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E. Cordero, F. De Mari, K. Nowak, Anita Tabacco

Institutions: Polytechnic University of Turin, Drexel University

Published on: 01 Jan 2006

Topics: Semidirect product, Group (mathematics), Heisenberg group and Symplectic group

REPRODUCING GROUPS FOR THE METAPLECTIC REPRESENTATION

E. CORDERO, F. DE MARI, K. NOWAK, AND A. TABACCO

Abstract. We consider the (extended) metaplectic representation of the semidirect product G of the symplectic group and the Heisenberg group. By looking at the standard resolution of the identity formula and inspired by previous work [5], [13], [4], we introduce the notion of admissible (reproducing) subgroup of G via the Wigner distribution. We prove some features of admissible groups and then exhibit an explicit example ($d = 2$) of such a group, in connection with wavelet theory.

1. Introduction.

Reproducing formulae based on, or inspired by, various versions of the resolution of the identity appear pervasively in the literature, from coherent states in physics[1] to group representations [6] and to wavelet and Gabor analysis [9]. In a very general and abstract sense, they can all be recast in a formula of the type

$$f = \int_H \langle f, \phi_h \rangle \, \phi_h \, dh, \qquad f \in \mathcal{H}, \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space and $h \mapsto \phi_h$ is an $\mathcal{H}$-valued measurable function on some measure space (H, dh). Of course, the cases of greatest interest concern Hilbert spaces of functions and measure spaces with additional structure such as Lie groups. A formula like (1.1) is known as *reproducing formula*.

This paper is part of an ongoing project that addresses several questions related to (1.1) in the case in which the ingredients are as follows. First, the Hilbert space is $L^2(\mathbb{R}^d)$. Secondly, H is a subgroup of the semidirect product G of the symplectic group and the Heisenberg group (of the appropriate dimensions). Thirdly, the map $h \mapsto \phi_h$ arises from

Date: January 15, 2005.

1991 *Mathematics Subject Classification.* 42C15, 17B45.

Key words and phrases. Metaplectic representation, symplectic group, wavelets, Wigner distribution.

This work was partially supported by the Progetto MIUR Cofinanziato 2002 "Analisi Armonica".

the restriction to H of the (extended) metaplectic representation μ_e of G as applied to a fixed and suitable "window" $\phi \in L^2(\mathbb{R}^d)$. Thus, at least formally, (1.1) can be written

$$f = \int_H \langle f, \mu_e(h)\phi\rangle \mu_e(h)\phi \, dh, \qquad f \in L^2(\mathbb{R}^d).$$

The main question is: for which subgroups H of G does there exist a window $\phi \in L^2(\mathbb{R}^d)$ such that the above reproducing formula holds *for all* $f \in L^2(\mathbb{R}^d)$? Clearly, one looks for invariants or other general properties that will decide whether a group H enjoys the property or not. Further, in the affirmative case, one seeks conditions that single out the "good" windows, namely those for which the formula holds. These questions are far from being fully answered. A complete classification of reproducing subgroups in the case $d = 1$ is given in [5] and many interesting facts have been proved in [13] in a somewhat different setting. Some new results in higher dimensions are in [4]. For the relevance of the extended metaplectic representation in the context of harmonic analysis in phase space or time-frequency analysis, the reader is referred to [8] and [9], and the references therein.

2. Admissible subgroups.

2.1. Symplectic group and metaplectic representation. The symplectic group is as usual

$$Sp(d, \mathbb{R}) = \left\{ g \in GL(2d, \mathbb{R}) : \ {}^t g J g = J \right\},$$

where $J = \left[\begin{smallmatrix} 0 & I_d \\ -I_d & 0 \end{smallmatrix}\right]$ defines the standard symplectic form

$$\omega(x, y) = \ {}^t x J y, \qquad x, y \in \mathbb{R}^{2d}. \tag{2.1}$$

The Lie algebra $\mathfrak{sp}(d, \mathbb{R})$ of $Sp(d, \mathbb{R})$ is therefore

$$\begin{aligned} \mathfrak{sp}(d, \mathbb{R}) &= \left\{ X \in \mathfrak{gl}(2d, \mathbb{R}) : \ {}^t X J + J X = 0 \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & -\,{}^t A \end{bmatrix} : A \in \mathfrak{gl}(d, \mathbb{R}), B, C \in \mathrm{Sym}(d, \mathbb{R}) \right\}. \end{aligned}$$

The bracket in $\mathfrak{sp}(d, \mathbb{R})$ is the commutator of matrices. The Cartan involution θ on $\mathfrak{sp}(d, \mathbb{R})$ is defined by $\theta X = -\,{}^t X$, and it decomposes $\mathfrak{sp}(d, \mathbb{R})$ into its $+1$ and -1 eigenspaces, namely

$$\begin{aligned} \mathfrak{k} &= \{ X \in \mathfrak{sp}(d, \mathbb{R}) : \theta X = X \} \\ &= \left\{ X \in \mathfrak{sp}(d, \mathbb{R}) : \begin{bmatrix} A & B \\ -B & -\,{}^t A \end{bmatrix} : A \in \mathfrak{so}(d, \mathbb{R}), B \in \mathrm{Sym}(n, \mathbb{R}) \right\} \end{aligned}$$

and

$$\begin{aligned}\mathfrak{p} &= \{X \in \mathfrak{sp}(d,\mathbb{R}) : \theta X = -X\} \\ &= \left\{ X \in \mathfrak{sp}(d,\mathbb{R}) : \begin{bmatrix} A & B \\ B & -\,{}^tA \end{bmatrix} : A,\ B \in \mathrm{Sym}(d,\mathbb{R}) \right\},\end{aligned}$$

