

## Representing Geometric Structures in $d$ Dimensions: Topology and Order\*

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**Abstract.** This work investigates data structures for representing and manipulating  $d$ -dimensional geometric objects for arbitrary  $d \geq 1$ . A class of geometric objects is defined, the “subdivided  $d$ -manifolds,” which is large enough to encompass many applications. A new representation is given for such objects, the “cell-tuple structure,” which provides direct access to topological structure, ordering information among cells, the topological dual, and boundaries.

The cell-tuple structure gives a simple, uniform representation of subdivided manifolds which unifies the existing work in the field and provides intuitive clarity in all dimensions. The dual subdivision, and boundaries, are represented consistently.

This work has direct applications in solid modeling, computer graphics, and computational geometry.

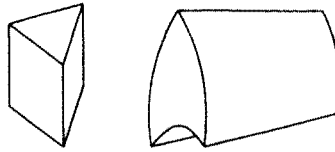
### 1. Introduction

Many computational models of physical and mathematical phenomena describe objects as a set of simple building blocks connected by some proximity relation. This allows the separation of the description of geometry (the “shape” of the building blocks) from topology (the relation between the building blocks). While there are a great variety of different geometric descriptions, many models use basically the same topological framework. The class of “subdivided  $d$ -manifolds” defined in Section 3 gives such a topological framework and allows a very general

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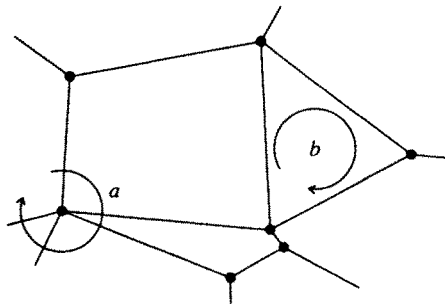
**Fig. 1.** Example of two topologically equivalent objects: Two subdivided manifolds are equivalent if there is a homeomorphism between them carrying  $k$ -cells onto  $k$ -cells (see Section 3).

geometric structure. The basic building blocks in subdivided manifolds are  $k$ -dimensional cells, where  $0 \leq k \leq d$ .

A basic requirement of any representation is that it represent topological structure up to equivalence. As an example, the two objects in Fig. 1 are considered to be equivalent. The definition of topological equivalence is given in Section 3.

It is also useful to have access to several other properties. A number of computational geometry algorithms use ordering information, for example: the only known optimal three-dimensional convex hull algorithm [29]; the divide-and-conquer method (one optimal algorithm) for building the two-dimensional Voronoi diagram [16], [19], [33]; and one way of describing the optimal algorithm for constructing the two-dimensional arrangement [6] (the other method is given in [13]). Figure 2 shows, within a subdivision of a surface: (a) an ordering of the edges and faces<sup>1</sup> incident to a vertex, and (b) an ordering of the vertices and edges incident to a face. Figure 3 shows, within a subdivision of a three-dimensional object: (a) an ordering of the faces and volumes incident to an edge, (b) an ordering of the edges and faces incident to both a vertex and a volume, and (c) an ordering of the vertices and edges incident to a face. These kinds of ordering are unified and generalized to  $d$ -dimensions in Section 7.

Given a subdivision of an object, it is sometimes useful to have access to the topological dual of that subdivision. A classic example in computational geometry is the duality between the Voronoi diagram and the Delaunay diagram. In rough



**Fig. 2.** Examples of order in two dimensions: (a) an ordering of the edges and faces incident to a vertex; (b) an ordering of the vertices and edges incident to a face.

<sup>1</sup> The term “face” is used in two different ways in this paper: to denote a two-dimensional cell in a two- or three-dimensional subdivided manifold; or to denote a subcell of a simplex.

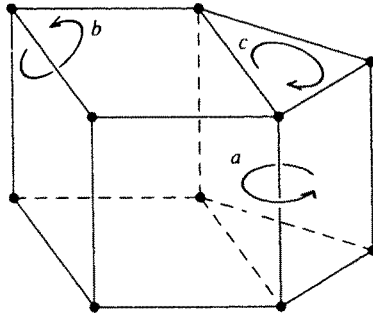


Fig. 3. Examples of order in three dimensions: (a) an ordering of the faces and volumes incident to an edge; (b) an ordering of the edges and faces incident to both a vertex and a volume; (c) an ordering of the vertices and edges incident to a face.

terms, the dual of a subdivided  $d$ -dimensional object is a subdivision of the same object produced by replacing every  $k$ -dimensional cell by a  $(d - k)$ -dimensional cell, while maintaining the corresponding incidence relations. An example is given in Fig. 4 (the original subdivision is depicted by solid lines and solid circles, while the dual is shown by dashed lines and open circles).

The objects encountered when modeling real-world phenomena often have boundaries, so to be useful in practice, a representation must be able to handle boundaries consistently. Figure 5 gives an example of an object and its boundary.

In the field of computational geometry, the most widely used and longest-standing representation of topological structure is the incidence graph [12], [15], [32]. This is a graph containing a node for every cell in an object, with arcs connecting nodes if the corresponding cells are incident and differ in dimension by one. The incidence graph does not provide any direct access to ordering information. Accessing ordering information requires searching in the graph, or the addition of auxiliary data structures, e.g., linked lists.

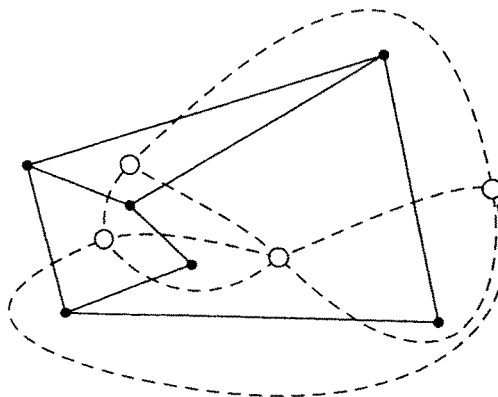


Fig. 4. Example of the dual. The original subdivision is depicted by solid lines and solid circles, while the dual is shown by dashed lines and open circles. Note the one-to-one correspondences between original faces and dual vertices, between original edges and dual edges, and between original vertices and dual faces.

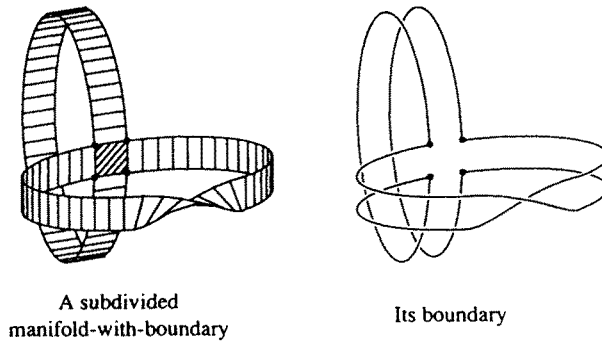


Fig. 5. Example of an object and its boundary.

In the case of two-dimensions, ordering information may be attached to edges; such edge-based schemes [2], [7], [25] work out nicely because an edge is always incident to at most two faces and to at most two vertices. Similar work appears in the context of constructive solid geometry, where one method of representing solid (three-dimensional) objects is to represent their boundaries (e.g., [1], [3], [11], and [23]). Other work on representing subdivided surfaces appears in [10], [14], [20], and [34]. For a comparative analysis of edge-based representations, see the work of Weiler [38], [39].

In 1984 Guibas and Stolifi [16] introduced the “edge algebra,” which uses a single basic unit (the directed, oriented edge), together with three simple operators, to represent the structure of, and ordering in, subdivisions of surfaces (2-manifolds). Faces and vertices are represented implicitly by “rings” of directed, oriented edges. The edge algebra can be characterized in an abstract setting as a finite set with three operators (following certain rules) acting on them. A one-to-one correspondence is shown between subdivided 2-manifolds and abstract edge algebras. The development includes an explicit representation of the dual subdivision, allowing symmetric access to the dual. Two primitive constructors are given for creating and combining edge algebras, capable of producing the edge algebra corresponding to any subdivided 2-manifold. The implementation of the edge algebra is called the “quad-edge data structure.”

In 1987 Dobkin and Laszlo [9] gave a generalization of this new idea to the three-dimensional case. They use a single basic unit, the “facet-edge,” along with a set of five operators for moving within the structure, and four constructors. A somewhat similar data structure was described by Buckley [5].

One of the main properties of the new approach opened up by Guibas, Stolfi, Dobkin, and Laszlo is that it represents cells implicitly (by certain sets of basic elements). For this reason, the representations developed in this new line of research will be referred to as “implicit-cell representations,” and the more conventional approaches will be referred to as “explicit-cell representations.”

This paper is concerned with generalizing the implicit-cell approach to the general case of arbitrary dimension. Similar work has been done independently by Lienhardt [21]; while the approach and the class of objects considered in his work is different from that in this work, the underlying idea is the same. The main

difference is that the present work defines and uses the class of subdivided manifolds, which allows a simple description of the basic elements and operators, and allows the results given by the three main theorems of this paper. The work of Tits [36] also contains a similar idea in an entirely different context, proving properties of mathematical objects called “buildings.”

The present work introduces the “cell-tuple structure,” a new representation of  $d$ -dimensional geometric objects, which contains in an easily accessible form the topological structure, ordering information, the dual, and boundaries. The basic elements are  $(d + 1)$ -tuples of cells, and the basic operators act on the components of these tuples in a natural way. Part of this work is the careful identification of an appropriate class of geometric objects, the subdivided manifolds, which encompass a large class of natural objects.

The main results are given as three theorems. The first theorem shows a relationship between sets of incident cells in a subdivided manifold and orbits of cell-tuples in its cell-tuple structure, which allows a simple description of many natural decompositions of objects and gives a unifying framework for existing representations. The second theorem shows that both the incidence graph and the cell-tuple structure are powerful enough to represent subdivided manifolds up to topological equivalence. The third theorem shows that circular orderings of cells exist in subdivided  $d$ -manifolds for all  $1 \leq k \leq d$ : given a  $(k - 2)$ -dimensional cell which is incident to a  $(k + 1)$ -dimensional cell, there is a simple closed path encountering each of the cells “between” them exactly once.

To prove the theorems, the “generalized barycentric subdivision” of a subdivided manifold is defined. This gives the connection between the “continuous” subdivided manifolds and the “discrete” cell-tuple structure, and is essential to understanding implicit-cell representations and seeing the basic similarity which they all share.

The cell-tuple structure may be implemented in several ways: as a set of abstract entities acted on by operators following “algebraic” rules; as a regular, arc-labeled graph; or as a data base acted on by relational queries. The “ring of cells” in any ordering can be obtained by a straightforward application of the basic operators. Each cell-tuple contains “positional” information, e.g., which endpoint of an edge you are near, which side of a face you are on, etc. Constructors are defined for creating and manipulating subdivided manifolds.

## 2. Problems with the Combinatorial Approach in Higher Dimensions

The representations of topological structure considered in this paper are combinatorial in nature, in that they are discrete structures representing relations between cells. One of the nice properties of the existing schemes in two dimensions (e.g., [16]) is that they are complete and unambiguous: a one-to-one correspondence may be established between all subdivided 2-manifolds (surfaces) and all valid instances of a combinatorial representation. While it would be nice to keep this property when generalizing to higher dimensions, it is unlikely that this will be possible.

While all surfaces may be characterized combinatorially by the classification theorem for surfaces [8], [35], there is no corresponding classification theorem for  $d$ -manifolds for any  $d \geq 3$ . A famous problem in mathematics is whether the Poincaré conjecture [28] is true or false: that among 3-manifolds the 3-sphere is characterized by having a trivial fundamental group.<sup>2</sup> Since there is currently no way to identify the 3-sphere combinatorially, it is unlikely that there is any simple way to decide if a combinatorial structure is a 4-manifold. The same holds for  $d > 4$ . Furthermore, two related results show that such an approach is not feasible. Markov [24] showed that the problem of recognizing whether two  $d$ -manifolds are homeomorphic (for  $d \geq 4$ ) is recursively unsolvable, and S. P. Novikov (see [37]) showed that the problem of recognizing a  $d$ -sphere (for  $d \geq 5$ ) is recursively unsolvable.

These results suggest that any combinatorial scheme which is general enough to represent all subdivided manifolds is also bound to allow nonmanifolds as well. Unfortunately, this implies that when building and manipulating such general combinatorial representations of geometric objects in higher dimensions, no matter what set of rules are imposed, the risk is run of constructing what Dobkin and Laszlo [9] aptly term “exquisite garbage.”

An alternative is to restrict the class of subdivided manifolds allowed; though it is not known how to determine whether or not an arbitrary combinatorial object may be realized as a subdivided manifold, in applications it is usually possible to use heuristics to ensure that manifold structure and other desired properties are maintained. (See [4] for further discussion of the combinatorial approach for manifolds and pseudomanifolds.)

### 3. Subdivided Manifolds

In geometric modeling, the objects being represented, and the building blocks which make them up, must be carefully chosen and well defined.

Manifolds are a natural choice for objects in higher dimensions because they are simply defined, there are useful mathematical results available, and they are general enough to include most objects which occur in real applications. Informally, a  $d$ -manifold is a topological space which looks locally like  $d$ -dimensional Euclidean space. (The reader is referred to [26] and [27], or [4], for topological background.) Many of the existing representations in two and three dimensions use manifolds or some restricted class of manifolds.

There are several possibilities for the basic building blocks. In this work, cells which are homeomorphic to open balls are used, where “self-intersecting” boundaries are not allowed. If self-intersecting boundaries were allowed, many of the results would still hold, but the description would not be as simple and clean. If more complicated building blocks are allowed (e.g., solid tori in three dimensions),

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<sup>2</sup> The (homotopic) fundamental group is a basic concept in algebraic topology, which may be found in introductory texts on the subject. It is not used (directly) in this paper.

then representing the structure of their boundaries is not enough to specify their topology; thus a separate data structure is required. Since a goal of this work is to provide a single data structure, this possibility is not considered.

For integers  $k > 0$  denote the **open unit  $k$ -ball** by  $\mathbb{B}^k$ , and the **closed unit  $k$ -ball** by  $\overline{\mathbb{B}^k}$ . For  $k = 0$ , let  $\mathbb{B}^0$  and  $\overline{\mathbb{B}^0}$  each consist of a single point. An **open  $k$ -cell** is a space which is homeomorphic with  $\mathbb{B}^k$  and a **closed  $k$ -cell** is a space which is homeomorphic with  $\overline{\mathbb{B}^k}$ . Both open and closed  $k$ -cells are said to have **dimension  $k$** . A nonempty Hausdorff space  $X$  is called a  **$d$ -manifold** if each point of  $X$  has a neighborhood homeomorphic with  $\mathbb{B}^d$ . Some authors include in their definition of a manifold that it have a countable basis, or that it be metrizable; neither restriction is added here. It is called a  **$d$ -manifold-with-boundary** if each point of  $X$  has a neighborhood homeomorphic with either  $\mathbb{B}^d$  or  $\mathbb{B}_{1/2}^d$ , where

$$\mathbb{B}_{1/2}^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid (x_1^2 + \dots + x_d^2)^{1/2} < 1 \text{ and } x_1 \geq 0\}.$$

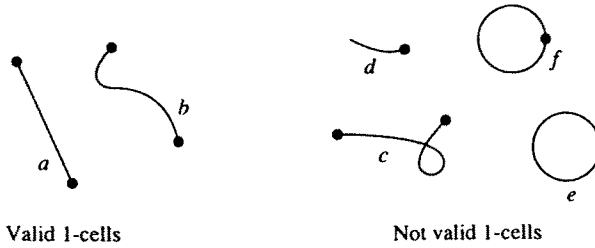
The class of manifolds-with-boundary includes manifolds (without boundaries), but not vice versa. A point  $x$  in a manifold-with-boundary  $X$  is called a **boundary point** if its neighborhoods (of the type mentioned in the definition) are homeomorphic to  $\mathbb{B}_{1/2}^d$ . The set of all boundary points of  $X$  is called the **boundary** of  $X$ , denoted  $\partial X$ . Points of  $X$  which are not boundary points are called **interior points**, and the set of all interior points is denoted by  $\text{Int } X$ . In this paper use is made of the boundary and interior of balls, simplices (defined in Section 4), and manifolds. Balls always have the subspace topology as a subspace of  $\mathbb{R}^k$ . If a closed  $k$ -ball is considered as a manifold-with-boundary, its boundary and interior as a manifold-with-boundary agree with its boundary and interior as a subspace of  $\mathbb{R}^k$ . Note that  $\mathbb{B}^k = \text{Int } \overline{\mathbb{B}^k}$ . The same observations hold for  $k$ -dimensional simplices. When the interior or boundary of a general manifold-with-boundary are taken, the manifold-with-boundary definition will always apply.

Define a **subdivided  $d$ -manifold** as a pair  $(M, C)$ , where  $M$  is a  $d$ -manifold and  $C = \{c_\alpha\}_{\alpha \in I_C}$  is a finite collection of disjoint subsets of  $M$  whose union is  $M$  such that for each  $\alpha \in I_C$ :

- There is an integer  $k$  and an embedding  $f_\alpha: \overline{\mathbb{B}^k} \rightarrow M$  such that  $f_\alpha(\mathbb{B}^k) = c_\alpha$  and  $f_\alpha(\overline{\mathbb{B}^k}) = \bar{c}_\alpha$ .
- $\bar{c}_\alpha - c_\alpha$  is equal to a union of cells from  $C$ .

