

$$\psi_{\mu,\tau}(n_w t) = \frac{1}{|W_\mu|} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0) \left(\sum_{x \in \Phi_w^{-1}(w_0)} \mu(t^x) \right), \tag{9.2}$$

where $\tau = \mu \rtimes \sigma$. Since $S_\rho(n_w t)$ is the trace of $\rho(n_w t)$, there appear only the terms corresponding to the weights fixed by w:

$$S_\rho(n_w t) = \sum_{\substack{v=(i_1,\dots,i_n)\\ w(v)=v}} c_v t_1^{i_1} \cdots t_n^{i_n}, \tag{9.3}$$

where c_v is the trace of $\rho(n_w)$ on the weight space V_v of weight v.

Since

$$\int_T (t_1^{a_1} \cdots t_n^{a_n})\overline{(t_1^{b_1} \cdots t_n^{b_n})}\, dt = \begin{cases} 1 & a_i = b_i \ (1 \le i \le n) \\ 0 & \text{otherwise}, \end{cases} \tag{9.4}$$

we have

$$\int_T \mu(t^x)\overline{S_\rho(n_w t)}\, dt = c_{x\cdot\mu}. \tag{9.5}$$

For $x \in \Phi_w^{-1}(w_0)$, w stabilizes the weight $x \cdot \mu$, since $x^{-1}wx = w_0 \in W_\mu$ and hence $(wx)\cdot\mu = (xw_0)\cdot\mu = x\cdot\mu$. If $x, x' \in \Phi_w^{-1}(w_0)$, then $x' = zx$ for some element z of the centralizer of w. Since the action of z transforms $V_{x\cdot\mu}$ to $V_{x'\cdot\mu}$ and this transformation commutes with the action of w, the trace of $\rho(w)$ on $V_{x\cdot\mu}$ is equal to that on $V_{x'\cdot\mu}$. Hence we have

$$c_{x\cdot\mu} = c_{x'\cdot\mu}.$$

We denote this by $m(\rho, \mu; w, w_0)$. Then by (9.1), (9.2), (9.5) and (4.3) we have

$$\begin{aligned} [\rho|_N : \theta_{\mu,\tau}] &= \frac{1}{n!}\frac{1}{|W_\mu|}\sum_{w \in \mathfrak{S}_n}\sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0)|Z_{\mathfrak{S}_n}(w)|m(\rho,\mu;w,w_0) \\ &= \sum_{w \in \mathfrak{S}_n} \frac{|Z_{\mathfrak{S}_n}(w)|}{n!|W_\mu|} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0) m(\rho,\mu;w,w_0) \end{aligned} \tag{9.6}$$

The problem is now reduced to determination of $m(\rho, \mu; w, w_0)$. Since an element $n_w t \in N$ is diagonalizable, $m(\rho, \mu; w, w_0)$ is equal to the Schur function evaluated at the eigenvalues $\varepsilon_{w,t} = (\varepsilon_1, \dots, \varepsilon_n)$ of $n_w t$. To calculate this we use the Jacobi-Trudy identity:

$$\begin{aligned} s_\rho &= \det(h_{\rho_i - i + j})_{1 \le i,j \le n} \\ &= \begin{vmatrix} h_{\rho_1} & h_{\rho_1+1} & \cdots & h_{\rho_1-1+n} \\ h_{\rho_2-1} & h_{\rho_2} & \cdots & h_{\rho_2-2+n} \\ \vdots & & \ddots & \vdots \\ h_{\rho_n-n+1} & h_{\rho_n-n+2} & \cdots & h_{\rho_n} \end{vmatrix}, \end{aligned} \tag{9.7}$$

where $\rho = (\rho_1, \dots, \rho_n)$ and h_i is the ith complete symmetric function: the sum of all distinct monomials of degree i.

We have

$$\begin{aligned} m(\rho, \mu; w, w_0) &= \int_T \mu(t^x)\overline{S_\rho(n_w t)}\, dt \\ &= \int_T \mu(t^x)\left(\overline{\sum_{\gamma \in \mathfrak{S}_n} \operatorname{sgn}(\gamma) h_{\rho_1 - 1 + \gamma(1)}(\varepsilon_{w,t}) \cdots h_{\rho_n - n + \gamma(n)}(\varepsilon_{w,t})}\right) dt \\ &= \sum_{\gamma \in \mathfrak{S}_n} \operatorname{sgn}(\gamma) \int_T \mu(t^x)\overline{h_{\rho_1 - 1 + \gamma(1)}(\varepsilon_{w,t}) \cdots h_{\rho_n - n + \gamma(n)}(\varepsilon_{w,t})}\, dt \end{aligned} \tag{9.8}$$

Since h_i is the restriction to T of the character of representation of $GL(n, \mathbf{C})$ on the ith symmetric tensor of $\mathbf{C}^n$, we next study the action of $n_w t$ on the weight space of $x \cdot \mu$ in the tensor product $S^{\rho_1 - 1 + \gamma(1)} \otimes \cdots \otimes S^{\rho_n - n + \gamma(n)}$, where S^k is kth symmetric tensor of $\mathbf{C}^n$. Let

$$\{e_{i_1} \cdots e_{i_k} \mid 1 \le i_1 \le \cdots \le i_k \le n\}$$

be a basis of S^k, where $e_1, \dots, e_n$ are the standard basis of $\mathbf{C}^n$ and $e_{i_1} \cdots e_{i_k}$ is the symmetric tensor product of $e_{i_1}, \dots, e_{i_k}$.

