Representations of Two-parameter Quantum Orthogonal and Symplectic Groups

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Abstract. We investigate the finite-dimensional representation theory of two-parameter quantum orthogonal and symplectic groups that we found in [BGH] under the assumption that rs^{-1} is not a root of unity and extend some results [BW1, BW2] obtained for type A to types B, C and D. We construct the corresponding R-matrices and the quantum Casimir operators, by which we prove that the complete reducibility Theorem also holds for the categories of finite-dimensional weight modules for types B, C, D.

1. Preliminaries: Two-parameter Quantum Groups for Classical Types

Let $\mathbb{K} \supset \mathbb{Q}(r, s)$ denote an algebraically closed field, where the two-parameters r, s are nonzero complex numbers satisfying $r^2 \neq s^2$.

In this section, we recall the definitions of the two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ from [BW1], and for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ from [BGH]. Let Ψ be a finite root system of a simple Lie algebra $\mathfrak{g}$ of rank n with Π a base of simple roots. Regard Ψ as a subset of a Euclidean space $E = \mathbb{R}^n$ with an inner product $(\,,)$. Let $\epsilon_1, \cdots, \epsilon_n$ denote an orthonormal basis of E. We need the following data on (prime) root systems.

Type A:

$$\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n\},$$
$$\Psi = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n+1\}.$$

Type B:

$$\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\} \cup \{\alpha_n = \epsilon_n\},$$
$$\Psi = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n\} \cup \{\pm\epsilon_i \mid 1 \leq i \leq n\}.$$

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Type C:

$$\Pi=\{\alpha_i=\epsilon_i-\epsilon_{i+1}\mid 1\le i<n\}\cup\{\alpha_n=2\epsilon_n\},$$
$$\Psi=\{\pm\epsilon_i\pm\epsilon_j\mid 1\le i\ne j\le n\}\cup\{2\epsilon_i\mid 1\le i\le n\}.$$

Type D:

$$\Pi=\{\alpha_i=\epsilon_i-\epsilon_{i+1}\mid 1\le i<n\}\cup\{\alpha_n=\epsilon_{n-1}+\epsilon_n\},$$
$$\Psi=\{\pm\epsilon_i\pm\epsilon_j\mid 1\le i\ne j\le n\}.$$

In the cases of type A, C and D, we set $r_i=r^{\frac{(\alpha_i,\alpha_i)}{2}}$, $s_i=s^{\frac{(\alpha_i,\alpha_i)}{2}}$; while for type B, we set $r_i=r^{(\alpha_i,\alpha_i)}$, $s_i=s^{(\alpha_i,\alpha_i)}$.

Assigned to Π, there are two sets of mutually-commutative symbols $W=\{\omega_i^{\pm1}\mid 1\le i\le n\}$ and $W'=\{\omega_i'^{\pm1}\mid 1\le i\le n\}$. Define a pairing $\langle\,,\rangle: W'\times W\longrightarrow\mathbb{K}$ as follows:

$$(1_A)\qquad \langle\omega_i',\omega_j\rangle=r^{(\epsilon_j,\alpha_i)}s^{(\epsilon_{j+1},\alpha_i)},\qquad i\le n{+}1,\ j\le n,\qquad \textit{for}\quad \mathfrak{sl}_{n+1},$$

$$(1_B)\qquad \langle\omega_i',\omega_j\rangle=\begin{cases} r^{2(\epsilon_j,\alpha_i)}s^{2(\epsilon_{j+1},\alpha_i)}, & i\le n,\ j<n,\\ r^{2(\epsilon_n,\alpha_i)}, & i<n,\ j=n,\\ r^{(\epsilon_n,\alpha_n)}s^{-(\epsilon_n,\alpha_n)}, & i=j=n.\end{cases}\qquad \textit{for}\quad \mathfrak{so}_{2n+1},$$

$$(1_C)\qquad \langle\omega_i',\omega_j\rangle=\begin{cases} r^{(\epsilon_j,\alpha_i)}s^{(\epsilon_{j+1},\alpha_i)}, & i\le n,\ j<n,\\ r^{2(\epsilon_n,\alpha_i)}, & i<n,\ j=n,\\ r^{(\epsilon_n,\alpha_n)}s^{-(\epsilon_n,\alpha_n)}, & i=j=n.\end{cases}\qquad \textit{for}\quad \mathfrak{sp}_{2n},$$

$$(1_D)\qquad \langle\omega_i',\omega_j\rangle=\begin{cases} r^{(\epsilon_j,\alpha_i)}s^{(\epsilon_{j+1},\alpha_i)}, & i\le n,\ j<n,\\ r^{(\epsilon_{n-1},\alpha_i)}s^{-(\epsilon_n,\alpha_i)}, & i\ne n-1,\ j=n,\\ r^{(\epsilon_n,\alpha_{n-1})}s^{-(\epsilon_{n-1},\alpha_{n-1})}, & i=n-1,\ j=n.\end{cases}\qquad \textit{for}\quad \mathfrak{so}_{2n},$$

$$(2)\qquad \langle\omega_i'^{\pm1},\omega_j^{-1}\rangle=\langle\omega_i'^{\pm1},\omega_j\rangle^{-1}=\langle\omega_i',\omega_j\rangle^{\mp1},\qquad \textit{for any}\quad \mathfrak{g}.$$

LEMMA 1.1. *For the prime root systems of the Lie algebras $\mathfrak{g}=\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, and $\mathfrak{sp}_{2n}$, there hold the identities:*

$$(\epsilon_{j+1},\alpha_i)=-(\epsilon_i,\alpha_j),\qquad (i,j<n),\qquad \textit{for any}\quad \mathfrak{g},$$

$$(\epsilon_{j+1},\alpha_n)=\begin{cases}-(\epsilon_n,\alpha_j), & (j<n), \quad \textit{for}\quad \mathfrak{g}=\mathfrak{so}_{2n+1},\\ -2(\epsilon_n,\alpha_j), & (j<n), \quad \textit{for}\quad \mathfrak{g}=\mathfrak{sp}_{2n},\end{cases}$$

$$(\epsilon_j,\alpha_n)=\begin{cases}-(\epsilon_n,\alpha_{j-1}), & (j\le n,\ j\ne n-1),\\ (\epsilon_{n-1},\alpha_{n-1}), & (j=n-1)\end{cases}\qquad \textit{for}\quad \mathfrak{g}=\mathfrak{so}_{2n}.$$

□

Observe that Lemma 1.1 ensures the compatibility of the defining relations of the two-parameter quantum groups defined below.

Let $U_{r,s}(\mathfrak{g})$ be the unital associative algebra over $\mathbb{K}$ generated by symbols $e_i, f_i, \omega_i^{\pm1}$, $\omega_i'^{\pm1}$ $(1\le i\le n)$, subject to the following relations $(X1)$—$(X4)$:

$(X1)$ $\omega_i^{\pm1}\omega_j'^{\pm1}=\omega_j'^{\pm1}\omega_i{}^{\pm1}$, $\qquad \omega_i^{\pm1}\omega_i{}^{\mp1}=1=\omega_i'^{\pm1}\omega_i'^{\mp1}$.

$(X2)$ For $1 \leq i,\, j \leq n$, we have

$$\omega_j\, e_i\, \omega_j^{-1} = \langle \omega_i', \omega_j \rangle\, e_i, \qquad \omega_j\, f_i\, \omega_j^{-1} = \langle \omega_i', \omega_j \rangle^{-1}\, f_i,$$
$$\omega_j'\, e_i\, {\omega_j'}^{-1} = \langle \omega_j', \omega_i \rangle^{-1}\, e_i, \qquad \omega_j'\, f_i\, {\omega_j'}^{-1} = \langle \omega_j', \omega_i \rangle\, f_i.$$

$(X3)$ For $1 \leq i,\, j \leq n$, we have

$$[\, e_i, f_j\,] = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i}.$$

$(X4)$ For any $i \neq j$, we have the (r,s)-Serre relations:

$$\big(\mathrm{ad}_l\, e_i\big)^{1-a_{ij}}\,(e_j) = 0,$$
$$\big(\mathrm{ad}_r\, f_i\big)^{1-a_{ij}}\,(f_j) = 0,$$

where the definitions of the left-adjoint action $\mathrm{ad}_l\, e_i$ and the right-adjoint action $\mathrm{ad}_r\, f_i$ are given in the following sense:

$$\mathrm{ad}_l\, a\,(b) = \sum_{(a)} a_{(1)}\, b\, S(a_{(2)}), \quad \mathrm{ad}_r\, a\,(b) = \sum_{(a)} S(a_{(1)})\, b\, a_{(2)}, \quad \forall\, a, b \in U_{r,s}(\mathfrak{g}),$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ is given by Proposition 1.2 below.

The following fact is straightforward.

Proposition 1.2. *The algebra* $U_{r,s}(\mathfrak{g})$ ($\mathfrak{g} = \mathfrak{sl}_{n+1},\ \mathfrak{so}_{2n+1},\ \mathfrak{sp}_{2n}$*, or* $\mathfrak{so}_{2n}$) *is a Hopf algebra under the comultiplication, the counit and the antipode defined below:*

$$\Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1}, \qquad \Delta({\omega_i'}^{\pm 1}) = {\omega_i'}^{\pm 1} \otimes {\omega_i'}^{\pm 1},$$
$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i',$$
$$\varepsilon(\omega_i^{\pm}) = \varepsilon({\omega_i'}^{\pm 1}) = 1, \qquad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$
$$S(\omega_i^{\pm 1}) = \omega_i^{\mp 1}, \qquad S({\omega_i'}^{\pm 1}) = {\omega_i'}^{\mp 1},$$
$$S(e_i) = -\omega_i^{-1} e_i, \qquad S(f_i) = -f_i\, {\omega_i'}^{-1}.$$

□

Remark 1.3. When $r = s^{-1} = q$, Hopf algebra $U_{r,s}(\mathfrak{g})$ modulo the Hopf ideal generated by the elements $\omega_i' - \omega_i^{-1}$ $(1 \leq i \leq n)$, is just the quantum groups $U_q(\mathfrak{g})$ of Drinfel'd-Jimbo type.

Definition 1.4. *A skew-dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ is a bilinear form* $\langle\, ,\rangle : \mathcal{U} \times \mathcal{A} \longrightarrow \mathbb{K}$ *such that*

$$\langle f, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(f), \qquad \langle 1_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a),$$
$$\langle f, a_1 a_2 \rangle = \langle \Delta_{\mathcal{U}}^{op}(f), a_1 \otimes a_2 \rangle, \qquad \langle f_1 f_2, a \rangle = \langle f_1 \otimes f_2, \Delta_{\mathcal{A}}(a) \rangle,$$

for all $f,\, f_1,\, f_2 \in \mathcal{U}$*, and* $a,\, a_1,\, a_2 \in \mathcal{A}$*, where* $\varepsilon_{\mathcal{U}}$ *and* $\varepsilon_{\mathcal{A}}$ *denote the counits of* $\mathcal{U}$ *and* $\mathcal{A}$*, respectively, and* $\Delta_{\mathcal{U}}$ *and* $\Delta_{\mathcal{A}}$ *are their respective comultiplications.*

Let $\mathcal{B} = B(\mathfrak{g})$ (resp. $\mathcal{B}' = B'(\mathfrak{g})$) denote the Hopf subalgebra of $U = U_{r,s}(\mathfrak{g})$ generated by e_j, $\omega_j^{\pm 1}$ (resp. f_j, ${\omega_j'}^{\pm 1}$) with $1 \leq j \leq n$ for $\mathfrak{g} = \mathfrak{sl}_{n+1}$, and with $1 \leq j \leq n$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, and $\mathfrak{sp}_{2n}$, respectively. The following result was obtained for the type A case by [BW1], and for the types B, C and D cases by [BGH].