It is not hard to show that this definition is equivalent to saying that  $(M, C)$  is a finite, regular CW complex in which  $M$  is a  $d$ -manifold. This fact is used in the proof of Lemma 1, and puts the definition of subdivided manifolds in the proper mathematical context. The definition of subdivided  $d$ -manifold-with-boundary is the same as that for subdivided  $d$ -manifold, except that  $M$  is a  $d$ -manifold-with-boundary.

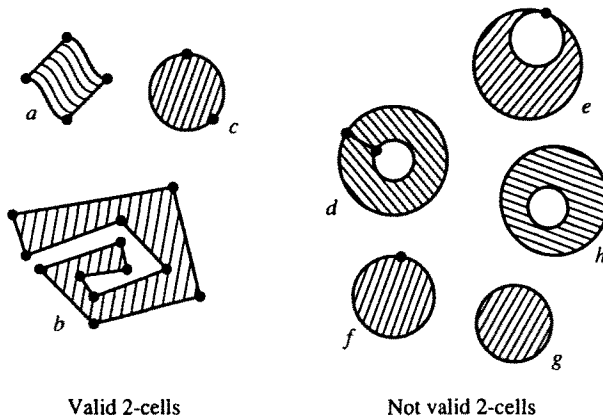
By this definition, each  $c_\alpha \in C$  is an open  $k$ -cell for some  $k$  between 0 and  $d$ , inclusive. The open cells in subdivided manifolds are referred to simply as cells. Examples of 1-cells in a subdivided manifold are given in Fig. 6, and examples of 2-cells in a subdivided manifold are given in Fig. 7.



**Fig. 6.** 1-cells in a subdivided manifold. Dots represent 0-cells and curved lines represent 1-cells. Observe that (a) and (b) each consist of an open 1-cell and two distinct boundary 0-cells, hence are homeomorphic to  $\overline{\mathbb{B}^1}$ . The curve in (c) crosses itself, hence is not a 1-cell. The boundary of the 1-cell in (d) should consist of two 0-cells; one is absent. The closed curve in (e) is not a 1-cell. The 1-cell in (f), when taken with the single 0-cell which contains its boundary, is not homeomorphic to  $\overline{\mathbb{B}^1}$  (its boundary “self-intersects”).

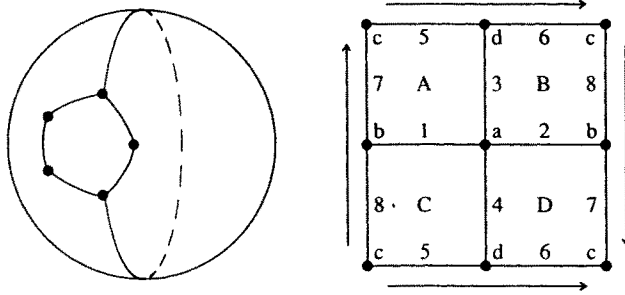
Figure 8 shows a subdivision of a 2-sphere containing five vertices (0-cells), six edges (1-cells), and three faces (2-cells), and a subdivision of a Klein bottle containing four vertices, eight edges, and four faces. Consider the object shown in Fig. 3 obtained by identifying a face of a cube with the square side of a triangular prism. The result is a subdivided 3-manifold-with-boundary consisting of ten 0-cells, seventeen 1-cells, ten 2-cells, and two 3-cells. This object could be made into a subdivided 3-manifold (without boundary) by adding another 3-cell whose boundary matches the boundary of the given object; this may be embedded in  $\mathbb{R}^4$ .

The **boundary** of a cell is  $\partial_\alpha = \bar{c}_\alpha - c_\alpha$ . The function  $f_\alpha$  is called the **characteristic function** for  $\alpha$ . Since the characteristic functions are homeomorphisms, the boundary of a  $k$ -cell  $c_\alpha$  is homeomorphic to the  $(k - 1)$ -sphere, i.e., the cells do not have



**Fig. 7.** 2-cells in a subdivided manifold. Dots represent 0-cells, curved lines represent 1-cells, and shaded areas represent 2-cells. Observe that (a), (b), and (c) each consist of an open 2-cell whose boundary is a path of nonoverlapping 0-cells and 1-cells, hence are homeomorphic to  $\overline{\mathbb{B}^2}$ . The cells making up the boundary of the 2-cells in (d) and (e) do not form a simple path (their boundaries “self-intersect”). The single 1-cell in the boundary of the 2-cell in (f) is not a valid 1-cell, as its boundary self-intersects. The closed curve making up the boundary of the 2-cell in (g) is not a 1-cell. The shaded area in (h) is not a 2-cell, as it is not homeomorphic to  $\overline{\mathbb{B}^2}$ .





**Fig. 8.** Two subdivided 2-manifolds. On the left is a subdivision of a 2-sphere containing five vertices (0-cells), six edges (1-cells), and three faces (2-cells). On the right is a subdivision of a Klein bottle containing four vertices, eight edges, and four faces, obtained by identifying the top and bottom sides of the square (edges 5 and 6), and the left and right sides of the square (edges 7 and 8), using the orientation given by the arrows.

self-intersecting boundaries. The dimension of a cell  $c_\alpha \in C$  is denoted by  $\dim(c_\alpha)$ . Let  $c_{\alpha_k}$  be a  $k$ -cell and let  $c_{\alpha_l}$  be an  $l$ -cell. If  $c_{\alpha_k} \subseteq c_{\alpha_l}$  write  $c_{\alpha_k} < c_{\alpha_l}$ , and say that  $c_{\alpha_k}$  is a **subcell** of  $c_{\alpha_l}$ . The cells  $c_{\alpha_k}$  and  $c_{\alpha_l}$  are **incident** if  $c_{\alpha_k} < c_{\alpha_l}$  or  $c_{\alpha_l} < c_{\alpha_k}$ . Note that  $<$  is a strict partial ordering of  $C$ . A sequence  $\alpha_{i_0}, \dots, \alpha_{i_k}$  of cells satisfying  $\alpha_{i_0} < \dots < \alpha_{i_k}$  is called an **ascending chain** of cells. If  $c_{\alpha_k} < c_{\alpha_l}$  and  $l = k + 1$ , write  $c_{\alpha_k} < c_{\alpha_l}$ . If  $M$  is a manifold-with-boundary, a  $k$ -cell  $c_\alpha \subseteq \partial M$  is called a **boundary  $k$ -cell**.

Two subdivided  $d$ -manifolds (or subdivided  $d$ -manifolds-with-boundary) are **equivalent**,  $(M, C) \simeq (N, D)$ , if there is a homeomorphism between  $M$  and  $N$  carrying  $k$ -cells onto  $k$ -cells. An example of two equivalent subdivided 3-manifolds-with-boundary was given in Fig. 1.

For notational convenience, we assume the existence of a cell  $c_{\alpha_{-1}}$  of dimension  $-1$  and a cell  $c_{\alpha_{d+1}}$  of dimension  $d + 1$ , such that  $c_{\alpha_{-1}} < c_\alpha < c_{\alpha_{d+1}}$  for all  $c_\alpha \in C$ . It is stressed that this is notation only, and these cells are never actually used. For example, when we write  $c_{\alpha_{k-1}} < c_{\alpha_k} < c_{\alpha_{k+1}}$ , where  $\dim(c_{\alpha_i}) = i$ , it is understood that if  $k = 0$ , this simply means  $c_{\alpha_0} < c_{\alpha_1}$ , and if  $k = d$ , it simply means  $c_{\alpha_{d-1}} < c_{\alpha_d}$ .

The incidence graph can be defined using the partial ordering defined on cells. This graph has a node for every cell and arcs connecting two nodes if and only if the corresponding cells are incident and differ in dimension by one. It is convenient to refer to the nodes by the indices of the cells, so the node set for the incidence graph of  $(M, C)$  is the index set  $I_C$ . The **incidence graph** of  $(M, C)$  is  $\mathcal{P}_{(M, C)} = (I_C, <)$ , where  $\alpha_k < \alpha_l \Leftrightarrow c_{\alpha_k} < c_{\alpha_l}$ . It is assumed that associated with the incidence graph is a labeling of the index set giving the dimension of its associated cell,  $\dim(\alpha) = k \Leftrightarrow \dim(c_\alpha) = k$ .

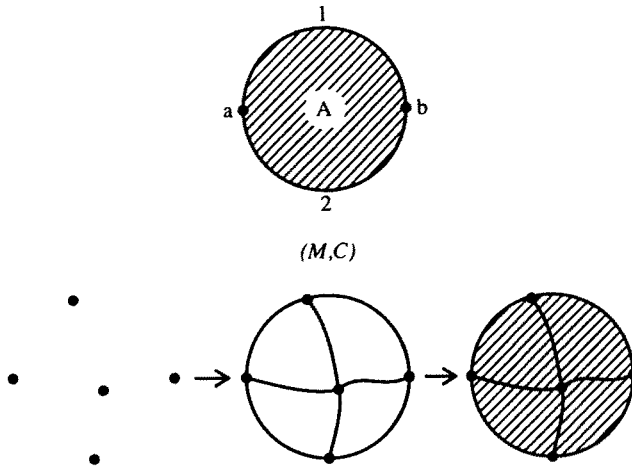
#### 4. The Generalized Barycentric Subdivision

The generalized barycentric subdivision of a subdivided manifold is defined in this section. An understanding of the generalized barycentric subdivision gives the right

vantage point for understanding the mathematics underlying this work, and for seeing the underlying similarities among all of the implicit-cell representations. It is also a crucial object in the proofs of the theorems given in Sections 6 and 7, giving a link between subdivided manifolds and their combinatorial representation. The definition of the generalized barycentric subdivision applies to subdivided manifolds-with-boundary as well as subdivided manifolds. Note that the term generalized barycentric subdivision is used for two different concepts in the algebraic topology literature. Our definition agrees with that in [22], but is different from that in [27].

Given a subdivided manifold  $(M, C)$ , the partial order defined by its incidence relation is used to define a simplicial complex. Lemma 1 shows the existence of a homeomorphism from this simplicial complex onto  $(M, C)$ , thus giving a triangulation of the manifold  $M$ . Furthermore, this triangulation is a refinement of  $(M, C)$  which allows the cells of  $C$  to be written in a very convenient form. (A refinement of  $(M, C)$  is a subdivided manifold  $(M, D)$  in which every cell of  $D$  is contained in a cell of  $C$ . In this particular case, the cells of  $D$  are the images of simplices under the homeomorphism.)

An intuitive description of the generalized barycentric subdivision is given now, by giving an informal sketch of a construction of the set of cells which make it up. For every cell  $c_\alpha \in C$  create a new vertex  $c_{sd}(\alpha)$  in  $c_\alpha$ . For every pair of cells  $c_{\alpha_{k_1}} < c_{\alpha_{k_2}}$  in  $C$ , add a new 1-cell  $c_{sd}(\alpha_{k_1}, \alpha_{k_2})$  in  $c_{\alpha_{k_2}}$  having as endpoints the new 0-cells  $c_{sd}(\alpha_{k_1})$  and  $c_{sd}(\alpha_{k_2})$ . For every ascending chain  $c_{\alpha_{k_1}} < c_{\alpha_{k_2}} < c_{\alpha_{k_3}}$  of three cells, add a 2-cell in  $c_{\alpha_{k_3}}$  which has as its boundary the 0-cells  $c_{sd}(\alpha_{k_1})$ ,  $c_{sd}(\alpha_{k_2})$ , and  $c_{sd}(\alpha_{k_3})$  and the 1-cells  $c_{sd}(\alpha_{k_1}, \alpha_{k_2})$ ,  $c_{sd}(\alpha_{k_1}, \alpha_{k_3})$ , and  $c_{sd}(\alpha_{k_2}, \alpha_{k_3})$ . Continue “filling in” cells of increasing dimension in this fashion, until finally new  $d$ -cells are added for all maximal ascending chains of cells in the partial order on  $C$ . The set of newly defined cells gives the generalized barycentric subdivision. (See Fig. 9.)



**Fig. 9.** Example of construction of the generalized barycentric subdivision. At the top is a subdivided 2-manifold-with-boundary. The bottom sequence shows the construction. First five new vertices are added, one in each of the five cells; then eight edges are added, one for each pair of incident cells; and finally four faces are added, one for each full ascending chain of cells.

A  $k$ -**simplex**,  $k \geq 0$ , is the convex hull of  $k + 1$  affinely independent points in  $\mathbb{R}^d$ . A **face** of a simplex is a simplex formed by a subset of the simplex's vertices. The following notation is used to identify particular simplices: if  $v_1, \dots, v_l$  are points in  $\mathbb{R}^d$ ,  $\sigma(v_1, \dots, v_l)$  denotes the simplex defined by  $v_1, \dots, v_l$ . A **simplicial complex**  $K$  in  $\mathbb{R}^d$  is a collection of simplices in  $\mathbb{R}^d$  such that every face of a simplex of  $K$  is in  $K$  and the intersection of any two simplices of  $K$  is a face of each of them. The **polytope** (or **underlying space**) of a simplicial complex  $K$  is  $|K| = \bigcup_{\sigma \in K} \sigma$ . A **triangulation** of a space  $X$  is a simplicial complex  $K$  together with a homeomorphism  $h: |K| \rightarrow X$ .

An **abstract simplicial complex**  $\mathcal{A}$  is a collection of finite nonempty sets, such that if  $A$  is an element of the collection, so is every subset of  $A$ . The **vertex scheme** of a simplicial complex is the abstract simplicial complex defined by the vertex sets of its simplices. A **geometric realization** of an abstract simplicial complex  $\mathcal{A}$  is any simplicial complex whose vertex scheme is  $\mathcal{A}$ .

If  $(M, C)$  is a subdivided  $d$ -manifold or a subdivided  $d$ -manifold-with-boundary, where  $C = \{c_\alpha\}_{\alpha \in I_C}$ , define the abstract simplicial complex

$$\mathcal{A}_{(M,C)} = \{ \{ \alpha_{i_0}, \dots, \alpha_{i_k} \} \mid \alpha_{i_0} < \dots < \alpha_{i_k}, \alpha_{i_j} \in I_C \}.$$

For example,  $\mathcal{A}_{(M,C)}$  for the subdivided manifold shown in Fig. 9 is

$$\{ a, b, 1, 2, A, a1, a2, b1, b2, aA, bA, 1A, 2A, a1A, a2A, b1A, b2A \}.$$

Let  $K_{(M,C)}$  be any geometric realization of  $\mathcal{A}_{(M,C)}$ , so that the vertex  $v_\alpha \in K_{(M,C)}$  corresponds to  $\{ \alpha_i \} \in \mathcal{A}_{(M,C)}$ . This always exists by Theorem 3.1(a) of [27]. It may be assumed that  $K_{(M,C)}$  is embedded in  $\mathbb{R}^n$  for some  $n$  (see the proof of Theorem 3.1(a) of [27]). If  $\psi: |K_{(M,C)}| \rightarrow M$  is a homeomorphism and  $\alpha_{i_0} < \dots < \alpha_{i_k}$ , define  $c_\psi(\alpha_{i_0}, \dots, \alpha_{i_k}) = \psi(\text{Int } \sigma(v_{\alpha_{i_0}}, \dots, v_{\alpha_{i_k}}))$ ; thus  $c_\psi(\alpha_{i_0}, \dots, \alpha_{i_k})$  is the image under  $\psi$  of the interior of the simplex  $\sigma(v_{\alpha_{i_0}}, \dots, v_{\alpha_{i_k}})$ . Recall that the interior of a vertex is equal to the vertex, so that  $c_\psi(\alpha_{i_0})$  is a point in  $M$ .

**Lemma 1.** *If  $(M, C)$  is a subdivided  $d$ -manifold or a subdivided  $d$ -manifold-with-boundary, then there exists a homeomorphism  $\psi_{(M,C)}: |K_{(M,C)}| \rightarrow M$  such that*

$$c_\alpha = \bigcup_{\alpha_0 < \dots < \alpha_l < \alpha} c_{\psi_{(M,C)}}(\alpha_0, \dots, \alpha_l, \alpha).$$

In the statement of this lemma, and in the remainder of the paper, the following convention holds: the set of sequences over which the union is taken includes the sequence consisting only of  $\alpha$ .

*Proof.* The proof is omitted for brevity. It involves a formalization of the construction given earlier in this section, and may be found in [4]. □

For a given  $(M, C)$ , let  $\psi_{(M,C)}$  be a homeomorphism satisfying Lemma 1, and

define  $c_{sd}(\alpha_{i_0}, \dots, \alpha_{i_k}) = c_{\psi_{(M,C)}}(\alpha_{i_0}, \dots, \alpha_{i_k})$ . Let

$$C_{sd} = \{c_{sd}(\alpha_{i_0}, \dots, \alpha_{i_k}) \mid \alpha_{i_0} < \dots < \alpha_{i_k} \in C\}.$$

$(M, C_{sd})$  is the **generalized barycentric subdivision** of  $(M, C)$ .  $(M, C_{sd})$  is a subdivided  $d$ -manifold or subdivided  $d$ -manifold-with-boundary itself, and each cell  $c_{sd}(\alpha_{i_0}, \dots, \alpha_{i_k})$  of  $C_{sd}$  is the image under  $\psi_{(M,C)}$  of the interior of the simplex  $\sigma(v_{\alpha_{i_0}}, \dots, v_{\alpha_{i_k}})$ . Because characteristic maps are not unique, the barycentric subdivision is not unique, but is unique up to equivalence.

