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Representing braids by automorphisms

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Abstract. Based on a normal form for braid group elements suggested by Dehornoy, we prove several representations of braid groups by automorphisms of a free group to be faithful. This includes a simple proof of the standard Artin's representation being faithful.

1 Introduction

Braid groups need no introduction; we just refer to the monograph [1] for the background. Some notation has to be reminded though. We denote the braid group on n strands by B_n; this group has a standard presentation $\langle \sigma_1, ..., \sigma_{n-1} | \ \sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$; $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \le i \le n-2\rangle$. We shall call elements of B_n *braids*, as opposed to *braid words* that are elements of the ambient free group on $\sigma_1, ..., \sigma_{n-1}$. We say that two braid words are *equivalent* if they represent the same braid.

There is a well-known representation (due to Artin) of the group B_n in the group $Aut(F_n)$ of automorphisms of the free group F_n (see e.g. [1, p.25]). Let F_n be generated by $x_1, ..., x_n$. Then the automorphism $\hat{\sigma}_i$ corresponding to the braid generator σ_i, takes x_i to $x_i x_{i+1} x_i^{-1}$, x_{i+1} to x_i, and fixes all other free generators.

More recently, Wada [6] has discovered several other representations of the group B_n by automorphisms of F_n. Some of them are obviously non-faithful; two of the remaining 4 are conjugate, which leaves us with the following 3 interesting representations:

(1) This is actually an infinite series of representations generalizing the standard Artin's representation. For an arbitrary non-zero integer k, the automorphism $\hat{\sigma}_i^{(k)}$ corresponding to the braid generator σ_i, takes x_i to $x_i^k x_{i+1} x_i^{-k}$, x_{i+1} to x_i, and fixes all other free generators.

(2) Here the automorphism $\hat{\sigma}_i$ corresponding to the braid generator σ_i, takes x_i to $x_i x_{i+1}^{-1} x_i$, x_{i+1} to x_i, and fixes all other free generators.

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(3) Here $\hat{\sigma}_i$ takes x_i to $x_i^2 x_{i+1}$, x_{i+1} to $x_{i+1}^{-1} x_i^{-1} x_{i+1}$, and fixes all other free generators.

In this paper, we prove the following

Theorem A. Each of the representations (1), (2), (3) above is faithful.

Our (very easy) proof is based on part (a) of the following theorem of Dehornoy [3], [4]. At the same time, our proof establishes part (b) of Dehornoy's theorem. In fact, each of the faithfulness results of Theorem A gives a simple proof of part (b) of Theorem B. In particular, the argument involving Wada's representation (3) seems to give the easiest proof of (b) known so far.

We call a braid word u σ_1-nonnegative if there are no occurences of σ_1^{-1} in u, and σ_1-negative if there are no occurences of σ_1 with a positive exponent in u. Then:

Theorem B. (Dehornoy [3], [4])
(a) Every braid word is equivalent to either a σ_1-nonnegative or a σ_1-negative braid word.
(b) If u is a σ_1-nonnegative braid word with at least one occurence of σ_1, then u is not equivalent to the empty word.

It is hoped that Dehornoy's normal form for braid group elements can be useful in proving other representations of braid groups to be faithful. We note at this point that part of our Theorem A follows from [2, Theorem 7], because it is proved there that, if N_m is the normal closure of the n elements $x_1^m, \ldots, x_n^m$ in F_n, where $m \geq 2$, then the induced action of the standard Artin's representation on F_n/N_m is faithful. Therefore, if a representation $\phi : B_n \to Aut(F_n)$ induces the same action on F_n/N_m for some m as Artin's representation does (which requires, in particular, N_m being invariant under $\phi(B_n)$), then this ϕ must be faithful, too. Using this result, one can establish faithfulness of, say, representations (1) above for $k \neq 2$. However, the combination of the two conditions (N_m being invariant under $\phi(B_n)$ and ϕ inducing the same action as Artin's representation on F_n/N_m) appears to be rather restrictive, and is unlikely to be satisfied by most representations. For example, the representation (3) above satisfies the former condition for $m = 2$, but does not satisfy the latter. Our method based on Dehornoy's normal form therefore appears to be more flexible. The referee has pointed out that Larue [5] has used a method similar to ours to show that the standard Artin's representation is faithful.

In the concluding Section 3, we show that different Wada's representations have different images in $Aut(F_n)$, with one possible exception. A probably difficult question is whether or not different Wada's representations are conjugate. For example, take two representations φ and ψ of type (1), where $\varphi : \sigma_i \to \hat{\sigma}_i^{(k)}$; $\psi : \sigma_i \to \hat{\sigma}_i^{(-k)}$ for some non-zero integer k. Then the images $\varphi(B_n)$ and $\psi(B_n)$ are conjugate by the automorphism that takes every free generator x_i to its inverse. This might be the only instance of different representations of the types (1)–(3) being conjugate, but I was not able to prove that.

