Representative Yield Curve Shocks
and Stress Testing

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March 2008

In this paper we propose a systematic procedure to identify a set of representative yield curve shocks and use them for stress-testing purposes. We first fit a factor model to actual bond yields and estimate the main shape factors of the yield curve. We then partition the factors into non-overlapping sets of representative shocks. The key feature of our procedure is that it provides a wide variety of yield curve shocks including typical, uncommon, and extreme ones. We apply our methodology to a variety of bond strategies using actual U.S. yields.

Keywords: Risk Management, Yield curve; Interest Rate Risk; Stress-Testing

JEL Classification Codes: G1; E4; C5

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1   Introduction

Stress testing is a vital component of any sound risk-management system. It consists of measuring the impact of pre-specified shocks on the value of a given portfolio. The recent financial market turmoil, sparked by losses related to U.S. subprime mortgage securities, has imposed a major review of stress testing procedures on the modern financial market system. Banks have been early proponents of stress testing in order to assess potential vulnerabilities in their trading portfolios. This risk management component has been strengthen even further in the Basel II accord (Basel Committee on Banking Supervision, BCBS, 2006). In particular, regulatory capital needs to be sufficient to cover the results of the bank's stress testing. Moreover, insurances (International Association of Insurance Supervisors, 2003) and life and pension companies (Jorgensen, 2007) are also required by their respective regulatory autorities to carry out scenario-based stress tests. Furthermore, a growing number of prime brokerage firms and hedge funds conduct stress testing on a regular basis. Commonly used events for historical scenarios are the large U.S. stock market declines in October 1987, the Asian financial crisis of 1997, the financial market fluctuations surrounding the Russian default of 1998, and financial market developments following the September 11, 2001, terrorist attacks in the United States. The 2005 international survey of stress testing at major financial institutions conducted by the Bank for International Settlements (Basel Committee on the Global Financial System, 2005) reveals that *interest rate* stress testing is by far the most popular type of test. Around 94% of the surveyed institutions report using stress testing based on interest rates. Furthermore, the number of scenarios on interest rates is three times higher than those on equity, foreign exchange, or credit, respectively, and more than 10 times higher than those on commodity or property, respectively.

In practice, the determination of the shocks relies either on historical or hypothetical shocks. Specifically, historical scenarios are based on significant market events experienced in the past, e.g., the U.S. interest rate rise observed in 2003, whereas the hypothetical scenarios rely on plausible market events that have not yet happened. According to the BIS 2005 survey, historical and hypothetical scenarios are equally popular at international major financial institutions. For instance, the Basel II guidelines suggest that, for G-10 currencies, banks should consider either a parallel rate change of ± 200 basis points or the changes implied by the 1st and 99th percentiles of historically observed interest rate changes over at least five years (BCBS, 2001, 2006). While stress tests do not put exact figures on the probability of scenarios, scenarios still need to be plausible. The evaluation

of scenario plausibility calls for, at least, a rough idea of probability with which given scenarios will occur (Berkowitz, 2000). Furthermore, supervisors also expect institutions to examine multiple shocks that include yield curve twists, inversions, and other relevant shocks in evaluating the appropriate level of their interest rate exposures. Interest rates of different maturities behave quite independently of each other. As a wide range of shifts in the shape of the yield curve are observed in practice, the quality and the effectiveness of interest rate risk management depends on the ability to identify relevant yield curve shocks. In theory, the short end of the yield curve remains under the direct control of the central bank, whereas longer yields are risk-adjusted averages of expected future short rates which reflects public perception of monetary policy. Factors affecting the shape of the yield curve include macroeconomic conditions (Ang and Piazzesi, 2003, and Diebold, Rudebusch and Aruoba, 2006), monetary policy (Mankiw and Miron, 1986, Rudebusch, 1995, Piazzesi, 2005, and Ang, Boivin and Song, 2008), and interest rate volatility (Litterman, Scheinkman and Weiss, 1991, and Dai, Singleton and Yang, 2007). The goal of this paper is to propose a procedure leading to homogeneous and representative yield curve shocks. More precisely we want to answer at the following questions: 1) How do yield curves move in reality? 2) Can we describe interest rate movements systematically? 3) Is it possible to generate risk scenarios systematically? 4) How can we reduce user dependence and operational risk?

Principal Component Analysis (PCA) is often proposed as a tractable method for computing risk scenarios. This statistical technique is often used to extract the key factors driving interest rates (see Litterman and Scheinkman, 1991). It is also supported by Cochrane and Piazzesi (2005), where it shows that a linear combination of the forward rates is a common factor for forecasting future excess returns on bonds at all maturities. Several authors have proposed to combine movements in principal components to produce scenarios as a method to generate scenarios of "large" historical changes in term structures, i.e. stress-scenarios (see Loretan, 1997 and Rodrigues, 1997). The approach involves creating a separate scenario for each possible combination of changes in the princpal components, say N_k for $k = 1, 2, 3$ which are selected using either observed values in the tails of the empirical distribution or multiples of the standard deviation (with an assumption of elliptical distributions). Thus, with three principal components there are $N_1 \times N_2 \times N_3$ possible scenarios. This approach, however, has at least two shortcommings. Firstly, as pointed out by Diebold and Li (2006), PCA have unappealing features, including that they cannot be used to produce yields at maturities other than those observed in the data. Secondly, Fung and Hsieh (1996) have shown that during periods of large interest rate moves, the change in the shape of the yield curve is usually correlated to the level of interest rate itself. This

fact means that specifying separate "shocks" in each of the directions given by the retained principal components is not appropriate to genetare stress scenarios. Ironically, this is also the key scenario of concern from the risk management perspective. This paper answers at these two shortcomings. Our approach consists of two stages.

In the first stage, we fit a factor model to actual bond yields and estimate the main shape factors of the yield curve. For instance, the Nelson and Siegel (1987) model that was recently re-interpreted by Diebold and Li (2006) as a modern linear three-factor model of level slope and curvature characterizes the movement of these unobservable factors and the associated factor loadings. Several authors have proposed extensions of the functional form that was initially proposed by Nelson and Siegel (1987) which sometimes has difficulty in fitting the entire term structure of interest rates. For instance, Svensson (1994) extends Nelson-Siegel to allow for two humps in the yield curve. Björk and Christensen (1999) consider generalizations of the Nelson-Siegel model to maintain consistency with arbitrage-free pricing for certain short-rate processes.

In the second stage of our method, we partition the factors into non-overlapping sets of representative yield curve shocks by means of cluster analysis. This is a statistical method that separates a sample population into natural groups according to measures that define the characteristics of the population. Within a specified factor model, factor changes are natural dimensions for grouping yield curve changes into clusters. To do so, we use information obtained from a one-dimensional projection pursuit algorithm based on directions obtained by both maximizing and minimizing the kurtosis coefficient of the projected data (Pena and Prieto, 2001b). Minimizing the kurtosis coefficient implies maximizing the bimodality of the projections that will lead to breaking the sample into two large clusters. Differently, maximizing the kurtosis coefficient implies detecting groups of outliers in the projections that will lead to the identification of clusters that are clearly separated from the rest along some specific projections. This result is consistent with the dual interpretation of the standard fourth-moment coefficient of kurtosis as measuring tail heaviness and lack of bimodality (see Balanda and MacGillivray, 1988). A small number of outliers will produce heavy tails and a larger kurtosis coefficient. But if we increase the amount of outliers, we can start introducing bimodality and the kurtosis coefficient may decrease. A key advantage is that it provides a consistent framework for providing a wide variety of historical interest rate shocks, including typical, uncommon, and extreme ones. We then evaluate impacts of these shocks on several bond portfolio strategies and show that they have very different impacts on the portfolios' values and returns.