PROPOSITION 1.5. *There exists a unique skew-dual pairing* $\langle\,,\rangle : \mathcal{B}' \times \mathcal{B} \longrightarrow \mathbb{K}$ *of the Hopf subalgebras* $\mathcal{B}$ *and* $\mathcal{B}'$ *in* $U_{r,s}(\mathfrak{g})$, *for* $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, *or* $\mathfrak{sp}_{2n}$ *such that* $\langle f_i, e_j\rangle = \frac{\delta_{ij}}{s_i - r_i}$, *and the conditions* (1_X) (*where* $X = A$, B, C, *or* D) *and* (2) *are satisfied, and all other pairs of generators are* 0. *Moreover, we have* $\langle S(a), S(b)\rangle = \langle a, b\rangle$ *for* $a \in \mathcal{B}'$, $b \in \mathcal{B}$. □

DEFINITION 1.6. *For any two skew-paired Hopf algebras* $\mathcal{A}$ *and* $\mathcal{U}$ *by a skew-dual pairing* $\langle\,,\rangle$, *one may form the Drinfel'd double* $\mathcal{D}(\mathcal{A},\mathcal{U})$ *as in* [KS, 8.2], *which is a Hopf algebra whose underlying coalgebra is* $\mathcal{A} \otimes \mathcal{U}$ *with the tensor product coalgebra structure, and whose algebra structure is defined by*

$$(a\otimes f)(a'\otimes f') = \sum \langle S_{\mathcal{U}}(f_{(1)}), a'_{(1)}\rangle \langle f_{(3)}, a'_{(3)}\rangle\, aa'_{(2)} \otimes f_{(2)}f', \tag{3}$$

for $a, a' \in \mathcal{A}$ *and* $f, f' \in \mathcal{U}$. *The antipode* S *is given by*

$$S(a\otimes f) = (1 \otimes S_{\mathcal{U}}(f))(S_{\mathcal{A}}(a)\otimes 1).$$

Clearly, both mappings $\mathcal{A} \ni a \mapsto a\otimes 1 \in \mathcal{D}(\mathcal{A},\mathcal{U})$ and $\mathcal{U} \ni f \mapsto 1\otimes f \in \mathcal{D}(\mathcal{A},\mathcal{U})$ are injective Hopf algebra homomorphisms. Let us denote the image $a \otimes 1$ (resp. $1 \otimes f$) of a (resp. f) in $\mathcal{D}(\mathcal{A},\mathcal{U})$ by $\hat{a}$ (resp. $\hat{f}$). By (3), we have the following cross commutation relations between elements $\hat{a}$ (for $a \in \mathcal{A}$) and $\hat{f}$ (for $f \in \mathcal{U}$) in the algebra $\mathcal{D}(\mathcal{A},\mathcal{U})$:

$$\hat{f}\,\hat{a} = \sum \langle S_{\mathcal{U}}(f_{(1)}), a_{(1)}\rangle\, \langle f_{(3)}, a_{(3)}\rangle\; \hat{a}_{(2)}\hat{f}_{(2)}, \tag{4}$$

$$\sum \langle f_{(1)}, a_{(1)}\rangle\, \hat{f}_{(2)}\, \hat{a}_{(2)} = \sum \hat{a}_{(1)}\, \hat{f}_{(1)}\, \langle f_{(2)}, a_{(2)}\rangle. \tag{5}$$

In fact, as an algebra the double $\mathcal{D}(\mathcal{A},\mathcal{U})$ is the universal algebra generated by the algebras $\mathcal{A}$ and $\mathcal{U}$ with cross relations (4) or, equivalently, (5).

THEOREM 1.7 ([BW1, BGH]). *The two-parameter quantum group* $U = U_{r,s}(\mathfrak{g})$ *is isomorphic to the Drinfel'd quantum double* $\mathcal{D}(\mathcal{B},\mathcal{B}')$, *for* $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, *or* $\mathfrak{sp}_{2n}$. □

Let us denote $U_{r,s}(\mathfrak{n})$ (resp. $U_{r,s}(\mathfrak{n}^-)$) the subalgebra of $\mathcal{B}$ (resp. $\mathcal{B}'$) generated by e_i (resp. f_i) for all $i \le n$. Let

$$U^0 = \mathbb{K}\,[\omega_1^{\pm 1}, \cdots, \omega_n^{\pm 1}, {\omega_1'}^{\pm 1}, \cdots, {\omega_n'}^{\pm 1}],$$
$$U_0 = \mathbb{K}\,[\omega_1^{\pm 1}, \cdots, \omega_n^{\pm 1}], \qquad U_0' = \mathbb{K}\,[{\omega_1'}^{\pm 1}, \cdots, {\omega_n'}^{\pm 1}]$$

denote the respective Laurent polynomial subalgebras of $U_{r,s}(\mathfrak{g})$, $\mathcal{B}$, and $\mathcal{B}'$. Clearly, $U^0 = U_0U_0' = U_0'U_0$. Thus, by definition, we have $\mathcal{B} = U_{r,s}(\mathfrak{n}) \rtimes U_0$, and $\mathcal{B}' = U_0' \ltimes U_{r,s}(\mathfrak{n}^-)$, such that the double $\mathcal{D}(\mathcal{B},\mathcal{B}') \cong U_{r,s}(\mathfrak{n}) \otimes U^0 \otimes U_{r,s}(\mathfrak{n}^-)$, as vector spaces.

Let $\langle\,|\,\rangle_0 : \mathcal{B} \times \mathcal{B}' \longrightarrow \mathbb{K}$ denote the skew-dual pairing given by $\langle b\,|\,b'\rangle_0 = \langle S(b'), b\rangle$. Then, via a variation of its Drinfel'd double structure, we obtain the standard triangular decomposition of $U_{r,s}(\mathfrak{g})$ in [BGH, Corollary 2.6] as follows.

COROLLARY 1.8. $U_{r,s}(\mathfrak{g}) \cong U_{r,s}(\mathfrak{n}^-) \otimes U^0 \otimes U_{r,s}(\mathfrak{n})$, *as vector spaces. In particular, it induces* $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes U_0 \otimes U_q(\mathfrak{n})$, *as vector spaces.* □

Let $Q = \mathbb{Z}\Psi$ denote the root lattice and set $Q^+ = \sum_{i=1}^n \mathbb{Z}_{\ge 0}\alpha_i$. Then for any $\zeta = \sum_{i=1}^n \zeta_i\alpha_i \in Q$, we denote

$$\omega_\zeta = \omega_1^{\zeta_1}\cdots\omega_n^{\zeta_n}, \qquad \omega_\zeta' = (\omega_1')^{\zeta_1}\cdots(\omega_n')^{\zeta_n}. \tag{6}$$

The following Q-graded structure on $U_{r,s}(\mathfrak{g})$ is necessary to develop to its weight representation theory discussed in the sequel.

COROLLARY 1.9 ([BGH, Corollary 2.7]). *For any* $\zeta = \sum_{i=1}^{n} \zeta_i \alpha_i \in Q$, *the defining relations* $(X2)$ *in* $U_{r,s}(\mathfrak{g})$ *take the form below:*

$$\omega_\zeta\, e_i\, \omega_\zeta^{-1} = \langle \omega_i', \omega_\zeta \rangle\, e_i, \qquad \omega_\zeta\, f_i\, \omega_\zeta^{-1} = \langle \omega_i', \omega_\zeta \rangle^{-1} f_i,$$
$$\omega_\zeta'\, e_i\, {\omega_\zeta'}^{-1} = \langle \omega_\zeta', \omega_i \rangle^{-1} e_i, \qquad \omega_\zeta'\, f_i\, {\omega_\zeta'}^{-1} = \langle \omega_\zeta', \omega_i \rangle\, f_i.$$

Then $U_{r,s}(\mathfrak{n}^\pm) = \bigoplus_{\eta \in Q^+} U_{r,s}^{\pm\eta}(\mathfrak{n}^\pm)$ *is* $Q^\pm$*-graded, where*

$$U_{r,s}^{\eta}(\mathfrak{n}^\pm) = \left\{ a \in U_{r,s}(\mathfrak{n}^\pm) \;\middle|\; \omega_\zeta\, a\, \omega_\zeta^{-1} = \langle \omega_\eta', \omega_\zeta \rangle\, a,\ \omega_\zeta'\, a\, {\omega_\zeta'}^{-1} = \langle \omega_\zeta', \omega_\eta \rangle^{-1} a \right\},$$

for $\eta \in Q^+ \cup Q^-$.

Moreover, $U = \bigoplus_{\eta \in Q} U_{r,s}^{\eta}(\mathfrak{g})$ *is* Q*-graded such that*

$$U_{r,s}^{\eta}(\mathfrak{g}) = \Big\{ \sum F_\alpha \omega_\mu' \omega_\nu E_\beta \in U \;\Big|\; \omega_\zeta\, (F_\alpha \omega_\mu' \omega_\nu E_\beta)\, \omega_\zeta^{-1} = \langle \omega_{\beta-\alpha}', \omega_\zeta \rangle\, F_\alpha \omega_\mu' \omega_\nu E_\beta,$$
$$\omega_\zeta'\, (F_\alpha \omega_\mu' \omega_\nu E_\beta)\, {\omega_\zeta'}^{-1} = \langle \omega_\zeta', \omega_{\beta-\alpha} \rangle^{-1} F_\alpha \omega_\mu' \omega_\nu E_\beta, \text{ with } \beta - \alpha = \eta \Big\},$$

where F_α *(resp.* E_β*) is a certain monomial* $f_{i_1} \cdots f_{i_l}$ *(resp.* $e_{j_1} \cdots e_{j_m}$*) such that* $\alpha_{i_1} + \cdots + \alpha_{i_l} = \alpha$ *(resp.* $\alpha_{j_1} + \cdots + \alpha_{j_m} = \beta$*).* □

2. Finite-Dimensional Weight Representation Theory and Category $\mathcal{O}$

As we know, the standard triangular decomposition of $U_{r,s}(\mathfrak{g})$ suggests that $U_{r,s}(\mathfrak{g})$ possesses highest weight representation theory. Indeed, this has been developed by Benkart and Witherspoon in [BW2] for $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$. In principle, one can expect the same theory to be valid as well for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$. To establish this, we will follow Benkart and Witherspoon's main ideas. However, to treat these cases in a unified fashion, we need to have better insights here and there to generalize the techniques used in the type A case. Throughout the article, we assume that $\mathbb{K}$ is an algebraically closed field containing $\mathbb{Q}(r,s)$ as a subfield and rs^{-1} is not a root of unity.

Let Λ be the weight lattice of $\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$, respectively. We adopt similar notions and notations in [BW1]. Associated to any $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra U^0 over $\mathbb{K}$ generated by the elements ω_i, ω_i' $(1 \le i \le n)$ to $\mathbb{K}$ given by

$$\hat{\lambda}(\omega_i) = \langle \omega_\lambda', \omega_i \rangle, \qquad \hat{\lambda}(\omega_i') = \langle \omega_i', \omega_\lambda \rangle^{-1}, \tag{1}$$

here we extend the definition of $\langle\,,\rangle$ from $\lambda \in Q$ to $\lambda \in \Lambda$ via taking appropriate half-integer powers when necessary, observing that $\Lambda \subseteq \bigoplus_{i=1}^{n} \frac{1}{2}\mathbb{Z}\alpha_i \subseteq \bigoplus_{i=1}^{n} \frac{1}{2}\mathbb{Z}\epsilon_i$.