2 Proof of Theorem A

Theorem A will be proved if we establish the following

Lemma. Let $\sigma \to \hat{\sigma}$ be any of Wada's representations. Suppose σ is a σ_1-positive braid word of the form $\sigma_1\sigma'$. Then $\hat{\sigma}(x_1)$ has at least 2 occurences of $x_1^{\pm 1}$.

Proof. **(1)** We start with Wada's representation of type (1). Since we assume that σ is of the form $\sigma_1\sigma'$, we have the automorphism $\hat{\sigma}_1^{(k)}$ applied first, hence $\hat{\sigma}_1^{(k)}(x_1) = x_1^k x_2 x_1^{-k}$ already has at least 2 occurences of $x_1^{\pm 1}$.

Then, any $\hat{\sigma}_i^{(k)}$ with $i \geq 2$ does not change existing occurences of x_1 and does not introduce any new ones. Thus, we have to only concern ourselves with how $\hat{\sigma}_1^{(k)}$ acts on an element of the free group of the form $w = x_1^k u x_1^{-k}$, where u neither starts nor ends with $x_1^{\pm 1}$. We are going to show that $\hat{\sigma}_1^{(k)}(w)$ has the same form (with different u, perhaps), i.e., that x_1^k on the left and x_1^{-k} on the right cannot cancel after $\hat{\sigma}_1^{(k)}$ is applied. Because of the symmetry, we are going to consider x_1^k on the left only. Consider 2 cases:

(a) $u = x_m^s u'$, where $m \geq 3$, $s \neq 0$, and u' does not start with $x_m^{\pm 1}$. Then $\hat{\sigma}_1^{(k)}(w) = x_1^k x_2^k x_1^{-k} x_m^s \hat{\sigma}_1^{(k)}(u') x_1^k x_2^{-k} x_1^{-k}$, and x_1^k on the left does not cancel. Indeed, for a cancellation process to start, there must be a cancellation between x_m^s and $\hat{\sigma}_1^{(k)}(u')$, i.e., $\hat{\sigma}_1^{(k)}(u')$ should start with $x_m^{\pm 1}$. Since u' itself does *not* start with $x_m^{\pm 1}$ and $\hat{\sigma}_1^{(k)}$ does not affect occurences of x_m, that could only mean that some initial fragment of u' became the empty word after $\hat{\sigma}_1^{(k)}$ was applied. But this is impossible because $\hat{\sigma}_1^{(k)}$ is an automorphism.

(b) $u = x_2^s u'$, where $s \neq 0$, and u' does not start with $x_2^{\pm 1}$. Then $\hat{\sigma}_1^{(k)}(w) = x_1^k x_2^k x_1^{-k} x_1^s \hat{\sigma}_1^{(k)}(u')$, where there is no cancellation between x_1^s and $\hat{\sigma}_1^{(k)}(u')$ because $\hat{\sigma}_1^{(k)}(u')$ cannot start with $x_1^{\pm 1}$. To have the cancellation process get to

x_2^k, we must have $k = s$. Then, to cancel all of x_2^k, we must have $\hat{\sigma}_1^{(k)}(u')$ start with x_2^{-k}, which is impossible. Indeed, if u' starts with $x_m^{\pm 1}$, $m \geq 3$, then, in order to have $\hat{\sigma}_1^{(k)}(u')$ start with $x_2^{\pm 1}$, we must have some non-empty fragment of u' between $x_m^{\pm 1}$ and $x_m^{\mp 1}$ become the empty word after $\hat{\sigma}_1^{(k)}$ is applied, which is impossible because $\hat{\sigma}_1^{(k)}$ is an automorphism. If u' starts with $x_1^{\pm 1}$, then the obvious inductive argument implies that $\hat{\sigma}_1^{(k)}(u')$ should start with $x_1^{\pm 1}$ as well. (Note that the length of u' is smaller than that of w).

Thus, in either case, x_2^k cannot cancel, and therefore, $\hat{\sigma}_1^{(k)}(w)$ has the same form as w. $\square$

(2) For Wada's representation of type (2), the proof goes along exactly the same lines.

(3) Finally, consider Wada's representation of type (3). Again, we have the automorphism $\hat{\sigma}_1$ applied first, hence $\hat{\sigma}_1(x_1) = x_1^2 x_2$ already has 2 occurences of x_1.

Also, any $\hat{\sigma}_i$ with $i \geq 2$ does not change existing occurences of x_1 and does not introduce any new ones, so we have to only concern ourselves with how $\hat{\sigma}_1$ acts on an element of the free group of the form $w = x_1^2 u$, where u does not start with $x_1^{\pm 1}$. We are going to show that $\hat{\sigma}_1(w)$ has the same form. Again, there are 2 cases:

(a) $u = x_m^{\pm 1} u'$, where $m \geq 3$. Then $\hat{\sigma}_1(w) = x_1^2 x_2 x_m^{\pm 1} \hat{\sigma}_1(u')$, hence x_1^2 on the left does not cancel. (If u' does not start with $x_m^{\mp 1}$, then neither does $\hat{\sigma}_1(u')$. Therefore, for cancellation between u' and $\hat{\sigma}_1(u')$ to occur, some fragment of u' must be mapped to the empty word, which is impossible since $\hat{\sigma}_1$ is an automorphism).