The generalized barycentric subdivision allows the use of theorems from algebraic topology about triangulated manifolds in the proofs of the main results, and may be used when extending those results to manifolds-with-boundary. Writing cells in  $C$  as unions of cells in  $C_{sd}$  is useful in proofs, and allows the definition of the dual in a straightforward manner in Section 8.

Note that the proof of Lemma 1 depends on the basic building blocks being open cells and on the characteristic functions being homeomorphisms. The definition of the generalized barycentric subdivision depends on Lemma 1. The proof of Lemma 2 in the next section makes use of the generalized barycentric subdivision, and Lemma 2 is necessary for the definition of the cell-tuple structure. In short, the results in this paper depend critically on the particular definition of subdivided manifolds.

Two facts about  $K_{(M,C)}$  for manifolds and manifolds-with-boundary are used at various times in the remainder of the paper. The first says that in a triangulated manifold or manifold-with-boundary, every simplex is a face of a  $d$ -simplex. This follows from Exercise 35.4 of [27].

**Fact 1.** *If  $\sigma \in K_{(M,C)}$ , then:*

- (a)  $\sigma$  is a  $d$ -simplex, or
- (b)  $\sigma$  is contained in a  $d$ -simplex.

Since  $K_{(M,C)}$  is homeomorphic to  $M$ , and is trivially triangulated, it is a triangulated manifold. The following fact says that a  $(d - 1)$ -simplex in  $K_{(M,C)}$  is in the boundary of  $|K_{(M,C)}|$  if and only if it is incident to exactly one  $d$ -simplex (otherwise it is incident to exactly two  $d$ -simplices). This fact follows from Exercise 35.4 of [27].

**Fact 2.** *If  $\sigma \in K_{(M,C)}$  is a  $(d - 1)$ -simplex, then:*

- (a)  $\sigma \not\subseteq \partial|K_{(M,C)}| \Rightarrow \sigma$  is contained in exactly two  $d$ -simplices.
- (b)  $\sigma \subseteq \partial|K_{(M,C)}| \Rightarrow \sigma$  is contained in exactly one  $d$ -simplex.

### 5. The Cell-Tuple Structure

The basic elements of the cell-tuple structure are now defined. If  $(M, C)$  is a subdivided  $d$ -manifold, a  $(d + 1)$ -tuple  $(c_{\alpha_0}, \dots, c_{\alpha_d})$  of cells from  $C$  such that  $c_{\alpha_i} \in C$ ,

$c_{\alpha_0} < \dots < c_{\alpha_d}$  is called a **cell-tuple**. Tuples are conceptually simple and change in a uniform way with dimension. This simplicity and uniformity carries over to the cell-tuple structure. The set of all cell-tuples for  $(M, C)$  denoted by

$$T_{(M,C)} = \{(c_{\alpha_0}, \dots, c_{\alpha_d}) \mid c_{\alpha_i} \in C, c_{\alpha_0} < \dots < c_{\alpha_d}\}.$$

The  $k$ -component of a cell-tuple  $t = (c_{\alpha_0}, \dots, c_{\alpha_d})$  is denoted by  $t_k = c_{\alpha_k}$ .

The basic operators act on components of the cell-tuples. Before defining these operators the following lemma is needed, which shows that if a component of a given cell-tuple is specified, then there is a unique cell-tuple which agrees with the given cell-tuple on all components except the specified one.

**Lemma 2.** *If  $(M, C)$  is a subdivided  $d$ -manifold,  $t \in T_{(M,C)}$ , and  $0 \leq k \leq d$ , then there is a unique  $t' \in T_{(M,C)}$  such that  $t'_k \neq t_k$  and  $t'_i = t_i$  for all  $i \neq k$ .*

Recall from Section 4 that  $v_{\alpha_k}$  denotes the vertex of  $K_{(M,C)}$  corresponding to  $c_{\alpha_k} \in C$ . In this section  $\sigma(\alpha_1, \dots, \alpha_i)$  is used as shorthand for the simplex  $\sigma(v_{\alpha_1}, \dots, v_{\alpha_i})$  corresponding to the chain of cells  $c_{\alpha_1} < \dots < c_{\alpha_i} \in C$ .

*Proof.* Let  $t \in T_{(M,C)}$  and  $0 \leq k \leq d$ . If  $t = (c_{\alpha_0}, \dots, c_{\alpha_{k-1}}, c_{\alpha_k}, c_{\alpha_{k+1}}, \dots, c_{\alpha_d})$ , let  $\sigma = \sigma(\alpha_0, \dots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \dots, \alpha_d)$  and let  $\sigma' = \sigma(\alpha_0, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_d)$ . By Fact 2(a), there is a  $d$ -simplex  $\sigma'' \neq \sigma$  which contains  $\sigma'$ . There must exist a  $k$ -cell  $c_{\alpha'_k}$  such that  $\sigma'' = \sigma(\alpha_0, \dots, \alpha_{k-1}, \alpha'_k, \alpha_{k+1}, \dots, \alpha_d)$ . Let

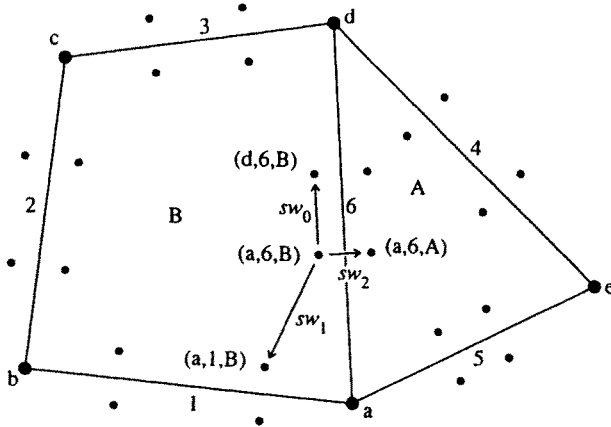
$$t' = (c_{\alpha_0}, \dots, c_{\alpha_{k-1}}, c_{\alpha'_k}, c_{\alpha_{k+1}}, \dots, c_{\alpha_d}). \quad \square$$

This guarantees that the following definition of the basic operators is well defined. For  $0 \leq k \leq d$ , define  $switch_k: T_{(M,C)} \rightarrow T_{(M,C)}$  by  $switch_k(t) = t'$ , where  $t'$  is such that  $t'_k \neq t_k$  and  $t'_i = t_i$  for  $i \neq k$ , as given by Lemma 2. The result of  $switch_k(t)$  is to return the cell-tuple which agrees with  $t$  on all but the  $k$ th component.

When showing examples of the cell-tuple structure, cell-tuples are drawn as dots located within their  $d$ -dimensional component, and near to their other components. See Fig. 10 for examples of  $switch_0(t)$ ,  $switch_1(t)$ , and  $switch_2(t)$ , where  $t = (a, 6, B)$ .

The **cell-tuple structure**  $\mathcal{F}_{(M,C)}$  for  $(M, C)$  is the pair  $(T_{(M,C)}, \{switch_k\}_{0 \leq k \leq d})$ . Abusing notation,  $switch$  will often be taken to represent the set of operators  $\{switch_k\}_{0 \leq k \leq d}$ , e.g., the cell-tuple structure will be written as  $(T_{(M,C)}, switch)$ .

It is useful, and helpful for intuition, to think of the cell-tuple structure as an undirected arc-labeled graph  $G_{(M,C)}$ , which has a node for each cell-tuple and a  $k$ -arc between nodes whose cell-tuples are related by  $switch_k$ . That is,  $G_{(M,C)} = (V, E)$ , where  $V = T_{(M,C)}$ ,  $E_k = \{(t^1, t^2) \mid switch_k(t^1) = t^2\}$ , and  $E = \bigcup_{0 \leq k \leq d} E_k$ . An arc  $e \in E$  is labeled with  $k$  if  $e \in E_k$ . These undirected arcs are well defined, because Lemma 2 implies that  $switch_k(t^1) = t^2$  if and only if  $switch_k(t^2) = t^1$ . This labeling is well defined, because the  $E_k$ 's are disjoint:  $k \neq l \Rightarrow [switch_k(t)]_k \neq [switch_l(t)]_k$ . Note that every node of this graph is incident to exactly one arc labeled  $k$  for



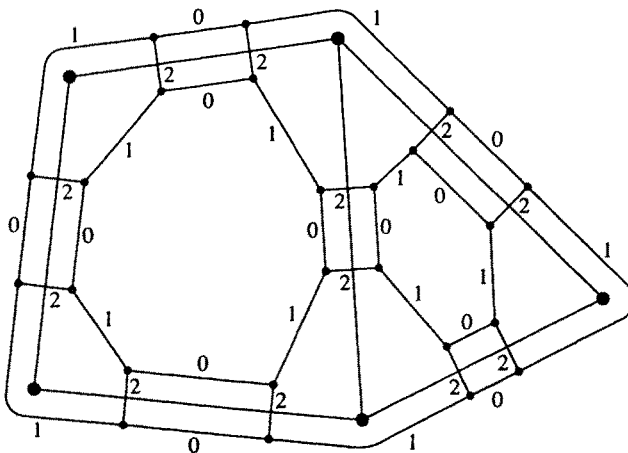
**Fig. 10.** Examples of *switch*. Cell-tuples are depicted as dots, within the face which is their 2-component, near the edge which is their 1-component, toward the vertex which is their 0-component, e.g., cell-tuple  $(a, 6, B)$  is shown as a dot in face  $B$ , near edge 6, toward vertex  $a$ . The arrows indicate the application of  $switch_k$  to  $(a, 6, B)$ :  $switch_0((a, 6, B)) = (d, 6, B)$ ,  $switch_1((a, 6, B)) = (a, 1, B)$ , and  $switch_2((a, 6, B)) = (a, 6, A)$ .

$0 \leq k \leq d$ . Figure 11 shows  $G_{(M,C)}$  for the subdivision of the 2-sphere given in Fig. 8.

If  $w = w_1 \cdots w_l \in \{0, \dots, d\}^*$ , define

$$switch_w(t) = \begin{cases} switch_{w_l}(switch_{w_{l-1}}(\cdots (switch_{w_2}(switch_{w_1}(t)) \cdots))) & \text{if } w \neq \lambda, \\ t & \text{if } w = \lambda. \end{cases}$$

If  $J \subseteq \{0, \dots, d\}^*$ , let  $switch_J(t) = \{switch_w(t) \mid w \in J\}$ . If  $I \subseteq \{0, \dots, d\}$ ,  $switch_{I^*}(t)$  is called the  $I$ -orbit of  $t$ . This may be thought of as the set of all cell-tuples



**Fig. 11.** Example of  $G_{(M,C)}$ . Nodes corresponding to cell-tuples are depicted by dots as in Fig. 10. There is an arc labeled  $k$  for every pair of cell-tuples related by  $switch_k$ .

obtainable from  $t$  by repeated applications of  $switch_k$  operations, where the  $k$ 's are in  $I$ . The set of  $I$ -orbits for all  $t \in T_{(M,C)}$  partitions  $T_{(M,C)}$  into a set of equivalence classes. Each subset  $I$  gives a unique partition.

In terms of the graphical interpretation of the cell-tuple structure an  $I$ -orbit of  $t$  may be understood as the set of all cell-tuples in  $G_{(M,C)}$  reachable from  $t$  by following paths whose arcs are labeled by elements of  $I$ . Another way of thinking of this is by considering the subgraph of  $G_{(M,C)}$  consisting of all arcs whose labels are in  $I$ . The  $I$ -orbit of  $t$  is just the connected component of this subgraph containing  $t$ . The connected components of this subgraph are exactly the set of  $I$ -orbits of  $T_{(M,C)}$ , hence are equal to the equivalence classes generated by the  $I$ -orbits.

If  $c_\alpha$  is a  $k$ -cell of  $C$ , the set of cell-tuples having  $c_\alpha$  as a component,

$$assoc(c_\alpha) = \{t \in T_{(M,C)} \mid t_k = c_\alpha\},$$

is called the set of **associated cell-tuples** of  $c_\alpha$ . If  $c_{\alpha_0} < \dots < c_{\alpha_l}$ , such that  $\dim(c_{\alpha_j}) = i_j$  for  $j \in \{0, \dots, l\}$  the set of cell-tuples

$$assoc(c_{\alpha_0}, \dots, c_{\alpha_l}) = \{t \in T_{(M,C)} \mid t_{i_j} = c_{\alpha_j}, 0 \leq j \leq l\}$$

is called the set of **associated cell-tuples** of  $c_{\alpha_0}, \dots, c_{\alpha_l}$ . This gives a correspondence between chains of cells in  $C$  and subsets of  $T_{(M,C)}$ . For  $0 \leq k \leq d$ , the associated sets of the  $k$ -cells in  $C$  form a partition of  $T_{(M,C)}$ . This is developed to give a fundamental relationship between cells in  $C$  and orbits in  $\mathcal{T}_{(M,C)}$ , when Theorem 1 and Corollary 3 are proved in Section 6.

Two cell-tuple structures are **equivalent**,  $\mathcal{T}_{(M,C)} \simeq \mathcal{T}_{(N,D)}$ , if there is a bijection  $j: T_{(M,C)} \rightarrow T_{(N,D)}$  which preserves the  $switch$  operation, i.e.,

$$switch_k(j(t)) = j(assoc_k(t))$$

for all  $t \in T_{(M,C)}$ ,  $0 \leq k \leq d$ .

The following lemma shows that for any ascending chain of cells there is a cell-tuple which has all of the cells in that chain as components. This is equivalent to showing that any ascending chain of cells may be extended to a maximal ascending chain.

**Lemma 3.** *If  $(M, C)$  is a subdivided  $d$ -manifold or a subdivided  $d$ -manifold-with-boundary and  $c_{\alpha_0} < \dots < c_{\alpha_l}$ , where  $l \geq 0$ ,  $c_{\alpha_j} \in C$  and  $\dim(c_{\alpha_j}) = i_j$ , then there exists  $t \in T_{(M,C)}$  such that  $t_{i_j} = c_{\alpha_j}$  for  $j = 0, \dots, l$ .*

*Proof.* If  $l = d$ , let  $t = (c_{\alpha_0}, \dots, c_{\alpha_l})$ . If  $l < d$ , let  $\sigma = \sigma(\alpha_{i_0}, \dots, \alpha_{i_l})$ . By Fact 1 there exists a  $d$ -simplex  $\sigma(\alpha_0, \dots, \alpha_d) \in K_{(M,C)}$  containing  $\sigma$ . Let  $t = (c_{\alpha_0}, \dots, c_{\alpha_l})$ .  $\square$

A similar result to that of Lemma 2 can be phrased without reference to the cell-tuple structure. Given an ascending chain of three cells, where the difference in dimension between each pair of consecutive cells is one, there is a unique cell

which is incident to both the first and third cells (taken in ascending order) but is different from the middle cell. Recall (from Section 3) that the assumption of the existence of cells  $c_{\alpha_{-1}}$  and  $c_{\alpha_{d+1}}$  is made for notational convenience.

**Corollary 1.** *If  $(M, C)$  is a subdivided  $d$ -manifold,  $c_{\alpha_{k-1}} \prec c_{\alpha_k} \prec c_{\alpha_{k+1}}$ , where  $0 \leq k \leq d$ ,  $c_{\alpha_k} \in C$  and  $\dim(c_{\alpha_k}) = k$ , then there is a unique  $c_{\alpha'_k} \in C$  such that  $c_{\alpha'_k} \neq c_{\alpha_k}$  and  $c_{\alpha_{k-1}} \prec c_{\alpha'_k} \prec c_{\alpha_{k+1}}$ .*

*Proof.* By Lemma 3, there is a  $t \in T_{(M,C)}$  such that  $t_i = c_{\alpha_i}$  for  $i \in \{0, \dots, d\} \cap \{k-1, k, k+1\}$ . By Lemma 2 there exists a unique  $t'$  such that  $t'_i = t_i \Leftrightarrow i \neq k$ . Let  $c_{\alpha'_k} = t'_k$ . This satisfies the statement of the lemma. It must be shown that  $c_{\alpha'_k}$  is unique. Suppose there exists  $c_{\alpha''_k}$  which is not equal to  $c_{\alpha_k}$  or  $c_{\alpha'_k}$ , such that  $c_{\alpha_{k-1}} \prec c_{\alpha''_k} \prec c_{\alpha_{k+1}}$ . Then the cell-tuple  $(t_0, \dots, t_{k-1}, c_{\alpha''_k}, t_{k+1}, \dots, t_d)$  is not equal to  $t$  or  $t'$ , contradicting Lemma 2.  $\square$

It is useful to define a *switch* operator on triples, which is possible given Corollary 1. Let

$$S = \{(c_{\alpha_{k-1}}, c_{\alpha_k}, c_{\alpha_{k+1}}) \mid c_{\alpha_{k-1}} \prec c_{\alpha_k} \prec c_{\alpha_{k+1}}, 0 \leq k \leq d, \text{ and } c_{\alpha_{k-1}}, c_{\alpha_k}, c_{\alpha_{k+1}} \in C\}.$$

Define *switch*:  $S \rightarrow C$  by  $switch(c_{\alpha_{k-1}}, c_{\alpha_k}, c_{\alpha_{k+1}}) = c_{\alpha'_k}$ , where  $c_{\alpha'_k}$  is given by Corollary 1.