LEMMA 9.1. *The trace of $\rho(n_w t)$ on the space of weight $x \cdot \mu$ is equal to that of $\rho(n_{w_0} t)$ on the space of weight μ.*

PROOF. This follow from $x^{-1} w x = w_0$. □

By this lemma, we may consider the action of w_0 on the space of weight μ in the tensor product $S^{q_1} \otimes \cdots \otimes S^{q_n}$.

Recall $\mu(t) = (t_1 \cdots t_{n_1})^{f_1} \cdots (t_{n - n_r + 1} \cdots t_n)^{f_r}$ (see (7.5)) and $W_\mu = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$. Write $w_0 = w_1 \cdots w_r$ with $w_i \in \mathfrak{S}_{n_i}$. Let $w_i = w_{i1} \cdots w_{ik_i}$ be a cyclic factorization and l_{ij} the order of w_{ij}.

Let $(1^{p_{i1}}, \dots, n_i^{p_{in_i}})$ be the cycle type of w_i $(1 \le i \le r)$. Then $n_i = \sum_{j=1}^{n_i} j p_{ij}$.

The set

$$\left\{ (e_1^{\alpha_{11}} \cdot e_2^{\alpha_{12}} \cdots e_n^{\alpha_{1n}}) \otimes \cdots \otimes (e_1^{\alpha_{n1}} \cdot e_2^{\alpha_{n2}} \cdots e_n^{\alpha_{nn}}) \;\middle|\; \begin{aligned} q_i &= \sum_{j=1}^n \alpha_{ij} \\ \mu_j &= \sum_{i=1}^n \alpha_{ij} \end{aligned} \right\} \tag{9.9}$$

gives a basis of the space of weight μ in $S^{q_1} \otimes \cdots \otimes S^{q_n}$, where $\mu(t) = t_1^{\mu_1} \cdots t_n^{\mu_n}$. The action of w_0 on the weight space induces a permutation of the columns of the $n \times n$ matrix (α_{ij}). Hence the condition that w_0 fixes an element of the basis is equivalent to the condition

$$\alpha_{sj_1} = \alpha_{sj_2} = \cdots, \quad 1 \leq s \leq n \tag{9.10}$$

for any cycle $(j_1, j_2, \ldots)$ of w_i, $1 \leq i \leq r$. Thus we put these integers determined for cycles of w_i in the form of $n \times p_i$ matrix

$$A_i = (a_{st}^i)_{\substack{1 \leq s \leq n,\\ 1 \leq t \leq p_i}} \quad 1 \leq i \leq r,$$

where $p_i = \sum_{j=1}^{n_i} p_{ij}$ is the number of cycles of w_i.

Let

$$b_s^i = a_{s1}^i l_{i1} + \cdots + a_{sp_i}^i l_{ik_i}, \quad 1 \leq i \leq r,\ 1 \leq s \leq n.$$

Then the integers a_{st}^i and b_s^i satisfy the conditions

$$f_i = \sum_{k=1}^{n} a_{kt}^i, \quad 1 \leq i \leq r,\ 1 \leq t \leq p_i \tag{9.11}$$

$$q_s = \sum_{k=1}^{r} b_s^k, \quad 1 \leq s \leq n. \tag{9.12}$$

Example 9.2. Let $n = 5$, $(q_1, q_2) = (5, 3)$ and $\mu(t) = (t_1 t_2 t_3)^2 (t_4 t_5) : r = 2$, $(f_1, f_2) = (2, 1)$. Put

$$w_0 = (1)(23)(45), \quad w_1 = (1)(23), \quad w_2 = (45).$$

The matrices (α_{ij}) corresponding to the vector of weight μ in $S^5 \otimes S^3$ are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \ldots,$$

where we omit the last 3 rows. Among them, w_0 fixes only the above two matrices. Hence for the former we have

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (b_1^1, b_2^1) = (3, 3), \quad (b_1^2, b_2^2) = (2, 0),$$

and for the latter

$$A_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (b_1^1, b_2^1) = (5, 1), \quad (b_1^2, b_2^2) = (0, 2).$$

LEMMA 9.3.