respectively. Here $\mathfrak{so}(d,\mathbb{R})$ denotes the Lie algebra of $d \times d$ skew-symmetric matrices. Thus $\mathfrak{sp}(d,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, a direct sum of vector spaces. An immediate count gives $\dim \mathfrak{sp}(d,\mathbb{R}) = d(2d+1)$, $\dim \mathfrak{k} = d^2$ and $\dim \mathfrak{p} = d^2 + d$. The bracket relations $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ show that $\mathfrak{k}$ is a subalgebra, while $\mathfrak{p}$ is not. Moreover, $\mathfrak{k}$ is the Lie algebra of the connected Lie subgroup K defined as the fix-point set of the Cartan involution $\Theta g = {}^tg^{-1}$ of $Sp(d,\mathbb{R})$ (observe that $d\Theta = \theta$). In practice,

$$K = Sp(d,\mathbb{R}) \cap SO(2d) \simeq U(d)$$

is the unique maximal compact subgroup of $Sp(d,\mathbb{R})$, up to conjugation. If $d = 2$ the full Lie algebra has dimension 10, whereas $\mathfrak{k}$ is given by

$$\mathfrak{k} = \left\{ X \in \mathfrak{sp}(2,\mathbb{R}) : \begin{bmatrix} bJ & \Sigma \\ -\Sigma & bJ \end{bmatrix} : b \in \mathbb{R},\ \Sigma \in \mathrm{Sym}(2,\mathbb{R}) \right\} \tag{2.2}$$

and has dimension 4.

The metaplectic representation μ of (the two-sheeted cover of) the symplectic group arises as intertwining operator between the standard Schrödinger representation ρ of the Heisenberg group $\mathbb{H}^d$ and the representation that is obtained from it by composing ρ with the action of $Sp(d,\mathbb{R})$ by automorphisms on $\mathbb{H}^d$ (see e.g.[8]). We briefly review its construction.

The Heisenberg group $\mathbb{H}^d$ is obtained by defining on $\mathbb{R}^{2d+1}$ the product

$$(z,t) \cdot (z',t') = (z + z', t + t' + \frac{1}{2}\omega(z,z')),$$

where ω is given in (2.1). We denote the translation and modulation operators on $L^2(\mathbb{R}^d)$ by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i \langle \xi, t\rangle} f(t).$$

The Schrödinger representation of the group $\mathbb{H}^d$ on $L^2(\mathbb{R}^d)$ is then defined by

$$\rho(x,\xi,t)f(y) = e^{2\pi i t} e^{\pi i\langle x,\xi\rangle} e^{2\pi i\langle \xi, y\rangle} f(y-x) = e^{2\pi i t} e^{\pi i\langle x,\xi\rangle} T_x M_\xi f(t),$$

where we write $z = (x,\xi)$ when we separate the space components x from the frequency components ξ of a point z in phase space $\mathbb{R}^{2d}$. The symplectic group acts on $\mathbb{H}^d$ via automorphisms that leave the center $\mathbb{R} = \{(0,t) : t \in \mathbb{R}\}$ of $\mathbb{H}^d$ pointwise fixed. The action $\varphi : Sp(d,\mathbb{R}) \times \mathbb{H}^d \to \mathbb{H}^d$ is given by $\varphi(A,(z,t)) = A \cdot (z,t)$, where

$$A \cdot (z,t) = (Az,t)\,.$$

Therefore, for any fixed $A \in Sp(d,\mathbb{R})$ there is a representation

$$\rho_A : \mathbb{H}^d \to \mathcal{U}(L^2(\mathbb{R}^d)), \qquad (z,t) \mapsto \rho\,(A \cdot (z,t))$$

whose restriction to the center is a multiple of the identity. By the Stone-von Neumann theorem, $\rho_A \simeq \rho$. Hence there exists an intertwining unitary operator $\mu(A)$ for the two representations, namely

$$\rho_A = \mu(A) \circ \rho \circ \mu(A)^{-1}.$$

By Schur's lemma, μ is determined up to a phase factor. It turns out that the phase ambiguity is really a sign, so that μ lifts to a representation of the (double cover of the) symplectic group. It is the famous metaplectic or Shale-Weil representation. The representations ρ and μ can be combined and give rise to the extended metaplectic representation of the group

$$G = \mathbb{H}^d \times_\varphi Sp(d,\mathbb{R}),$$

the semidirect product of $\mathbb{H}^d$ and $Sp(d,\mathbb{R})$. The group law on G is

$$((z,t),A) \cdot ((z',t'),A') = ((z,t) \cdot (Az',t'),AA')$$

and the extended metaplectic representation μ_e of G is

$$\mu_e\,((z,t),A) = \rho(z,t) \circ \mu(A).$$

For elements $Sp(d,\mathbb{R})$ in special form, the metaplectic representation can be computed explicitly in a simple way. For $f \in L^2(\mathbb{R}^d)$ we have

$$\mu\left(\begin{bmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{bmatrix}\right) f(x) = (\det A)^{-1/2} f(A^{-1}x) \tag{2.3}$$

$$\mu\left(\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\right) f(x) = \pm e^{-i\pi\langle Cx,x\rangle} f(x) \tag{2.4}$$

$$\mu\,(J) = i^{d/2}\mathcal{F}^{-1}, \tag{2.5}$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i\langle x,\xi\rangle}\,dx, \qquad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

2.2. **Reproducing and admissible groups.** The point of this paper is to look at the reproducing formulae that arise by restricting μ_e to subgroups H of G. A slight simplification in our formalism comes from the observation that the reproducing formula (2.6) is insensitive to phase factors: if we replace $\mu_e(h)\phi$ with $e^{is}\mu_e(h)\phi$ the formula is unchanged, for any $s \in \mathbb{R}$. The role of the center of the Heisenberg group is thus irrelevant, so that the "true" group under consideration is $\mathbb{R}^{2d} \times_\varphi Sp(d, \mathbb{R})$, which we denote again by G.

We write dh for the left Haar measure of the group H. Also, we shall always assume that the Haar measure of a compact group is normalized so that the total mass of the group is one.

Definition 2.1. We say that a connected Lie subgroup H of $G = \mathbb{R}^{2d} \times_\varphi Sp(d, \mathbb{R})$ is a *reproducing group* for μ_e if there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that

$$f = \int_H \langle f, \mu_e(h)\phi\rangle \mu_e(h)\phi \, dh, \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{2.6}$$

Notice that we do require formula (2.6) to hold for all functions in $L^2(\mathbb{R}^d)$ for the same "window" ϕ, but we do not require the restriction of μ_e to H to be irreducible.