Let M be a U-module of dimension $d < \infty$ where $U = U_{r,s}(\mathfrak{g})$. As $\mathbb{K}$ is algebraically closed, by linear algebra, we have

$$M = \bigoplus_{\chi} M_\chi,$$

where each $\chi : U^0 \longrightarrow \mathbb{K}$ is an algebra homomorphism, and M_χ is the generalized eigenspace given by

$$M_\chi = \left\{ m \in M \mid (\omega_i - \chi(\omega_i)1)^d m = 0 = (\omega_i' - \chi(\omega_i')1)^d m, \ \forall\, i \right\}. \tag{2}$$

When $M_\chi \neq 0$ we say that χ is a weight and M_χ is the corresponding weight space. In the case when M decomposes into genuine eigenspaces relative to U^0, we say that U^0 acts semisimply on M.

Relations in $(X2)$ imply

$$e_j M_\chi \subseteq M_{\chi\cdot\widehat{\alpha_j}}\,, \qquad f_j M_\chi \subseteq M_{\chi\cdot(\widehat{-\alpha_j})}\,, \tag{3}$$

where $\widehat{\alpha_j}$ is as in (1), and $\chi\cdot\psi$ is the homomorphism with values $(\chi\cdot\psi)(\omega_i) = \chi(\omega_i)\psi(\omega_i)$ and $(\chi\cdot\psi)(\omega_i') = \chi(\omega_i')\psi(\omega_i')$. In fact, if $(\omega_i - \chi(\omega_i)1)^k m = 0$, then $(\omega_i - \chi(\omega_i)\langle\omega_j', \omega_i\rangle 1)^k e_j m = 0$, and similarly for ω_i' and for f_j. On the one hand, (3) means that the sum of the eigenspaces is a submodule of M, and so if M is simple, the sum must be M itself, meanwhile we may replace the power d in (2) by 1, that is, U^0 acts semisimply on each simple M. On the other hand, a direct consequence of (3) is that for each simple M there is a homomorphism χ so that all the weights of M are of the form $\chi\cdot\hat{\zeta}$, where $\zeta\in Q$.

When all the weights of a module M are of the form $\hat{\lambda}$, where $\lambda\in\Lambda$, we say that M has weights in Λ. Any simple U-module having one weight in Λ has all its weights in Λ.

The observation below, which arises from Benkart and Witherspoon [BW2, Proposition 3.5] in the case when $\mathfrak{g} = \mathfrak{gl}_n$, or $\mathfrak{sl}_n$, also holds in our cases when $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$.

Lemma 2.1. *For $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, suppose that $\hat{\zeta} = \hat{\eta}$, where $\zeta, \eta\in\Lambda$. Assume that rs^{-1} is not a root of unity, then $\zeta = \eta$.*

Proof. The proof for $\mathfrak{g} = \mathfrak{sl}_n$ was given in [BW1, Proposition 3.5]. We now give the proof case by case for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$, respectively.

For $\zeta = \sum_{i=1}^n \zeta_i\alpha_i \in \Lambda$, by definition, we have

$$\text{(B)}\qquad \hat{\zeta}(\omega_i) = \langle\omega_\zeta', \omega_i\rangle = \begin{cases} r^{2(\epsilon_i,\zeta)}s^{2(\epsilon_{i+1},\zeta)}, & i<n,\\ r^{2(\epsilon_n,\zeta)}(rs)^{-\zeta_n}. & i=n;\end{cases}$$

$$\hat{\zeta}(\omega_i') = \langle\omega_i', \omega_\zeta\rangle^{-1} = \begin{cases} r^{2(\epsilon_{i+1},\zeta)}s^{2(\epsilon_i,\zeta)}, & i<n,\\ s^{2(\epsilon_n,\zeta)}(rs)^{-\zeta_n}. & i=n.\end{cases}$$

$$\text{(C)}\qquad \hat{\zeta}(\omega_i) = \langle\omega_\zeta', \omega_i\rangle = \begin{cases} r^{(\epsilon_i,\zeta)}s^{(\epsilon_{i+1},\zeta)}, & i<n,\\ r^{2(\epsilon_n,\zeta)}(rs)^{-2\zeta_n}. & i=n;\end{cases}$$

$$\hat{\zeta}(\omega_i') = \langle\omega_i', \omega_\zeta\rangle^{-1} = \begin{cases} r^{(\epsilon_{i+1},\zeta)}s^{(\epsilon_i,\zeta)}, & i<n,\\ s^{2(\epsilon_n,\zeta)}(rs)^{-2\zeta_n}. & i=n.\end{cases}$$

$$\text{(D)}\qquad \hat{\zeta}(\omega_i) = \langle\omega_\zeta', \omega_i\rangle = \begin{cases} r^{(\epsilon_i,\zeta)}s^{(\epsilon_{i+1},\zeta)}, & i<n,\\ r^{(\epsilon_{n-1},\zeta)}s^{-(\epsilon_n,\zeta)}(rs)^{-2\zeta_{n-1}}. & i=n;\end{cases}$$

$$\hat{\zeta}(\omega_i') = \langle\omega_i', \omega_\zeta\rangle^{-1} = \begin{cases} r^{(\epsilon_{i+1},\zeta)}s^{(\epsilon_i,\zeta)}, & i<n,\\ r^{-(\epsilon_n,\zeta)}s^{(\epsilon_{n-1},\zeta)}(rs)^{-2\zeta_{n-1}}. & i=n.\end{cases}$$

Denote $\mu = \zeta - \eta$, from $\hat{\zeta}(\omega_n) = \hat{\eta}(\omega_n)$ and $\hat{\zeta}(\omega_n') = \hat{\eta}(\omega_n')$, in the type B or C case, we get $r^{2(\epsilon_n,\mu)}(rs)^{-\mu_n} = 1$, $s^{2(\epsilon_n,\mu)}(rs)^{-\mu_n} = 1$; or $r^{2(\epsilon_n,\mu)}(rs)^{-2\mu_n} = 1$, $s^{2(\epsilon_n,\mu)}(rs)^{-2\mu_n} = 1$. So $(rs^{-1})^{2(\epsilon_n,\mu)} = 1$ which, together with the assumption, means the integer $2(\epsilon_n,\mu) = 0$, that is,

$$\mu_{n-1} = \mu_n, \quad (\textit{for type } \mathrm{B}), \quad \textit{or} \quad \mu_{n-1} = 2\mu_n, \quad (\textit{for type } \mathrm{C}). \tag{4}$$

Again from $\hat{\zeta}(\omega_{n-1}) = \hat{\eta}(\omega_{n-1})$ and $\hat{\zeta}(\omega'_{n-1}) = \hat{\eta}(\omega'_{n-1})$, in the type B or C case, we get $(\alpha_{n-1}, \mu) = 0$, that is,

$$(5) \qquad \mu_{n-2} = \mu_n, \quad (\textit{for type } \mathrm{B}), \quad \textit{or} \quad \mu_{n-2} = 2\mu_n, \quad (\textit{for type } \mathrm{C}).$$

But similar to the deduction in the case of type A (see [BW1]), noting $\mu_0 = 0$, we have

$$(6) \qquad \mu_{i+2} - \mu_{i+1} - \mu_i + \mu_{i-1} = 0, \quad (i = 1, 2, \cdots, n-2),$$

$$(7) \qquad \mu_{2k} = k\mu_2, \qquad \mu_{2k+1} = k\mu_2 + \mu_1.$$

Thus, by (4), (5) & (6), we get $\mu_n = \mu_{n-1} = \cdots = \mu_1 = \mu_0 = 0$ in the type B case. For the type C case, if $n = 2m$, by (5) & (7), we get $\mu_{n-2} = (m-1)\mu_2 = 2\mu_n = 2m\mu_2$, i.e., $\mu_2 = 0$, so $\mu_n = 0$; if $n - 1 = 2m$, then by (4), (5), & (7), we get $m\mu_2 = \mu_{n-1} = \mu_{n-2} = (m-1)\mu_2 + \mu_1$, i.e., $\mu_2 = \mu_1$, again by (4) & (7), we get $\mu_2 = 0$, so $\mu_n = 0$, which is reduced to the precondition of the proof in the type A case. Hence, using the same argument as in the case of type A ([BW1]), we have $\mu = 0$. Therefore, $\zeta = \eta$ in both cases B and C.

For the type D case, from $\hat{\zeta}(\omega_i) = \hat{\eta}(\omega_i)$ and $\hat{\zeta}(\omega'_i) = \hat{\eta}(\omega'_i)$ for $i = n-1, n$, we have $(rs^{-1})^{(\alpha_{n-1}, \mu)} = 1$ and $(rs^{-1})^{(\alpha_n, \mu)} = 1$, that means, together with the assumption, the integers $(\alpha_{n-1}, \mu) = 0$ and $(\alpha_n, \mu) = 0$. So we get $\mu_{n-2} = 2\mu_{n-1} = 2\mu_n$. If $n = 2m$, then $(m-1)\mu_2 = \mu_{n-2} = 2m\mu_2$, i.e., $\mu_2 = 0$. If $n - 1 = 2m$, applying (7) to $\mu_{n-1} = \mu_n$, we get $\mu_1 = 0$; applying (7) to $\mu_{n-2} = 2\mu_{n-1}$, we get $\mu_2 = 0$. So we have $\mu_n = 0$ for any n. Using the same proof as in the case of type A, we obtain $\mu = 0$, i.e., $\zeta = \eta$. □

Remark 2.2. Lemma 2.1 indicates that under the assumption that rs^{-1} is not a root of unity, we may simplify the notation by writing M_λ for the weight space rather than writing $M_{\hat{\lambda}}$ for $\lambda \in \Lambda$. So it makes sense to let (3) take the classical form: $e_j M_\lambda \subseteq M_{\lambda+\alpha_j}$ and $f_j M_\lambda \subseteq M_{\lambda-\alpha_j}$.

Similar to the proof of [BW2, Corollary 3.14], we have

Corollary 2.3. *Let M be a finite-dimensional $U_{r,s}(\mathfrak{g})$-module for $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$. Assume that rs^{-1} is not a root of unity, then the elements e_i, f_i $(1 \le i \le n)$ act nilpotently on M.* □

Obviously, when rs^{-1} is not a root of unity, a finite-dimensional simple U-module is a highest weight module by Corollary 2.3 and (3).

We state the definition of the category $\mathcal{O}$ of weight U-modules as in [BW1, Section 4].

Definition 2.4. *Let $\mathcal{O}$ denote the category of modules M for $U_{r,s}(\mathfrak{g})$ (where $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$) which satisfy the following conditions:*

($\mathcal{O}1$) *U^0 acts semisimply on M, and the set $\mathrm{wt}(M)$ of weights of M belongs to Λ : $M = \bigoplus_{\lambda \in \mathrm{wt}(M)} M_\lambda$, where $M_\lambda = \{\, m \in M \mid \omega_i.m = \langle \omega'_\lambda, \omega_i \rangle\, m,\ \omega'_i.m = \langle \omega'_i, \omega_\lambda \rangle^{-1} m,\ \forall\, i\, \}$;*

($\mathcal{O}2$) *$\dim_{\mathbb{K}} M_\lambda < \infty$ for all $\lambda \in \mathrm{wt}(M)$;*

($\mathcal{O}3$) *$\mathrm{wt}(M) \subseteq \cup_{\mu \in F}(\mu - Q^+)$ for some finite set $F \subset \Lambda$.*

The morphisms in $\mathcal{O}$ are U-module homomorphisms.

Actually, the category $\mathcal{O}$ just focuses on the class of the so-called *type* 1 U-modules like in the case of Drinfel'd-Jimbo quantum groups (see [J], [Jo], [KS]), which is closed under taking sub-object or sub-quotient object, making finite direct sum and taking tensor product.