Several practical advantages of our stress testing framework need to be emphasized. When equipped with our spectrum of representative shocks, risk managers can easily measure the trading loss associated with each type of shocks. In particular, our approach does not rely on extreme shocks only, i.e., shocks with the lowest occurrence probability. Depending on the nature of the trading portfolio (i.e., type of securities, maturities, and positions: long vs. short), it may be the case that some more frequent shocks will cause a bigger loss than extreme shocks. Furthermore, our method is flexible enough to accommodate any horizon for the shocks (e.g. 1 day, 3 month, 1 year). Finally, although our main focus in this paper is on interest risk management, our methodology can be implemented with other type of data, such as foreign exchange rate or credit spread data.

The remainder of the paper proceeds as follows. Section I describes the identification procedures for representative yield curve shocks. Section II presents an empirical analysis based on the U.S. term structure of interest rates. Section III offers a summary and concluding comments.

2 Methodology

In this section we propose a general two-step procedure for generating reliable historical yield curve shocks which is computationally feasible and can account for the dependence of interest rates at all available maturities.

2.1 Parsimonious Model of Bond Yields

Numerous studies have found that shifts or changes in the shape of the yield curve are attributable to three unobservable factors, named level, slope and curvature (Litterman and Scheinkman, 1991). In order to characterize the movement of these unobservable factors, Nelson and Siegel (1987) proposed the parsimonious yield curve model:

$$y_t\left(\tau\right) = b_{1t} + b_{2t}\left(\frac{1 - e^{-\lambda_t \tau}}{\lambda_t \tau}\right) - b_{3t}e^{-\lambda_t \tau}$$

where $y_t\left(\tau\right)$ denotes the continuously-compounded zero-coupon nominal yield at maturity τ, and b_{1t}, b_{2t}, b_{3t} and λ_t are (time-varying) parameters.

The Nelson-Siegel model can generate a variety of yield curve shapes including up-

ward sloping, downward sloping, humped, and inversely humped. Recently, this model has been re-interpreted by Diebold and Li (2006) as a modern linear three-factor model. The corresponding yield curve is:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right).$$

The advantage of this representation is that we can easily give economic interpretations to the parameters β_{1t}, β_{2t} and β_{3t}. In particular, we can interpret them as a level factor and two shape factors: a slope factor, and a curvature factor, respectively. To see this, note that the loading on β_{1t} is 1, a constant that does not depend on the maturity. Thus β_{1t} affects yields at different maturities equally and hence can be regarded as a level factor. The loading associated with β_{2t} starts at 1 but decays monotonically to 0. Thus β_{2t} affects primarily short-term yields and hence changes the slope of the yield curve. Finally, factor β_{3t} has loading $\left(\frac{1 - e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right)$, which starts at 0, increases, and then decays. Thus β_{3t} has largest impact on medium-term yields and hence moves the curvature of the yield curve. Rather than estimating the three factors by nonlinear least squares, Diebold and Li (2006) fix the value of λ (with maturities measured in months) and estimate the model for each period using ordinary least squares. They argue that this not only greatly simplifies the estimation, but likely results in more trustworthy estimates of the level, slope and curvature factors. More precisely, they set $\lambda = 0.0609$ (and with maturity measured in years rather than months: $\lambda = 0.0609 \times 12 = 0.7308$) precisely the value where the loading on the curvature factor reaches it maximum on the assumption that the curvature of the yield curve reaches its maximum at 30 months. In short, we can express the yield curve at any point of time as a linear combination of the level, slope and curvature factors, the dynamics of which drive the dynamics of the entire yield curve.

A number of authors have proposed extensions to the Nelson-Siegel model that enhance flexibility. Svensson (1994) extends Nelson-Siegel to allow for two decay parameters. The instantaneous forward rate curve:

$$f_t(\tau) = \beta_{1t} + \beta_{2t}e^{-\lambda_{1t}\tau} + \beta_{3t}\lambda_{1t}\tau e^{-\lambda_{1t}\tau} + \beta_{4t}\lambda_{2t}\tau e^{-\lambda_{2t}\tau}$$

which implies the yield curve:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1 - e^{-\lambda_{1t}\tau}}{\lambda_{1t}\tau}\right) + \beta_{3t}\left(\frac{1 - e^{-\lambda_{1t}\tau}}{\lambda_{1t}\tau} - e^{-\lambda_{1t}\tau}\right) + \beta_{4t}\left(\frac{1 - e^{-\lambda_{2t}\tau}}{\lambda_{2t}\tau} - e^{-\lambda_{2t}\tau}\right).$$

The Svensson model allows for up to two humps in the yield curve, whereas the original Nelson-Siegel model allows only one. The advantage of this representation is that we can easily give economic interpretations. In particular, we can interpret the factors as a level factor and three shape factors: a slope factor, and two curvature (hump) factors, respectively. The first hump (the 3rd term) is often located at relatively short horizons, which in many cases is needed to capture the effects of near term monetary policy expectations (along with the decay component, the 2nd term). The second hump (the 4th term) is typically located at much longer horizons. As a generalization of the Diebold and Li' model, rather than estimating the three factors by nonlinear least squares, it is possible to fix the value of λ_1 and λ_2 (with maturities measured in years) and estimate the model for each period using ordinary least squares. As with the Diebold and Li'model, it can be argued that this not only greatly simplifies the estimation, but likely results in more trustworthy estimates of the level, slope and the two hump factors. More precisely, it is possible to set $\lambda_1 = \lambda = 0.7308$ (respectively $\lambda_2 = 0.08$) precisely the value where the loading on the third (respectively fourth) factor reaches its maximum, i.e., at 2.5 years (respectively 22.5 years).

Other models generalize the Nelson-Siegel approach to maintain consistency with arbitrage-free pricing for certain short-rate processes. Björk and Christensen (1999) show that in the Heath-Jarrow-Morton (1992) framework with deterministic volatility, Nelson-Siegel forward-rate dynamics are inconsistent with standard interest rate processes, such as those of Ho and Lee (1986) and Hull and White (1990). Björk and Christensen (1999), however, show that a five-factor variant of the Nelson-Siegel forward rate curve:

$$f_t\left(\tau\right) = \beta_{1t} + \beta_{2t}\tau + \beta_{3t}e^{-\lambda\tau} + \beta_{4t}\tau e^{-\lambda\tau} + \beta_{5t}e^{-2\lambda\tau}$$

which implies the yield curve:

$$y_t\left(\tau\right) = \beta_{1t} + \beta_{2t}\left(\frac{\tau}{2}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{4t}\left(\frac{1-e^{-\lambda\tau}}{\lambda^2\tau} - \frac{e^{-\lambda\tau}}{\lambda}\right) + \beta_{5t}\left(\frac{1-e^{-2\lambda\tau}}{2\lambda\tau}\right)$$

is consistent not only with Ho-Lee and Hull-White, but also with the two-factor models studied in Heath, Jarrow and Morton (1992) under deterministic volatility. Our empirical investigation reveals that it is possible to fix the λ at a value equals to 0.29.