In [13], the authors consider subgroups D of $GL(d, \mathbb{R})$ and their actions on $\mathbb{R}^d$. Motivated by the analysis of the case in which D is the "$ax+b$" group, they say that a subgroup D is *admissible* if there exists a Borel measurable $h \in L^1(\mathbb{R}^d)$ such that $h \geq 0$ and

$$\int_D h(xa)\, d\mu(a) = 1 \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

If D is the "$ax+b$" group, then any admissible wavelet ψ (in the usual Calderón sense) gives a function $h = |\hat{\psi}|^2$ for which the above formula holds, showing that "$ax+b$" is admissible. In [4], in the same context that we are considering here, we introduce a similar notion of admissibility via the Wigner distribution and prove a sufficient condition for a subgroup to be reproducing. We briefly explain this issue. The cross-Wigner distribution $W_{f,g}$ of $f, g \in L^2(\mathbb{R}^d)$ is defined by

$$W_{f,g}(x,\xi) = \int e^{-2\pi i\langle \xi, y\rangle} f(x + \frac{y}{2})\overline{g(x - \frac{y}{2})}\, dy. \tag{2.7}$$

The quadratic expression $W_f := W_{f,f}$ is usually called the Wigner distribution of f. A crucial property of W is that it intertwines the (extended) metaplectic representation and the affine action on $\mathbb{R}^{2d}$ [8]. In other words:

$$W_{\mu_e(g)\phi}(x,\xi) = W_\phi\left(g^{-1} \cdot (x,\xi)\right), \qquad g \in G,$$

where the affine action $g \cdot (x, \xi)$ is defined by

$$g \cdot (x, \xi) = ((q, p), A) \cdot (x, \xi) = A\, {}^t\!(x, \xi) + {}^t\!(q, p). \tag{2.8}$$

The following result is proved in [4]:

Theorem 2.2. *If there exists a function ϕ such that the mapping*

$$h \to W_{\mu_e(h)\phi}(x, \xi) = W_\phi(h^{-1} \cdot (x, \xi)) \tag{2.9}$$

is in $L^1(H)$ for a.e. $(x, \xi) \in \mathbb{R}^{2d}$, and such that

$$\int_H |W_{\mu_e(h)\phi}(x, \xi)|\, dh \le M \qquad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}, \tag{2.10}$$

then (2.6) *holds for all $f \in L^2(\mathbb{R}^d)$ if and only if the following admissibility condition is satisfied:*

$$\int_H W_\phi(h^{-1} \cdot (x, \xi))\, dh = 1 \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}. \tag{2.11}$$

The above discussion and Theorem 2.2 justify the following definition.

Definition 2.3. We say that a connected Lie subgroup H of $G = \mathbb{R}^{2d} \times_\varphi Sp(d, \mathbb{R})$ is an *admissible group* for μ_e if there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that

$$\int_H W_\phi(h^{-1} \cdot (x, \xi))\, dh = 1 \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}. \tag{2.12}$$

3. Geometric features.

The main geometric properties related to admissibility are described in [13, Prop. 2.3]. The arguments can be easily adapted to our setting and yield the following

Theorem 3.1. *Let $H \subset G$ be admissible. Then*

- (i) *The stabilizer* $\mathrm{Stab}_z(H)$ *of the H-action on phase space is compact for a.e. $z \in \mathbb{R}^{2d}$;*
- (ii) *If H satisfies* (2.10) *and $H \subset Sp(d, \mathbb{R})$, then H is not unimodular.*

Here, as customary, $\mathrm{Stab}_z(H) = \{h \in H : h \cdot z = z\}$ and $g \cdot z$ is defined in (2.8). Likewise, we adopt the usual notation for orbits: $\mathcal{O}_z = \{h \cdot z : h \in H\}$, whenever there is no ambiguity about the group H acting on z.

Remark 3.2. (i) If the subgroup H has finite measure, then (2.6) holds if and only if condition (2.11) is satisfied. This is because for all $\phi \in L^2(\mathbb{R}^d)$ the Wigner distribution $W_\phi \in L^\infty$, and there exists $M > 0$ such that $|W_\phi(x,\xi)| \leq M$. Therefore

$$\int_{\mathbb{R}^{2d}} \int_H |W_{\mu_e(h)\phi}(x,\xi)\, W_f(x,\xi)|\, dh dx d\xi \leq M \left(\int_H dh \right) \int_{\mathbb{R}^{2d}} |W_f(x,\xi)| dx d\xi.$$

The only compact subgroups of G, however, are of the form $\{0\} \times K$ with K compact in $Sp(d,\mathbb{R})$. This is because the projection of G onto $\mathbb{R}^{2d}$ is a continuous group homomorphism and $\{0\}$ is the only compact additive subgroup of $\mathbb{R}^{2d}$. Hence, item (ii) of Theorem 3.1 applies and any such group cannot be admissible, hence reproducing.
(ii) The requirement $H \subset Sp(d,\mathbb{R})$ is essential for the non-unimodularity of the group H. In fact, one can find reproducing subgroups H of $\mathbb{R}^{2d} \times_\varphi Sp(d,\mathbb{R})$ that are unimodular. This case, for instance, arises in Gabor's analysis, where the subgroup is $H = \mathbb{R}^{2d}$, $\mu_e(q,p)\phi = T_q M_p \phi$ and $\phi(t) = 2^{d/4} e^{-\pi t^2}$.

The main contribution of this paper concerning the geometry of admissible groups is Corollary 3.3 below, where we give a dimension bound in the case $d = 2$. Its proof is postponed to Section 3.2 because it needs the Lie-algebraic considerations of the next section.

Corollary 3.3. *If* $H \subset Sp(2,\mathbb{R})$ *is admissible, then* $\dim(H) \leq 5$.