Let V^ψ be the one-dimensional $\mathcal{B}$-module on which e_i acts as multiplication by 0 ($1 \le i \le n$), and U^0 acts via ψ, an algebra homomorphism from U^0 to $\mathbb{K}$. As usual, we can define the Verma module $M(\psi)$ with highest weight ψ to be the U-module induced from V^ψ, that is,

$$M(\psi) = U \otimes_{\mathcal{B}} V^\psi.$$

Set $v_\psi = 1 \otimes v \in M(\psi)$, where $v (\neq 0) \in V^\psi$. Then $e_i.v_\psi = 0$ ($1 \le i \le n$) and $a.v_\psi = \psi(a)\, v_\psi$ for any $a \in U^0$ by construction. By Corollary 1.8, $M(\psi) \cong U_{r,s}(\mathfrak{n}^-) \otimes v_\psi$. Corollary 1.9 indicates that each Verma module $M(\psi) \in \mathrm{Ob}(\mathcal{O})$ if and only if $\psi \in \hat{\Lambda}$.

Let N' be a proper submodule of $M(\psi)$, then (3) implies that

$$N' \subset \sum_{\mu \in Q^+ - \{0\}} M(\psi)_{\psi \cdot (\widehat{-\mu})},$$

as $M(\psi)_\psi = \mathbb{K} v_\psi$ generates $M(\psi)$. Hence, $M(\psi)$ has a unique maximal submodule N, namely the sum of all proper submodules, and a unique simple quotient, $L(\psi)$. Actually, all finite-dimensional simple U-modules are of this form, as the Theorem below indicates (which was proved by Benkart and Witherspoon [BW2, Theorem 2.1] in the case when $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{sl}_n$, but still holds with the same proof for our cases of $\mathfrak{g}$).

Theorem 2.5. *For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$, let M be a $U_{r,s}(\mathfrak{g})$-module, on which U^0 acts semisimply and which contains an element $m \in M_\psi$ ($\psi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$) such that $e_i.m = 0$ for all i. Then there is a unique homomorphism of $U_{r,s}(\mathfrak{g})$-modules $F : M(\psi) \longrightarrow M$ with $F(v_\psi) = m$. In particular, if rs^{-1} is not a root of unity and M is a finite-dimensional simple $U_{r,s}(\mathfrak{g})$-module, then $M \cong L(\psi)$ for some weight ψ.* □

As in [BW2, Lemma 2.3], it is easy to verify the commutation relations below.

Lemma 2.6. *For $m \ge 1$, set $[m]_i = \frac{r_i^m - s_i^m}{r_i - s_i}$. Then for $1 \le i \le n$, we have*

$$e_i f_i^m = f_i^m e_i + [m]_i\, f_i^{m-1} \frac{r_i^{1-m}\omega_i - s_i^{1-m}\omega_i'}{r_i - s_i},$$
$$e_i^m f_i = f_i e_i^m + [m]_i\, e_i^{m-1} \frac{s_i^{1-m}\omega_i - r_i^{1-m}\omega_i'}{r_i - s_i}.$$

□

Set $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, for any simple root $\alpha \in \Pi$, then for any $\lambda \in \Lambda$, $(\lambda, \alpha^\vee) \in \mathbb{Z}$ by definition. Let $\Lambda^+ \subset \Lambda$ be the subset of dominant weights, that is, $\Lambda^+ = \{\, \lambda \in \Lambda \mid (\lambda, \alpha_i^\vee) \ge 0, \text{ for } 1 \le i \le n \,\}$.

Similar to [BW2, Lemma 2.4] in the type A case, we have

Lemma 2.7. *For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, assume that rs^{-1} is not a root of unity. Let M be a nonzero finite-dimensional $U_{r,s}(\mathfrak{g})$-module on which U^0 acts semisimply. Suppose there is some nonzero vector $v \in M_\lambda$ with $\lambda \in \Lambda$ such that $e_i.v = 0$ for all i ($1 \le i \le n$). Then $\lambda \in \Lambda^+$.*

PROOF. It suffices to prove that $(\lambda, \alpha_n^\vee) \geq 0$, as the proof of $(\lambda, \alpha_i^\vee) \geq 0$ $(1 \leq i < n)$ is the same as that of [BW2, Lemma 2.4].

Since f_n acts nilpotently on M by Corollary 2.3, there is some integer $m \geq 0$ such that $f_n^{m+1}.v = 0$ but $f_n^m.v \neq 0$. Applying e_n to $f_n^{m+1}.v = 0$, using Lemma 2.6 and the fact that $e_n.v = 0$, we get $r_n^{-m}\hat{\lambda}(\omega_n) = s_n^{-m}\hat{\lambda}(\omega_n')$. Equivalently,

$$r_n^{-m} r^{2(\epsilon_n,\lambda)}(rs)^{-\lambda_n} = s_n^{-m} s^{2(\epsilon_n,\lambda)}(rs)^{-\lambda_n}, \qquad (\textit{for type } \mathrm{B})$$

$$r_n^{-m} r^{2(\epsilon_n,\lambda)}(rs)^{-2\lambda_n} = s_n^{-m} s^{2(\epsilon_n,\lambda)}(rs)^{-2\lambda_n}, \qquad (\textit{for type } \mathrm{C})$$

$$r^{-m} r^{(\epsilon_{n-1},\lambda)} s^{-(\epsilon_n,\lambda)}(rs)^{-2\lambda_{n-1}} = s^{-m} r^{-(\epsilon_n,\lambda)} s^{(\epsilon_{n-1},\lambda)}(rs)^{-2\lambda_{n-1}}, \qquad (\textit{for type } \mathrm{D})$$

or equivalently,

$$(r_n s_n^{-1})^{-m+(\lambda,\alpha_n^\vee)} = 1, \quad (\textit{for types } \mathrm{B, C, D}).$$

The assumption of rs^{-1} forces $(\lambda, \alpha_n^\vee) = m \geq 0$. Therefore, $\lambda \in \Lambda^+$. □

COROLLARY 2.8. *For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, assume that rs^{-1} is not a root of unity, then any finite-dimensional simple $U_{r,s}(\mathfrak{g})$-module with weights in Λ is isomorphic to $L(\lambda)$ for some $\lambda \in \Lambda^+$.* □

The representation theory of $U_{r,s}(\mathfrak{sl}_2)$, developed by Benkart and Witherspoon in [BW2], plays a crucial role in the classification of finite-dimensional simple modules for $U_{r,s}(\mathfrak{sl}_n)$ (see [BW2, Section 2]) like in the classical case of the simple Lie algebras or in the quantized case of the Drinfel'd-Jimbo quantum groups. Note the observation arising from the structure constants of $U_{r,s}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$: for any vertex i from the corresponding Dynkin diagram of type B, C, or D, respectively, $\langle \omega_i', \omega_i \rangle = r_i s_i^{-1}$ always holds. This fact guarantees that even in the two-parameter quantum orthogonal or symplectic groups $U_{r,s}(\mathfrak{g})$, there exist isomorphic copies of $U_{r,s}(\mathfrak{sl}_2)$ as well. This suggests that these quantum groups possess a familiar finite-dimensional (weight) representation theory provided that rs^{-1} is not a root of unity.

Now let us recall the representation theory for $U_{r,s}(\mathfrak{sl}_2)$. The first two assertions of the following Proposition comes from [BW2, Proposition 2.8 (i)], the last one may be regarded as an intrinsic generalization of [BW2, Proposition 2.8 (ii)] with a deep insight.

PROPOSITION 2.9. *Assume that rs^{-1} is not a root of unity. For $U = U_{r,s}(\mathfrak{sl}_2)$ generated by e, f, ω and ω', for a given $\phi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$, set $\phi = \phi(\omega)$, $\phi' = \phi(\omega')$, and in the Verma module $M(\phi)$, put $v_j = f^j/[j]! \otimes v_\phi$ for $j \geq 0$. Then*

(i) *$M(\phi)$ is a simple U-module if and only if $\phi \cdot r^{-j} - \phi' \cdot s^{-j} \neq 0$ for any $j \geq 0$.*

(ii) *If $\phi(\omega') = \phi(\omega)(rs^{-1})^{-m}$ for some integer $m \geq 0$, then $\mathrm{Span}_{\mathbb{K}}\{ v_j \mid j \geq m+1 \} \cong M(\phi-(m+1)\alpha)$ is the unique maximal submodule of $M(\phi)$. The quotient is the $(m+1)$-dimensional simple module $L(\phi)$ spanned by vectors $v_0, v_1, \cdots, v_m$ and having U-action given by*

$$\begin{aligned} \omega.v_j = \phi \cdot (rs^{-1})^{-j} v_j, \qquad \omega'.v_j = \phi \cdot (rs^{-1})^{-(m-j)} v_j, \\ e.v_j = \phi \cdot r^{-m}[m+1-j]\, v_{j-1}, \quad (v_{-1} = 0) \\ f.v_j = [j+1]\, v_{j+1}. \quad (v_{m+1} = 0) \end{aligned} \tag{8}$$

Any $(m+1)$-dimensional simple U-module is isomorphic to $L(\phi)$ for some such ϕ.

(iii) *If* $\nu=\nu_1\lambda_1+\cdots+\nu_n\lambda_n\in\Lambda^+$, *where* λ_i *is the* i*-th fundamental weight for* $\mathfrak{g}$, *then* $\hat{\nu}(\omega_i')=\hat{\nu}(\omega_i)(r_is_i^{-1})^{-\nu_i}$, *and the* U_i*-module* $L(\nu_i\lambda_i)$ *is* (ν_i+1)*-dimensional and has* U_i*-action given by* (8) *with* $\phi_i=\hat{\nu}(\omega_i)$, *where* U_i *is the copy of* $U_{r,s}(\mathfrak{sl}_2)$ *in* $U_{r,s}(\mathfrak{g})$ *corresponding to the* i*-th vertex of the Dynkin diagram.*

Proof. For the proof of the last assertion, it suffices to show that there hold

$$\frac{\hat{\nu}(\omega_i')}{\hat{\nu}(\omega_i)}=(r_is_i^{-1})^{-(\alpha_i^\vee,\,\nu)}=\frac{\widehat{\nu_i\lambda_i}(\omega_i')}{\widehat{\nu_i\lambda_i}(\omega_i)},\quad (\textit{for any }\ i) \tag{9}$$

for $\mathfrak{g}=\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$.

In the type A case, we have $\lambda_i=\epsilon_1+\cdots+\epsilon_i$ for $1\le i\le n$ and $\lambda_n=0$. By definition,

$$\begin{aligned}\frac{\hat{\nu}(\omega_i')}{\hat{\nu}(\omega_i)}&=\frac{s^{(\epsilon_i,\nu)}r^{(\epsilon_{i+1},\nu)}}{r^{(\epsilon_i,\nu)}s^{(\epsilon_{i+1},\nu)}}=(rs^{-1})^{-(\alpha_i,\nu)}=(rs^{-1})^{-\nu_i}\\&=\frac{s^{(\epsilon_i,\nu_i\lambda_i)}r^{(\epsilon_{i+1},\nu_i\lambda_i)}}{r^{(\epsilon_i,\nu_i\lambda_i)}s^{(\epsilon_{i+1},\nu_i\lambda_i)}}=\frac{\widehat{\nu_i\lambda_i}(\omega_i')}{\widehat{\nu_i\lambda_i}(\omega_i)}.\end{aligned}$$

For types B, C and D, it suffices to consider types B_2, C_2 and D_4, respectively.