2.2 Projection-Pursuit Approach

Projection pursuit (Friedman and Tukey, 1974), a technique developed in statistics for finding 'interesting' projections of multidimensional data, can also be useful for optimal visualization of the clustering structure of the data.[1],[2] The objective of projection pursuit algorithms is to find interesting features of high-dimensional data in low-dimensional spaces via projections obtained by maximizing or minimizing an objective function termed the projection index, which depends on the data and the projection vector. It is commonly assumed that the most interesting projections are the farthest ones from normality, showing some unexpected structure such as clusters, outliers or nonlinear relationships among the variables. General reviews of projection pursuit techniques have been given by Huber (1985), Jones and Sibson (1987), and Posse (1995). In basic (one-dimensional) projection pursuit, we try to find directions such that the projection of the data in that direction has an 'interesting' distribution, i.e., displays some structure. It has been argued by Huber (1985) and by Jones and Sibson (1987) that the Gaussian distribution is the least interesting one, and that the most interesting directions are those that show the least Gaussian distribution. In particular they suggested that the standardized absolute cummulants can be useful for cluster detection. Peña and Prieto (2001a), Peña and Prieto (2001b) and Galeano, Pena and Tsay (2006) proposed a procedure for multivariate outlier detection based on projections that maximize or minimize the kurtosis coefficient of the projected data

Peña and Prieto (2001b) have given an intuitive explanation by considering an univariate sample of zero-mean variables of size n which is concentrated around two values but with different probabilities, for instance, n_1 observations take the value $-a$ and n_2 take the value a, with $n = n_1 + n_2$. Let $r = n_1 / n_2$, the kurtosis coefficient will be $k = (1 + r^3) / r(1 + r)$. This function has its minimum value at $r = 1$ (i.e., $n_1 = n_2$) and grows without limit either when $r \to 0$ or when $r \to \infty$. This result suggests that searching for directions where the kurtosis coefficient is minimized will tend to produce projections in which the sample is split into two bimodal distributions of about the same size. On the other hand, maximizing the

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[1]This approach naturally accompanies the technique of principal component analysis of the covariance structure of a random vector . In principal component analysis we are interested in finding the axes of the covariance ellipsoid. The index function is in this case the variance of a linear combination subject to the normalizing constraint.

[2]An approach for obtaining robust multivariate methods is projection pursuit (Huber, 1985). For example, in principal component analysis the first component is defined as that direction maximizing a measure of spread of the projected data on this direction. If a robust spread measure is considered, the resulting principal component is robust. Thus, robust estimation is done only in one dimension, namely in the direction of the projected data. A non-trivial task is finding the direction which maximizes an objective function, like a robust spread measure for robust principal component analysis. In this context, Croux and Ruiz-Gazen (2005) suggested to use each observation for the construction of candidate directions.

kurtosis coefficient will produce projections in which the data is split among groups of very different size: we have a central distribution with heavy tails owing to the small clusters of outliers. As a conclusion we can say that minimizing the kurtosis coefficient implies maximizing the bimodality of the projections, whereas maximizing the kurtosis coefficient implies detecting groups of outliers in the projections. Searching for bimodality will lead to breaking the sample into two large clusters that will be further analyzed. Searching for groups of outliers with respect to a central distribution will lead to the identification of clusters that are clearly separated from the rest along some specific projections.

3 Empirical Application

In our empirical analysis, we use the zero-coupon Treasury yield dataset of Gurkaynak, Sack and Wright (2007). Over the sample period November 1985 - March 2007, we obtain 5,318 daily yield curves with the following maturities: 1, 2, 3, 5, 7, 10, 15, 20, and 30 years. By looking at long maturities, we can check the behavior of our scenarios at longer horizons. In Figure 1, we provide a three-dimensional plot of our yield curve data. The large amount of temporal variation in the yield curve is visually important. In Table 1, we provide some descriptive statistics. According to the NBER, this sample period contain two major recessions and three major expansions.[3] Several major historical and economic events occurred during our period of analysis (such as the 1987 equity crash, the ERM crises of 1992-1993, the bond market crash of 1994, the 1994 peso crisis, and the 1997 east Asian crisis and also the Gulf war, the 9/11 terrorist attack), among which some strongly impacted U.S. interest rates. Moreover, in this sample there have been three different Federal Reserve chairmen : Paul A. Volcker (August 1979 - August 1987), Alan Greenspan (August 1987 - February 2006) and finally Ben Bernanke (February 2006 - present).

< Insert Figure 1 and Table 1>

3.1 Algorithms for Identifying Representative Yield Curve Schocks

If $y_t(\tau)$ denotes the continuously-compounded zero-coupon nominal yield at maturity τ, and $\beta_{1t}, \beta_{2t}...\beta_{pt}$ are (time-varying) parameters, we can write yield curve as a functional

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[3]The NBER peaks are 1990:07, and 2001:03, and the NBER troughs are 1991:03, and 2001:11.

form of those coefficients and the maturity.

$$y_t(\tau) = g\left(\beta_{1t}, ..., \beta_{pt}\right)$$

and

$$\Delta y_t(\tau) = y_t(\tau) - y_{t-\Delta t}(\tau) = f\left(\Delta\beta_{1t}, ..., \Delta\beta_{pt}\right)$$

Here we propose a sequential procedure for identify representative yield curve shocks based on the directions that mimimize and maximize the kurtosis coefficient of the projections. We refer to these directions as the optimal projections. The procedure is divided into two three steps: (1) Obtain the beta coeffcients, 2) obtain "interesting" directions, (2) search for clusters in the projected univariate time series. In the presence of multiple outliers, it would be of limited value if one considered only the projections that maximize or minimize the kurtosis coefficient because of the potential problem of masking effects. For instance, a projection might effectively reveal one outlier but almost eliminate the effects of other outliers. To overcome such a difficulty, we present an iterative procedure for analyzing a set of $2p$ orthogonal directions consisting of (a) the direction that maximizes the kurtosis coefficient, (b) the direction that minimizes the kurtosis coefficient, and (c) two sets of $p-1$ directions orthogonal to (a) and (b). Our motivation for using these orthogonal directions is twofold. First, the results of section 4 of Galeano, Pena and Tsay (2006), reveal that in some cases the directions of interest are orthogonal to those that maximize or minimize the kurtosis coefficient of the projected series; second, these directions ensure nonoverlapping information, so that if the effect of an outlier is almost hidden in one direction, then it may be revealed by one of the orthogonal directions.

3.1.1 Compute the projection directions

Following Pena and Prieto (2001a) let us assume we have a sample of size T valued in R^p denoted by $\beta_t = \left(\beta_{1t}, \beta_{2t}...\beta_{pt}\right)$, $t = 1, \ldots, T$. At the first iteration,

1. Denote by $y_t^{(1)} = \beta_t$ and calculate the mean vector $\bar{y}^{(1)}$ and the variance matrix S_1.

2. Find a direction d_1 that solves

$$\max k(d_1) = \frac{1}{T}\sum_{t=1}^{T}\left(d_1' y_t^{(1)} - d_1' \bar{y}^{(1)}\right) \quad \text{s.t.} \quad d_1' S_1 d_1 = 1.$$

that is a direction that maximizes the kurtosis coefficient of the projected data.