$$m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0)$$
$$= \# \left\{ (A_1, \ldots, A_r) \left| \begin{array}{l} A_i = (a^i_{st})_{\substack{1 \le s \le n, \\ 1 \le t \le p_i}} \quad 1 \le i \le r \\ f_i = \sum_{k=1}^{n} a^i_{kt}, \quad 1 \le i \le r,\ 1 \le t \le p_i \\ q_s = \sum_{k=1}^{r} b^k_s, \quad 1 \le s \le n \end{array} \right. \right\} \tag{9.13}$$

PROOF. The action of W_μ, the stabilizer of the weight μ, on $S^{q_1} \otimes \cdots \otimes S^{q_n}$ induces permutation of the elements u of the set given by (9.9), which also induces a permutation of columns of the matrix (α_{ij}). Hence the number of elements u of weight μ that are fixed by w_0 is equal to the number of the set of the matrices $(A_1, \ldots, A_r)$ satisfying (9.11) and (9.12), which is also equal to $m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0)$ by Lemma 9.1. □

REMARK 9.4. If $w_0, w_0' \in W_\mu \cap K_w$ are conjugate in W_μ, then

$$m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0) = m(S^{q_1} \otimes \cdots \otimes S^{q_n}, \mu; w, w_0').$$

We come to give a multiplicity formula.

THEOREM 9.5. *Let ρ be an irreducible representation of $U(n)$ and $\theta_{\mu,\tau}$ an irreducible representation of N, then*

$$[\rho|_N : \theta_{\mu,\tau}] = \sum_{w \in \mathfrak{S}_n} \frac{|Z_{\mathfrak{S}_n}(w)|}{n!|W_\mu|} \sum_{w_0 \in W_\mu \cap K_w} \phi_\sigma(w_0)$$
$$\times \sum_{\gamma \in \mathfrak{S}_n} \mathrm{sgn}(\gamma) m(S^{\rho_1 - 1 + \gamma(1)} \otimes \cdots \otimes S^{\rho_n - n + \gamma(n)}, \mu; w, w_0),$$

where $\tau = \mu \rtimes \sigma \in \mathrm{Irr}(N_\mu)$ (see Remark 2.4). The number $m(S^{\rho_1 - 1 + \gamma(1)} \otimes \cdots \otimes S^{\rho_n - n + \gamma(n)}, \mu; w, w_0)$ is given by (9.13).

PROOF. The result follows from (9.6), (9.8), Lemma 9.1 and 9.3. □

10. On the Irreducibility of the N-Span of a Weight Space for $GL(n, \mathbf{C})$

Every irreducible module of $\mathfrak{S}_n$ is afforded by the zero weight space of certain irreducible module of $SL(n, \mathbf{C})$ ([8], [9], [2]). We consider the similar problem for N and give a series of examples of irreducible modules V of $GL(n, \mathbf{C})$ and their weights μ such that the N-modules

$$\bigoplus_{w \in W/W_\mu} V_{w(\mu)}$$

are irreducible, where the summation is taken over a complete set of coset representatives of W/W_μ. We denote this N-module by $V_{W\mu}$.

Let $G = GL(n, \mathbf{C})$, T the group of all diagonal matrices in G, and N the normalizer of T in G, which is isomorphic to the semidirect product of T by the Weyl group $W \simeq \mathfrak{S}_n$. Let ρ be an irreducible rational representation of G afforded by V and V_μ the weight space with dominant weight μ of T given by

$$\mu(t) = (t_1 \cdots t_{n_1})^{f_1} (t_{n_1+1} \cdots t_{n_1+n_2})^{f_2} \cdots (t_{n-n_r+1} \cdots t_n)^{f_r},$$

where $t = (t_1, \ldots, t_n) \in T$ and $f_1 > \cdots > f_r \geq 0$, $n_1 + \cdots + n_r = n$. We regard μ as the Young diagram

$$(\underbrace{f_1, \ldots, f_1}_{n_1}, \underbrace{f_2, \ldots, f_2}_{n_2}, \ldots, \underbrace{f_r, \ldots, f_r}_{n_r}).$$

The stabilizer subgroup W_μ of μ in W is isomorphic to $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$. Let $\sigma^i = (\sigma_1^i, \ldots, \sigma_{n_i}^i)$ be a young diagram with $|\sigma^i| = n_i$, which corresponds to an irreducible representation of $\mathfrak{S}_{n_i}$. Then $\sigma = (\sigma^1, \ldots, \sigma^r)$ corresponds to an irreducible representation of W_μ. We will give a Young diagram corresponding to an irreducible representation of G whose restriction to N contains the irreducible representation $\theta_{\mu, \mu \rtimes \sigma}$ of N.