3.1. **Compact Lie subalgebras.**

We will prove below that there are only finitely many conjugation classes of (Lie algebras of) compact subgroups of $Sp(2,\mathbb{R})$ and that each of them has an explicit representative. Since the maximal compact subgroup K of $Sp(d,\mathbb{R})$ is the fixed point set of the involutive analytic automorphism Θ and it is compact and connected, the pair (G,K) is a Riemannian symmetric pair (see [10], p.209). Therefore, any compact connected Lie subgoup of G is conjugate to a subgroup of K (see [10], Theorem 2.1, Ch. VI). Consequently, if $\mathfrak{t} = \mathrm{Lie}(T)$ is the Lie algebra of the compact connected Lie subgroup T of G, then we may assume $T \subset K$ and $\mathfrak{t} \subset \mathfrak{k}$. In Proposition 3.4 below we classify the subalgebras of $\mathfrak{k}$ for $d = 2$, up to conjugation. In the sequel, we shall then refer to any such algebra as a canonical compact algebra.

First of all, we work out the bracket in $\mathfrak{k}$. For $T \in \mathfrak{k}$ we write $T = T_{b,\Sigma}$ using the parametrization introduced in (2.2). An immediate computation gives

$$[T_{b,\Sigma}, T_{b',\Sigma'}] = \begin{bmatrix} [\Sigma',\Sigma] & b[J,\Sigma'] + b'[\Sigma,J] \\ -b[J,\Sigma'] - b'[\Sigma,J] & [\Sigma',\Sigma] \end{bmatrix}. \tag{3.1}$$

We also write

$$\Sigma = \begin{bmatrix} m & n \\ n & p \end{bmatrix} = mE + nL + pF, \tag{3.2}$$

where evidently

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since

$$[\Sigma', \Sigma] = \{(p-m)n' - (p'-m')n\}\, J$$

$$[J, \Sigma] = \begin{bmatrix} 2n & p-m \\ p-m & -2n \end{bmatrix} = 2n(E-F) + (p-m)L, \tag{3.3}$$

it follows that if $T = T_{b,\Sigma}$, $T' = T_{b',\Sigma'}$ and if $[T, T'] = T_{b(T,T'),\Sigma(T,T')}$, then

$$b(T,T') = (p-m)n' - (p'-m')n \tag{3.4}$$

$$\Sigma(T,T') = 2(bn' - b'n)(E-F) + (b(p'-m') - b'(p-m))\, L. \tag{3.5}$$

Next we consider the action of $SO(2)$ on $\mathfrak{k}$. Indeed, if

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{3.6}$$

denotes the standard rotation by the (real) angle θ, then it is easily checked that

(3.7)

$$SO(2) \simeq \left\{ k_\theta = \begin{bmatrix} R_\theta & \\ & R_\theta \end{bmatrix} : \theta \in [0, 2\pi] \right\} \subset Sp(2,\mathbb{R}) \cap SO(4) = K.$$

The adjoint action of K on $\mathfrak{k}$ restricts to the following action of $SO(2)$:

$$\operatorname{Ad} k_\theta \left(\begin{bmatrix} bJ & \Sigma \\ -\Sigma & bJ \end{bmatrix} \right) = k_\theta \begin{bmatrix} bJ & \Sigma \\ -\Sigma & bJ \end{bmatrix} k_{-\theta} = \begin{bmatrix} bJ & \operatorname{Ad} R_\theta(\Sigma) \\ -\operatorname{Ad} R_\theta(\Sigma) & bJ \end{bmatrix},$$

because $\operatorname{Ad} R_\theta(J) = J$. Parametrizing 2×2 symmetric matrices as in (3.2), we have

$$\begin{bmatrix} m \\ n \\ p \end{bmatrix} \overset{\operatorname{Ad} R_\theta}{\longmapsto} \begin{bmatrix} m_\theta \\ n_\theta \\ p_\theta \end{bmatrix} = \begin{bmatrix} m(\cos\theta)^2 + p(\sin\theta)^2 + n\sin 2\theta \\ \frac{1}{2}(p-m)\sin 2\theta + n\cos 2\theta \\ m(\sin\theta)^2 + p(\cos\theta)^2 - n\sin 2\theta \end{bmatrix}.$$

Observe that under this action

$$\begin{bmatrix} p_\theta - m_\theta \\ 2n_\theta \end{bmatrix} = R_{-2\theta} \begin{bmatrix} p-m \\ 2n \end{bmatrix}.$$

Proposition 3.4. *Up to conjugation, the following is a complete list of the Lie algebras of the compact subgroups of $Sp(2,\mathbb{R})$.*

(i) *The* 1*-dimensional toral non-conjugate algebras*

$$\mathfrak{k}_{1,\lambda} = \left\{ t \begin{bmatrix} & C_\lambda \\ -C_\lambda & \end{bmatrix} : t \in \mathbb{R},\ C_\lambda = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right] \right\} \simeq \mathfrak{so}(2), \quad \lambda \in \mathbb{R};$$

(ii) *the (maximally compact)* 2*-dimensional Cartan subalgebra*

$$\mathfrak{k}_2 = \left\{ \begin{bmatrix} & D_y \\ -D_y & \end{bmatrix} : D_y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \right\} \simeq \mathfrak{so}(2) \times \mathfrak{so}(2);$$

(iii) *the* 3*-dimensional Lie algebra*

$$\mathfrak{k}_3 = \left\{ \begin{bmatrix} aJ & \Sigma_y \\ -\Sigma_y & aJ \end{bmatrix} : a \in \mathbb{R},\ \Sigma_y = \begin{bmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{bmatrix} \right\} \simeq \mathfrak{so}(3)$$

(iv) *the maximal compact* 4*-dimensional algebra*

$$\mathfrak{k}_{\max} = \left\{ \begin{bmatrix} aJ & \Sigma \\ -\Sigma & aJ \end{bmatrix} : a \in \mathbb{R},\ \Sigma \in \mathrm{Sym}_2(\mathbb{R}) \right\} = \mathfrak{sp}(2,\mathbb{R}) \cap \mathfrak{so}(4) \simeq \mathfrak{u}(2);$$