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3. Project the observation onto a subspace S_1-orthogonal to d_1, so

$$y_t^{(2)} = \left(I - \frac{d_1 d_1' S1}{d'1 S_1 d_1} \right) y_t^{(1)}.$$

4. Continue until $k = p$ the dimension of β_t.

Acting that way, we obtain p directions $d_1, \ldots, d_p$. Doing exactly the same way by minimizing the criteria given in step 2, will give another p directions. The second part of the algorithm is the clustering one.

3.1.2 Analysis of the univariate projections

Following Pena and Prieto (2001b) having $2p$ directions, we project the data on these directions and get $z_{kt} = \beta_t' d_k$ for k varying from 1 to $2p$. We have then $2p$ samples of real-valued data. The choice of cluster is then made according to the first-order spacings of each sample. For each direction k,

1. Compute the univariate projection $u_{kt} = \beta_t' d_k$ and standardize those observations denoting them z_{kt}.

2. Sort the z_{kt} to get order statistics $z_{k(t)}$ and transform them using the inverse of the standard normal distribution function $v_{kt} = \Phi^{-1}(z_{k(t)})$.

3. Compute the gap between consecutive values $w_{kt} = v_{k[t+1]} - v_{kt}$.

4. Search for the significant gaps in w_{kt}. Large gaps will indicate the presence of more than one cluster. If large gaps are found, label all the observations v_{kt} differently according to theirs positions in relation to the gaps

3.2 Empirical Results

Our dataset consists of a fitted function which smooths across maturities. Gurkaynak, Sack, and Wright (2007) estimate the Svennson (1994) six-parameter function for instantaneous forward rates. All of their yield, price, etc. "data" derive from this fitted function. Yield curve models will be evaluated by how closely they match the functional form, not necessarily by how well they match the underlying data. Three models are compared:

1. Nelson and Siegel model $y_t\left(\tau\right) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right)$ .with $\lambda = \lambda_t = 0.7308$

2. Svensson model $y_t\left(\tau\right) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_{1t}\tau}}{\lambda_{1t}\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_{1t}\tau}}{\lambda_{1t}\tau} - e^{-\lambda_{1t}\tau}\right) + \beta_{4t}\left(\frac{1-e^{-\lambda_{2t}\tau}}{\lambda_{2t}\tau} - e^{-\lambda_{2t}\tau}\right)$ with $\lambda_1 = \lambda_{1t} = 0.7308$ and $\lambda_2 = \lambda_{2t} = 0.08$

3. Bjork and Christensen $y_t\left(\tau\right) = \beta_{1t} + \beta_{2t}\left(\frac{\tau}{2}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{4t}\left(\frac{1-e^{-\lambda\tau}}{\lambda^2\tau} - \frac{e^{-\lambda\tau}}{\lambda}\right) + \beta_{5t}\left(\frac{1-e^{-2\lambda\tau}}{2\lambda\tau}\right)$ with $\lambda = 0.29$

Estimation results are presented in Figure 2. We see that the model proposed by Bjork and Christensen is the best one since, unlike competing models, the $R-$square is systematically close to one. Moreover in Figure 3 we plot at selected dates two yield curves estimated by Gurkaynak, Sack, and Wright (2006) and with the Bjork and Christensen model with $\lambda = 0.29$. In all cases, the two curves overlap almost perfectly.

<**Insert Figures 2 and 3**>

We then use the clustering procedure that is based on projection pursuit presented in Section 2.3 to partition the vectors of coefficient: $\Delta\beta_{it} = \beta_{i(t+j)} - \beta_{it}$, $i = 1, 2, 3, 4, 5$ and $j = 1, 25, 250$.

The corresponding clustering results is presented in Figure 4 with a one year horizon, in Figure 5 with a one month horizon, and Figure 6 with a one day horizon. Thirty groups are found by our procedure at the one day horizon, while approximatively twenty groups are found at longer horizons. An important feature is the coexistence of (1) large groups including typical shocks, (2) medium-sized groups of uncommon shocks, and (3) small groups with one or a few extreme shocks.

For comparison purpose we also use this clustering procedure on the yield curve shapes, i.e., on the vector of coefficients β_{it}, directly and plot the clustering results in Figure 7. As documented in many other research, the typical yield curve shape is upward-sloping, but it can take many other shapes such as flat, downward sloping, etc.

<**Insert Figures 4, 5, 6 and 7**>

Having identified the yield curve shape and change scenario clusters we now apply them to two bond portfolio strategies and compare their returns in these different scenarios. We will take monthly change as example. The first bond portfolio is an even-ladder portfolio that invest equal amount in the four coupon bonds with the following four maturities: 1 year, 5 year, 10 year and 20 years. The second bond portfolio is long-short portfolio that long equal amount in the 10 year and 20 year bonds and short equal amount in the 1 year and 5 year bond. This strategy of borrowing money in the short end of yield curve and investing it in the long end is a common strategy used by banks, mortgage companies and fixed income traders. The coupon rates are set as the interest rates on the most recent yield curve in our dataset, which is on March 29, 2007. This will make these bonds priced at par value on March 29, 2007. For example, we assume the 1 year coupon bond pays a fixed coupon rate equal to the 1 year zero rate on March 29, 2007, and the 5 year coupon bond pays a fixed coupon rate equal to the 5 year zero rate on March 29, 2007.

We assume the initial zero yield curve at current time t can take any one of the 34 identified zero yield curve shapes. For each shape $y(i)$, $i = 1, ..., 34$, we first use it to value the above four coupon bonds to calculate the initial investment amount. We then apply the 24 monthly yield curve changes $dy(j)$ to it to derive the new yield curve $y(i, j)$ at time $t + 1$: $y(i, j) = y(i) + dy(j)$ with $i = 1, ..., 34$ and $j = 1, ..., 24$. We then use this new yield curve to re-value the four coupon bonds at time $t + 1$ and calculate the portfolio's value and return. We plot the ladder portfolio's returns in Figure 8 and the long-short Portfolio's returns in Figure 9. Clearly the identified yield curve shape and change clusters have very different impacts on the bond portfolios' values and returns.

< **Insert Figures 8 and 9**>

4   Conclusion

The ability to recognize the different types of yield curve shocks and understand what they portend to the future is an essential market skill. In this paper, we propose a general methodology to build and analyze the changes of the shape of the term structure of interest rates. The statistical approach consists of two stages: a building stage and a classification stage. The building step relies on fitting yields by model, and the classifying step relies on partitioning the estimated coefficients by a clustering algorithm. We apply our methodology to the US term structure of interest rates over the last two decades, and spot several representative shocks. We then evaluate impacts of these shocks on several bond portfolio

strategies and show that they have very different impacts on the bond portfolios' values and returns.

The methodology presented in this paper could be employed in different contexts. Indeed, it could be used to generate representative shocks from FX or credit spread data.