This allows another approach to the representation of subdivided manifolds, which may be called the **augmented incidence graph**. To the standard incidence graph, add a new data structure which contains a pointer for every triple  $(c_{\alpha_{k-1}}, c_{\alpha_k}, c_{\alpha_{k+1}})$  in the set  $S$ , which points to  $switch(c_{\alpha_{k-1}}, c_{\alpha_k}, c_{\alpha_{k+1}})$ . This requires considerably less space than the full cell-tuple structure when the number of dimensions gets large, at some cost in the speed of *switch* queries and searches.

Yet another possible approach is to maintain a relational database of the set  $T_{(M,C)}$  of tuples, without explicitly representing the *switch* operators at all. If  $t$  is a cell-tuple,  $switch_k(t)$  may be found by determining the cell-tuple in the database satisfying the definition of *switch*. Similarly, the set of cell-tuples associated with a chain of cells may be found by a relational query, i.e., finding all cell-tuples in the database containing the specified cells in their appropriate components.

Thus there are a number of ways of viewing the cell-tuple structure, each leading to a different possible implementation: as a set of data elements and pointers directly representing the cell-tuples and *switch* operators; as an arc-labeled graph; as an augmented incidence graph; or as a relational database. The time and size requirements of these methods are discussed in [4].

## 6. A Basic Relationship Between Cells and Orbits

There is a beautiful relationship between sets of cells in a subdivided manifold and orbits of cell-tuples in its cell-tuple structure. This is stated and proved as Theorem 1 in this section. Besides being necessary for the proofs of the two

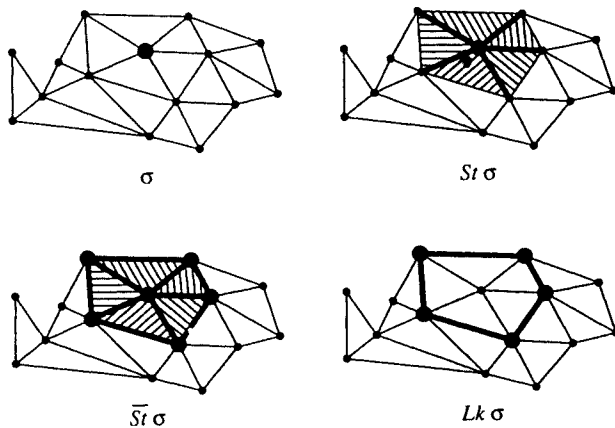


theorems in the next section, this relationship allows a concise description of commonly used elements of subdivided manifolds. An informal explanation of the theorem, along with examples relating to useful modeling techniques, are given immediately following the proof of Corollary 3.

Recall from Section 5 the definition of the  $I$ -orbit, the set of cell-tuples reachable from a given cell-tuple by applying sequences of  $switch_k$ 's, where the  $k$ 's are taken from  $I$ . If  $I$  is a subset of  $\{0, \dots, d\}$ , let  $\hat{I} = \{0, \dots, d\} - I$ . Thus if  $I = \{i_0, \dots, i_l\}$ , then  $\hat{I}$ -orbit( $t$ ) =  $switch_{\{(0, \dots, d) - \{i_0, \dots, i_l\}\}}(t)$ . If  $c_{\alpha_0} < \dots < c_{\alpha_l}$  is an ascending chain of cells, recall from Section 5 that the associated set of this chain of cells,  $assoc(c_{\alpha_0}, \dots, c_{\alpha_l})$  is the set of cell-tuples which contain these cells as components. If  $I = \{i_0, \dots, i_l\}$  is the set of dimensions of these cells, and  $t$  is any cell-tuple in the associated set, then Theorem 1 shows that the associated set is equal to the set of cell-tuples which may be obtained from  $t$  by repeated applications of  $switch_k$ , where  $k \notin I$ .

**Theorem 1.** *If  $c_{\alpha_0} < \dots < c_{\alpha_l}$  and  $t \in T_{(M,C)}$  such that  $t_{i_j} = c_{\alpha_j}$  for  $0 \leq j \leq l$ , then  $assoc(c_{\alpha_0}, \dots, c_{\alpha_l}) = switch_{\hat{I}}(t)$ , where  $I = \{i_0, \dots, i_l\}$ .*

The following definitions are needed in the proof. If  $\sigma$  is a simplex in  $K$ , then the **star** of  $\sigma$  is  $St \sigma = \{\text{Int } \sigma' \mid \sigma' \in K \text{ and } \sigma \text{ is a face of } \sigma'\}$ . The **closed star** of  $\sigma$  is  $\overline{St} \sigma = \{\sigma' \in K \mid \exists \sigma'' \in K \text{ s.t. } \sigma \text{ is a face of } \sigma'' \text{ and } \sigma' \text{ is a face of } \sigma''\}$ . The **link** of  $\sigma$  is  $Lk \sigma = \{\sigma' \in K \mid \sigma' \in \overline{St} \sigma \text{ and } \sigma' \cap \sigma = \emptyset\}$ . In words, if  $\sigma$  is a simplex in a simplicial complex  $K$ , then  $St \sigma$  is the set of the interiors of simplices in  $K$  which have  $\sigma$  as a face;  $\overline{St} \sigma$  is the set of simplices in  $K$  which have  $\sigma$  as a face, plus all of their faces; and  $Lk \sigma$  is the set of simplices in  $\overline{St} \sigma$  which are disjoint from  $\sigma$ . (See Fig. 12.) The  $k$ -skeleton of a simplicial complex is the union of all simplices of dimension less than or equal to  $k$ , and the  $k$ -skeleton of a subdivided manifold is the union of all cells of dimension less than or equal to  $k$ . The intuition behind the use of



**Fig. 12.** Examples of the star, closed star, and link. Here  $\sigma$  is a 0-simplex (point); its star  $St \sigma$  consists of itself, plus the interiors of the 1-simplices and 2-simplices (open line segments and triangles) incident to  $\sigma$ ; its closed star  $\overline{St} \sigma$  consists of all 2-simplices (closed triangles) incident to  $\sigma$  plus their subfaces; and its link  $Lk \sigma$  is the set of simplices in  $\overline{St} \sigma$  which are disjoint from  $\sigma$ .

the star, closed star, and link in what follows is that if  $c_\alpha$  is a  $k$ -cell in  $C$ , then, within the  $k$ -skeleton of  $K_{(M,C)}$ ,  $|St v_\alpha|$  corresponds to  $c_\alpha$ ,  $|\overline{St} v_\alpha|$  corresponds to  $\bar{c}_\alpha$ , and  $|Lk v_\alpha|$  corresponds to  $\dot{c}_\alpha$  (under the homeomorphism given by Lemma 1).

To show  $assoc(c_{\alpha_0}, \dots, c_{\alpha_i}) \subseteq switch_{F^*}(t)$  requires showing that the subgraph of  $G_{(M,C)}$  induced by  $assoc(c_{\alpha_0}, \dots, c_{\alpha_i})$  is connected. This is the only difficult part of the proof. The proof uses the fact that there is a one-to-one correspondence between the set of ascending chains of length  $l$  in  $C$  and the  $l$ -simplices of  $K_{(M,C)}$ , and the similar fact that there is a one-to-one correspondence between cell-tuples in  $T_{(M,C)}$  and  $d$ -simplices in  $K_{(M,C)}$ . These facts both follow from the definition of  $K_{(M,C)}$  (the graphical interpretation of the cell-tuple structure is equivalent to a labeled version of the 1-skeleton of the topological dual of the generalized barycentric subdivision). The ascending chain  $c_{\alpha_0} < \dots < c_{\alpha_i}$  corresponds to  $\sigma(\alpha_{i_0}, \dots, \alpha_{i_i})$ , and the cell-tuples in  $assoc(c_{\alpha_0}, \dots, c_{\alpha_i})$  correspond to the  $d$ -simplices in  $\overline{St} \sigma(\alpha_{i_0}, \dots, \alpha_{i_i})$ . Then the problem of showing that the subgraph of  $G_{(M,C)}$  induced by  $assoc(c_{\alpha_0}, \dots, c_{\alpha_i})$  is connected reduces to proving that there is a “path” of  $d$ -simplices in  $\overline{St} \sigma(\alpha_{i_0}, \dots, \alpha_{i_i})$  between any two  $d$ -simplices in  $\overline{St} \sigma(\alpha_{i_0}, \dots, \alpha_{i_i})$ .

It is useful to define  $switch$  on the  $d$ -simplices of  $K_{(M,C)}$ : given  $d$ -simplex  $\sigma = \sigma(\alpha_0, \dots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \dots, \alpha_d)$ ,  $0 \leq k \leq d$ , define  $switch_k(\sigma)$  to be the unique  $d$ -simplex  $\sigma(\alpha_0, \dots, \alpha_{k-1}, \alpha'_k, \alpha_{k+1}, \dots, \alpha_d)$  such that  $\alpha'_k \neq \alpha_k$ , i.e., if  $\sigma$  corresponds to  $t$ , then  $switch_k(\sigma)$  corresponds to  $switch_k(t)$ .

*Proof of Theorem 1.* To see that  $switch_{F^*}(t) \subseteq assoc(c_{\alpha_0}, \dots, c_{\alpha_i})$  is easy: If  $k \in \hat{I}$ , then  $switch_k$  does not change  $t_j$  for any  $j \in I$ . So if  $t' = switch_w(t)$ , for any  $w \in \hat{I}^*$ , then  $t'_j = t_j$ , for all  $j \in I$ , i.e.,  $t' \in assoc(t_{i_0}, \dots, t_{i_i})$ .

What must now be shown is that  $assoc(c_{\alpha_0}, \dots, c_{\alpha_i}) \subseteq switch_{F^*}(t)$ . First the following claim is proved:

If  $\tilde{\sigma}$  is a  $k$ -simplex in  $K_{(M,C)}$ ,  $0 \leq k \leq d$ , and if  $\sigma, \sigma'$  are  $d$ -simplices in  $\overline{St} \tilde{\sigma}$ , then there is a sequence  $\sigma = \sigma_0, \dots, \sigma_m = \sigma'$  of  $d$ -simplices of  $K_{(M,C)}$ , contained in  $\overline{St} \tilde{\sigma}$ , such that  $\sigma_j \cap \sigma_{j+1}$  is a  $(d - 1)$ -simplex for  $0 \leq j \leq m - 1$ .

To do so, use is made of Corollary 70.3 of [27]:

**Corollary 70.3.** *Let  $(X, A)$  be a compact triangulated relative homology  $d$ -manifold, with  $X - A$  connected. If  $\sigma$  and  $\sigma'$  are two  $d$ -simplices of  $X$  not in  $A$ , there is a sequence  $\sigma = \sigma_0, \sigma_1, \dots, \sigma_{m-1} = \sigma'$  of  $d$ -simplices not in  $A$ , such that  $\sigma_i \cap \sigma_{i+1}$  is a  $(d - 1)$ -simplex not in  $A$ , for each  $i$ .*

It will be shown that the hypotheses of this corollary are met if  $X = |K_{(M,C)}|$  and  $A = |K_{(M,C)} - St \tilde{\sigma}|$ . The claim then follows, since  $|St \tilde{\sigma}| = X - A$ , and the  $d$ -simplices in  $\overline{St} \tilde{\sigma}$  are exactly those which are not in  $K_{(M,C)} - St \tilde{\sigma}$ .

Here the pair notation  $(X, A)$  means that  $X$  and  $A$  are topological spaces such that  $A$  is a subspace of  $X$ . Since if  $X - A$  is a  $d$ -manifold, then  $(X, A)$  is a relative homology  $d$ -manifold (see comment on p. 374 of [27]), we only need to show that

$|St \tilde{\sigma}|$  is a  $d$ -manifold. This follows from the observation that any open subset  $Y$  of a  $d$ -manifold  $X$  is a  $d$ -manifold.

For a pair  $(X, A)$  to be compact means that  $X$  and  $A$  are compact. Since  $X = |K_{(M,C)}|$  and  $A = |K_{(M,C)} - St \tilde{\sigma}|$  are finite unions of  $d$ -simplices, they are compact. For a pair  $(X, A)$  to be triangulated means that  $X$  is triangulated by  $h: |K| \rightarrow X$  in such a way that  $A$  is triangulated by a subcomplex of  $K$  under  $h$ .  $K_{(M,C)}$  is a simplicial complex, hence  $X$  is trivially triangulated, and  $K_{(M,C)} - St \tilde{\sigma}$  is a subcomplex, so  $(X, A)$  is a triangulated pair. All that remains to be shown is that  $X - A = |St \tilde{\sigma}|$  is connected. Let  $x, y \in |St \tilde{\sigma}|$ , and let  $z \in \text{Int } \tilde{\sigma}$  (remember that  $z = \tilde{\sigma}$  if  $\dim(\tilde{\sigma}) = 0$ ). Let  $l_{xz}$  and  $l_{yz}$  be the line segments joining  $x$  and  $y$  with  $z$ , respectively. Both  $l_{xz}$  and  $l_{yz}$  are contained in  $|St \tilde{\sigma}|$ , so  $l_{xz} \cup l_{yz}$  is a path from  $x$  to  $y$  contained in  $St \tilde{\sigma}$ , so  $St \tilde{\sigma}$  is path connected, hence is connected. This finishes the proof that the hypotheses of Corollary 70.3 of [27] are met, and hence the claim is proven.

Apply the claim with  $\tilde{\sigma} = \sigma(\alpha_{i_0}, \dots, \alpha_{i_l})$ . Let  $\sigma_j$  be a simplex in the sequence  $\sigma_0, \dots, \sigma_m$ , where  $0 \leq j < m$ . By the definition of *switch* on the  $d$ -simplices of  $K_{(M,C)}$ ,  $\sigma_j \cap \sigma_{j+1}$  is a  $(d - 1)$ -simplex if and only if  $\sigma_{j+1} = \text{switch}_k(\sigma_j)$  for some  $0 \leq k \leq d$ . Since  $\sigma_j \subseteq \overline{St \tilde{\sigma}}$ , it must be that  $\tilde{\sigma} \subseteq \sigma_j$ , which gives that the vertices of  $\sigma_j$  labeled by  $i_0, \dots, i_l$  are also in  $\sigma_{j+1}$ . Therefore,  $\sigma_{j+1} = \text{switch}_k(\sigma_j)$  for some  $k \in \hat{I}$ .

Given  $t$  as in the statement of the lemma, and a  $t' \in \text{assoc}(t_{i_0}, \dots, t_{i_l})$ , let  $\sigma$  and  $\sigma'$  be the corresponding  $d$ -simplices in  $K_{(M,C)}$ , respectively. Then

$$\begin{aligned} t, t' \in \text{assoc}(t_{i_0}, \dots, t_{i_l}) &\Rightarrow \tilde{\sigma} \subseteq \sigma, \sigma' \\ &\Rightarrow \sigma, \sigma' \in \overline{St \tilde{\sigma}} \\ &\Rightarrow \exists \text{ a sequence } \sigma = \sigma_0, \dots, \sigma_m = \sigma' \text{ of} \\ &\quad d\text{-simplices in } \overline{St \tilde{\sigma}} \text{ such that } \sigma_j \cap \sigma_{j+1} \\ &\quad \text{is a } (d - 1)\text{-face for } j = 0, \dots, m - 1 \\ &\Rightarrow \exists w = w_0 \cdots w_{m-1} \in \hat{I}^* \text{ s.t. } \sigma' = \text{switch}_w(\sigma) \\ &\Rightarrow \exists w = w_0 \cdots w_{m-1} \in \hat{I}^* \text{ s.t. } t' = \text{switch}_w(t) \\ &\Rightarrow t' \in \text{switch}_{\hat{I}^*}(t). \end{aligned}$$

Thus  $\text{assoc}(t_{i_0}, \dots, t_{i_l}) \subseteq \text{switch}_{\hat{I}^*}(t)$ . □

It will be useful in the proof of Theorem 2 to think of *assoc* as a function taking ascending chains of cells to sets of cell-tuples. The next corollary shows that this function is injective.

**Corollary 2.** *If  $\alpha_{i_0} < \dots < \alpha_{i_m}$  and  $\alpha_{j_0} < \dots < \alpha_{j_n}$ , then*

$$\{\alpha_{i_0}, \dots, \alpha_{i_m}\} = \{\alpha_{j_0}, \dots, \alpha_{j_n}\} \Leftrightarrow \text{assoc}(c_{\alpha_{i_0}}, \dots, c_{\alpha_{i_m}}) = \text{assoc}(c_{\alpha_{j_0}}, \dots, c_{\alpha_{j_n}}).$$

*Proof.* ( $\Rightarrow$ ) From the definition of *assoc*.