Proof. (i) Clearly, the matter reduces to finding normal forms for vectors in $\mathfrak{k}$. These are well-known and follow from Williamson's classification of Hamiltonians (see [14] for the original paper and [2] for a modern account). The eigenvalues of $H \in \mathfrak{sp}(2,\mathbb{R})$ are of four possible kinds: zero eigenvalues, real pairs $(a,-a)$, quadruples $\pm a \pm ib$ and pairs of purely imaginary eigenvalues $(ib,-ib)$. By skew-symmetry, if $H \neq 0$ is conjugate to some element in $\mathfrak{k}$, then it has two pairs of imaginary eigenvalues, say $(ib_1,-ib_1)$ and $(ib_2,-ib_2)$, and one of b_1,b_2 must be non-zero. Different sets of eigenvalues correspond to non-conjugate matrices, and conversely, equal sets of eigenvalues correspond to conjugate matrices. Thus $H \in \mathfrak{k}$ is conjugate to a matrix of the form

$$H(b_1,b_2) = \begin{bmatrix} & b_1E + b_2F \\ -(b_1E + b_2F) & \end{bmatrix}.$$

because $H(b_1,b_2) \in \mathfrak{k}$ and has eigenvalues $\{\pm ib_1, \pm ib_2\}$. It is straightforward to check that $wH(b_1,b_2)w^{-1} = H(b_2,b_1)$, where

$$w = \begin{bmatrix} & L \\ -L & \end{bmatrix} \in K.$$

We may thus assume $b_1 \neq 0$, so that $H(b_1,b_2)$ spans $\mathfrak{k}_{1,b_2/b_1}$.

(ii) First of all, any 2-dimensional compact Lie group is abelian. Secondly, it is well-known [12] that all the maximal abelian subalgebras of $\mathfrak{k}$ have dimension 2. (This actually follows also from the next part of this proof, where it is shown that no subalgebra of dimension 3 is abelian.) We may thus appeal to the conjugacy of maximal tori within compact connected Lie groups: two maximal abelian subalgebras of the Lie algebra of a compact connected Lie group K are conjugate via

$\mathrm{Ad}(K)$ (see Theorem 4.34 in [12]). Hence there is a unique conjugacy class, that of

$$\mathfrak{t} = \left\{ \begin{bmatrix} & & a & \\ & & & b \\ -a & & & \\ & -b & & \end{bmatrix} : a, b \in \mathbb{R} \right\} = \mathfrak{k}_2. \tag{3.8}$$

This is the standard maximally compact Cartan subalgebra of $Sp(2, \mathbb{R})$.

(iii) Suppose now that $\mathfrak{t}$ is a 3-dimensional subalgebra of $\mathfrak{k}$. We shall denote by X, Y, Z a basis of $\mathfrak{t}$. First of all, observe that it is possible to assume $X = T_{1,\Sigma}$ and hence, by linear independence, $Y = T_{0,\Sigma_Y}$ and $Z = T_{0,\Sigma_Z}$. Otherwise, $X = T_{0,\Sigma_X}$, $Y = T_{0,\Sigma_Y}$ and $Z = T_{0,\Sigma_Z}$. Since X, Y, Z span a Lie algebra we could infer from (3.1) that $[\Sigma_X, \Sigma_Y] = [\Sigma_X, \Sigma_Z] = [\Sigma_Y, \Sigma_Z] = 0$. We would thus be given three mutually commuting diagonalizable matrices. and there would exist a basis in which they are all diagonal. Since they are 2×2, they could not be linearly independent, a contradiction. This proves our claim, so that until the end of the proof

$$X = T_{1,\Sigma}, \qquad Y = T_{0,\Sigma_Y}, \qquad Z = T_{0,\Sigma_Z}.$$

Next we look at $[\Sigma_Y, \Sigma_Z]$.

• Assume first that $[\Sigma_Y, \Sigma_Z] = 0$. By means of the $SO(2)$-action we can simultaneously diagonalize Σ_Y, Σ_Z by sending $X = T_{1,\Sigma}$ to another matrix of the same kind $T_{1,\Sigma'}$, that we rename $X = T_{1,\Sigma}$. Also, by linear independence, we assume that $Y = T_{0,E}$ and $Z = T_{0,F}$. Consequently, if we write Σ as in (3.2), subtracting if necessary $mY + pZ$ from X, we have $X = T_{1,nL}$ for some $n \in \mathbb{R}$. But then

$$[X, Y] = [\begin{bmatrix} J & nL \\ -nL & J \end{bmatrix}, \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}] = \begin{bmatrix} nJ & -L \\ L & nJ \end{bmatrix} = T_{n,-L}.$$

Now, E, F and L are linearly independent. Hence the only way in which $[X, Y]$ can be a linear combination of X, Y, Z is if $[X, Y]$ is actually equal to $nX = T_{n,n^2L}$, which in turn implies $n^2 = -1$. This is impossible and so this case does not arise. Thus:

• $[\Sigma_Y, \Sigma_Z] = \alpha J$ for some $\alpha \neq 0$. By (3.1) we have $[Y, Z] = T_{\alpha,0}$. Suppose first that $X = T_{1,\Sigma}$ with $\Sigma \neq 0$. Since $[Y, Z]$ must lie in $\mathfrak{t}$ and since

$$[Y, Z] = T_{\alpha,0} = \alpha T_{1,\Sigma} - \alpha T_{0,\Sigma} = \alpha X - \alpha T_{0,\Sigma},$$

$T_{0,\Sigma}$ must be a linear combination of Y and Z. But then we may again change basis and take $X' = X - T_{0,\Sigma} = T_{1,0}$. In other words we assume $X = T_{1,0}$. In this case $[X, Y] = T_{0,[J,\Sigma_Y]}$ and $[X, Z] = T_{0,[J,\Sigma_Z]}$.