(b) $u = x_2^s u'$, where $s \neq 0$, and u' does not start with $x_2^{\pm 1}$. Then $\hat{\sigma}_1(w) = x_1^2 x_2 x_1^2 x_2 x_2^{-1} x_1^{-s} x_2 \hat{\sigma}_1(u')$, and there is no cancellation between x_2 and $\hat{\sigma}_1(u')$ since $\hat{\sigma}_1(u')$ cannot start with x_2^{-1} unless u' starts with $x_2^{\pm 1}$.

Therefore, $\hat{\sigma}_1(w) = x_1^2 x_2 x_1^{2-s} x_2 \hat{\sigma}_1(u')$, and no matter what s is, $\hat{\sigma}_1(w)$ has the form $x_1^2 u$, where u does not start with $x_1^{\pm 1}$. $\square$

3 Images of Wada's representations

Here we prove the following

Proposition. Let φ and ψ be two different Wada's representations. Then the groups $\varphi(B_n)$ and $\psi(B_n)$ are different subgroups of $Aut(F_n)$ unless, perhaps,

φ and ψ are both of type (1) and $\varphi : \sigma_i \to \hat{\sigma}_i^{(k)}$; $\psi : \sigma_i \to \hat{\sigma}_i^{(-k)}$ for some non-zero integer k.

Proof. We have to consider several cases.

(1) Both φ and ψ are of type (1), so that $\varphi : \sigma_i \to \hat{\sigma}_i^{(k)}$; $\psi : \sigma_i \to \hat{\sigma}_i^{(s)}$ for some non-zero integers k, s. We assume that $k \neq \pm s$, so let $|k| > |s|$.

Consider the Magnus representation of the groups $\varphi(B_n)$ and $\psi(B_n)$ (for $k = 1$, it is also known as the Burau representation – see [1, p.102]). Under this representation, the automorphism $\hat{\sigma}_i^{(k)}$ is mapped onto the $n \times n$ matrix which differs from the identity matrix only by a 2×2 block with the top left corner in the (i, i)th place. This block is $\begin{pmatrix} 1 - t^k & t^k \\ 1 & 0 \end{pmatrix}$. Thus, the determinant of this matrix is $-t^k$, and therefore, the determinant of the matrix corresponding to an arbitrary braid under the composition of φ and the Magnus representation, is equal to $\pm t^{km}$ for some integer m.

Now if we take, say, $\psi(\sigma_1)$ and then apply the Magnus representation, we shall get a matrix with the determinant $-t^s$. A matrix like that cannot be a product of matrices with determinants of the form $\pm t^{km}$ since $|k| > |s|$. This completes the proof in case (1).

(2) φ is of type (1), and ψ is of type (2). Again, we apply the Magnus representation to both groups $\varphi(B_n)$ and $\psi(B_n)$. Note that this is possible since the *mapping group* of $\psi(B_n)$ is the same as that of $\varphi(B_n)$. (The mapping group of a single automorphism $\alpha : x_i \to y_i,\ 1 \leq i \leq n$, is the group with the presentation $\langle x_1, ..., x_n |\ x_i = y_i,\ 1 \leq i \leq n \rangle$. The mapping group of a group G of automorphisms has the set of relations which is the union of sets of relations for the mapping group of each individual automorphism in G).

If we apply the Magnus representation to $\psi(\sigma_i)$, we shall get a matrix which differs from the identity matrix only by the following 2×2 block with the top left corner in the (i, i)th place: $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. The determinant of this matrix is 1. Therefore, any matrix in the image of the Magnus representation of $\psi(B_n)$ has determinant 1. This completes the proof in case (2).

(3) φ is of type (1) or (2), and ψ is of type (3). In that case, $\varphi(B_n)$ and $\psi(B_n)$ are different subgroups of $Aut(F_n)$ because they have different presentations of their mapping groups. More accurately, the mapping group of $\varphi(B_n)$ has the presentation $\langle x_1, ..., x_n |\ x_1 = x_2 = ... = x_n \rangle$, whereas $\psi(B_n)$ has the presentation $\langle x_1, ..., x_n |\ x_1 = x_2^{-1} = ... = x_n^{(-1)^{n+1}} \rangle$. The groups themselves are isomorphic, yet the presentations are different. Now we argue as follows.

Consider, say, $\varphi(\sigma_1)$. The mapping group of this automorphism has the presentation $\langle x_1, ..., x_n | \, x_1 = x_2 \rangle$. Suppose $\varphi(\sigma_1) \in \psi(B_n)$. That means, in particular, that by adding some elements to $\{x_1 x_2^{-1}\}$, we can get a set of elements of a free group whose normal closure is the same as that of $\{x_1 x_2, ..., x_1 x_n\}$. But this is impossible since the normal closure of the union of these two sets contains, say, the element x_1^2, which none of the two normal closures alone does. □

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