( $\Leftarrow$ ) The proof is by contradiction. Assume without loss of generality that

$\dim(\alpha_{i_k}) = i_k$  and  $\dim(\alpha_{j_k}) = j_k$ . Suppose  $\{\alpha_{i_0}, \dots, \alpha_{i_m}\} \neq \{\alpha_{j_0}, \dots, \alpha_{j_n}\}$ . Without loss of generality, assume that there exists a  $k$  such that  $\alpha_{i_k} \notin \{\alpha_{j_0}, \dots, \alpha_{j_n}\}$ . By Lemma 3, there exists  $t \in \text{assoc}(c_{a_0}, \dots, c_{a_m})$ . If  $t \notin \text{assoc}(c_{a_0}, \dots, c_{a_n})$  we are done. If  $t \in \text{assoc}(c_{a_0}, \dots, c_{a_n})$ , let  $t' = \text{switch}_{i_k}(t)$ . Now,  $t' \in \text{assoc}(c_{a_0}, \dots, c_{a_n})$ , but  $t' \notin \text{assoc}(c_{a_0}, \dots, c_{a_m})$ .  $\square$

If  $0 \leq k \leq d$ , let  $\hat{k} = \{0, \dots, d\} - \{k\}$ . Thus

$$\hat{k}\text{-orbit}(t) = \text{switch}_{\{1, \dots, k-1, k+1, \dots, d\}}(t).$$

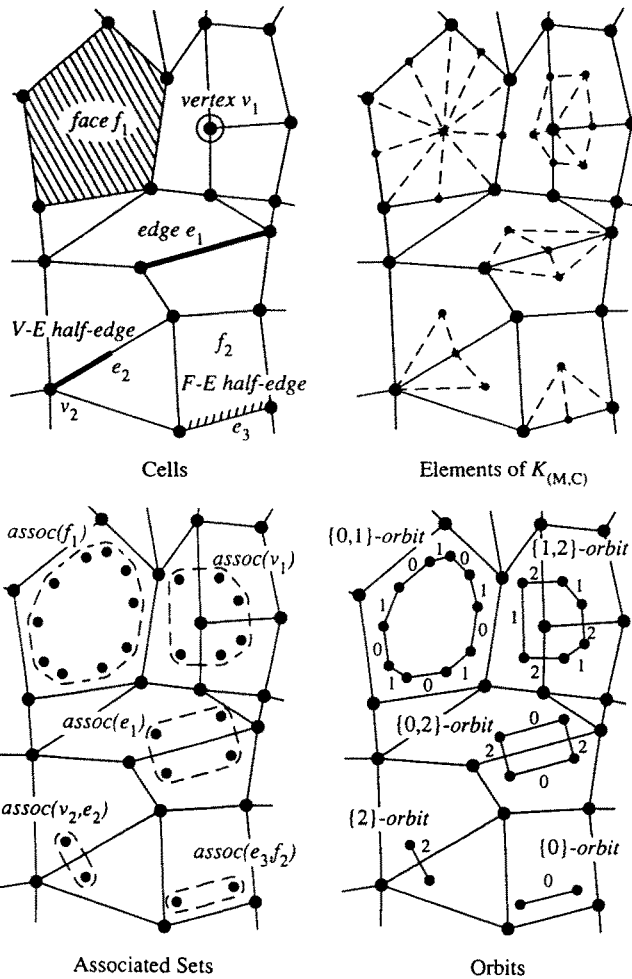
The next corollary shows how the cells of a subdivided manifold are represented in  $\mathcal{T}_{(M,C)}$ . It follows from Corollary 2 and Theorem 1.

**Corollary 3.** *There is a one-to-one correspondence between  $k$ -cells and  $\hat{k}$ -orbits.*

An informal explanation of the relationships given by Theorem 1 and Corollary 3 is now given, along with some pictorial examples which show the importance of this result. Recall the graph  $G_{(M,C)}$  for the cell-tuple structure given in Section 5: its node set is  $T_{(M,C)}$ , and two nodes are connected by an arc labeled  $k$  if the corresponding cell-tuples are related by  $\text{switch}_k$ , where  $0 \leq k \leq d$ . Now fix  $0 \leq k \leq d$ , and suppose all of the arcs labeled  $k$  are eliminated from  $G_{(M,C)}$ . There is a one-to-one correspondence between the  $k$ -cells of  $C$  and the connected components in this new graph. Furthermore, consider a sequence of increasing integers  $0 \leq k_1 < \dots < k_l \leq d$ . Suppose all of the arcs labeled with  $k_1, \dots, k_l$  are eliminated from  $G_{(M,C)}$ . There is a one-to-one correspondence between ascending chains of cells  $c_{a_{k_1}} < \dots < c_{a_{k_l}}$ , where the dimension of each  $c_{a_k}$  is  $k_j$ , and the connected components in this new graph.

The relationship between a subdivided manifold, its generalized barycentric subdivision, and the cell-tuple structure provides a general framework for many of the entities which have been devised in existing representations. The sets of  $d$ -simplices in the generalized barycentric subdivision which correspond to the cell-tuples in an orbit may be thought of as a “realization” of the entity, and these “realizations” will generally break up the object into something which is almost a partition (there is usually some measure-zero overlap between the entities). Figure 13 gives several examples in two dimensions, and Fig. 14 gives several examples in three dimensions (see Section 11 for the definitions of V-E and F-E half-edges).

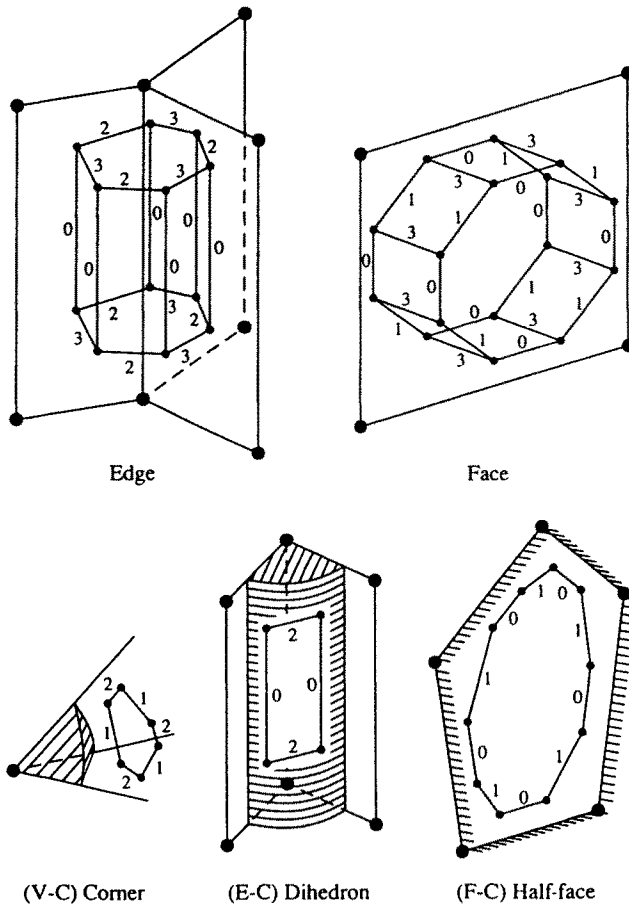
Theorem 1 gives a direct, efficient method of accessing all cell-tuples associated with a cell or chain of cells, and thus a straightforward method of accessing cells incident to a given cell or chain of cells. This is accomplished by doing a search within the graph  $G_{(M,C)}$ , using arcs in  $\hat{I}$ , where  $I$  is the set of dimensions of the specified cells. This search can be done in time linear in the number of cell-tuples in the associated set if a depth-first search is used (assuming that each  $\text{switch}$  query takes constant time). For instance, to find all cells in the boundary of a  $k$ -cell  $c_{a_k}$ , given a cell-tuple such that  $t_k = c_{a_k}$ , perform a search of  $\text{switch}_{\{0, \dots, k-1\}}(t)$ . Starting with an empty list, as each new cell-tuple  $t'$  is encountered, add to the current list



**Fig. 13.** Examples of Theorem 1 in two dimensions. The cells  $v_1$ ,  $e_1$ , and  $f_1$  and ascending chains  $v_2 < e_2$  and  $e_3 < f_2$  shown in the upper left have the associated sets shown in the lower left. At the lower right are shown the orbits specified by Theorem 1 for these cells and chains. Observe that in each case the associated set equals the orbit, as stated by the theorem. In the upper right are shown the 2-simplices in the generalized barycentric subdivision incident to each of  $v_1$ ,  $e_1$ , and  $f_1$ , and to both  $v_2$  and  $e_2$ , and to both  $e_3$  and  $f_2$ . Note the one-to-one correspondence between these 2-simplices and the cell-tuples in the lower illustrations.

of boundary cells the components  $t'_0$  through  $t'_{k-1}$  which have not yet been found (keeping a “found” bit for each cell makes this efficient).

The following lemma, which has an algebraic flavor, is of interest in and of itself, and is useful when defining constructors and in connecting the cell-tuple structure to others’ work. The properties, or rules, given in this lemma are analogous to the rules given for the quad-edge and facet-edge data structures. Rule (ct4) essentially shows the existence of circular orderings which is given in full detail by Theorem 3.



**Fig. 14.** Examples of Theorem 1 in three dimensions. The upper left illustration shows the  $\{0, 2, 3\}$ -orbit corresponding to the associated set of cell-tuples of a 1-cell; the upper right illustration shows the  $\{0, 1, 3\}$ -orbit corresponding to the associated set of cell-tuples of a 2-cell; the lower left illustration shows the  $\{1, 2\}$ -orbit corresponding to a “corner” defined by an ascending chain consisting of a 0-cell and a 3-cell; the lower middle illustration shows the  $\{0, 2\}$ -orbit corresponding to a “dihedron” defined by an ascending chain consisting of a 1-cell and a 3-cell; the lower right illustration shows the  $\{0, 1\}$ -orbit corresponding to a “half-face” defined by an ascending chain consisting of a 2-cell and a 3-cell.

**Corollary 4.** *If  $t \in T_{(M,C)}$  and  $i \neq j \in \{0, \dots, d\}$ , then:*

- (ct1)  $switch_i(t) \neq t$ ,
- (ct2)  $switch_{ij}(t) \neq t$ ,
- (ct3)  $switch_{i^2}(t) = t$ ,
- (ct4) *if  $j = i \pm 1$ ,  $\exists m \geq 2$  such that  $switch_{(ij)^m}(t) = t$ ,*
- (ct5) *if  $j \neq i \pm 1$ , then  $switch_{(ij)^2}(t) = t$ .*

*Proof.* Properties (ct1), (ct2), and (ct3) follow from the definition of *switch*.

Let  $t \in T_{(M,C)}$  and  $I = \{0, \dots, d\} - \{i, j\}$ . Let  $c_{\alpha_k} = t_{i_k}$  for  $i_k \in I$ , so that

$$\{c_{\alpha_0}, \dots, c_{\alpha_{d-2}}\} = \{t_k | k \in I\}.$$

The subgraph of  $G_{(M,C)}$  induced by  $\text{assoc}(c_{\alpha_0}, \dots, c_{\alpha_{d-2}})$  is connected (this follows from the proof of Theorem 1). Since every node is incident to exactly one arc labeled  $i$  and one labeled  $j$ , this subgraph is a cycle, with alternating arcs labeled  $i$  and  $j$ . If  $j = i \pm 1$ , this gives (ct4). If  $j \neq i \pm 1$ , then there are exactly two  $i$ -cells satisfying  $t_{i-1} < c_\alpha < t_{i+1}$ , namely  $c_\alpha = t_i$  and  $c_\alpha = [\text{switch}_i(t)]_i$ ; and there are exactly two  $j$ -cells satisfying  $t_{j-1} < c_{\alpha'} < t_{j+1}$ , namely  $c_{\alpha'} = t_j$  and  $c_{\alpha'} = [\text{switch}_j(t)]_j$ . Thus by the definition of associated sets,  $\text{assoc}(c_{\alpha_0}, \dots, c_{\alpha_{d-2}})$  has exactly four elements, formed by the four possible choices for  $c_\alpha$  and  $c_{\alpha'}$ . This gives (ct5). □

### 7. Representation of Incidence and Order

Two theorems are proved in this section. The first shows that the incidence graph and the cell-tuple structure are each powerful enough to represent subdivided manifolds up to equivalence. The second shows the existence of many orderings of cells within subdivided manifolds, and shows that these orderings are directly accessible using the cell-tuple structure.

Recall from Section 3 the definition of the incidence graph  $\mathcal{P}_{(M,C)}$  for a subdivided manifold  $(M, C)$ . Two incidence graphs are equivalent if there exists a bijective map between their nodes which preserves order ( $<$ ) and the dimension label ( $\text{dim}$ ).

**Theorem 2.** *If  $(M, C)$  and  $(N, D)$  are subdivided  $d$ -manifolds, then the following are equivalent:*

- (1)  $(M, C) \simeq (N, D)$ ,
- (2)  $\mathcal{P}_{(M,C)} \simeq \mathcal{P}_{(N,D)}$ ,
- (3)  $\mathcal{F}_{(M,C)} \simeq \mathcal{F}_{(N,D)}$ .

*Proof.* The proof is organized as follows:

$$\begin{array}{ccccc} (M, C) \simeq (N, D) & \Rightarrow & \mathcal{P}_{(M,C)} \simeq \mathcal{P}_{(N,D)} & \Leftrightarrow & \mathcal{F}_{(M,C)} \simeq \mathcal{F}_{(N,D)} \\ & & \uparrow & & \downarrow \\ & & K_{(M,C)} \simeq K_{(N,D)} & \Leftarrow & \mathcal{A}_{(M,C)} \simeq \mathcal{A}_{(N,D)} \end{array}$$

If  $I_C$  is an index set for a set of cells  $C$ , define the *index function*  $\text{index}: C \rightarrow I_C$  by  $\text{index}(c_\alpha) = \alpha$ , i.e.,  $\text{index}(c) = \alpha \Leftrightarrow c = c_\alpha$ . In this proof the index sets are  $C = \{c_\alpha\}_{\alpha \in I_C}$  and  $D = \{d_\beta\}_{\beta \in I_D}$ .

$(M, C) \simeq (N, D) \Rightarrow \mathcal{P}_{(M,C)} \simeq \mathcal{P}_{(N,D)}$ . Let  $h: M \rightarrow N$  give an equivalence between  $(M, C)$  and  $(N, D)$ , i.e.,  $h$  is a homeomorphism which takes  $k$ -cells onto  $k$ -cells. Define  $i: I_C \rightarrow I_D$  by  $i(\alpha) = \text{index}(h(c_\alpha))$ . The following shows that  $i: I_C \rightarrow I_D$  is an equivalence between  $\mathcal{P}_{(M,C)}$  and  $\mathcal{P}_{(N,D)}$ .  $i$  is well defined because  $h$  takes  $k$ -cells

onto  $k$ -cells.  $i$  is injective because

$$\alpha_1 \neq \alpha_2 \Rightarrow c_{\alpha_1} \neq c_{\alpha_2} \Rightarrow h(c_{\alpha_1}) \neq h(c_{\alpha_2}) \Rightarrow i(\alpha_1) \neq i(\alpha_2).$$

To see that  $i$  is surjective, let  $\alpha = \text{index}(h^{-1}(d_\beta))$  for any  $\beta \in I_D$ ; then  $i(\alpha) = \text{index}(h(h^{-1}(d_\beta))) = \beta$ .  $i$  preserves  $\dim$  because

$$\dim(\alpha) = \dim(c_\alpha) = \dim(h(c_\alpha)) = \dim(i(\alpha)).$$

That  $i$  preserves  $<$  can be seen as follows:

$$\alpha_1 < \alpha_2 \Leftrightarrow c_{\alpha_1} < c_{\alpha_2} \Leftrightarrow h(c_{\alpha_1}) < h(c_{\alpha_2}) \Leftrightarrow i(\alpha_1) < i(\alpha_2).$$

$\mathcal{P}_{(M,C)} \simeq \mathcal{P}_{(N,D)} \Rightarrow \mathcal{A}_{(M,C)} \simeq \mathcal{A}_{(N,D)} \Rightarrow K_{(M,C)} \simeq K_{(N,D)} \Rightarrow (M,C) \simeq (N,D)$ . Let  $i: I_C \rightarrow I_D$  give an equivalence between  $\mathcal{P}_{(M,C)}$  and  $\mathcal{P}_{(N,D)}$ , i.e.,  $i$  is bijective, and preserves  $<$  and  $\dim$ . It must be shown that  $i$  preserves simplices, hence gives an equivalence between  $\mathcal{A}_{(M,C)}$  and  $\mathcal{A}_{(N,D)}$ . If  $\{\alpha_0, \dots, \alpha_k\} \in \mathcal{A}_{(M,C)}$ , then there exists an ordering  $\alpha_{i_0}, \dots, \alpha_{i_k}$  of the elements of  $\{\alpha_0, \dots, \alpha_k\}$  so that  $\alpha_{i_0} < \dots < \alpha_{i_k}$ . Since  $i$  preserves  $<$ , this implies that  $i(\alpha_{i_0}) < \dots < i(\alpha_{i_k})$ . So  $\{i(\alpha_{i_0}), \dots, i(\alpha_{i_k})\} \in \mathcal{A}_{(N,D)}$ . The same argument applied to  $i^{-1}$  shows that  $\{\beta_0, \dots, \beta_k\} \in \mathcal{A}_{(N,D)}$  implies  $\{i^{-1}(\beta_0), \dots, i^{-1}(\beta_k)\} \in \mathcal{A}_{(M,C)}$ . Thus  $\{\alpha_0, \dots, \alpha_k\} \in \mathcal{A}_{(M,C)}$  if and only if  $\{i(\alpha_{i_0}), \dots, i(\alpha_{i_k})\} \in \mathcal{A}_{(N,D)}$ .