In the type B_2 case, we have $\lambda_1=\epsilon_1$, $\lambda_2=\frac{1}{2}(\epsilon_1+\epsilon_2)$. By the defining formula (B) in Lemma 2.1, for $i=1$, it follows directly from the argument in the type A case; while for $i=2$, we get

$$\frac{\hat{\nu}(\omega_2')}{\hat{\nu}(\omega_2)}=\frac{s^{2(\epsilon_2,\nu)}}{r^{2(\epsilon_2,\nu)}}=(rs^{-1})^{-(\alpha_2^\vee,\,\nu)}=\frac{s^{2(\epsilon_2,\nu_2\lambda_2)}}{r^{2(\epsilon_2,\nu_2\lambda_2)}}=\frac{\widehat{\nu_2\lambda_2}(\omega_2')}{\widehat{\nu_2\lambda_2}(\omega_2)}.$$

In the type C_2 case, we have $\lambda_1=\epsilon_1$, $\lambda_2=\epsilon_1+\epsilon_2$. It suffices to consider the case $i=2$. Similarly, we have

$$\frac{\hat{\nu}(\omega_2')}{\hat{\nu}(\omega_2)}=\frac{s^{2(\epsilon_2,\nu)}}{r^{2(\epsilon_2,\nu)}}=(r_2s_2^{-1})^{-(\alpha_2^\vee,\,\nu)}=\frac{s^{2(\epsilon_2,\nu_2\lambda_2)}}{r^{2(\epsilon_2,\nu_2\lambda_2)}}=\frac{\widehat{\nu_2\lambda_2}(\omega_2')}{\widehat{\nu_2\lambda_2}(\omega_2)}.$$

In the type D_4 case, we have $\lambda_1=\epsilon_1$, $\lambda_2=\epsilon_1+\epsilon_2$, $\lambda_3=\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)$, $\lambda_4=\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)$. It suffices to consider the cases $i=3$, 4. By the formula (D) in Lemma 2.1, we have

$$\frac{\hat{\nu}(\omega_i')}{\hat{\nu}(\omega_i)}=(rs^{-1})^{(\alpha_i,\,\nu)}=\frac{\widehat{\nu_i\lambda_i}(\omega_i')}{\widehat{\nu_i\lambda_i}(\omega_i)},$$

for $i=3$, 4.

The proof is completed. □

Proposition 2.9 (iii) and its proof imply the following result.

Corollary 2.10. *Assume that* rs^{-1} *is not a root of unity and* $\lambda\in\Lambda^+$, *set* $\nu_i=(\lambda,\alpha_i^\vee)$, *then each vector* $f_i^{\nu_i+1}.v_\lambda$ *in the Verma* U*-module* $M(\lambda)$ *generates the Verma submodule* $M(\lambda-(\nu_i+1)\alpha_i)$ *for all* i, *where* $\mathfrak{g}=\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ *or* $\mathfrak{sp}_{2n}$.

Proof. It follows from a direct calculation of $e_if_i^{\nu_i+1}.v_\lambda=0$ by Lemma 2.6 and (9). □

More generally, we have

PROPOSITION 2.11. *Let $M(\lambda)$ be a Verma module with $\lambda \in \Lambda^+$. Then for every element ω of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$, there exists a Verma submodule in $M(\lambda)$ with highest weight*

$$\lambda_\omega = \omega(\lambda + \rho) - \rho, \tag{10}$$

where ρ is the half-sum of all positive roots of $\mathfrak{g}$. Every simple U-module as a composition factor of $M(\lambda)$ determines a highest weight module in $\mathcal{O}$. These highest weights are of the form (10).

PROOF. The proof of this proposition is analogous to that of the corresponding assertion in the classical theory (see Dixmier [D]). □

LEMMA 2.12. *For any simple U-module $L(\lambda)$ with $\lambda \in \Lambda^+$, take any $\beta = \sum_{i=1}^n m_i\alpha_i \in Q^+$ such that $m_i \le (\lambda, \alpha_i^\vee)$, $\forall\, i$, then the linear mapping $U_{r,s}^{-\beta}(\mathfrak{n}^-) \ni x \mapsto x.v_\lambda$ is injective.*

PROOF. By the definition of the Verma module, it is enough to show that $\lambda - \beta$ is not a weight of the maximal U-submodule N. This follows from Proposition 2.11, because no set of weights $\{\, \lambda_\omega - \sum_{i=1}^n n_i\alpha_i \mid n_i \in \mathbb{Z}^+ \,\}$, $\omega \in \mathcal{W} - \{1\}$ contains $\lambda - \beta$. □

LEMMA 2.13. *If an element $a \in U_{r,s}^{-\beta}(\mathfrak{n}^-)$ satisfies the relations $e_i a = a e_i$ for $i = 1, 2, \cdots, n$, then we have $a = 0$. If $f_i b = b f_i$, $i = 1, 2, \cdots, n$, for some $b \in U_{r,s}^{\beta}(\mathfrak{n})$, then $b = 0$.*

PROOF. Write $\beta = \sum_{i=1}^n m_i\alpha_i \in Q^+$, and take a dominant weight $\lambda \in \Lambda^+$ such that $(\lambda, \alpha_i^\vee) \ge m_i$ for all i. Consider the simple U-module $L(\lambda)$ with highest weight vector v_λ. Since $(e_i a).v_\lambda = (a e_i).v_\lambda = 0$ for all i, the vector $a.v_\lambda$ generates a proper submodule of $L(\lambda)$. Thus $a.v_\lambda = 0$, as $L(\lambda)$ is simple. Hence $a = 0$ by Lemma 2.12.

In order to prove the second assertion, we introduce a $\mathbb{Q}$-algebra isomorphism $\theta : U_{r,s}(\mathfrak{g}) \longrightarrow U_{r,s}(\mathfrak{g})$ defined by

$$\begin{aligned} \theta(r) &= s^{-1}, \qquad \theta(s) = r^{-1}, \\ \theta(\omega_i) &= \omega_i', \qquad \theta(\omega_i') = \omega_i, \\ \theta(e_i) &= f_i, \qquad \theta(f_i) = (r_i s_i) e_i. \end{aligned} \tag{11}$$

In fact, we can find that the image of θ is $\mathbb{Q}$-algebraically isomorphic to the associated quantum group $U_{s^{-1},r^{-1}}(\mathfrak{g})$, i.e., $\mathrm{Im}(\theta) \cong (U_{s^{-1},r^{-1}}(\mathfrak{g}), \langle\,|\,\rangle)$, where the pairing $\langle \omega_i' | \, \omega_j \rangle$ is defined via substituting (r, s) by (s^{-1}, r^{-1}) in the defining formula for $\langle \omega_i', \omega_j \rangle$ (see formulae (1_X) and (2) in Section 1).

Now applying the $\mathbb{Q}$-algebra isomorphism θ to the equation $f_i b = b f_i$, we get $\theta(b) = 0$, by the first assertion. Hence, $b = 0$. □

Returning to the pairing $\langle\,,\rangle :\ \mathcal{B}' \times \mathcal{B} \longrightarrow \mathbb{K}$ in Proposition 1.5, and combining with the Q-gradation on U (see Corollary 1.9), we have

PROPOSITION 2.14. *For any $\beta \in Q^+$, the restriction of the pairing $\langle\,,\rangle$ in Proposition 1.5 to $\mathcal{B}'^{-\beta} \times \mathcal{B}^\beta$ is nondegenerate.*

PROOF. We have to show that for any $a \in \mathcal{B}'^{-\beta}$ such that $\langle a, b \rangle = 0$ for some $b \in \mathcal{B}^\beta$, implies that $b = 0$. This will be proved by induction with respect to the usual ordering of Q_+. If β is a simple root, then it is true by formula (2) in Section

1. Let $\beta > 0$ with $\mathrm{ht}(\beta) > 1$ and suppose that it holds for all $\gamma \in Q^+$ such that $\beta - \gamma \in Q^+$.

Note that using the defining properties of skew-dual pairing and the comultiplication in U (see Proposition 1.2), we may check by induction:

$$\langle c\,\omega'_\nu, \omega_\mu\, d\rangle = \langle \omega'_\nu, \omega_\mu\rangle\, \langle c, d\rangle, \qquad \forall\, c \in U_{r,s}(\mathfrak{n}^-),\ d \in U_{r,s}(\mathfrak{n}), \tag{12}$$

$$\langle c, d\rangle = 0, \qquad c \in U^{-\sigma}_{r,s}(\mathfrak{n}^-),\ d \in U^{\delta}_{r,s}(\mathfrak{n}),\ \sigma,\, \delta \in Q^+,\ \sigma \neq \delta. \tag{13}$$

It suffices to assume that $b \in U^{\beta}_{r,s}(\mathfrak{n})$. By Proposition 1.2, we can write

$$\Delta(b) = \sum_{0 \le \gamma \le \beta} (\omega_\gamma \otimes 1)\, b_\gamma, \qquad b_\gamma \in U^{\gamma}_{r,s}(\mathfrak{n}) \otimes U^{\beta-\gamma}_{r,s}(\mathfrak{n}), \tag{14}$$

where $b_0 = b \otimes 1$ and $b_\beta = 1 \otimes b$. Let $\gamma \in Q^+$, $0 < \gamma < \beta$, $x \in \mathcal{B}'^{-\gamma}$ and $y \in \mathcal{B}'^{-(\beta-\gamma)}$.

By (2), (12) & (13), we have

$$0 = \langle xy, b\rangle = \langle x \otimes y, \Delta(b)\rangle = \langle x \otimes y, (\omega_\gamma \otimes 1)\, b_\gamma\rangle = \langle x \otimes y, b_\gamma\rangle. \tag{15}$$

By assumption, for any $\gamma' < \beta$ the restriction of $\langle\,,\rangle$ to $\mathcal{B}'^{-\gamma'} \times \mathcal{B}^{\gamma'}$ is nondegenerate, so is its extension to a bilinear form on $[\,\mathcal{B}'^{-\gamma} \otimes \mathcal{B}'^{-(\beta-\gamma)}\,] \times [\,\mathcal{B}^{\gamma} \otimes \mathcal{B}^{\beta-\gamma}\,]$. Hence it follows from (15) that $b_\gamma = 0$. Because of (14) this means that $\Delta(b) = b \otimes 1 + \omega_\beta \otimes b$. By (13), together with $\Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i$, we get $\hat{f}_i\, \hat{b} = \hat{b}\, \hat{f}_i$, and then $f_i\, b = b\, f_i$ for any i, after using φ (see the proof of [BGH, Theorem 2.5]). Thus, by Lemma 2.12, $b = 0$.

Similar reasoning indicates that for any $b \in \mathcal{B}^{\beta}$ such that $\langle a, b\rangle = 0$ for some $a \in \mathcal{B}'^{-\beta}$ implies that $a = 0$. □

In what follows, we consider the finite-dimensionality question of the simple $U_{r,s}(\mathfrak{g})$-modules $L(\lambda)$ with $\lambda \in \Lambda^+$. This problem has been solved by Benkart and Witherspoon in [BW2, Section 2] in the case when $\mathfrak{g} = \mathfrak{gl}_n$, or $\mathfrak{sl}_n$. The same idea can be used to prove that $M(\lambda)$ has a $U_{r,s}(\mathfrak{g})$-submodule $M'(\lambda)$ of finite codimension, as $L(\lambda)$ is the quotient of $M(\lambda)$ by its unique maximal submodule, where $M'(\lambda)$ is defined by

$$M'(\lambda) = \sum_{i=1}^{n} U_{r,s}(\mathfrak{g}) f_i^{k_i+1}.v_\lambda \cong \sum_{i=1}^{n} M\big(\lambda - (k_i{+}1)\alpha_i\big), \tag{16}$$

where $k_i = (\lambda, \alpha_i^\vee)$ for all i. That is, to prove the module $L'(\lambda) = M(\lambda)/M'(\lambda)$ is nonzero and finite-dimensional. $L'(\lambda) \neq 0$ is clear, since any weight in $M'(\lambda)$ is less than or equal to $\lambda - (k_i{+}1)\alpha_i$ for some i, $v_\lambda \notin M'(\lambda)$.