Therefore, the subspace $\operatorname{span}\{\Sigma_Y, \Sigma_Z\}$ of $\operatorname{Sym}(2,\mathbb{R})$ must be invariant under $\operatorname{ad} J$, namely under the linear map

$$\begin{bmatrix} m \\ n \\ p \end{bmatrix} \mapsto \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} m \\ n \\ p \end{bmatrix}.$$

Clearly, $\operatorname{ad} J$ has 0 as the only real eigenvalue with eigenvector ${}^t[1,0,1]$. Its orthogonal complement $\mathcal{S} = \{\, {}^t[y,z,-y] : z, y \in \mathbb{R}\}$ coincides with the image of $\operatorname{ad} J$ and has no proper invariant subspaces. Hence $\mathcal{S}$ is the only invariant plane. Therefore $\operatorname{span}\{\Sigma_Y, \Sigma_Z\} = \mathcal{S}$ and we conclude that

$$\mathfrak{t} = \left\{ \begin{bmatrix} & x & y & z \\ -x & & z & -y \\ -y & -z & & x \\ -z & y & -x & \end{bmatrix} : x, y, z \in \mathbb{R} \right\} = \mathfrak{k}_3$$

is the only 3 dimensional subalgebra of $\mathfrak{k}$, up to conjugation. This concludes the proof of the because the case of 4-dimensional algebras has already been discussed. $\square$

3.2. **Proof of Corollary 3.3.** If $z = 0$, $\operatorname{Stab}_0(H) = H$. Let $z \neq 0$ and assume that $T = \operatorname{Stab}_z(H) \subset Sp(2,\mathbb{R})$ is compact. First, we show that $\dim(T) \leq 1$. We may assume that $\mathfrak{t} = \operatorname{Lie}(T)$ is one of the canonical compact algebras listed in Proposition 3.4. Indeed, given $\mathfrak{t}$, there exists a $g \in Sp(2,\mathbb{R})$ such that $\operatorname{Ad} g(\mathfrak{t})$ is canonical. Thus for all $X \in \mathfrak{t}$, we have

$$\exp(\operatorname{Ad} g(tX))gz = g\exp(tX)z = gz \qquad \forall t \in \mathbb{R}$$

because $\exp(tX) \in T$. Hence $\operatorname{Ad} g(\mathfrak{t}) \subset \operatorname{Lie}(\operatorname{Stab}_{gz}(H))$. Let $\mathfrak{t} = \operatorname{Lie}(T)$ denote a canonical algebra. Now $X \in \mathfrak{t}$ if and only if $\exp(tX)z = z$ for all $t \in \mathbb{R}$. This implies

$$0 = \frac{d}{dt} \exp(tX)z\Big|_{t=0} = Xz.$$

It is an easy computation to check that, if $\dim(\mathfrak{t}) = 2, 3, 4$, then there exists a non-zero $X \in \mathfrak{t}$ with $\ker(X) = 0$, so that $z \notin \ker(X)$. Therefore $\dim(\operatorname{Stab}_z(H)) \leq 1$ for every z for which $\operatorname{Stab}_z(H)$ is compact, that is for a.e. $z \in \mathbb{R}^4$ by virtue of Theorem 3.1. Finally, since $\mathcal{O}_z \subset R^4$, $\dim(\mathcal{O}_z) \leq 4$ and

$$4 \geq \dim(\mathcal{O}_z) = \dim(H) - \dim(\operatorname{Stab}_z(H)) \geq \dim(H) - 1,$$

that is, $\dim(H) \leq 5$. $\square$

4. The Translation-Dilation-Sheering group (TDS).

We prove that the following 3-dimensional triangular group

$$H = \left\{ A_{t,\ell,y} := \begin{bmatrix} t^{-1/2} S_{\ell/2} & 0 \\ t^{-1/2} B_y & t^{1/2}\, {}^t S_{-\ell/2} \end{bmatrix} : t > 0, \ell \in \mathbb{R},\ y \in \mathbb{R}^2 \right\}, \tag{4.1}$$

is a reproducing subgroup of $Sp(2, \mathbb{R})$, where

$$B_y = \begin{bmatrix} 0 & y_1 \\ y_1 & y_2 \end{bmatrix}, \quad y = (y_1, y_2) \in \mathbb{R}^2; \quad S_\ell = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}, \quad \ell \in \mathbb{R}. \tag{4.2}$$

The matrix S_ℓ is called *sheering* matrix. We call H the TDS group, because we prove in Theorem 4.2 that the restriction of μ to it is eqivalent to the wavelet representation (4.3) considered in [7] consisting of translation, dilation and sheering operators. This construction gives rise to the so-called "contourlet frames".

In [4] we show that the TDS group is admissible via Theorem 2.2. Here we give a second, direct proof. First, though, we review the connection between this subgroup of $Sp(2, \mathbb{R})$ and the 2-dimensional wavelet theory in [7] alluded to above. The sheering operator on functions is given by

$$(S_\ell f)(x) = f\left({}^t S_\ell x\right) \qquad f \in L^2(\mathbb{R}^2),$$

where S_ℓ is as in (4.2). These are the ingredients of the contourlet frames. As for curvelets, one allows dilation and translation operations, but the angular selectivity is achieved by a sheering operation rather than a rotation [7].

Let L denote the 2-dimensional subgroup of $Sp(2, \mathbb{R})$ given by

$$L = \left\{ \begin{bmatrix} t & 0 \\ -\ell t & t \end{bmatrix} : t > 0,\ \ell \in \mathbb{R} \right\} \subset Sp(2, \mathbb{R}).$$

We consider its natural action on $\mathbb{R}^2$, that is the semidirect product $H = \mathbb{R}^2 \times_\varphi L$. This action has two open orbits $\mathcal{O}_+$ and $\mathcal{O}_-$ in $\mathbb{R}^2$, where $\mathcal{O}_+ = \{(x_1, x_2)\,;\, x_2 > 0\}$ and $\mathcal{O}_- = \{(x_1, x_2)\,;\, x_2 < 0\}$. The wavelet representation ν is given by

$$\nu(t, y, \ell) f = (T_y D_t S_\ell)\, f, \qquad f \in L^2(\mathbb{R}^2), \tag{4.3}$$

but it is more convenient to view ν in the frequency domain, namely

$$\pi(t, y, \ell) f(u) = (\mathcal{F} \circ \nu(t, y, \ell) f)(u) = e^{-2\pi i \langle y, u \rangle} D_{-t}\, {}^t S_{-\ell} f\,(u)\,. \tag{4.4}$$

We have $\pi = \pi_{\mathcal{O}_+} \oplus \pi_{\mathcal{O}_-}$, where $\pi_{\mathcal{O}_+}$ and $\pi_{\mathcal{O}_-}$ are the subrepresentations of π obtained by restriction to $L^2(\mathcal{O}_+)$ and $L^2(\mathcal{O}_-)$, respectively. For

$\pi_{\mathcal{O}_+}$, the admissibility condition for a wavelet ϕ such that $\hat{\phi} \in L^2(\mathcal{O}_+)$ is

$$\int_0^\infty \int_{\mathbb{R}} \left| \frac{\hat{\phi}(\xi_1, \xi_2)}{\xi_2} \right|^2 d\xi_1 d\xi_2 < \infty$$

and similarly for $\pi_{\mathcal{O}_-}$ (see [3] for more details).