In fact,  $i: I_C \rightarrow I_D$  is also an equivalence between  $K_{(M,C)}$  and  $K_{(N,D)}$ . This follows from Theorem 3.1(b) of [27], when it is noted that the isomorphism produced there preserves labels: the isomorphism  $f: |K_{(M,C)}| \rightarrow |K_{(N,D)}|$  produced by the theorem is a bijective simplicial map which has the property that  $f(v_\alpha) = v_{i(\alpha)}$ , so  $\text{label}(v_\alpha) = \dim(\alpha) = \dim(i(\alpha)) = \text{label}(v_{i(\alpha)}) = \text{label}(f(v_\alpha))$ .

Let  $\psi_M: |K_{(M,C)}| \rightarrow M$  and  $\psi_N: |K_{(N,D)}| \rightarrow N$  be maps giving barycentric subdivisions of  $(M,C)$  and  $(N,D)$ , respectively, which exist by Lemma 1. Define  $h: M \rightarrow N$  by  $h = \psi_N \circ f \circ \psi_M^{-1}$ . This is a composition of homeomorphisms, so is a homeomorphism. It will be shown that  $h$  takes  $k$ -cells onto  $k$ -cells. Define  $j: I_C \rightarrow I_D$  by  $j(\alpha) = \beta \Leftrightarrow f(v_\alpha) = v_\beta$ . This is well defined since  $f$  is an equivalence. Let  $c_\alpha$  be a  $k$ -cell in  $C$ . Then

$$\begin{aligned} h(c_\alpha) &= h\left(\bigcup_{\alpha_0 < \dots < \alpha_1 < \alpha} c_{sd}(\alpha_0, \dots, \alpha_1, \alpha)\right) \\ &= \bigcup_{\alpha_0 < \dots < \alpha_1 < \alpha} \psi_N \circ f \circ \psi_M^{-1}(c_{sd}(\alpha_0, \dots, \alpha_1, \alpha)) \\ &= \bigcup_{\alpha_0 < \dots < \alpha_1 < \alpha} \psi_N \circ f(\text{Int } \sigma(\alpha_0, \dots, \alpha_1, \alpha)) \\ &= \bigcup_{j(\alpha_0) < \dots < j(\alpha_1) < j(\alpha)} \psi_N(\text{Int } \sigma(j(\alpha_0), \dots, j(\alpha_1), j(\alpha))) \\ &= \bigcup_{j(\alpha_0) < \dots < j(\alpha_1) < j(\alpha)} c_{sd}(j(\alpha_0), \dots, j(\alpha_1), j(\alpha)) \\ &= d_{j(\alpha)}, \end{aligned}$$

which is a  $k$ -cell in  $D$ .



The first and last equalities use Lemma 1. The rest follow from the definition of  $h$ , and the fact that  $f$  is an equivalence.

$\mathcal{P}_{(M,C)} \simeq \mathcal{P}_{(N,D)} \Rightarrow \mathcal{T}_{(M,C)} \simeq \mathcal{T}_{(N,D)}$ . Let  $i: I_C \rightarrow I_D$  be an equivalence between  $\mathcal{P}_{(M,C)}$  and  $\mathcal{P}_{(N,D)}$ . Define  $j: T_{(M,C)} \rightarrow T_{(N,D)}$  by

$$j((c_{\alpha_0}, \dots, c_{\alpha_d})) = (d_{i(\alpha_0)}, \dots, d_{i(\alpha_d)}).$$

( $d_{i(\alpha_i)}$  may be thought of as shorthand for  $d_{i(\text{index}(c_{\alpha_i}))}$ .) It must be shown that  $j$  is well defined, bijective, and preserves *switch*.

To see that  $j$  is well defined:

$$\begin{aligned} (c_{\alpha_0}, \dots, c_{\alpha_d}) \in T_{(M,C)} &\Leftrightarrow \alpha_0 < \dots < \alpha_d \\ &\Leftrightarrow i(\alpha_0) < \dots < i(\alpha_d) \\ &\Leftrightarrow (d_{i(\alpha_0)}, \dots, d_{i(\alpha_d)}) \in T_{(N,D)}. \end{aligned}$$

To see that  $j$  is injective: if  $t \neq t' \in T_{(M,C)}$ , then there is a  $k$  such that  $t_k \neq t'_k$ , and  $i(\text{index}(t_k)) \neq i(\text{index}(t'_k)) \Rightarrow [j(t)]_k \neq [j(t')]_k \Rightarrow j(t) \neq j(t')$ . To see that  $j$  is surjective: if  $(d_{\beta_0}, \dots, d_{\beta_d}) \in T_{(N,D)}$ , then

$$\beta_0 < \dots < \beta_d \Rightarrow i^{-1}(\beta_0) < \dots < i^{-1}(\beta_d) \Rightarrow (c_{i^{-1}(\beta_0)}, \dots, c_{i^{-1}(\beta_d)}) \in T_{(M,C)},$$

and  $j((c_{i^{-1}(\beta_0)}, \dots, c_{i^{-1}(\beta_d)})) = (d_{\beta_0}, \dots, d_{\beta_d})$ . To see that  $j$  preserves *switch*: let  $0 \leq k \leq d$  and  $t \in T_{(M,C)}$ . Then

$$\begin{aligned} t' = \text{switch}_k(t) &\Leftrightarrow (t_i = t'_i \Leftrightarrow i \neq k) \\ &\Leftrightarrow (\text{index}(t_i) \neq \text{index}(t'_i) \Leftrightarrow i \neq k) \\ &\Leftrightarrow (i(\text{index}(t_i)) \neq i(\text{index}(t'_i)) \Leftrightarrow i \neq k) \\ &\Leftrightarrow ([j(t)]_i = [j(t')]_i \Leftrightarrow i \neq k) \\ &\Leftrightarrow j(t') = \text{switch}_k(j(t)). \end{aligned}$$

$\mathcal{T}_{(M,C)} \simeq \mathcal{T}_{(N,D)} \Rightarrow \mathcal{P}_{(M,C)} \simeq \mathcal{P}_{(N,D)}$ . Let  $j: T_{(M,C)} \rightarrow T_{(N,D)}$  give an equivalence between  $\mathcal{T}_{(M,C)}$  and  $\mathcal{T}_{(N,D)}$ , i.e.,  $j$  is bijective, and  $\text{switch}_k(j(t)) = j(\text{switch}_k(t))$ ,  $\forall t \in T_{(M,C)}$ ,  $0 \leq k \leq d$ . An equivalence  $i: I_C \rightarrow I_D$  will be defined.

The following fact will be needed: if  $J \subseteq \{0, \dots, d\}^*$ , then

$$\text{switch}_j(j(t)) = j(\text{switch}_j(t))$$

for all  $t \in T_{(M,C)}$ . This can be seen by the following argument. If  $w \in J$ , then  $\text{switch}_w(j(t)) = j(\text{switch}_w(t))$  by repeated applications of the definition of equivalence. Thus

$$\begin{aligned} t' \in \text{switch}_j(j(t)) &\Leftrightarrow t' = \text{switch}_w(j(t)) && \text{for some } w \in J \\ &\Leftrightarrow t' = j(\text{switch}_w(t)) && \text{for some } w \in J \\ &\Leftrightarrow t' \in j(\text{switch}_j(t)). \end{aligned}$$

Let  $c_\alpha$  be a  $k$ -cell, and let  $t \in T_{(M,C)}$  be a cell-tuple such that  $t_k = c_\alpha$  (which exists by Lemma 3). Now,  $assoc(c_\alpha) = switch_k(t)$ , by Theorem 1. Putting this together with the fact given in the previous paragraph and Corollary 2 gives

$$j(assoc(c_\alpha)) = j(switch_k(t)) = switch_k(j(t)) = assoc(d_\beta)$$

for a unique  $k$ -cell  $d_\beta \in D$ .

By this argument, it is possible to define  $i: I_C \rightarrow I_D$  by

$$i(\alpha) = index(assoc^{-1}(j(assoc(c_\alpha)))).$$

This also gives that  $\dim$  is preserved ( $\dim(\alpha) = k = \dim(\beta)$ ). It must be shown that  $i$  is bijective and preserves  $<$ . To see that  $i$  is bijective: recall from Corollary 2 that  $assoc$  (as a function from  $I_C$  to  $2^{T(M,C)}$ ) is bijective with its range, by definition  $index$  is bijective, and it is given that  $j$  is bijective. To see that  $i$  preserves  $<$ : first note that if  $\beta = i(\alpha)$ , then  $\beta = index(assoc^{-1}(j(assoc(c_\alpha))))$  implies  $assoc(d_\beta) = j(assoc(c_\alpha))$ . Now suppose  $\alpha_1, \alpha_2 \in I_C$ ,  $k_1 = \dim(\alpha_1)$ , and  $k_2 = \dim(\alpha_2)$ . Let  $\beta_1 = i(\alpha_1)$  and  $\beta_2 = i(\alpha_2)$ . Then

$$\begin{aligned} \alpha_1 < \alpha_2 &\Leftrightarrow k_1 < k_2 \quad \text{and} \quad \exists t \in T_{(M,C)} \text{ s.t. } t_{k_1} = c_{\alpha_1}, t_{k_2} = c_{\alpha_2} \\ &\Leftrightarrow k_1 < k_2 \quad \text{and} \quad assoc(c_{\alpha_1}) \cap assoc(c_{\alpha_2}) \neq \emptyset \\ &\Leftrightarrow k_1 < k_2 \quad \text{and} \quad j(assoc(c_{\alpha_1})) \cap j(assoc(c_{\alpha_2})) \neq \emptyset \\ &\Leftrightarrow k_1 < k_2 \quad \text{and} \quad assoc(d_{\beta_1}) \cap assoc(d_{\beta_2}) \neq \emptyset \\ &\Leftrightarrow k_1 < k_2 \quad \text{and} \quad \exists t \in T_{(N,D)} \text{ s.t. } t_{k_1} = d_{\beta_1}, t_{k_2} = d_{\beta_2} \\ &\Leftrightarrow \beta_1 < \beta_2. \end{aligned} \quad \square$$

The next theorem shows the existence of many orderings of cells within subdivided manifolds, and shows that these orderings are directly accessible using the cell-tuple structure. Recall the examples of ordering given in Section 1. Figure 2 gave examples in two dimensions: Fig. 2(a) showed an ordering of the 1-cells and 2-cells incident to a 0-cell, and Fig. 2(b) showed an ordering of the 0-cells and 1-cells incident to a 2-cell. Figure 3 gave examples in three dimensions: Fig. 3(a) showed an ordering of the 2- and 3-cells incident to a 1-cell, Fig. 3(b) showed an ordering of the 1- and 2-cells incident to both a 0-cell and a 3-cell, and Fig. 3(c) showed an ordering of the 0- and 1-cells incident to a 2-cell. Observe that in each of these five cases, the ordering is of the set of all  $(k - 1)$ - and  $k$ -cells containing a  $(k - 2)$ -cell, or within a  $(k + 1)$ -cell, or both.

All of these cases can be put into one framework, by thinking of fixing both a  $(k - 2)$ -cell and a  $(k + 1)$ -cell containing it. This works for  $1 \leq k \leq d$ . In the case of  $k = 1$  the  $(k - 2)$ -cell is not needed, and in the case of  $k = d$  the  $(k + 1)$ -cell is not needed. Again the convention of including extra cells  $c_{\alpha_{-1}}$  and  $c_{\alpha_{d+1}}$  of dimension  $-1$  and  $d + 1$ , respectively, is used for notational convenience.

If  $c_{\alpha_{k-2}} < c_{\alpha_{k+1}}$ , where  $1 \leq k \leq d$ ,  $c_{\alpha_{k-2}}$  is a  $(k - 2)$ -cell and  $c_{\alpha_{k+1}}$  is a  $(k + 1)$ -cell, let

$$S(c_{\alpha_{k-2}}, c_{\alpha_{k+1}}) = \{c_\alpha \mid c_{\alpha_{k-2}} < c_\alpha < c_{\alpha_{k+1}}\}.$$

When it is clear from the context,  $S$  is written for  $S(c_{\alpha_{k-2}}, c_{\alpha_{k+1}})$ . It is this set  $S$  which can be ordered. The idea is that given a  $(k - 2)$ -cell contained in the boundary of a  $(k + 1)$ -cell, all of the cells “between” them may be put into a circular order “around” the  $(k - 2)$ -cell, such that this order alternates between  $(k - 1)$ -cells and  $k$ -cells, consecutive pairs of  $k$ -cells share the  $(k - 1)$ -cell between them, and consecutive pairs of  $(k - 1)$ -cells share the  $k$ -cell between them. The reason this idea is not more obvious when trying to generalize from two and three dimensions is that of the five cases  $d = 2, k = 1$  (as in Fig. 2(b)),  $d = 2, k = 2$  (as in Fig. 2(a)),  $d = 3, k = 1$  (as in Fig. 3(c)),  $d = 3, k = 2$  (as in Fig. 3(b)), and  $d = 3, k = 3$  (as in Fig. 3(a)), only one involves fixing two cells, namely  $d = 3, k = 2$ .

Let  $m = \#(S)$ . A **circular ordering** of  $S$  is an ordering  $c_{\eta_0}, \dots, c_{\eta_{m-1}}$  of the elements of  $S$  such that:

- (1)  $c_{\eta_i}$  is a  $(k - 1)$ -cell if  $i$  is even, and is a  $k$ -cell if  $i$  is odd,
- (2)  $c_{\eta_{i-1 \bmod m}}$  and  $c_{\eta_{i+1 \bmod m}}$  share  $c_{\eta_i}$  for  $0 \leq i \leq m - 1$

(by *share* is meant “are each incident to”).

Theorem 3 shows the existence of ordering information for all dimensions from 0 to  $d$ , and shows that the cell-tuple structure gives immediate access to it via the *switch* operators (the incidence graph does not offer such direct access).

**Theorem 3.** *There exists a circular ordering of  $S(c_{\alpha_{k-2}}, c_{\alpha_{k+1}})$ . Furthermore, the switch structure directly represents this ordering information:*

- (A) *There exists a  $t^0 \in T_{(M,C)}$  such that  $t_{k-2}^0 = c_{\alpha_{k-2}}$  and  $t_{k+1}^0 = c_{\alpha_{k+1}}$ . For any such  $t^0$ , if the sequence of cell-tuples  $t^0, \dots, t^{m-1}$  is defined by*

$$t^i = \begin{cases} \text{switch}_k(t^{i-1}), & i \text{ even,} \\ \text{switch}_{k-1}(t^{i-1}), & i \text{ odd,} \end{cases}$$

*then the sequence  $c_{\eta_0}, \dots, c_{\eta_{m-1}}$  defined by*

$$c_{\eta_i} = \begin{cases} t_{k-1}^i, & i \text{ even, } 0 \leq i \leq m - 2, \\ t_k^i, & i \text{ odd, } 1 \leq i \leq m - 1, \end{cases}$$

*gives a circular ordering of  $S(c_{\alpha_{k-2}}, c_{\alpha_{k+1}})$ .*

- (B) *There exist  $c_{\eta_0}, c_{\eta_1} \in C$  such that  $c_{\alpha_{k-2}} < c_{\eta_0} < c_{\eta_1} < c_{\alpha_{k+1}}$ . For any such  $c_{\eta_0}$  and  $c_{\eta_1}$ , if the sequence  $c_{\eta_0}, \dots, c_{\eta_{m-1}}$  is defined by*

$$c_{\eta_i} = \begin{cases} \text{switch}_{k-1}(c_{\alpha_{k-2}}, c_{\eta_{i-2}}, c_{\eta_{i-1}}), & i \text{ even, } 0 \leq i \leq m - 2, \\ \text{switch}_k(c_{\eta_{i-1}}, c_{\eta_{i-2}}, c_{\alpha_{k+1}}), & i \text{ odd, } 1 \leq i \leq m - 1, \end{cases}$$

*gives a circular ordering of  $S(c_{\alpha_{k-2}}, c_{\alpha_{k+1}})$ .*

*Proof.* Let  $t^0 = (c_{\alpha_0}, \dots, c_{\alpha_d}) \in T_{(M,C)}$  such that  $t_{k-2}^0 = c_{\alpha_{k-2}}$  and  $t_{k+1}^0 = c_{\alpha_{k+1}}$ , which is possible by Lemma 3. Let  $T' = \text{assoc}(c_{\alpha_0}, \dots, c_{\alpha_{k-2}}, c_{\alpha_{k+1}}, \dots, c_{\alpha_d})$ . Then  $T' = \text{switch}_{(k-1, k)}(t^0)$  by Theorem 1.