Lemma 2.15. (i) ([BW2, Lemma 2.10]) *The elements e_j, f_j $(1 \le j \le n)$ act locally nilpotently on $U_{r,s}(\mathfrak{g})$-module $L'(\lambda)$.*

(ii) ([BW2, Lemma 2.11]) *Assume that rs^{-1} is not a root of unity, $V = \bigoplus_{j \in \mathbb{Z}^+} V_{\lambda - j\alpha} \in \mathrm{Ob}(\mathcal{O})$ is a $U_{r,s}(\mathfrak{sl}_2)$-module for some weight $\lambda \in \Lambda$. If e, f act locally nilpotently on V, then $\dim_{\mathbb{K}} V < \infty$, and the weights of V are preserved under the simple reflection taking α to $-\alpha$.*

Proof. The proof of (i) is parallel to the type A case; the second part assertion is direct from [BW2]. □

Proposition 2.16. *Assume that rs^{-1} is not a root of unity. Then for the $U_{r,s}(\mathfrak{g})$-module $L'(\lambda) \in \mathrm{Ob}(\mathcal{O})$ with $\lambda \in \Lambda^+$, we have $\dim_{\mathbb{K}} L'(\lambda) < \infty$, so $\dim_{\mathbb{K}} L(\lambda) < \infty$.*

PROOF. Consider $L'(\lambda)$ as a U_i-module, where U_i is the copy generated by e_i, f_i, ω_i, ω_i'. For μ a weight of $L'(\lambda)$, applying Lemma 2.15 to the U_i-module

$$L_i'(\mu) = U_i.L'(\lambda)_\mu = \bigoplus_{j\in\mathbb{Z}^+} L_i'(\mu)_{\lambda'-j\alpha_i}$$

for some weight $\lambda' \leq \lambda$, we get that the simple reflection w_i preserves the weights of $L_i'(\mu)$, so $w_i(\mu)$ is a weight of $L'(\lambda)$. That is, the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ preserves the set of weights of $L'(\lambda)$. From Lie theory, we know that each $\mathcal{W}$-orbit only contains one dominant weight. But there are only finitely many dominant weights $\leq \lambda$, and as each weight space of $L'(\lambda)$ is of finite-dimension, we have $\dim_{\mathbb{K}} L'(\lambda) < \infty$. Thereby, $\dim_{\mathbb{K}} L(\lambda) < \infty$. □

For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$, Corollary 2.8 and Proposition 2.16 imply the following

COROLLARY 2.17. *A finite-dimensional simple object in the category $\mathcal{O}$ is precisely a $U_{r,s}(\mathfrak{g})$-module $L(\lambda)$ for some $\lambda \in \Lambda^+$, and $L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$.* □

Finite-dimensional simple (weight) modules of generic type. As noted in [BW2, Section 2], for $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{sl}_n$, Benkart and Witherspoon gave a description of a classification of finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules. We find that a similar structural feature for finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules also holds when $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$, after modifying some of the treatments.

Given a one-dimensional $U_{r,s}(\mathfrak{g})$-module L, Theorem 2.5 indicates that $L = L(\chi)$ for some $\chi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$ with the elements e_i, f_i $(1 \leq i \leq n)$ trivially acting on $L(\chi)$. Relation $(X3)$ $(X = B, C, D)$ gives

$$\chi(\omega_i) = \chi(\omega_i'), \qquad (1 \leq i \leq n). \tag{17}$$

Conversely, if $\chi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$ satisfies the equation (17), then Proposition 2.9 (ii) guarantees $\dim_{\mathbb{K}} L(\chi) = 1$. We denote by L_χ the one-dimensional $U_{r,s}(\mathfrak{g})$-module $L(\chi)$.

The following Lemma was proved by Benkart and Witherspoon in the case of type A. We will give a unified proof for the classical types of $\mathfrak{g}$ based on an intrinsic observation in Proposition 2.9 (ii) & (iii).

LEMMA 2.18. *Assume rs^{-1} is not a root of unity. Given a finite-dimensional simple $U_{r,s}(\mathfrak{g})$-module $L(\psi)$ with highest weight ψ, there exists a pair (χ, λ), where $\chi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$ such that* (17) *holds, and $\lambda \in \Lambda^+$, so that $\psi = \chi \cdot \hat{\lambda}$, and $\mathrm{wt}\big(L(\psi)\big) \subseteq \chi \cdot \hat{\Lambda}$.*

PROOF. As $L(\psi)$ is finite-dimensional and simple, for each pair of eigenvalues $(\psi(\omega_i), \psi(\omega_i'))$ when considering $L(\psi)$ as a U_i-module (where U_i is a $U_{r,s}(\mathfrak{sl}_2)$-copy of $U_{r,s}(\mathfrak{g})$), Proposition 2.9 (ii) tells us that there exists a nonnegative integer ν_i for each index i such that $\psi(\omega_i') = \psi(\omega_i)(r_i s_i^{-1})^{-\nu_i}$. Set $\lambda = \sum_{i=1}^n \nu_i \lambda_i$ where λ_i is the ith fundamental weight of $\mathfrak{g}$, then $\lambda \in \Lambda^+$.

Now we take $\chi(\omega_i) = \psi(\omega_i)\hat{\lambda}_i(\omega_i)^{-1}$ and $\chi(\omega_i') = \psi(\omega_i')\hat{\lambda}_i(\omega_i')^{-1}$, that is, $\chi = \psi \cdot \hat{\lambda}_i^{-1} \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$ and satisfies

$$\begin{aligned}\chi(\omega_i') &= \psi(\omega_i')\hat{\lambda}_i^{-1}(\omega_i') = \psi(\omega_i)(r_i s_i^{-1})^{-\nu_i}\hat{\lambda}_i^{-1}(\omega_i') \\ &= \psi(\omega_i)\hat{\lambda}^{-1}(\omega_i) \qquad \big(\text{ by } (9)\ \big) \\ &= \chi(\omega_i),\end{aligned}$$

as required. The last assertion that $\mathrm{wt}(L(\psi)) \subseteq \chi \cdot \hat{\Lambda}$ is quite clear. □

Similar to [BW2, Theorem 2.19], for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$, we have the classification Theorem for finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules as follows.

THEOREM 2.19. *Let rs^{-1} be a non-root of unity. Each finite-dimensional simple $U_{r,s}(\mathfrak{g})$-module $L(\psi)$ with $\psi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$ is isomorphic to $L_\chi \otimes L(\lambda)$, where $\chi \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$ with $\chi(\omega_i) = \chi(\omega_i')$ $(1 \le i \le n)$ and $\lambda \in \Lambda^+$.* □

3. R-matrices, Quantum Casimir Operators, Complete Reducibility

For any two objects M, $M' \in \mathrm{Ob}(\mathcal{O})$, Benkart and Witherspoon in [BW1, Section 4] constructed a $U_{r,s}(\mathfrak{sl}_n)$-module isomorphism

$$R_{M',M} : M' \otimes M \longrightarrow M \otimes M'$$

by a remarkable method due to Jantzen [J, Chap. 7] for the quantum groups $U_q(\mathfrak{g})$ of Drinfel'd-Jimbo type.

The aim of this section is to generalize this result to the setting of $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, $\mathfrak{sp}_{2n}$.

Noting that the weight lattice

$$\Lambda \subseteq \bigoplus_{i=1}^{n} \frac{1}{2}\mathbb{Z}\alpha_i \subseteq \bigoplus_{i=1}^{n} \frac{1}{2}\mathbb{Z}\epsilon_i,$$

as it was done in formula (1) of Section 2, for $\lambda \in \Lambda$, we have an algebra homomorphism $\hat{\lambda} \in \mathrm{Hom}_{\mathrm{Alg}}(U^0, \mathbb{K})$. Furthermore, we extend the pairing $\langle\, ,\, \rangle$ to $\Lambda \times \Lambda$, such that for any $\lambda = \sum_{i=1}^n p_i\alpha_i$, $\mu = \sum_{i=1}^n q_i\alpha_i \in \Lambda$ with p_i, $q_i \in \frac{1}{2}\mathbb{Z}$, we define

$$\langle \omega_\lambda', \omega_\mu \rangle = \Pi_{i=1}^n \hat{\lambda}(\omega_i)^{q_i}, \tag{1}$$

which is well-defined in the algebraically closed field $\mathbb{K}$.

Now we define the map $f : \Lambda \times \Lambda \longrightarrow \mathbb{K}^*$ by

$$f(\lambda, \mu) = \langle \omega_\mu', \omega_\lambda \rangle^{-1}, \tag{2}$$

which satisfies

$$\begin{gathered} f(\lambda + \mu, \nu) = f(\lambda, \nu)\, f(\mu, \nu), \\ f(\lambda, \mu + \nu) = f(\lambda, \mu)\, f(\lambda, \nu), \\ f(\alpha_i, \mu) = \langle \omega_\mu', \omega_i \rangle^{-1}, \qquad f(\lambda, \alpha_i) = \langle \omega_i', \omega_\lambda \rangle^{-1}. \end{gathered} \tag{3}$$

And we define the linear transformation $\tilde{f} = \tilde{f}_{M,M'} : M \otimes M' \longrightarrow M \otimes M'$ by $\tilde{f}(m \otimes m') = f(\lambda, \mu)\,(m \otimes m')$ for $m \in M_\lambda$ and $m' \in M_\mu'$.

Owing to $\Delta(e_i)=e_i\otimes 1+\omega_i\otimes e_i$, we have $\Delta(x)\in\sum_{0\le\nu\le\zeta}U_{r,s}^{\zeta-\nu}(\mathfrak{n})\omega_\nu\otimes U_{r,s}^{\nu}(\mathfrak{n})$, for all $x\in U_{r,s}^{\zeta}(\mathfrak{n})$, by induction. For each i, the expression of $\Delta(x)$ defines two skew-derivations $\partial_i,\ {}_i\partial:\ U_{r,s}^{\zeta}(\mathfrak{n})\longrightarrow U_{r,s}^{\zeta-\alpha_i}(\mathfrak{n})$ such that

$$(4)\qquad \begin{aligned}\Delta(x)&=x\otimes 1+\sum_{i=1}^{n}\partial_i(x)\,\omega_i\otimes e_i+\textit{the rest},\\ \Delta(x)&=\omega_\zeta\otimes x+\sum_{i=1}^{n}e_i\,\omega_{\zeta-\alpha_i}\otimes\ {}_i\partial(x)+\ \textit{the rest},\end{aligned}$$

where in each case "*the rest*" refers to terms involving products of more than one e_j in the second (resp. first) factor. More precisely, parallel to [BW1, Lemma 4.6] or comparing with [KS, Lemmas 6.14, 6.17], we have

LEMMA 3.1. *For all $x\in U_{r,s}^{\zeta}(\mathfrak{n})$, $x'\in U_{r,s}^{\zeta'}(\mathfrak{n})$, and $y\in U_{r,s}(\mathfrak{n}^-)$, the following hold:*

(i) $\partial_i(xx')=\langle\omega'_{\zeta'},\omega_i\rangle\,\partial_i(x)\,x'+x\,\partial_i(x')$.
(ii) ${}_i\partial(xx')=\ {}_i\partial(x)\,x'+\langle\omega'_i,\omega_\zeta\rangle\,x\,{}_i\partial(x')$.
(iii) $\langle f_iy,\,x\rangle=\langle f_i,e_i\rangle\,\langle y,\,{}_i\partial(x)\rangle=(s_i-r_i)^{-1}\langle y,\,{}_i\partial(x)\rangle$.
(iv) $\langle yf_i,\,x\rangle=\langle f_i,e_i\rangle\,\langle y,\,\partial_i(x)\rangle=(s_i-r_i)^{-1}\langle y,\,\partial_i(x)\rangle$.
(v) $f_ix-xf_i=(s_i-r_i)^{-1}\big(\partial_i(x)\,\omega_i-\omega'_i\,{}_i\partial(x)\big)$. □