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5   Annexes

Table 1: Descriptive Statistics

Maturity	1Y	2Y	3Y	5Y	7Y	10Y	15Y	20Y	30Y
Min	0.8960	1.0912	1.3948	2.0788	2.6955	3.4111	4.1523	4.3355	4.1330
Max	9.8020	9.7331	9.6595	9.7455	9.9496	10.1800	10.2760	10.4480	10.7620
Mean	5.1165	5.4074	5.6267	5.9615	6.2197	6.5086	6.7896	6.8925	6.8068
Median	5.4277	5.5960	5.7289	5.9451	6.1416	6.3574	6.5944	6.7393	6.7814
Mode	2.0918	6.2567	2.5404	5.8595	5.0003	6.1788	4.8452	5.0323	4.5046
Std-Dev	1.9897	1.8946	1.8079	1.6902	1.6140	1.5331	1.4480	1.4101	1.4435

Note: Descriptive statistics are computed from the 5,318 daily observations from 1985/11/25 to 2007/03/29 for bond yields, for maturities ranging from one year to 30 years.
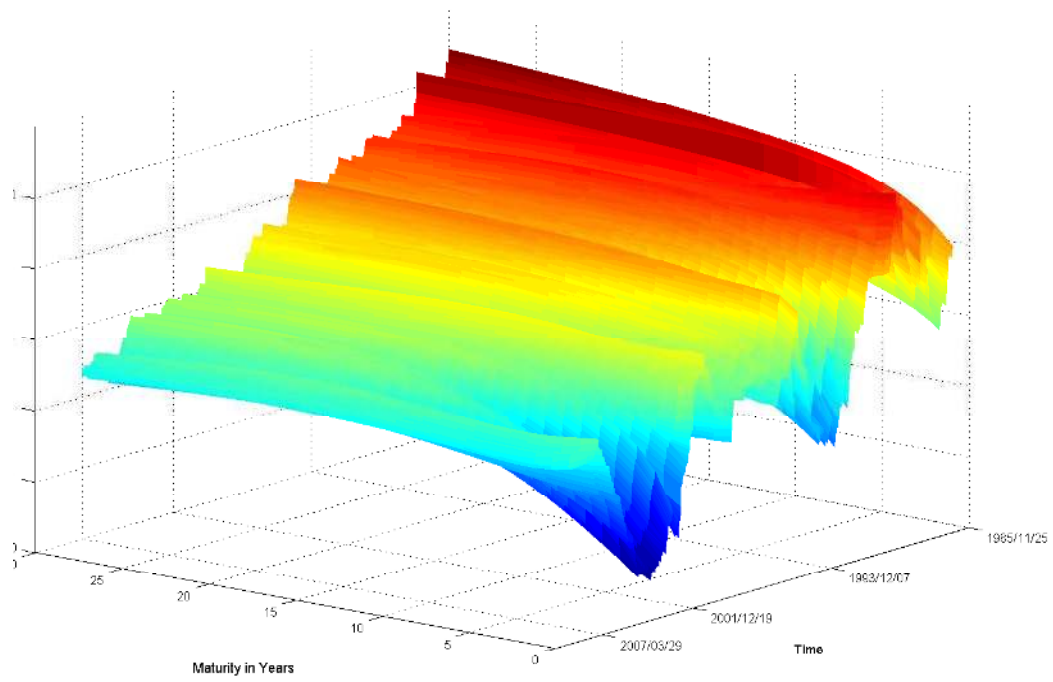