Recall from Section 5 the definition of  $G_{(M,C)}$ , the graph whose nodes are cell-tuples and whose labeled arcs represent the  $switch_k$  operators. Let  $G$  be the subgraph of  $G_{(M,C)}$  induced by  $T'$ , i.e.,  $G = (V, E)$ , where  $V = T'$  and  $E = E_{k-1} \cup E_k$ :

$$E = \{(t^1, t^2) | t^1, t^2 \in T' \text{ and } e \text{ is labeled with } k-1 \text{ or } k\}.$$

Every  $t \in V$  is incident to exactly one arc labeled  $k-1$  and one labeled  $k$ , and  $G$  is connected, because  $T' = switch_{(k-1,k)}(t^0)$ . So  $G$  is a simple circuit, whose arcs are labeled alternately by  $k-1$  and  $k$ . Let

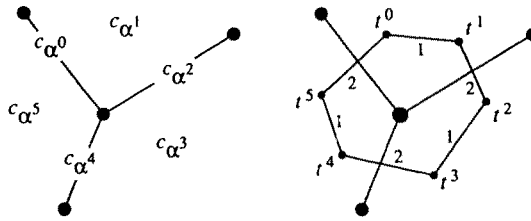
$$\begin{aligned} t^1 &= switch_{k-1}(t^0), \\ t^2 &= switch_k(t^1), \\ t^3 &= switch_{k-1}(t^2), \\ t^4 &= switch_k(t^3), \\ &\vdots \\ t^{m-2} &= switch_k(t^{m-3}), \\ t^{m-1} &= switch_{k-1}(t^{m-2}). \end{aligned}$$

Then  $T' = \{t^0, \dots, t^{m-1}\}$ . Let

$$\begin{aligned} c_{\eta_0} &= t_{k-1}^0, \\ c_{\eta_1} &= t_k^1, \\ c_{\eta_2} &= t_{k-1}^2, \\ c_{\eta_3} &= t_k^3, \\ &\vdots \\ c_{\eta_{m-2}} &= t_{k-1}^{m-2}, \\ c_{\eta_{m-1}} &= t_k^{m-1}. \end{aligned}$$

Let  $S = S(c_{\alpha_{k-2}}, c_{\alpha_{k+1}})$ . It must be shown that  $S = \{c_{\eta_0}, \dots, c_{\eta_{m-1}}\}$ . If  $c_\alpha \in S$ , then  $c_{\alpha_{k-2}} < c_\alpha < c_{\alpha_{k+1}}$  implies that there exists a  $t \in T'$  such that either  $t_{k-1} = c_\alpha$  or  $t_k = c_\alpha$ , so  $c_\alpha \in \{c_{\eta_0}, \dots, c_{\eta_{m-1}}\}$ . This shows that  $S \subset \{c_{\eta_0}, \dots, c_{\eta_{m-1}}\}$ . If  $0 \leq i \leq m-1$ , then  $c_{\alpha_{k-2}} < c_{\eta_i} < c_{\alpha_{k+1}}$  implies  $c_{\eta_i} \in S$ , i.e.,  $\{c_{\eta_0}, \dots, c_{\eta_{m-1}}\} \subset S$ . Property (1) of a circular ordering follows from  $c_{\eta_i} = t_{k-1}^i$  if  $i$  is even,  $c_{\eta_i} = t_k^i$  if  $i$  is odd. Property (2) follows from the definition of  $switch$ . Parts (A) and (B) of the theorem follow.  $\square$

To access the cells in a circular ordering starting at a given cell-tuple, apply  $switch_{k-1,k}$  repeatedly, taking the  $(k-1)$ - and/or  $k$ -components of the resulting cell-tuples to get the  $(k-1)$ - and/or  $k$ -cells in the ordering. Referring to Fig. 2, in



**Fig. 15.** Example of Theorem 3. On the left, the sequence  $c_{\alpha^0}, \dots, c_{\alpha^5}$  gives a circular ordering of the set of 1-cells and 2-cells incident to a 0-cell. On the right, the sequence of cell-tuples  $t^0, \dots, t^5$  may be obtained by starting with  $t^0$ , and alternately applying  $switch_1$  and  $switch_2$ . This “ring” of cell-tuples may be used to obtain the sequence of cells  $c_{\alpha^0}, \dots, c_{\alpha^5}$ .

(a) the 1-cells and 2-cells incident to a 0-cell are accessed by applying  $switch_{12}$ , and in (b) the 0- and 1-cells incident to a 2-cell are accessed by applying  $switch_{01}$ . Referring to Fig. 3, in (a) the 2- and 3-cells incident to a 1-cell are accessed by applying  $switch_{23}$ , in (b) the 1- and 2-cells incident to both a 0-cell and a 3-cell are accessed by applying  $switch_{12}$ , and in (c) the 0- and 1-cells incident to a 2-cell are accessed by applying  $switch_{01}$ . An example in two dimensions for the set of 1-cells and 2-cells incident to a 0-cell is given in Fig. 15.

### 8. The Dual

An example of the topological dual of a subdivided 2-manifold was given in Fig. 4. Essentially, the dual of a subdivided  $d$ -dimensional object is a subdivision of the same object produced by replacing every  $k$ -dimensional cell by a  $(d - k)$ -dimensional cell, while maintaining the corresponding incidence relations. Suppose  $(M, C)$  is a subdivided  $d$ -manifold. Recall from Lemma 1:

$$c_{\alpha} = \bigcup_{\alpha_0 < \dots < \alpha_l < \alpha} c_{sd}(\alpha_0, \dots, \alpha_l, \alpha).$$

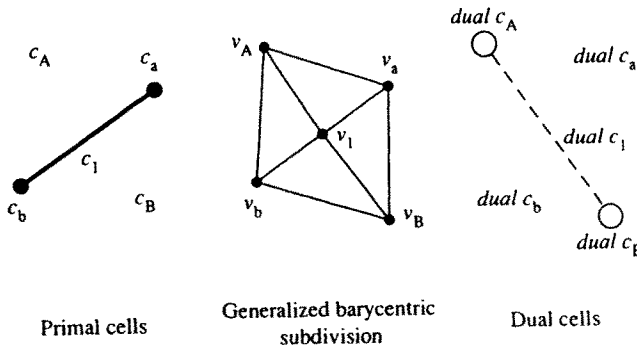
If  $c_{\alpha} \in C$ , then the dual cell for  $c_{\alpha}$  is

$$dual\ c_{\alpha} = \bigcup_{\alpha < \alpha_0 < \dots < \alpha_l} c_{sd}(\alpha, \alpha_0, \dots, \alpha_l).$$

Examples of dual cells are shown in Fig. 16. The **dual complex** of  $(M, C)$  is  $(M, C^{dual})$ , where  $C^{dual} = \{dual\ c_{\alpha}\}_{\alpha \in I_C}$ .

Since the characteristic functions for  $C$  are not unique, the cells in  $C_{sd}$  are not unique. Hence the dual is not unique, but it is unique up to equivalence.

From the definition it follows that there is a one-to-one correspondence between the original cells and the dual cells, and that incidence and the orderings are preserved. The incidence graph for  $(M, C^{dual})$  may be obtained from the incidence graph for  $(M, C)$  by “turning it upside-down,” and  $(M, C^{dual})$  is represented implicitly by the incidence graph for  $(M, C)$ .



**Fig. 16.** Example of dual cells.  $c_a, c_b$  are 0-cells,  $c_1$  is a 1-cell, and  $c_A, c_B$  are 2-cells. The middle illustration shows the simplicial cells in the generalized barycentric subdivision associated with  $c_1$ , e.g. the vertex  $v_1$  is the 0-cell  $c_{sd}(1)$ , and the open triangle defined by  $v_a, v_1$ , and  $v_b$  is the 2-cell  $c_{sd}(a, 1, B)$ . Thus  $c_1 = \{c_{sd}(1)\} \cup \{c_{sd}(a, 1)\} \cup \{c_{sd}(b, 1)\}$  and  $dual\ c_1 = \{c_{sd}(1)\} \cup \{c_{sd}(1, A)\} \cup \{c_{sd}(1, B)\}$ .

There is a one-to-one correspondence between  $T_{(M,C)}$  and  $T_{(M,C^{dual})}$ ; define  $switch_R: T_{(M,C)} \rightarrow T_{(M,C^{dual})}$  by  $switch_R((c_{\alpha_0}, \dots, c_{\alpha_d})) = (dual\ c_{\alpha_d}, \dots, dual\ c_{\alpha_0})$ . Note that  $switch_k$  in the dual corresponds to  $switch_{d-k}$  in the original subdivision, i.e.,  $switch_{RkR} = switch_{d-k}$ . Thus the cell-tuple structure for  $(M, C)$  represents the dual implicitly.

If the dual is to be represented explicitly, consider the set of tuples

$$T_{(M,C)} \cup T_{(M,C^{dual})},$$

acted on by  $switch_R$  and  $switch_k, 0 \leq k \leq d$ . Then the extension of Corollary 4 obtained by adding the following rule holds:

$$(ct6) \quad switch_{RiR}(t) = switch_{d-i}(t).$$

**9. Extension to Manifolds-with-Boundary**

Most of the results given so far have applied only to subdivided manifolds (without boundary). In generalizing the cell-tuple structure to the with-boundary case, the problem is that  $switch_d(t)$  is not defined if  $t_{d-1} \in \partial M$ .

Two approaches are considered for the extension of the earlier results to the with-boundary case. The first is to set  $switch$  to some special value in the undefined cases, indicating that an attempt was made to “pierce the boundary.” Then the definition of circular ordering may be modified so that when  $k = d$  and  $c_{\alpha_{k-2}} \in \partial M$ , the ordering is a simple path rather than a simple cycle. The theorems may be proved as before.

The second approach may be described intuitively as imagining that the space “outside” of the  $d$ -manifold-with-boundary  $(M, C)$  is simply another  $d$ -cell (though in general, the complement of a  $d$ -manifold-with-boundary embedded in  $\mathbb{R}^n$  is not a  $d$ -cell). An abstract object  $c_{\alpha_x}$  is added to  $C$ , such that, for every  $c_\alpha$  contained

in  $\partial M$ ,  $c_\alpha < c_{\alpha_x}$ . Now *switch* may be defined as before on the set  $C \cup \{c_{\alpha_x}\}$ , and all of the results for subdivided manifolds can be proved for subdivided manifolds-with-boundary (see [4] for details). In addition, this makes it possible to access the boundary as a subdivided  $(d - 1)$ -manifold immediately; the cell-tuple structure associated with  $c_{\alpha_x}$ , inherited from the cell-tuple structure for  $(M, C)$ , is exactly the cell-tuple structure for  $\partial M$ .

## 10. Constructors for $d$ -Dimensional Objects

So far subdivided manifolds have been treated as static, existing objects. A useful system must provide operators for creating, modifying, combining, separating, and destroying them. Such operators are termed constructors. One possible set of general constructors for subdivided manifolds, with and without boundary, which does not depend on a particular representation, is proposed in [4]. These operate on objects of various dimensions, allow changing of dimension, and are sufficient to construct any subdivided manifold. A very brief description is given here.

The basic creative constructor, *make\_vertex*, simply produces a vertex. Its inverse, *kill\_vertex*, removes an isolated vertex.

The constructor *lift* takes a subdivided  $(k - 1)$ -manifold homeomorphic to  $S^{k-1}$ , and “fills in the interior” of  $C$  with a  $k$ -cell  $c_{\alpha_k}$ . The constructor *unlift* is the inverse of *lift*.

The constructor *join* identifies a specified pair of equivalent cells, and all of their corresponding subcells. In general, the desired correspondence between the subcells in the boundaries of the two cells must be specified. One method is to specify for each cell a chain of boundary cells of decreasing dimension, i.e., for  $c_{\alpha_k}$ , give a chain  $c_{\alpha_k} > \dots > c_{\alpha_0}$ , and for  $c_{\alpha'_k}$ , give a chain  $c_{\alpha'_k} > \dots > c_{\alpha'_0}$ . Identifying the corresponding cells in these chains completely specifies how all the subcells of  $c_{\alpha_k}$  and  $c_{\alpha'_k}$  will be matched. This lends itself naturally to a cell-tuple implementation. The constructor *unjoin* is the inverse of *join*.

The constructor *split* is a generalization of the Euler operators which split edges and faces. Given a set of the subcells on the boundary of a  $k$ -cell  $c_{\alpha_k}$ , such that if taken as a subdivided  $(k - 2)$ -manifold their union is homeomorphic to  $S^{k-2}$ , *split* performs a *lift* on the subset, so as to split  $c_{\alpha_k}$  into two new  $k$ -cells. The constructor *unsplit* is the inverse of *split*.

It is not hard to show that any subdivided  $d$ -manifold can be built from this set of constructors, using a recursive construction. Build every  $d$ -cell by first building its boundary and then performing a lift; if  $d = 1$  building the boundary consists of two applications of *make\_vertex*. For every pair of  $d$ -cells which need to be adjacent, perform a *join* on the common pair of  $(d - 1)$ -cells in their boundaries.

These constructors may be implemented in any representation which allows access to incidence structure and ordering information. Implementations based on the incidence graph and the cell-tuple structure are discussed in [4]. To implement the proposed constructors using the cell-tuple structure, certain pairs of cell-tuples are “attached to” or “detached from” each other. The key to attaching or

detaching the proper cell-tuples is to traverse the edge-labeled *switch* graphs in the proper order, i.e., by exploring the corresponding orbits or matching cells in a prescribed lock-step order, and modifying the appropriate *switch* arcs to maintain the rules listed in Corollary 4.

The above discussion of constructors considers only topological structure. Maintaining the connection between topology and geometry, allowing access and easy updates of topology and geometry simultaneously, while also allowing changes to be made to either independently, is one of the most challenging problems for an implementation.

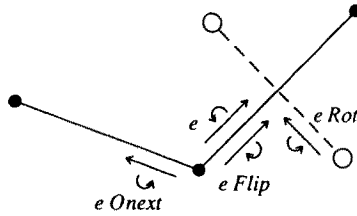
## 11. Relation to Other Implicit-Cell Representations

The relationship of the cell-tuple structure to half-edges, the quad-edge data structure, the facet-edge data structure,  $n$ - $G$ -maps, and chamber systems is examined in this section. In particular, it is discussed how the basic elements and the query operators correspond.

In the two-dimensional case, Baumgart [2] described the “winged-edge data structure.” The basic idea is to have two basic objects (winged-edges) associated with each edge, which point to each other and to adjacent winged-edges. The idea of “splitting” an edge in half has given rise to two different entities which have both been called “half-edges.” If one imagines splitting the edge lengthwise, a half-edge corresponds to an incident face-edge pair. Mäntylä [23] devotes a chapter to such half-edges. If one imagines splitting the edge by a “midpoint,” a half-edge corresponds to an incident vertex-edge pair. The term is used in this context in [21]. Weiler [38] distinguishes between the two possibilities for a half-edge by using the terms vertex-edge (V-E) structure and face-edge (F-E) structure (examples of V-E and F-E half-edges were included in Fig. 13.) Attaching pointers to half-edges allows direct access to ordering information. It follows from Theorem 1 that a V-E half-edge corresponds to a pair of cell-tuples related by  $switch_2$  (i.e., each vertex-edge pair corresponds to a 2-orbit), and an F-E half-edge corresponds to a pair of cell-tuples related by  $switch_0$  (i.e., each face-edge pair corresponds to a 0-orbit). Observe that the generalized barycentric subdivision gives a geometric realization to these half-edges, as indicated in Fig. 13.

Guibas and Stolfi [16] introduced the “edge algebra” for representing subdivisions of 2-manifolds (without boundaries) in which the boundaries of faces are allowed to self-intersect. For every edge in the original and dual subdivisions there are two possible directions of traversal and two ways of giving a local orientation to the surface. Taking all possible combinations of direction and orientation give four “directed, oriented edges” for each edge. These are the basic elements of the edge algebra. There are three operators, *Rot*, *Onext*, and *Flip*, which may be applied to directed, oriented edges. *Flip* reverses orientation, *Rot* essentially “rotates about the midpoint” (into the dual), and *Onext* rotates about the base (in the direction indicated by the orientation). These are illustrated in Fig. 17. It is proved that these operators obey a set of ten rules, and a one-to-one correspondence is established between their class of subdivisions and between the set of all “abstract



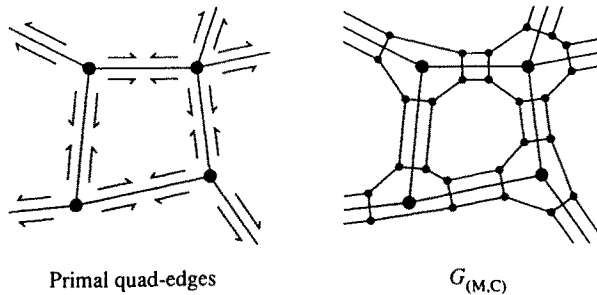


**Fig. 17.** Examples of *Rot*, *Onext*, and *Flip*. The straight arrows indicate direction along the edges, and the curved arrows indicate orientation, as the direction of rotation.

edge algebras” (systems of entities and operators obeying the ten rules). Ordering around vertices is available by applying a sequence of *Onext* operators, and ordering around the boundary of a face by a sequence of *Rot*<sup>2</sup>*Flip* operators.