We add rectangular diagram of length n_i to σ^i or $(\sigma^i)'$:

$$\gamma^i = \begin{cases} \sigma^i + (\underbrace{f_i, \ldots, f_i}_{n_i}) & f_i \text{ is odd}, \\ (\sigma^i)' + (\underbrace{f_i, \ldots, f_i}_{n_i}) & f_i \text{ is even}, \end{cases} \tag{10.1}$$

where $(\sigma^i)'$ is the transpose of σ^i. Put its transpose

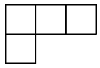
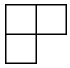
$$(\gamma^i)' = (\gamma_1^{i\prime}, \gamma_2^{i\prime}, \ldots)$$

and define a new Young diagram $\Gamma(\mu, \sigma)'$ by adding up jth columns of $\gamma^1, \ldots, \gamma^r$ for each j:

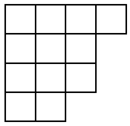
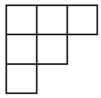
$$\Gamma(\mu, \sigma)' = (\hat{s}_1, \hat{s}_2, \ldots), \quad \hat{s}_j = \sum_{i=1}^{r} \gamma_j^{i\prime}.$$

Denote its transpose by $\Gamma(\mu, \sigma)$.

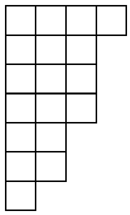
EXAMPLE 10.1. For $n = 7$, $\mu = (2, 2, 2, 2, 1, 1, 1)$, $(n_1, n_2) = (4, 3)$, $(f_1, f_2) = (2, 1)$. Let $\sigma^1 = (3, 1)$ and $\sigma^2 = (2, 1)$:

σ^1:  σ^2: 

Then $\gamma^1 = (4, 3, 3, 2)$ and $\gamma^2 = (3, 2, 1)$:

γ^1:  γ^2: 

and $\Gamma(\mu, \sigma) = (4, 3, 3, 3, 2, 2, 1)$:

$\Gamma(\mu, \sigma)$ 

PROPOSITION 10.2 (see [2] Proposition 5.1, 5.3). *For positive integers m and d, let λ be a Young diagram with $|\lambda| = m$. Let $\mu = (d, \dots, d)$ (m times) and U_μ be the weight space with weight μ of the irreducible representation of $GL(m, \mathbf{C})$ corresponding to the Young diagram $\lambda + (d-1, \dots, d-1)$ (m times). The representation of $\mathfrak{S}_m$ induced on U_μ is equivalent to $(\mathrm{sgn})^{\otimes(d-1)} \otimes \lambda$, where λ represents the irreducible representation of $\mathfrak{S}_m$ corresponding to the Young diagram λ.*

LEMMA 10.3. *Let ρ be an irreducible representation $(\det)^{-1} \otimes \Gamma(\mu, \sigma)$ of G and V_μ the weight space with weight μ. Then $\theta_{\mu, \mu \rtimes \sigma}$ is a subrepresentation of $V_{W\mu}$.*

PROOF. Put

$$t^{(1)} = (t_1, \dots, t_{n_1}), t^{(2)} = (t_{n_1+1}, \dots, t_{n_1+n_2}), \dots, t^{(r)} = (t_{n-n_r+1}, \dots, t_n),$$

then the Schur function $S_{\Gamma(\mu,\sigma)}(t_1, \dots, t_n)$ is the sum of the products of skew Schur functions ([10] I.5):

$$S_{\Gamma(\mu,\sigma)}(t) = \sum_{\boldsymbol{\nu}=(\nu^{(0)}, \dots, \nu^{(r)})} S_{\nu^{(1)}/\nu^{(0)}}(t^{(1)}) S_{\nu^{(2)}/\nu^{(1)}}(t^{(2)}) \cdots S_{\nu^{(r)}/\nu^{(r-1)}}(t^{(r)}), \tag{10.2}$$

where the summation is taken over all sequences $(\nu^{(0)}, \dots, \nu^{(r)})$ of Young diagrams such that $\nu^{(0)} = \emptyset$, $\nu^{(r)} = \Gamma(\mu, \sigma)$ and $\nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(r)}$. The skew Schur function is a sum of the Schur functions:

$$S_{\nu^{(i)}/\nu^{(i-1)}} = \sum_{\lambda} c^{\nu^{(i)}}_{\nu^{(i-1)}\lambda} S_\lambda, \tag{10.3}$$

where $c^{\nu^{(i)}}_{\nu^{(i-1)}\lambda}$ is the Littlewood-Richardson coefficient. Hence we have

$$\begin{aligned}&(t_1 \cdots t_n)^{-1} S_{\Gamma(\mu,\sigma)}(t)\\ &= \sum_{\boldsymbol{\nu}=(\nu^{(0)},\dots,\nu^{(r)})} \{(t_1 \cdots t_{n_1})^{-1} S_{\nu^{(1)}/\nu^{(0)}}(t^{(1)})\} \cdots \{(t_{n-n_r+1} \cdots t_n)^{-1} S_{\nu^{(r)}/\nu^{(r-1)}}(t^{(r)})\}.\end{aligned} \tag{10.4}$$