Also, for each i, the expression of $\Delta(y)$ for $y\in U_{r,s}^{-\zeta}(\mathfrak{n}^-)$ defines two skew-derivations $\partial_i,\ {}_i\partial:\ U_{r,s}^{-\zeta}(\mathfrak{n}^-)\longrightarrow U_{r,s}^{-\zeta+\alpha_i}(\mathfrak{n}^-)$ such that

$$(5)\qquad \begin{aligned}\Delta(y)&=y\otimes\omega'_\zeta+\sum_{i=1}^{n}\partial_i(y)\otimes f_i\,\omega'_{\zeta-\alpha_i}+\textit{the rest},\\ \Delta(y)&=1\otimes y+\sum_{i=1}^{n}f_i\otimes\ {}_i\partial(y)\,\omega'_i+\textit{the rest}.\end{aligned}$$

Parallel to [BW1, Lemma 4.8], we have

LEMMA 3.2. *For all $y\in U_{r,s}^{-\zeta}(\mathfrak{n}^-)$, $y'\in U_{r,s}^{-\zeta'}(\mathfrak{n}^-)$, and $x\in U_{r,s}(\mathfrak{n})$, the following hold:*

(i) $\partial_i(yy')=\partial_i(y)\,y'+\langle\omega'_\zeta,\omega_i\rangle\,y\,\partial_i(y')$.
(ii) ${}_i\partial(yy')=\langle\omega'_i,\omega_{\zeta'}\rangle\,{}_i\partial(y)\,y'+y\,{}_i\partial(y')$.
(iii) $\langle y,\,e_ix\rangle=\langle f_i,e_i\rangle\,\langle\partial_i(y),\,x\rangle=(s_i-r_i)^{-1}\langle\partial_i(y),\,x\rangle$.
(iv) $\langle y,\,xe_i\rangle=\langle f_i,e_i\rangle\,\langle\,{}_i\partial(y),\,x\rangle=(s_i-r_i)^{-1}\langle\,{}_i\partial(y),\,x\rangle$.
(v) $e_iy-ye_i=(r_i-s_i)^{-1}\big(\omega_i\,\partial_i(y)-\ {}_i\partial(y)\,\omega'_i\big)$. □

By Proposition 2.14, the spaces $U_{r,s}^{\zeta}(\mathfrak{n})$ and $U_{r,s}^{-\zeta}(\mathfrak{n}^-)$ are non-degenerately paired. We may select a basis $\{u_k^\zeta\}_{k=1}^{d_\zeta}$, ($d_\zeta=\dim U_{r,s}^{\zeta}(\mathfrak{n})$), for $U_{r,s}^{\zeta}(\mathfrak{n})$ and a dual basis $\{v_k^\zeta\}_{k=1}^{d_\zeta}$ for $U_{r,s}^{-\zeta}(\mathfrak{n}^-)$. Then for each $x\in U_{r,s}^{\zeta}(\mathfrak{n})$ and $y\in U_{r,s}^{-\zeta}(\mathfrak{n}^-)$, we have

$$(6)\qquad x=\sum_{k=1}^{d_\zeta}\langle v_k^\zeta,x\rangle\,u_k^\zeta,\qquad y=\sum_{k=1}^{d_\zeta}\langle y,u_k^\zeta\rangle\,v_k^\zeta.$$

For $\zeta\in Q^+=\bigoplus_{i=1}^{n}\mathbb{Z}^+\alpha_i$, we define

$$(7)\qquad \Theta_\zeta=\sum_{k=1}^{d_\zeta}v_k^\zeta\otimes u_k^\zeta.$$

Set $\Theta_\zeta = 0$ if $\zeta \notin Q^+$. Similar to [BW1, Lemma 4.10], for the cases when $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, we also have

LEMMA 3.3. *For $1 \le i \le n$, the following relations hold*
(i) $(\omega_i \otimes \omega_i)\,\Theta_\zeta = \Theta_\zeta\,(\omega_i \otimes \omega_i)$, $\qquad (\omega_i' \otimes \omega_i')\,\Theta_\zeta = \Theta_\zeta\,(\omega_i' \otimes \omega_i')$;
(ii) $(e_i \otimes 1)\,\Theta_\zeta + (\omega_i \otimes e_i)\,\Theta_{\zeta-\alpha_i} = \Theta_\zeta\,(e_i \otimes 1) + \Theta_{\zeta-\alpha_i}\,(\omega_i' \otimes e_i)$;
(iii) $(1 \otimes f_i)\,\Theta_\zeta + (f_i \otimes \omega_i')\,\Theta_{\zeta-\alpha_i} = \Theta_\zeta\,(1 \otimes f_i) + \Theta_{\zeta-\alpha_i}\,(f_i \otimes \omega_i)$. □

Now we define

$$\Theta = \sum_{\zeta \in Q^+} \Theta_\zeta. \tag{8}$$

Given $U_{r,s}(\mathfrak{g})$-module M and M' in $\mathcal{O}$, we apply Θ to their tensor product:

$$\Theta = \Theta_{M,M'} : M \otimes M' \longrightarrow M \otimes M'.$$

Note that $\Theta_\zeta : M_\lambda \otimes M'_\mu \longrightarrow M_{\lambda-\zeta} \otimes M'_{\mu+\zeta}$ for all λ, $\mu \in \Lambda$, and there are only finitely many $\zeta \in Q^+$ such that $M'_{\mu+\zeta} \neq 0$, thanks to condition ($\mathcal{O}3$). So Θ is a well-defined linear transformation on $M \otimes M'$. After appropriately ordering the chosen countable bases of weight vectors for both M and M', we see that each Θ_ζ with $\zeta > 0$ has a strictly triangular matrix, while $\Theta_0 = 1 \otimes 1$ acts as the identity transformation on $M \otimes M'$, hence $\Theta_{M,M'}$ is an invertible transformation.

THEOREM 3.4. *Let M and M' be $U_{r,s}(\mathfrak{g})$-modules in $\mathcal{O}$ where $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$. Then the map*

$$R_{M',M} = \Theta \circ \tilde{f} \circ P : M' \otimes M \longrightarrow M \otimes M'$$

is an isomorphism of $U_{r,s}(\mathfrak{g})$-modules, where $P : M' \otimes M \longrightarrow M \otimes M'$ is the flip map such that $P(m' \otimes m) = m \otimes m'$ for any $m \in M$, $m' \in M'$.

PROOF. Obviously, $R_{M',M}$ is invertible. It remains to show that $R_{M',M}$ is a $U_{r,s}(\mathfrak{g})$-module homomorphism, that is, to check that

$$\Delta(a) R_{M',M}(m' \otimes m) = R_{M',M} \Delta(a)(m' \otimes m) \tag{9}$$

holds for all $a \in U_{r,s}(\mathfrak{g})$, $m \in M_\lambda$ and $m' \in M'_\mu$. It suffices to verify (9) for the generators e_n, f_n, ω_n, ω_n', because the subalgebra generated by the first $4(n-1)$ generators e_i, f_i, ω_i, ω_i' $(1 \le i < n)$ is isomorphic to $U_{r,s}(\mathfrak{sl}_n)$, and this can be reduced to the proof of the type A case (see [BW1, Theorem 4.11]). We will present the computation just for $a = f_n$. Using Lemma 3.3 (iii), we get

$$\begin{aligned}
\text{LHS of (9)} &= f(\lambda,\mu)\Delta(f_n)\Theta(m \otimes m') \\
&= f(\lambda,\mu)(1 \otimes f_n)\big(\textstyle\sum \Theta_\zeta\big)(m \otimes m') \\
&\quad + f(\lambda,\mu)(f_n \otimes \omega_n')\big(\textstyle\sum \Theta_{\zeta-\alpha_n}\big)(m \otimes m') \\
&= f(\lambda,\mu)\big(\textstyle\sum \Theta_\zeta\big)(1 \otimes f_n)(m \otimes m') \\
&\quad + f(\lambda,\mu)\big(\textstyle\sum \Theta_{\zeta-\alpha_n}\big)(f_n \otimes \omega_n)(m \otimes m') \\
&= f(\lambda,\mu)\langle \omega_n', \omega_\lambda \rangle \big(\textstyle\sum \Theta_\zeta\big)(\omega_n' m \otimes f_n m') \\
&\quad + f(\lambda,\mu)\langle \omega_\mu', \omega_n \rangle \big(\textstyle\sum \Theta_{\zeta-\alpha_n}\big)(f_n m \otimes m').
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
\text{RHS of (9)} &= R_{M',M}(m'\otimes f_n m + f_n m'\otimes \omega_n' m)\\
&= (\Theta\circ\tilde f)(f_n m\otimes m' + \omega_n' m\otimes f_n m')\\
&= f(\lambda-\alpha_n,\mu)\Theta(f_n m\otimes m') + f(\lambda,\mu-\alpha_n)\Theta(\omega_n' m\otimes f_n m')\\
&= f(\lambda-\alpha_n,\mu)\big(\sum\Theta_\zeta\big)(f_n\otimes 1)(m\otimes m')\\
&\qquad + f(\lambda,\mu-\alpha_n)\big(\sum\Theta_\zeta\big)(\omega_n'\otimes f_n)(m\otimes m').
\end{aligned}
$$

Thus (3) indicates that (9) holds. □

REMARK 3.5. Similar to the treatment in [BW1, Section 5] for the type A case, we can prove the maps $R_{M',M}$ satisfy the quantum Yang-Baxter equation for our cases. That is, given three $U_{r,s}(\mathfrak{g})$-modules M, M', M'' in $\mathcal{O}$, we have $R_{12}\circ R_{23}\circ R_{12} = R_{23}\circ R_{12}\circ R_{23}$ as maps from $M\otimes M'\otimes M''$ to $M''\otimes M'\otimes M$ (see [BW1, Theorem 5.4]). On the other hand, we also can prove the hexagon identities (see [BW1, Theorem 5.7]) for the maps $R_{M',M}$ by the same approach. Consequently, $\mathcal{O}$ is a braided monoidal category with braiding $R = R_{M',M}$ for each pair of modules M', M in $\mathcal{O}$.

Quantum Casimir operators and complete reducibility. The $U_{r,s}(\mathfrak{g})$-module isomorphisms $R_{M',M}$ constructed in Theorem 3.4, which are called the R-matrices, are mainly determined by Θ. For the expression (7) of Θ_ζ, we set

$$\Omega_\zeta = \sum_k S(v_k^\zeta)u_k^\zeta, \qquad \Omega_\zeta' = \theta(\Omega_\zeta), \tag{10}$$

$$\Omega = \sum_{\zeta\in Q^+}\Omega_\zeta, \qquad \Omega' = \sum_{\zeta\in Q^+}\Omega_\zeta', \tag{11}$$

where θ is the $\mathbb{Q}$-algebra isomorphism of $U_{r,s}(\mathfrak{g})$ into its associated quantum group $U_{s^{-1},r^{-1}}(\mathfrak{g})$ (for definition, see [BGH]) introduced in the formula (11) in Section 2. Obviously, Θ_ζ, Ω_ζ, Ω and Ω' are independent of the choice of bases $\{u_k^\zeta\}$ and $\{v_k^\zeta\}$. Ω preserves the weight spaces of any $M\in\mathcal{O}$.