Figure 1: The sample consists of data yield data from November'25 1985 to March'29 2007 at maturities of 1 year to 30 years.
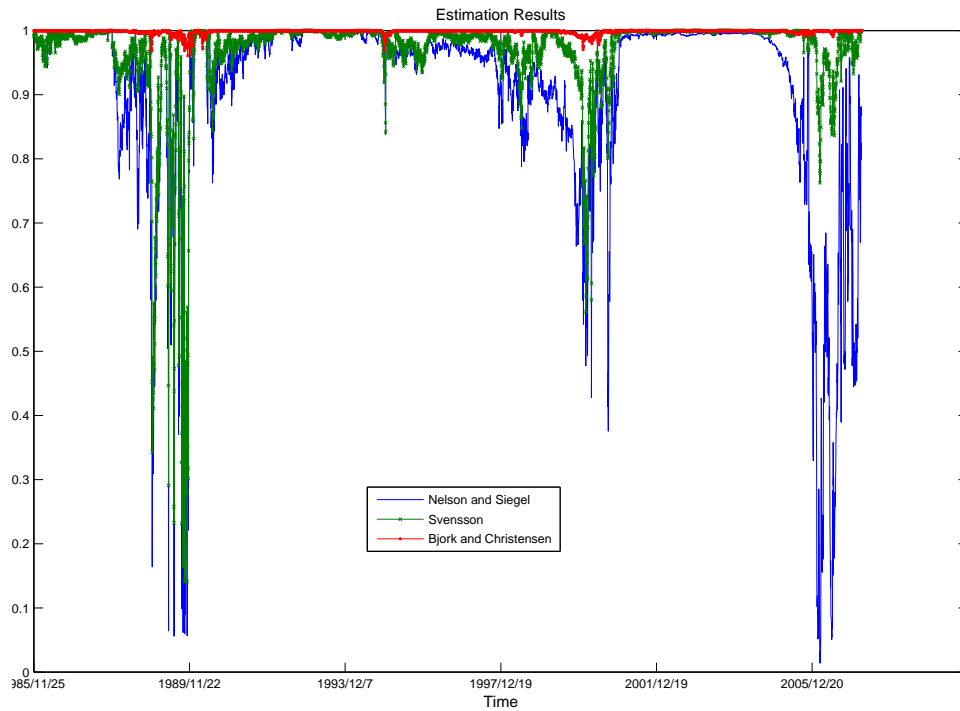
Figure 2: We plot the R-Square from 1) Nelson and Siegel curves fitted day-by-day with $\lambda = 0.7308$. 2) Svensson curves fitted day-by-day with $\lambda_1 = 0.7308$ and $\lambda_2 = 0.08$. 3) Bjork and Christensen curves fitted day-by-day with $\lambda = 0.29$. The sample consists of daily yield data from 1985/11/25 to 2007/03/29 as estimated by Gurkaynak, Sack, and Wright (2006). Maturities spanned include 1-2-3-5-7-10-15-20-30 years.
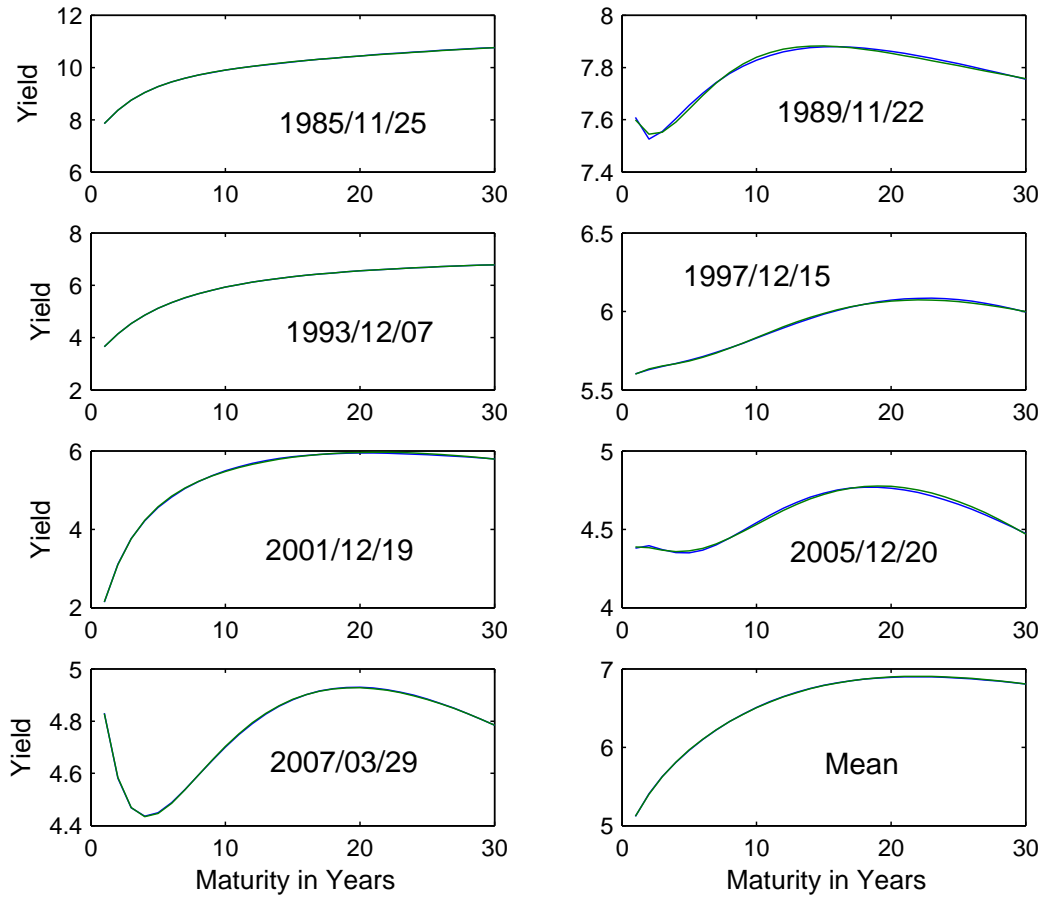
Figure 3: Selected fitted yield curves. We plot fitted Bjork-Christensen'yield curves for selected dates, together with fitted Svesson'yield curves estimated by Gurkaynak, Sack and Wright (2006).