This gives the decomposition of the restriction $\det^{-1} \otimes \Gamma(\mu, \sigma)|_{G_\mu}$. Here G_μ is the Levi subgroup of the standard parabolic subgroup corresponding to W_μ:

$$G_\mu = GL(n_1, \mathbf{C}) \times \cdots \times GL(n_r, \mathbf{C}).$$

As in §6, let $(\det^{-1} \otimes \Gamma(\mu, \sigma))_\mu$ be the sum of all irreducible subrepresentations of $\det^{-1} \otimes \Gamma(\mu, \sigma)|_{G_\mu}$ having μ as its weight. Then the character of $(\det^{-1} \otimes \Gamma(\mu, \sigma))_\mu$ is given by

$$\sum_{\substack{\boldsymbol{\nu}=(\nu^{(0)},\dots,\nu^{(r)})\\ |\nu^{(i)}/\nu^{(i-1)}|=(f_i+1)n_i}} \{(t_1 \cdots t_{n_1})^{-1} S_{\nu^{(1)}/\nu^{(0)}}(t^{(1)})\} \cdots \{(t_{n-n_r+1} \cdots t_n)^{-1} S_{\nu^{(r)}/\nu^{(r-1)}}(t^{(r)})\}. \tag{10.5}$$

Put

$$\hat{s}^{(k)}_j = \sum_{i=1}^{k} \gamma^{i\prime}_j$$

and define $\Gamma^{(k)}(\mu, \sigma)'$ by

$$\Gamma^{(k)}(\mu, \sigma)' = (\hat{s}^{(k)}_1, \hat{s}^{(k)}_2, \dots), \quad \Gamma^{(0)}(\mu, \sigma)' = \emptyset.$$

Denote by $\Gamma^{(k)}(\mu, \sigma)$ its transpose. Then we have a sequence of Young diagrams:

$$\emptyset = \Gamma^{(0)}(\mu, \sigma) \subset \Gamma^{(1)}(\mu, \sigma) \subset \cdots \subset \Gamma^{(r)}(\mu, \sigma) = \Gamma(\mu, \sigma),$$

which appears in the summation (10.5), since

$$|\Gamma^{(i)}(\mu, \sigma)/\Gamma^{(i-1)}(\mu, \sigma)| = (f_i + 1)n_i.$$

Note that this is equal to $|\gamma^i|$. We next show

$$c^{\Gamma^{(i)}(\mu,\sigma)}_{\Gamma^{(i-1)}(\mu,\sigma)\,\gamma^i} \neq 0. \tag{10.6}$$

The number $c^{\Gamma^{(i)}(\mu,\sigma)}_{\Gamma^{(i-1)}(\mu,\sigma)\,\gamma^i}$ is given by the Littlewood-Richardson rule (see [10]): the number of column-strict skew tableaux T of shape $\Gamma^{(i)}(\mu,\sigma) - \Gamma^{(i-1)}(\mu,\sigma)$ and of weight γ^i such that the word $w(T)$ is a lattice permutation. Here the word $w(T)$ is obtained by reading the entries of T from right to left in each row, starting with the top row and proceeding downward. The word $w(T) = (a_1,\dots,a_k)$ is called a lattice permutation if for $1 \le j \le k$, in the first j elements of $w(T)$, the number of occurrence of i is not less than the number of occurrence of $i+1$ for each i.

By the definition of γ^i and $\Gamma^{(i)}(\mu,\sigma)$, if we fill each of the columns of the skew diagram $\Gamma^{(i)}(\mu,\sigma) - \Gamma^{(i-1)}(\mu,\sigma)$ with the numbers

$$b_1 = n_1 + n_2 + \cdots + n_{i-1} + 1, \dots, b_{n_i} = n_1 + \cdots + n_i$$

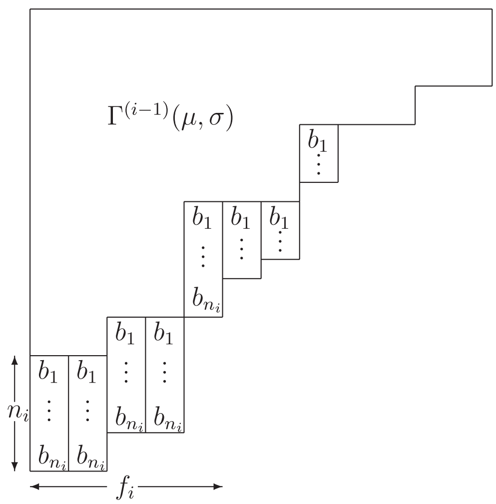


Figure 1: skew tableau $\Gamma^{(i)}(\mu,\sigma) - \Gamma^{(i-1)}(\mu,\sigma)$

from the top to the bottom in that order, then we have a column-strict skew tableau T of shape $\Gamma^{(i)}(\mu,\sigma) - \Gamma^{(i-1)}(\mu,\sigma)$ and of weight γ^i. Since the number b_i $(i > 1)$ lies just below the number b_{i-1} in T, the number of the occurrence of b_{i-1} is not less than that of b_i in $w(T)$, the word $w(T)$ is a lattice permutation (see Figure 1). Thus we have (10.6).