DEFINITION 3.6. *The element Ω is called a quantum Casimir element for the two-parameter quantum group $U_{r,s}(\mathfrak{g})$.*

PROPOSITION 3.7. *Let ψ and φ be the algebra automorphisms of $U_{r,s}(\mathfrak{g})$ such that $\psi(\omega_i)=\omega_i$, $\psi(\omega_i')=\omega_i'$, $\psi(e_i)=\omega_i'\omega_i^{-1}e_i$, $\psi(f_i)=f_i\omega_i'^{-1}\omega_i$ and $\varphi(\omega_i)=\omega_i$, $\varphi(\omega_i')=\omega_i'$, $\varphi(e_i)=e_i\omega_i^{-1}\omega_i'$, $\varphi(f_i)=\omega_i\omega_i'^{-1}f_i$. Then*

$$\psi(a)\,\Omega = \Omega\,a, \qquad \varphi(a)\,\Omega' = \Omega'\,a, \quad \textit{for}\ \ a\in U_{r,s}(\mathfrak{g}). \tag{12}$$

PROOF. Since ψ is an algebra automorphism, it is enough to prove the first assertion for the generators $a=\omega_i$, ω_i', e_i, f_i. For $a=\omega_i$ or ω_i', it is obviously true. Applying the mapping $\mathfrak{m}\circ(S\otimes 1)$ to both sides of Lemma 3.3 (ii) & (iii) (where $\mathfrak{m}$ is the product of $U_{r,s}(\mathfrak{g})$ and S is its antipode) and summing over $\zeta\in Q^+$ we obtain $\Omega\,e_i = \omega_i'\omega_i^{-1}e_i\Omega$ and $\Omega\,f_i = f_i\omega_i'^{-1}\omega_i\Omega$. This means that $\Omega\,e_i=\psi(e_i)\,\Omega$ and $\Omega\,f_i=\psi(f_i)\,\Omega$. Applying the automorphism θ we get the assertion for Ω'. □

COROLLARY 3.8. *For $M\in\mathrm{Ob}(\mathcal{O})$, assume that $m\in M_\lambda$. Then*

(i) $\Omega\,e_i.m = (r_is_i^{-1})^{-(\lambda+\alpha_i,\alpha_i^\vee)}e_i\,\Omega.m$,

(ii) $\Omega\,f_i.m = (r_is_i^{-1})^{(\lambda,\alpha_i^\vee)}f_i\,\Omega.m$.

PROOF. By Proposition 3.7, for $m \in M_\lambda$, we have

$$\psi(e_i)\,\Omega.m = \langle \omega'_{\lambda+\alpha_i}, \omega_i\rangle^{-1}\langle \omega'_i, \omega_{\lambda+\alpha_i}\rangle^{-1} e_i\,\Omega.m,$$
$$\psi(f_i)\,\Omega.m = \langle \omega'_\lambda, \omega_i\rangle\,\langle \omega'_i, \omega_\lambda\rangle\, f_i\,\Omega.m.$$

Using formulas (B), (C), & (D) in Lemma 2.1, we can conclude the required result. □

REMARK 3.9. According to Section 1, we have made a convention: we have $r_i = r^{(\alpha_i,\alpha_i)}$, $s_i = s^{(\alpha_i,\alpha_i)}$ only in the type B case,so $(r_i s_i^{-1})^{(\lambda,\alpha_i^\vee)} = (rs^{-1})^{2(\lambda,\alpha_i)}$ for any i. However, for any other case, we always have $(r_i s_i^{-1})^{(\lambda,\alpha_i^\vee)} = (rs^{-1})^{(\lambda,\alpha_i)}$ for any i since $r_i = r^{\frac{(\alpha_i,\alpha_i)}{2}}$, $s_i = s^{\frac{(\alpha_i,\alpha_i)}{2}}$. Based on this observation, we make the following definition.

DEFINITION 3.10. *For $M \in \mathrm{Ob}(\mathcal{O})$, define a linear operator $\omega : M \longrightarrow M$ by setting*

$$\omega.v_\mu = (rs^{-1})^{\frac{\Delta_{X,B}}{2}(\mu+\rho,\mu+\rho)} v_\mu, \quad \textit{for} \;\; v_\mu \in M_\mu, \tag{13}$$

where ρ is the half-sum of all positive roots of $\mathfrak{g}$, and $\Delta_{X,B} = 2$ if $X = B$, otherwise, $\Delta_{X,B}$ will take value 1.

PROPOSITION 3.11. *Assume that the Verma module $M(\lambda) \in \mathrm{Ob}(\mathcal{O})$, then the operator $\Omega\omega$ is a multiple of the identity operator, that is,*

$$\Omega\omega = (rs^{-1})^{\frac{\Delta_{X,B}}{2}(\lambda+\rho,\lambda+\rho)} I. \tag{14}$$

PROOF. Let v_λ be a highest weight vector of the Verma module $M(\lambda)$. Then $M(\lambda) = U_{r,s}(\mathfrak{n}^-)v_\lambda = \sum_{\beta\in Q^+} U_{r,s}^{-\beta}(\mathfrak{n}^-)v_\lambda$. For $f_\beta \in U_{r,s}^{-\beta}(\mathfrak{n}^-)$, denote $v_{\lambda-\beta} := f_\beta.v_\lambda$, which is a weight vector of weight $\lambda-\beta$. We claim that

$$\Omega\omega.f_i.v_{\lambda-\beta} = f_i.\Omega\omega.v_{\lambda-\beta}, \tag{15}$$

for any $\beta \in Q^+$ and any i. Indeed, noting that

$$\frac{1}{2}\big[\,(\sigma-\alpha_i+\rho,\sigma-\alpha_i+\rho) - (\sigma+\rho,\sigma+\rho)\,\big] + (\sigma,\alpha_i) = \frac{1}{2}\big[\,(\alpha_i,\alpha_i) - 2(\alpha_i,\rho)\,\big] = 0,$$

and setting $\lambda-\beta=\sigma$, we have

$$\begin{aligned}\Omega\omega.f_i.v_{\lambda-\beta} &= (\Omega f_i)(rs^{-1})^{\Delta_{X,B}c}\,\omega.v_{\lambda-\beta}\\ &= \big(f_i{\omega'_i}^{-1}\omega_i\Omega\big)\,(rs^{-1})^{\Delta_{X,B}c}\,\omega.v_{\lambda-\beta}\\ &= f_i(rs^{-1})^{\Delta_{X,B}(\lambda-\beta,\alpha_i)}(rs^{-1})^{\Delta_{X,B}c}\,\Omega\omega.v_{\lambda-\beta}\\ &= f_i\,\Omega\omega.v_{\lambda-\beta},\end{aligned}$$

where $c = \frac{1}{2}\big[\,(\lambda-\beta-\alpha_i+\rho,\lambda-\beta-\alpha_i+\rho) - (\lambda-\beta+\rho,\lambda-\beta+\rho)\,\big]$. (15) yields

$$\begin{aligned}\Omega\omega.f_\beta.v_\lambda &= f_\beta.\Omega\omega.v_\lambda\\ &= (rs^{-1})^{\frac{\Delta_{X,B}}{2}(\lambda+\rho,\lambda+\rho)} f_\beta.\Omega.v_\lambda\\ &= (rs^{-1})^{\frac{\Delta_{X,B}}{2}(\lambda+\rho,\lambda+\rho)} f_\beta.\Omega_0.e_\lambda\\ &= (rs^{-1})^{\frac{\Delta_{X,B}}{2}(\lambda+\rho,\lambda+\rho)} f_\beta.v_\lambda, \quad (\Omega_0 = 1).\end{aligned}$$

So the relation (14) follows. □

Corollary 3.12. (i) *For the simple $U_{r,s}(\mathfrak{g})$-module $L(\lambda)\in \mathrm{Ob}(\mathcal{O})$, there holds*

$$\Omega\omega=(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\lambda+\rho,\lambda+\rho)}I.$$

(ii) *For each finite-dimensional $M\in \mathrm{Ob}(\mathcal{O})$, the eigenvalues of the operator $(\Omega\omega)|_M$ are integral powers of $(rs^{-1})^{\frac{1}{2}}$.*

Proof. (i) is evident. For (ii), as $M\in \mathrm{Ob}(\mathcal{O})$ is finite-dimensional, it has a composition series whose factors are finite-dimensional simple $U_{r,s}(\mathfrak{g})$-modules in $\mathcal{O}$, on which $\Omega\omega$ acts as multiplication by $(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\mu+\rho,\mu+\rho)}$ for some $\mu\in\Lambda^+$, as indicated by (i) and Corollary 2.8. After taking an appropriate basis of M compatible with a chosen composition series, the acting matrix of $(\Omega\omega)|_M$ has the required property. □

From Corollary 3.8 and Definition 3.10, we have a further result as follows.

Theorem 3.13. *The operator $\Omega\omega: M\longrightarrow M$ commutes with the action of $U_{r,s}(\mathfrak{g})$ on any module $M\in \mathrm{Ob}(\mathcal{O})$, where $\mathfrak{g}=\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$, or $\mathfrak{sp}_{2n}$.*

Proof. At first, it needs to show that $\Omega\omega$ commutes with e_i, f_i $(1\le i\le n)$. For $m\in M_\mu$, by Corollary 3.8 and Definition 3.10, we get

$$\begin{aligned}\Omega\omega.(e_i.m)&=(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\mu+\alpha_i+\rho,\,\mu+\alpha_i+\rho)}\Omega e_i.m\\&=(rs^{-1})^{\Delta_{X,B}[\,\frac{1}{2}(\mu+\alpha_i+\rho,\,\mu+\alpha_i+\rho)-(\mu+\alpha_i,\,\alpha_i)\,]}e_i\Omega.m\\&=(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\mu+\rho,\,\mu+\rho)}e_i\Omega.m\\&=e_i.(\Omega\omega.m).\\\Omega\omega.(f_i.m)&=(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\mu-\alpha_i+\rho,\,\mu-\alpha_i+\rho)}\Omega f_i.m\\&=(rs^{-1})^{\Delta_{X,B}[\,\frac{1}{2}(\mu-\alpha_i+\rho,\,\mu-\alpha_i+\rho)+(\mu,\,\alpha_i)\,]}f_i\Omega.m\\&=(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\mu+\rho,\,\mu+\rho)}f_i\Omega.m\\&=f_i.(\Omega\omega.m).\end{aligned}$$

Obviously, $\Omega\omega$ commutes with the action of ω_i, ω_i' $(1\le i\le n)$, for it preserves the weight spaces of M. □

The following Lemma is due to [BW2, Lemma 3.7] for the case of $\mathfrak{g}=\mathfrak{gl}_{n+1}$, or $\mathfrak{sl}_{n+1}$, which is still valid in our cases.

Lemma 3.14. *Assume that rs^{-1} is not a root of unity. Let M be a nonzero finite-dimensional quotient of the Verma $U_{r,s}(\mathfrak{g})$-module $M(\lambda)\in \mathrm{Ob}(\mathcal{O})$. Then M is simple. In particular, $L'(\lambda)=L(\lambda)$ for $\lambda\in\Lambda^+$.*

Proof. Lemma 2.6 means $\lambda\in\Lambda^+$. The proof is based on the counter-evidence method and Proposition 3.11, which is the same as that of [BW2, Lemma 3.7], with slight differences: for the function $g(\lambda)$ used in the proof there we use $(rs^{-1})^{\frac{\Delta_{X,B}}{2}(\lambda+\rho,\lambda+\rho)}$ instead, noting the fact from Lie algebra theory (see [D], or [K]) that for any weight $\mu\le\lambda$ where $\lambda\in\Lambda^+$, $(\lambda+\rho,\lambda+\rho)=(\mu+\rho,\mu+\rho)$ if and only if $\mu=\lambda$. □

Based on the above results, using a similar argument due to Kac [K] in the proof of complete reducibility of category $\mathcal{O}$ for affine Kac-Moody Lie algebras (or comparing with the proof of [BW2, Theorem 3.8] in the spirit of Lusztig [L1]), we have

THEOREM 3.15. *Assume that rs^{-1} is a non-root of unity. For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$, let M be a nonzero finite-dimensional $U_{r,s}(\mathfrak{g})$-module on which U^0 acts semisimply. Then M is completely reducible.* □

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