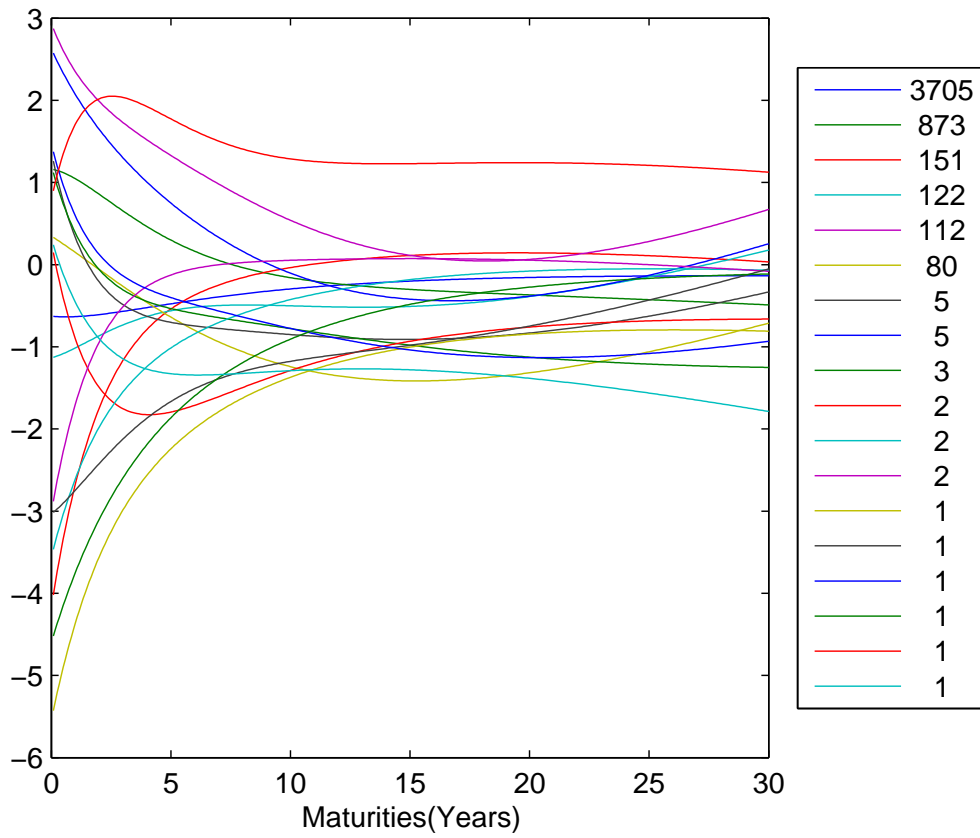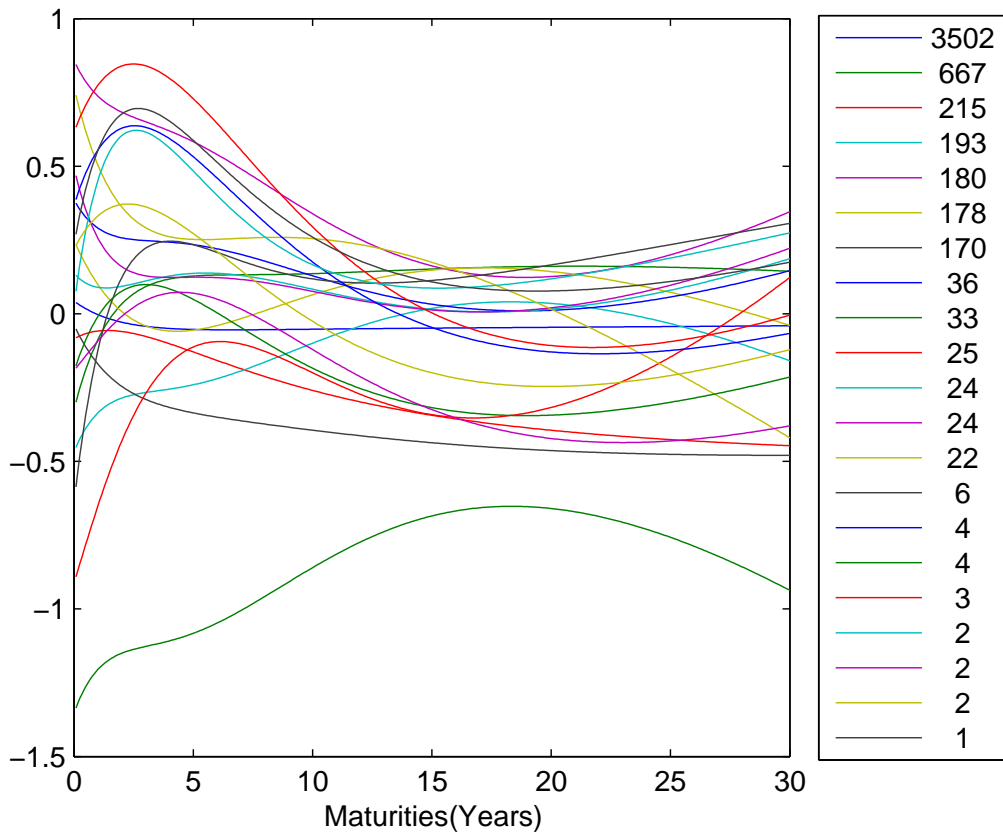
Figure 4: Estimated shocks at a one year horizon. We present the 18 scenarios generated by the procedure calculated as the median of the observations associated with their occurrences. The sample consists of daily yield data from 1985/11/25 to 2007/03/29. Maturities spanned include 1 year to 30 years.
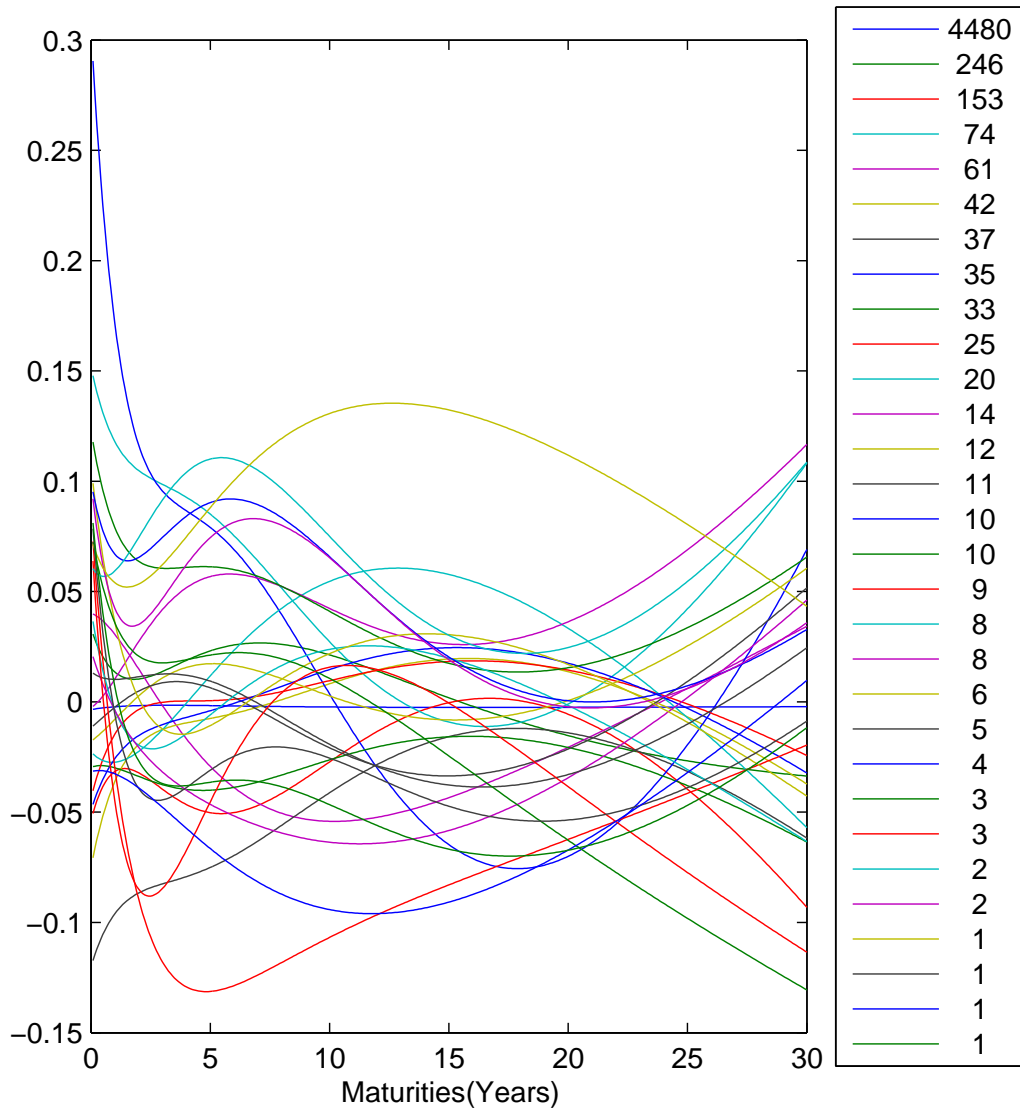
Figure 5: Estimated shocks at a one month horizon. We present the 21 scenarios generated by the procedure calculated as the median of the observations associated with their occurrences. The sample consists of daily yield data from 1985/11/25 to 2007/03/29. Maturities spanned include 1 year to 30 years.