It follows from (10.3), (10.5) and (10.6) that the irreducible representation $(\det^{-1} \otimes \gamma^1) \times \cdots \times (\det^{-1} \otimes \gamma^r)$ of G_μ is a subrepresentation of $(\det^{-1} \otimes \Gamma(\mu,\sigma))_\mu$.

By Proposition 10.2 the representation of $\mathfrak{S}_{n_i}$ induced on the weight space with weight $(f_i, \dots, f_i)$ (n_i times) of $\det^{-1} \otimes \gamma^i$ is equivalent to

$$\begin{cases} \mathrm{sgn}^{\otimes f_i+1} \otimes \sigma^i & f_i \text{ is odd,} \\ \mathrm{sgn}^{\otimes f_i+1} \otimes (\sigma^i)' & f_i \text{ is even} \end{cases}$$

which is equivalent to the irreducible representation of $\mathfrak{S}_{n_i}$ corresponding to σ^i in both cases. Hence the representation of W_μ induced on the weight space with weight μ of the representation $(\det^{-1} \otimes \gamma^1) \times \cdots \times (\det^{-1} \otimes \gamma^r)$ of G_μ affords the irreducible representation σ as a subrepresentation and the lemma is proved. □

Proposition 10.4. *Let V_μ be the weight space with weight μ of the representation* $\det^{-1} \otimes \Gamma(\mu, \sigma)$ *of* $GL(n, \mathbf{C})$. *If $\gamma^i_{n_i} \geq \gamma^{i+1}_1$ for $1 \leq i \leq r-1$, then V_{W_μ} is the irreducible module of N affording $\theta_{\mu, \mu \rtimes \sigma}$.*

Proof. It follows from Lemma 10.3 that $\theta_{\mu, \mu \rtimes \sigma}$ is a subrepresentation of V_{W_μ}. We have only to show $\dim V_\mu = \deg \sigma$.

The dimension of V_μ is equal to the number of the column-strict tableaux of shape $\Gamma(\mu, \sigma)$ and of weight $\mu + (1, \dots, 1)$ (n times). Put

$$I_1 = \{1, 2, \dots n_1\}, I_2 = \{n_1 + 1, \dots, n_1 + n_2\}, \dots, I_r = \{n - n_r + 1, \dots, n\}.$$

Since the length $\gamma^i_{n_i}$ of the last row of γ^i is greater than or equal to the length γ^{i+1}_1 of the first row of γ^{i+1}, the diagram $\Gamma(\mu, \sigma)$ is obtained simply by putting γ^i on γ^{i+1} (Figure 2).

The integer i in the column-strict tableau should be in the first i rows of $\Gamma(\mu, \sigma)$. Thus the elements of I_1 should be placed in the first n_1 rows, whose shape is just γ^1. Since $|\gamma^i| = (f_i + 1)n_i$ and every element of I_1 appears exactly $f_i + 1$ times in T, the first n_1 rows are filled only with the numbers of I_1 and any element of I_1 doesn't occur in the ith rows for $i > n_1$. The elements of I_2 should be placed in the next n_2 rows, whose shape is γ^2, and so on.

The diagram of γ^i is obtained by adjoining that of σ^i or $(\sigma^i)'$ to the right of a rectangular diagram $(f_i, \dots, f_i)$ (n_i times). The first f_i columns of γ^i are in the rectangular part and filled with all member of I_i. Since every element of I_i appears exactly f_i times in the rectangular part of γ^i, the number of column-strict tableaux of shape γ^i and of weight $(f_i + 1, \dots, f_i + 1)$ (n_i times) is equal to the number of those of shape σ^i (or $(\sigma^i)'$) and of weight $(1, \dots, 1)$ (n_i times), which is nothing but the degree of irreducible representation of $\mathfrak{S}_{n_i}$ corresponding to the Young diagram σ^i (or $(\sigma^i)'$). Hence we have

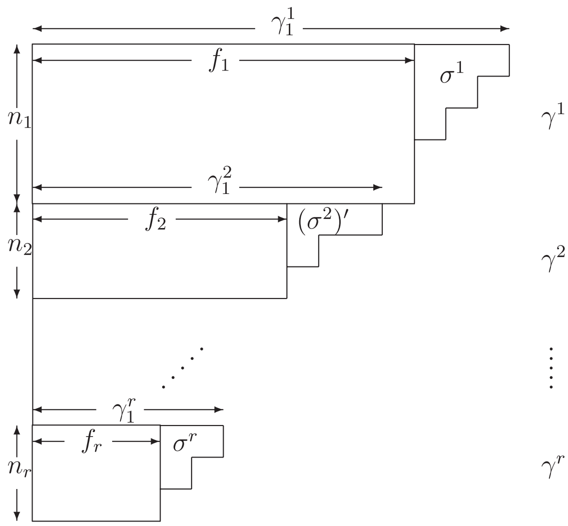


Figure 2: $\Gamma(\mu,\sigma)$

$$\dim V_\mu = \prod_{i=1}^{r} \deg \sigma^i = \deg \sigma,$$

which finishes the proof. □

We show another examples of irreducible representations of G and their weights μ such that V_{W_μ} affords $\theta_{\mu,\mu\rtimes\sigma}$.