If $y = (y_1, y_2) \in \mathbb{R}^2$, we put $B_y = \left[\begin{smallmatrix} 0 & y_1 \\ y_1 & y_2 \end{smallmatrix}\right]$, a 2×2 symmetric matrix. Then, we check that $H = G_0 G_1$, where

$$G_0 = \left\{ g_0(t, y) = \begin{bmatrix} t^{-1/2} & 0 \\ t^{-1/2} S_y & t^{1/2} \end{bmatrix} : t > 0, y \in \mathbb{R}^2 \right\}$$
$$G_1 = \left\{ g_1(\ell) = \begin{bmatrix} S_\ell & 0 \\ 0 & S_{-\ell} \end{bmatrix} : \ell \in \mathbb{R} \right\}.$$

Indeed, for $y = (y_1, y_2) \in \mathbb{R}^2$, it is straightforward to see that:

$$g_1(\ell) g_0(t, (y_1, y_2)) g_1(\ell)^{-1} = g_0(t, (y_1, y_2 - 2\ell y_1)). \tag{4.5}$$

This means that G_1 normalizes G_0 and hence that $H = G_0 G_1$ is a semidirect product, with product law given by

$$g(t, (y_1, y_2), \ell) g(r, (z_1, z_2), s) = g(tr, (y_1 + t z_1, y_2 + t z_2 - 2\ell t z_1), s + \ell).$$

Since G_0 is normal in H, one has the obvious isomorphism $H/G_0 \simeq G_1$.

The computation of the left Haar measure on H is a straightforward exercise:

$$dh(t, (y_1, y_2), \ell) = \frac{dt}{t^3}\, dy_1\, dy_2\, d\ell. \tag{4.6}$$

In order to compute the metaplectic representation on H, we observe first that the matrix $A_{t,y,\ell}$ in (4.1) can be written as the product of a diagonal matrix $D_{t,\ell}$ and a lower triangular matrix $L_{t,y,\ell}$ as follows

$$A_{t,y,\ell} = D_{t,\ell} L_{t,y,\ell} = \begin{bmatrix} t^{-1/2} S_{\ell/2} & 0 \\ 0 & t^{1/2}\, {}^t S_{-\ell/2} \end{bmatrix} \begin{bmatrix} I & 0 \\ t^{-1}\, {}^t S_{\ell/2} B_y S_{\ell/2} & I \end{bmatrix}.$$

We then use the fact that μ is a representation and formulae (2.3) and (2.4) to obtain that for $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \mu(A_{t,y,\ell}) f(x) &= \mu(D_{t,\ell} L_{t,y,\ell}) f(x) = t^{1/2} (L_{t,y,\ell} f)(t^{1/2} S_{-\ell/2} x) \\ &= t^{1/2} e^{-i\pi \langle {}^t S_{\ell/2} B_y x, S_{-\ell/2} x \rangle} f(t^{1/2} S_{-\ell/2} x) \\ &= t^{1/2} e^{-i\pi \langle B_y x, x \rangle} f(t^{1/2} S_{-\ell/2} x). \end{aligned} \tag{4.7}$$

In the following we denote

$$\mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}, \qquad \dot{\mathbb{R}}^2_+ = \{(x_1, x_2) : x_1 \neq 0,\, x_2 > 0\}.$$

Similarly we define $\mathbb{R}^2_-$ and $\dot{\mathbb{R}}^2_-$. We shall be concerned with the mapping

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2 \qquad x \mapsto \left(x_1 x_2, \frac{x_2^2}{2} \right), \tag{4.8}$$

whose properties are summarized in the following elementary proposition.

Proposition 4.1. *The mapping* (4.8) *defines diffeomorphisms* $\Psi : \dot{\mathbb{R}}^2_\pm \to \dot{\mathbb{R}}^2_+$ *and is such that* $\Psi(-x) = \Psi(x)$. *Further, it satisfies:*

(a) *the Jacobian of* Ψ *at* $x = (x_1, x_2) \in \dot{\mathbb{R}}^2_\pm$ *is* $J_\Psi(x) = x_2^2$*;*
(b) *the Jacobian of* Ψ^{-1} *at* $u = (u_1, u_2) = \Psi(x_1, x_2)$ *is* $J_{\Psi^{-1}}(u) = (2u_2)^{-1} = x_2^{-2}$*;*
(c) $\Psi^{-1}(t^2 S_{2\ell} u) = t S_\ell \Psi^{-1}(u)$ *for every* $t > 0$ *and every* $u \in \dot{\mathbb{R}}^2_\pm$*;*
(d) $\langle B_y x, x \rangle = 2\langle y, \Psi(x) \rangle$ *for every* $x \in \dot{\mathbb{R}}^2_\pm$ *and every* $y \in \mathbb{R}^2$.

The following theorem shows the equivalence between the wavelet representation π and the metaplectic representation μ.

Theorem 4.2. *Let* $u \in \dot{\mathbb{R}}^2_+$*, and extend the map*

$$\mathcal{Q} f(u) = |2u_2|^{-1/2} f(\Psi^{-1}(u_1, u_2))$$

as an even function to $\mathbb{R}^2 \setminus \{x_1 x_2 = 0\}$. *This is an isometry of* $L^2_{\text{even}}(\mathbb{R}^2)$ *onto itself that intertwines the representations* π *and* μ*, that is* $\pi(g) \circ \mathcal{Q} = \mathcal{Q} \circ \mu(g)$ *for every* $g \in H$.