There is a one-to-one correspondence between primal directed, oriented edges in the edge algebra and the cell-tuples in the cell-tuple structure. Given a tuple, the corresponding directed, oriented edge is given by:  $t_1$  tells which edge is involved,  $t_0$  gives the direction (as the origin of travel), and  $t_2$  gives the orientation (as the direction of rotation, i.e., which face *Onext* rotates through). Conversely, given a directed, oriented edge, the edge specifies a 1-cell, the direction gives a 0-cell (the endpoint moved away from), and the orientation gives a 2-cell (the face encountered during a positive rotation). In Fig. 18 this correspondence is indicated by location.

There is a similar correspondence between dual directed, oriented edges and the dual cell-tuples in  $T_{(M,C^{dual})}$  (described in Section 8). The set of cell-tuples  $T_{(M,C)} \cup T_{(M,C^{dual})}$  corresponds to the set of all directed, oriented edges in the edge algebra. Let  $\wp$  be a map from  $T_{(M,C)} \cup T_{(M,C^{dual})}$  which gives this correspondence. If  $e = \wp(t)$ , then  $e$  *Onext* may be “rewritten” as  $switch_{12}(t)$ , (i.e.,  $\wp(t)Onext = \wp(switch_{12}(t))$ ),  $e$  *Flip* as  $switch_2(t)$ , and  $e$  *Rot* as  $switch_{2R}(t)$ . Conversely,  $switch_0(t)$  may be rewritten as  $e$  *Rot*<sup>2</sup> *Flip*,  $switch_1(t)$  as  $e$  *Onext* *Flip*, and  $switch_2(t)$  as  $e$  *Flip*. The two quad-edge constructors, *MakeEdge* and *Splice*, may be rewritten in terms of the cell-tuple constructors mentioned in Section 10 (see [4]).



**Fig. 18.** Pictorial comparison between directed, oriented edges and cell-tuples. On the left are the quad-edges associated with edges in the original subdivision; the side on which the half-heads appear indicates orientation. On the right are the cell-tuples for the same subdivision, along with the arcs of the graph corresponding to the cell-tuple structure.

It is interesting to note that Guibas and Stolfi introduce a refinement of the original subdivision, called the “completion,” which is exactly the generalized barycentric subdivision in the two-dimensional case.

Dobkin and Laszlo [9] created a data structure for representing complexes of 3-cells in an analogous fashion to that of Guibas and Stolfi’s two-dimensional data structure. The class of 3-complexes allowed in [9] are subdivisions of spaces homeomorphic to  $\mathbb{B}^3$  or  $\overline{\mathbb{B}^3}$ , where the cell boundaries are allowed to self-intersect (though it is possible that others may be generated by using their constructive operators). The basic units are pairs of the form  $(e, f)$  such that the edge  $e$  is incident to the face  $f$ . Since the (topological) dual of an edge is a face, and vice versa, there is a one-to-one correspondence between facet-edges in the primal and dual subdivisions. As in the quad-edge data structure, the idea of orientation is now added. There are two possible directions to rotate around an edge, and two possible directions to traverse the boundary of a face. With each pair are associated four objects called facet-edge pairs (or simply facet-edges), corresponding to the four possible ways of assigning these directions.

There are five operators for moving among facet-edges: *Enext*, *Fnext*, *Clock*, *Rev*, and *Sdual*. *Enext* gives a pair with the same orientation and face component, but with the next edge around the face in the direction given by the face’s orientation. *Fnext* gives a pair with the same orientation and edge component, but with the next face around the edge in the direction given by the edge’s orientation. *Clock* and *Rev* leave both components unchanged. *Clock* changes both orientations, while *Rev* changes only the edge’s orientation. *Sdual* gives the corresponding facet-edge in the dual, such that the orientation of the dual face matches that of the primal edge, and the orientation of the dual edge matches that of the primal face. The first four of these operators are shown pictorially in Fig. 19. The set of all facet-edges, taken with the five operators above, constitutes the facet-edge data structure. The four facet-edge constructors, *make\_facet\_edge*, *splice\_facets*, *splice\_edges*, and *transfer*, may be rewritten in terms of the cell-tuple constructors mentioned in Section 10 (see [4]).

As in the case of the edge algebra, there is a one-to-one correspondence between the cell-tuples in the cell-tuple structure for a subdivided 3-manifold and the primal facet-edges in the facet-edge data structure:  $t_1$  gives the edge,  $t_2$  gives the facet,  $t_0$

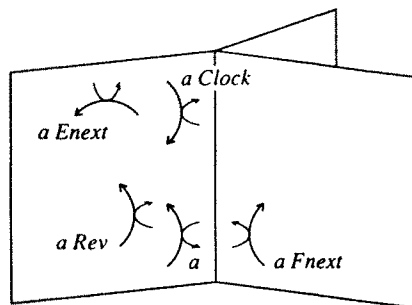


Fig. 19. A facet-edge pair and primitive operators. The larger curved arrows indicate orientation within a face, and the smaller curved arrows indicate orientation about an edge.

gives the direction within the face, and  $t_3$  gives the direction around the edge. This same correspondence holds for the dual. If the facet-edge  $a$  corresponds to the cell-tuple  $t$ , then the facet-edge operators can be rewritten in terms of the  $switch_k$  operators as follows:  $a$  Clock as  $switch_{03}(t)$ ,  $a$  Enext as  $switch_{01}(t)$ ,  $a$  Fnext as  $switch_{23}(t)$ ,  $a$  Rev as  $switch_3(t)$ , and  $a$  Sdual as  $switch_R(t)$ . The  $switch$  operators may also be written in terms of the facet-edge operators.

Two other researchers have described, independently of each other and of this work, representations of  $d$ -dimensional objects which are similar to the cell-tuple structure. Lienhardt is a computer scientist whose work with “ $n$ -G-maps” is concerned with representations of subdivisions of higher-dimensional objects suitable for computational applications, while Tits is a mathematician, who developed “chamber systems” as a tool for working with mathematical objects called “buildings.”

The idea underlying  $n$ -G-maps is basically the same as that of the cell-tuple structure. The definition of an  $n$ -G-map (taken from [21]) is:

**Definition 1.** Let  $n \geq 0$ ; an  $n$ -G-map is defined by an  $(n + 2)$ -tuple

$$G = (B, \alpha_0, \alpha_1, \dots, \alpha_n),$$

such that:

- $B$  is a finite, nonempty set of darts;
- $\alpha_0, \alpha_1, \dots, \alpha_n$  are involutions on  $B$  (i.e.,  $\forall i, 0 \leq i \leq n, \forall b \in B, \alpha_i^2(b) = b$ ), such that:
  - $\forall i \in \{0, \dots, n - 1\}$ ,  $\alpha_i$  is an involution without fixed points

$$(i.e., \forall i, 0 \leq i \leq n - 1, \forall b \in B, \alpha_i(b) \neq b);$$

- $\forall i \in \{0, \dots, n - 2\}$ ,  $\forall j \in \{i + 2, \dots, n\}$ ,  $\alpha_i \alpha_j$  is an involution.

If  $\alpha_n$  is an involution without fixed points,  $G$  is without boundaries, or closed, else  $G$  is with boundary, or open.

Here,  $n$  is the dimension. To associate an  $n$ -G-map with a topological object, first consider  $n = 1$ . In this case, darts are defined to be V-E half-edges. Two half-edges “gathered” by  $\alpha_0$  form an edge, and two edges are adjacent at an endpoint if the respective half-edges are gathered by  $\alpha_1$ . Thus a 1-G-map corresponds to a set of edges forming cycles and paths. A connected closed 1-G-map corresponds to a single cycle, and a connected open 1-G-map corresponds to a single path. (See Fig. 20.) In two dimensions, a face is defined by “a simple elementary cycle of edges.” These faces may be “sewn” together to form two-dimensional subdivisions. In general, any connected closed  $(n - 1)$ -G-map is allowed to be the boundary of an  $n$ -cell, and such cells may be sewn together to create  $n$ -dimensional subdivisions. Lienhardt discusses the orientability of  $n$ -G-maps and gives a definition for the dual of an  $n$ -G-map. In [20] he shows how to compute the number of boundaries, the Euler characteristic, orientability factor, and genus of 2-G-maps (representing surfaces).

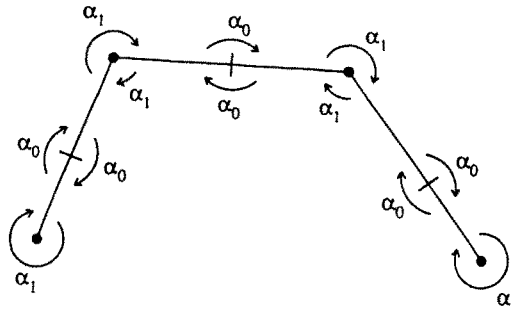


Fig. 20. Example of a 1-G-map. The three edges are divided into half-edges, or darts, by the small perpendicular marks near their midpoints. Darts are gathered into edges by the  $\alpha_0$  operators, and two edges are adjacent if they contain darts which are gathered by  $\alpha_1$  operators.

The topological objects represented by “ $n$ -G-maps” (termed subdivisions) form a much larger class of partitions of topological spaces than subdivided manifolds. What is referred to as a cell is very general. For instance, a torus or a Klein bottle might be the boundary of a three-dimensional cell. So the underlying spaces created are very general, i.e., are not manifolds, or any class of “familiar” space. The advantage to this is that allowed (gathering) operations always give another valid object. The disadvantage is that less can be said about them, e.g., the ordering results presented in this paper do not apply.

Taking the cell-tuple structure as an abstract system obeying Corollary 4, the rules followed by each are basically the same:  $\alpha_i$  acts on  $B$  in  $n$ -G-maps in the same way that  $switch_i$  acts on  $T_{(M,C)}$ . The only difference is the way that boundaries are handled. In an  $n$ -G-map,  $b\alpha_n = b$  if  $b$  is a boundary dart, whereas in the cell-tuple structure, either  $switch_n(t)$  is equal to a special value, or to another “outside” cell-tuple. If the first method for handling boundaries in the cell-tuple is taken, setting  $switch_n(t)$  to be equal to  $t$  would be just as easy to detect as setting it to an arbitrary value, i.e., this would make the cell-tuple structure agree completely with  $n$ -G-maps in this abstract framework. Recall that, in this approach, the boundary is represented only implicitly. The definition of  $n$ -G-map does not include any explicit representation of the boundary as an  $(n - 1)$ -G-map, as was given in the second approach for handling the boundary in the cell-tuple structure, so direct access to the boundary as a subdivided  $(d - 1)$ -manifold is not possible.

Tits’ work [36] is concerned with the characterization of combinatorial “buildings,” which arose out of the geometric study of semisimple Lie groups and semisimple algebraic groups. His work is developed in a much more general setting than that given here, and is not concerned with subdivided manifolds. He does not consider the representation of subdivided manifolds or ordering, and does not consider computational issues.

The concept of chamber system, when appropriately applied to a subdivided manifold  $(M, C)$ , is basically a labeled version of the 1-skeleton of the dual of the

barycentric subdivision of  $(M, C)$ . If a subdivided  $d$ -manifold  $(M, C)$  is considered as a geometry over the set  $0, \dots, d$ , so that the elements of the geometry are the cells of  $C$  and the labeling of the elements is the dimension of the cells, then the complex (in the context of Tits' work) associated with this geometry is equivalent to the abstract simplicial complex  $\mathcal{A}_{(M,C)}$  defined in Section 4. The chambers are then the  $d$ -simplices of  $\mathcal{A}_{(M,C)}$ . The partition  $\mathcal{P}_i$  in the associated chamber system is essentially a matching between chambers, which corresponds to the pairs of  $d$ -simplices associated by  $switch_i$ . ( $switch$  is defined on the  $d$ -simplices of  $K_{(M,C)}$  in Section 6. By the correspondence between the simplices of  $\mathcal{A}_{(M,C)}$  and  $K_{(M,C)}$ , this gives a definition of  $switch$  on  $\mathcal{A}_{(M,C)}$ .) By the one-to-one correspondence between the  $d$ -simplices of  $\mathcal{A}_{(M,C)}$  and the cell-tuples of  $T_{(M,C)}$ , there is a one-to-one correspondence between chambers and cell-tuples, so that a pair of chambers are in the partition  $\mathcal{P}_i$  if and only if the corresponding pair of cell-tuples are connected by  $switch_i$ .

Chamber systems are used by Huson and his coauthors [10], [14] in their study of tilings of the plane. Huson has developed an implementation of two-dimensional chamber systems as part of his work.

## 12. Directions for Future Work

Three possible directions in which further work might be pursued are briefly discussed: generalizing and extending the cell-tuple structure and similar representation schemes as an end in itself; using the cell-tuple structure to obtain new results in computational geometry; and the use of the cell-tuple structure in practical applications.

In this work the basic building blocks were cells whose boundaries were not self-intersecting, and the objects represented by their unions (the underlying spaces) were topological manifolds. A natural generalization of the problem is to consider larger classes of cells, or larger classes of underlying spaces, or both. If a larger class of cells is allowed (under suitable restrictions), the description of basic units as tuples does not work out as simply, but many of the basic ideas go through. Similarly, if a larger class of underlying spaces is allowed, i.e., nonmanifolds, the ordering result may be lost, but additional representational power may be gained. Lienhardt's work [21] on  $n$ - $G$ -maps deals with a very general class, a class which is larger than the class of all spaces which follow the rules given in Corollary 4. There may be a valuable class of objects between those represented by the cell-tuple structure and those by  $n$ - $G$ -maps, which have a simple representation and retain some ordering properties. Work on representation of nonmanifold structures include [17], [18], [30], [31], [39], [40], and [41].

As an example of where the cell-tuple structure might prove useful in obtaining new algorithms for open computational geometry problems, consider the computation of the convex hull of a set of points in  $\mathbb{R}^d$ . This is a much-studied problem (in  $\mathbb{R}^2$ , perhaps the most studied problem in computational geometry). The asymptotically best existing algorithms for computing the convex hull of  $n$  points in the general case of  $d$  dimensions run in time  $O(n^{\lfloor d/2 \rfloor})$  for even  $d \geq 4$ , and time

$O(n^{\lfloor d/2 \rfloor} \log n)$  for odd  $d > 4$ . The existing lower bound for  $d \geq 4$  is  $\Omega(n^{\lfloor d/2 \rfloor})$ , given by the size of the output, so closing the  $\log n$  gap for odd  $d > 4$  is an open problem. Another open problem for higher dimensions is to generalize the divide-and-conquer algorithm devised for the three-dimensional convex hull [29] to  $d \geq 4$ . (Buckley [5] gives a divide-and-conquer algorithm for the particular case of  $d = 4$ , but gives no analysis of the running time.) Edelsbrunner comments on this problem [12, p. 173]: “A crucial step in this generalization will be the formulation of order relations among the faces of the convex hull which can be exploited in ‘merging’ two separated polytopes.” Perhaps the ordering developed in this paper may help to produce such a generalization.

For the Voronoi diagram, there is a similar  $\log n$  gap for even  $d \geq 4$ . There is a lifting method which reduces the Voronoi diagram problem to the convex hull problem in one higher dimension, so an improved convex hull algorithm will also give an improved Voronoi diagram algorithm.

The cell-tuple structure may be thought of in several ways, as discussed in Section 5: as a set of data elements and pointers directly representing the cell-tuples and *switch* operators; as an arc-labeled graph; or as a relational database. Several hybrids are also possible, including the augmented incidence graph. Simplifications may be made if more is known about the particular objects being used in an application, e.g., perhaps all cells are simplicial. Building a useful system based on any of these strategies involves considerations of speed and memory, as well as deciding how to maintain the connection between geometry and topology. In any real application a geometry package with some degree of robustness, and some means of displaying the output, will have to be implemented. These are discussed in [4].

### 13. Conclusion

The primary contribution of this paper is the introduction of the cell-tuple structure, which gives a simple, easily implemented data structure for representing the topological structure of geometric objects, the ordering within such objects, their topological duals, and their boundaries. The class of objects studied, the subdivided manifolds, is general enough to encompass many applications. This work generalizes existing work in two and three dimensions to the general higher-dimensional case, and provides a simple unifying framework for existing work.

Part of this work, a contribution in itself, is the generalization of the ordering of cells familiar in two and three dimensions to the general higher-dimensional case, whose existence was proved by using the cell-tuple structure. In addition to describing the cell-tuple structure as a static representation, a set of possible constructors was proposed for creating and modifying subdivided manifolds.

It is hoped that the cell-tuple structure will lead to new theoretical results as well as providing a practical method of representing objects for modeling and graphics.

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