For a Young diagram $\lambda = (\lambda_1,\dots,\lambda_m)$ with $|\lambda| = m$ and an integer d with $d \geq \lambda_1$, define a Young diagram $\lambda^*(d)$ by

$$\lambda^*(d) = (d-\lambda_m, d-\lambda_{m-1},\dots,d-\lambda_1).$$

REMARK 10.5. The contragradient representation of the representation of $GL(m,\mathbf{C})$ corresponding to λ is $(\det^{-1})^{\otimes d} \otimes \lambda^*(d)$.

PROPOSITION 10.6 ([2], Proposition 5.1). *Let $\mu = (d-1,\dots,d-1)$ (m times) and U_μ be the weight space with weight μ of the irreducible representation of $GL(m,\mathbf{C})$ corresponding to the Young diagram $\lambda^*(d)$. The representation of $\mathfrak{S}_m$ induced on U_μ is equivalent to $(\mathrm{sgn})^{\otimes d} \otimes \lambda$, where λ represents the irreducible representation of $\mathfrak{S}_m$ corresponding to the Young diagram λ.*

For a weight μ and Young diagrams $\sigma^i = (\sigma^i_1, \sigma^i_2, \dots)$ $(1 \le i \le r)$, let a be the smallest numbet of the set

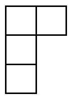
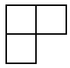
$$\{\{f_i + 1 - \sigma^i_1 \mid f_i \text{ is odd}, 1 \le i \le r\} \cup \{f_i + 1 - (\sigma^i)'_1 \mid f_i \text{ is even}, 1 \le i \le r\}\},$$

where $(\sigma^i)'$ is the transpose of σ^i. Put $e = -a$ (if $a < 0$) and $e = 0$ (if $a \ge 0$) so that $f_i + 1 + e - \sigma^i_1 \ge 0$ (if f_i is odd) and $f_i + 1 + e - (\sigma^i)'_1 \ge 0$ (if f_i is even). Then we can define a Young diagram $\tilde{\gamma}^i$ by

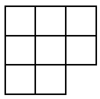
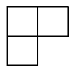
$$\tilde{\gamma}^i = \begin{cases} (\sigma^i)^*(f_i + 1 + e) & f_i \text{ is odd}, \\ ((\sigma^i)')^*(f_i + 1 + e) & f_i \text{ is even}. \end{cases}$$

For the diagrams $\tilde{\gamma}^i$ $(1 \le i \le r)$, we define a Young diagram $\tilde{\Gamma}(\mu, \sigma)$ as $\Gamma(\mu, \sigma)$.

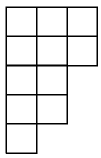
Example 10.7. Let $n = 7$, $(n_1, n_2) = (4, 3)$, $\mu = (2, 2, 2, 2, 1, 1, 1)$, $(f_1, f_2) = (2, 1)$ and $\sigma^1 = (2, 1, 1)$, $\sigma^2 = (2, 1)$. Then $e = 0$.

σ^1:  σ^2: 

Then $\tilde{\gamma}^1 = (3, 3, 2)$, $\tilde{\gamma}^2 = (2, 1)$.

$\tilde{\gamma}^1$:  $\tilde{\gamma}^2$: 

Thus we have $\tilde{\Gamma}(\mu, \sigma) = (3, 3, 2, 2, 1)$:

$\tilde{\Gamma}(\mu, \sigma)$: 

Proposition 10.8. *Let V_μ be the weight space with weight μ of the irreducible representation* $\det^{-e} \otimes \tilde{\Gamma}(\mu, \sigma)$ *of* G.

(i) $\theta_{\mu, \mu \rtimes \sigma}$ *is a subrepresentation of* $V_{W\mu}$.
(ii) *If* $\tilde{\gamma}^i_{n_i} \ge \tilde{\gamma}^{i+1}_1$ *for* $1 \le i \le r - 1$, *then* $V_{W\mu}$ *is the irreducible module of* N *affording* $\theta_{\mu, \mu \rtimes \sigma}$.

Proof. (i) By the same argument of the proof of Lemma 10.3, the irreducible representation $(\det^{-e} \otimes \tilde{\gamma}^1) \times \cdots \times (\det^{-e} \otimes \tilde{\gamma}^r)$ of G_μ is a subrepresentation of $((\det^{-e}) \otimes \tilde{\Gamma}(\mu, \sigma))_\mu$.