Figure 6: Estimated shocks at a one day horizon. We present the 30 scenarios generated by the procedure calculated as the median of the observations associated with their occurrences. The sample consists of daily yield data from 1985/11/25 to 2007/03/29. Maturities spanned include 1 year to 30 years.
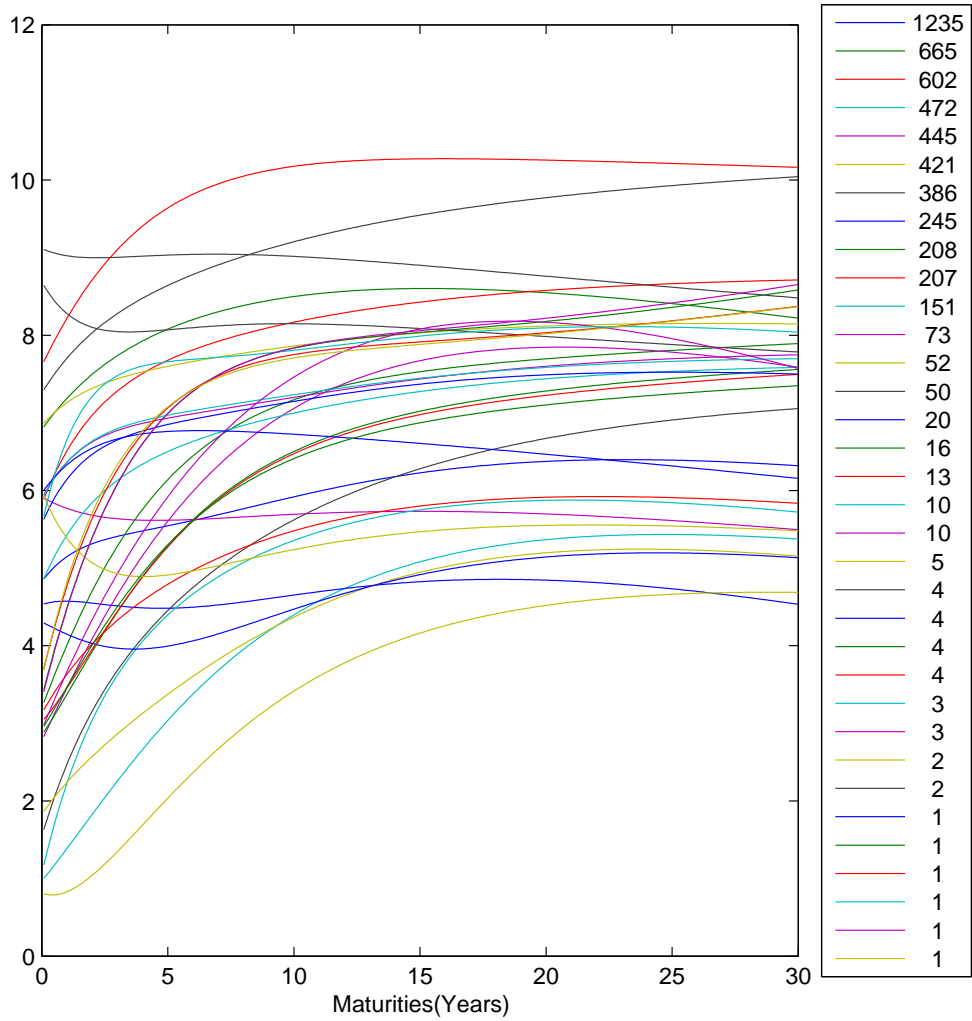
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Figure 7: Estimated Representative Yield Curves. We present the representative yield curves generated by the procedure calculated as the median of the observations associated with their occurrences. The sample consists of daily yield data from 1985/11/25 to 2007/03/29. Maturities spanned include 1 year to 30 years.
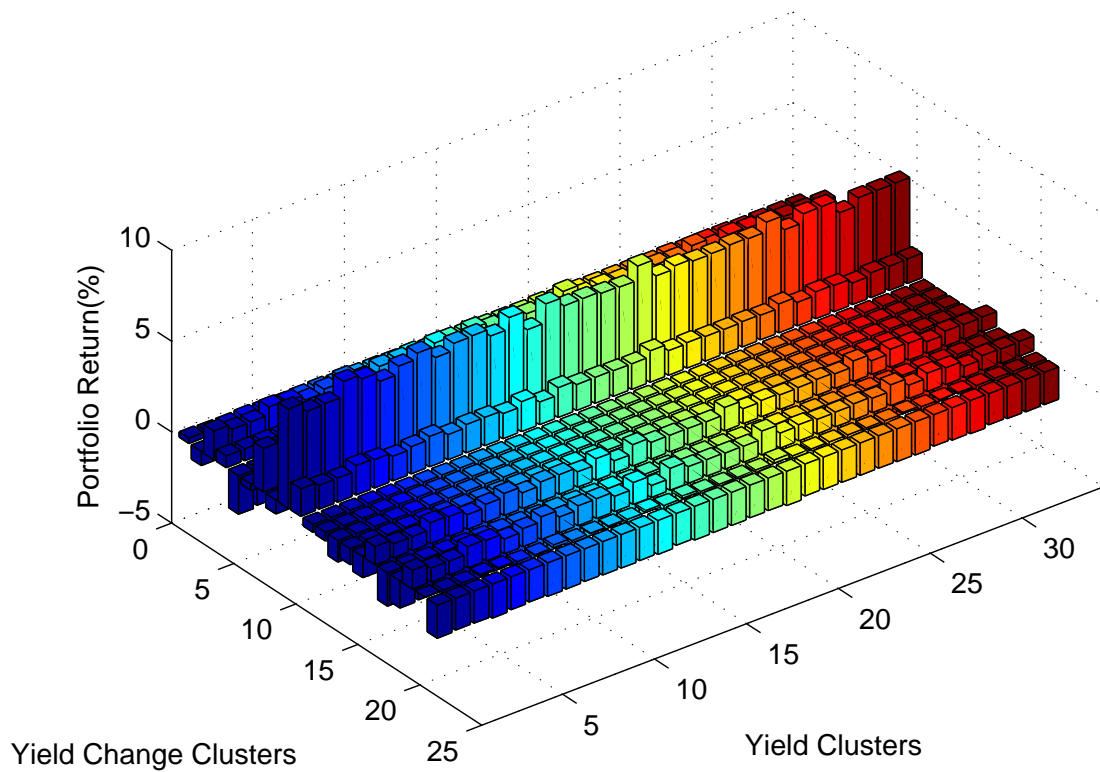
Figure 8: Ladder Bond Portfolio Returns. We present the one-month returns of a ladder bond portfolio that invests equal amount in the 1 year, 5 year, 10 year and 20 year coupon bonds under each yield curve clusters and each yield curve monthly change cluster. The clusters are sorted in descending order of number of occurrences.
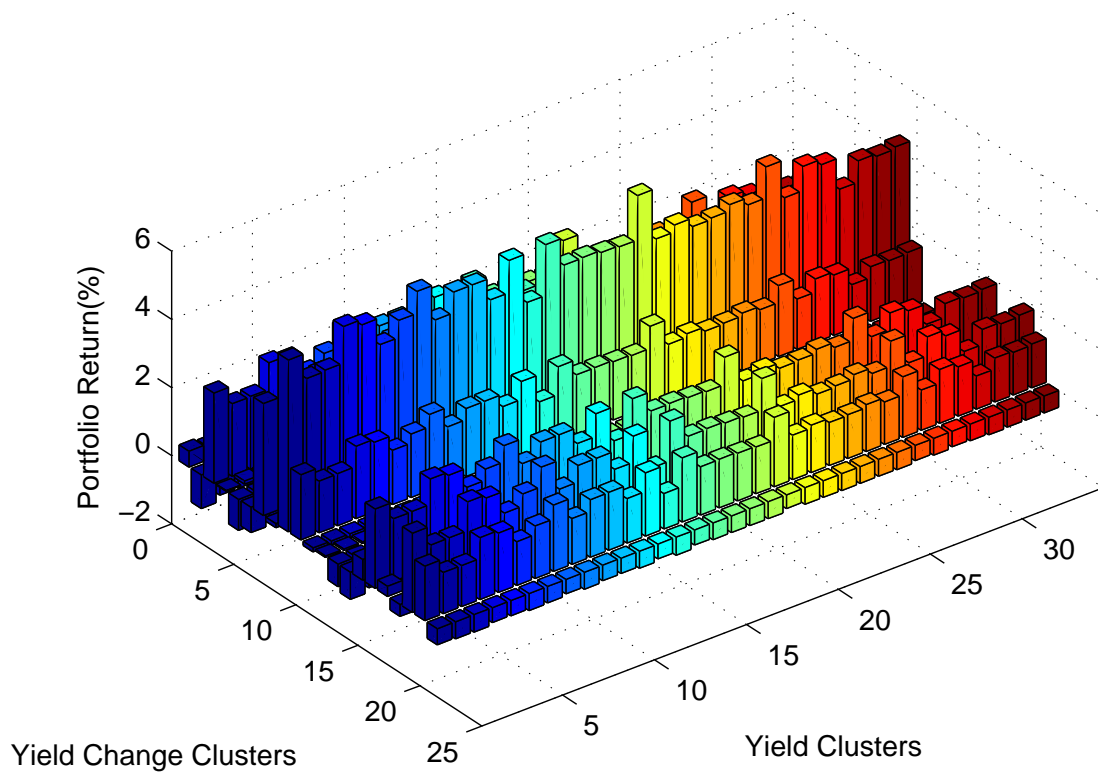
Figure 9: Long-Short Bond Portfolio Returns. We present the one-month returns of a long-short bond portfolio that shorts equal amount in the 1 year, 5 year and longs equal amount in 10 year and 20 year coupon bonds under each yield curve cluster and each yield curve monthly change cluster. The clusters are sorted in descending order of number of occurrences.