It follows from Proposition 10.6 that the representation of $\mathfrak{S}_{n_i}$ induced on the weight space with weight $(f_i, \dots, f_i)$ (n_i times) of $\tilde{\gamma}^i$ is equivalent to

$$\begin{cases} (\mathrm{sgn})^{\otimes f_i+1} \otimes \sigma^i & f_i \text{ is odd,} \\ (\mathrm{sgn})^{\otimes f_i+1} \otimes (\sigma^i)' & f_i \text{ is even.} \end{cases} \tag{10.7}$$

In both cases the representations are equivalent to the irreducible representation of $\mathfrak{S}_{n_i}$ corresponding to σ^i. Then the representation of W_μ induced on the weight space with weight μ of the representation $(\det^{-e} \otimes \tilde{\gamma}^1) \times \cdots \times (\det^{-e} \otimes \tilde{\gamma}^r)$ of G_μ affords the irreducible representation σ as a subrepresentation. Hence V_{W_μ} affords $\theta_{\mu, \mu \rtimes \sigma}$ as its subrepresentation.

(ii) Since $\theta_{\mu, \mu \rtimes \sigma}$ is a subrepresentation of V_{W_μ} by (i), we have only to show $\dim V_\mu = \deg \sigma$. By the assumption and the same argument of the proof in Proposition 10.4, we have to show that the number of the column-strict tableau of shape $\tilde{\gamma}^i$ and of weight $(f_i + e, \dots, f_i + e)$ (n_i times) is equal to the degree of the irreducible representation σ^i of $\mathfrak{S}_{n_i}$. However this follows from Proposition 10.6. $\square$

REMARK 10.9. The representation $\tilde{\Gamma}(\mu, \sigma)$ in the Example 10.7 satisfies the condition of the Proposition 10.8 (ii), but not that of Proposition 10.4.

References

[1] Adams, J. F., Lectures on Exceptional Lie Groups, Chicago Lectures in Math. Ser., The Univ. Chicago Press, 1996.

[2] Ariki, S., Terada, I. and Matsuzawa, J., Representations of Weyl Groups on Zero Weight Spaces of G-modules, Algebraic and Topological Theories, Kinokuniya, (1985), 546–568.

[3] Clifford, A. H., Representations induced in an invariant subgroup, Annals of Mathematics, **38**, No. 3, (1937), 533–550.

[4] Dedecker, P., Le foncteur Hom non abelian. Applications de la notion de poulpe, C. R. Acad. Sci. Paris **258** (1964), 1117–1120, Les foncteurs $\mathscr{E}xt_\pi$, $\boldsymbol{H}^2_\pi$ et H^2_π non abéliens, 4891–4894; Premier dérivé du foncteur Hom non abélien, **259** (1964), 2054–2057; **260** (1965), 4137–4139.

[5] Giraud, J., Cohomologie non abélienne, C. R. Acad. Sci. Paris **260** (1965), 2392–2394, 2666–2668.

[6] Giraud, J., Cohomologie non abélienne, Springer (1971).

[7] Curtis, C. W. and Reiner, I., Methods of Representation Theory: with applications to finite groups and orders, vol. 1, Wiley, 1990.

[8] Gutkin, E. A., Representation of the Weyl group in the space of vectors of zero weight. (in Russian) Uspehi Mat. Nauk. **28** (1973), no. 5 (173), 237–238. English summary in MR 53, #13423.

[9] Kostant, B., On Macdonald's η-function formula, the Laplacian and generalized exponents, Advances in Math., **20** (1975), 257–285.

[10] Macdonald, I. G., Symmetric Functions and Hall Polynomials, second edition, Oxford Univ. Press, 1995.

[11] Mackey, J. W., Unitary representations of group extensions. I, Acta Math., **99** (1958), 265–311.

[12] Matsuzawa, J., The relationship between irreducible representations of complex simple Lie groups and normalizer subgroups of maximal torus, (in Japanese), master's thesis, University of Tokyo, 1985.
[13] Matsuzawa, J., On the generalized exponents of classical Lie groups, Communications in Alg., **16** (1988), 2579–2623.
[14] Naruse, H., Characters of Party algebras, preprint.
[15] Nishiyama, K., Restriction of the irreducible representations of GL_n to the symmetric group S_n, preprint.
[16] Springer, T. A., Nonabelian H^2 in Galois cohomology, Algebraic groups and discontinuous subgroups, Proc. Symp. Pure Math., **9**, Amer. Math. Soc. 1966 pp. 164–182.
[17] Springer, T. A., Linear Algebraic Groups, 2nd edition, Birkhäuser, 1991.
[18] Takahashi, M., Representations of normalizer subgroups of maximal tori of the classical group of type C, preprint.
[19] Wigner, E., On unitary representations of the inhomogeneous Lorentz group, Ann. of Math., **40** (1939), 149–204.

Department of Mathematics
Nara Women's University
Kitauoya Nishimachi
Nara 630-8506
Japan
E-mail address: matsuzawa@cc.nara-wu.ac.jp

Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki 305-8571
Japan
E-mail address: makotot@math.tsukuba.ac.jp