Proof. Let $f \in L^2_{\text{even}}(\mathbb{R}^2)$. Then, by item (b) in Proposition 4.1

$$\begin{aligned}
\|\mathcal{Q} f\|_2^2 &= \int_{\mathbb{R}^2} |\mathcal{Q} f(u)|^2 \, du \\
&= 2 \int_{\mathbb{R}^2_+} \frac{1}{2u_2} \left| f(\Psi^{-1}(u_1, u_2)) \right|^2 \, du_1 du_2 \\
&= 2 \int_{\mathbb{R}^2_+} |f(x_1, x_2)|^2 \, dx_1 dx_2 \\
&= \|f\|_2^2.
\end{aligned}$$

Thus $\mathcal{Q}$ is an isometry. By (4.4) and item (c) in Proposition 4.1

$$\begin{aligned}
\pi(t, y, \ell) (\mathcal{Q} f)(u) &= t e^{-2\pi i \langle y, u \rangle} \, \mathcal{Q} f\,(t S_{-\ell} u) \\
&= \frac{t}{|2t u_2|^{1/2}} e^{-2\pi i \langle y, u \rangle} \, f\left(\Psi^{-1}\left(t S_{-\ell} u\right)\right) \\
&= \frac{t^{1/2}}{|2u_2|^{1/2}} e^{-2\pi i \langle y, u \rangle} \, f\left(t^{1/2} S_{-\ell/2} \Psi^{-1}(u)\right).
\end{aligned}$$

Finally, by (4.8) and item (d) in Proposition 4.1

$$\begin{aligned}\mathcal{Q}\left(\mu(t,y,\ell)f\right)(u) &= \frac{1}{|2u_2|^{1/2}}\left(\mu(t,y,\ell)f\right)\left(\Psi^{-1}(u)\right) \\ &= \frac{t^{1/2}}{|2u_2|^{1/2}} e^{-i\pi\langle B_y\Psi^{-1}(u),\Psi^{-1}(u)\rangle} f\left(t^{1/2}S_{-\ell/2}\Psi^{-1}(u)\right) \\ &= \frac{t^{1/2}}{|2u_2|^{1/2}} e^{-2i\pi\langle y,u\rangle} f\left(t^{1/2}S_{-\ell/2}\Psi^{-1}(u)\right),\end{aligned}$$

as desired. ∎

Theorem 4.3. *The identity*

$$\int_H |\langle f,\mu(h)\phi\rangle|^2\,dh = c_\phi\,\|f\|_2^2, \tag{4.9}$$

holds for every $f \in L^2(\mathbb{R}^2)$ if and only if the function ϕ satisfies the following two admissibility conditions:

$$c_\phi = 4\int_{\mathbb{R}^2_+} |\phi(x)|^2 \frac{dx}{x_2^4} = 4\int_{\mathbb{R}^2_+} |\phi(-x)|^2 \frac{dx}{x_2^4} \tag{4.10}$$

and

$$\int_{\mathbb{R}^2_+} \phi(x)\overline{\phi(-x)}\,\frac{dx}{x_2^4} = 0. \tag{4.11}$$

We need the following introductory equality of Plancherel type.

Lemma 4.4. *Let $h \in L^2(\mathbb{R}^2)$ be a function which vanishes outside some annulus $c < \|x\| < C$, with $0 < c < C < \infty$. Then*

$$\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} h(x)e^{2\pi i\langle y,\Psi(x)\rangle}\,dx\right|^2 dy = \int_{\mathbb{R}^2_+} |h(x)+h(-x)|^2 \frac{dx}{x_2^2}.$$

Proof. By Proposition 4.8, denoting Ψ the mapping $(x_1, x_2) \mapsto (x_1x_2, \frac{1}{2}x_2^2)$ regardless of the domain on which we look at it, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} h(x)e^{2\pi i\langle y,\Psi(x)\rangle}\,dx &= \left(\int_{\dot{\mathbb{R}}^2_+} + \int_{\dot{\mathbb{R}}^2_-}\right) h(x)e^{2\pi i\langle y,\Phi(x)\rangle}\,dx \\
&= \int_{\dot{\mathbb{R}}^2_+} h(x)e^{2\pi i\langle y,\Psi(x)\rangle}\,dx + \int_{\dot{\mathbb{R}}^2_+} h(-x)e^{2\pi i\langle y,\Psi(-x)\rangle}\,dx \\
&= \int_{\dot{\mathbb{R}}^2_+} [h(x)+h(-x)]\,e^{2\pi i\langle y,\Psi(x)\rangle}\,dx \\
&= \int_{\dot{\mathbb{R}}^2_+} \left[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))\right] e^{2\pi i\langle y,u\rangle}\,\frac{du}{2u_2} \\
&= \int_{\mathbb{R}^2} \frac{\chi_+(u)}{2u_2}\left[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))\right] e^{2\pi i\langle y,u\rangle}\,du,
\end{aligned}$$

where χ_+ is the characteristic function of $\dot{\mathbb{R}}^2_+$. By the Plancherel formula we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2}\frac{\chi_+(u)}{2u_2}\left[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))\right] e^{2\pi i\langle y,u\rangle}\,du\right|^2 dy \\
&\quad= \int_{\mathbb{R}^2}\left|\frac{\chi_+(u)}{2u_2}\left[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))\right]\right|^2 du \\
&\quad= \int_{\dot{\mathbb{R}}^2_+}\left|h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))\right|^2\,\frac{du}{4u_2^2} \\
&\quad= \int_{\dot{\mathbb{R}}^2_+}|h(x)+h(-x)|^2\,\frac{dx}{x_2^2},
\end{aligned}$$

as desired. $\square$

Proof of Theorem 4.3. Using (4.6) for the left Haar measure and (4.7) for the metaplectic representation, the left-hand side of (4.9) becomes

$$\begin{aligned}
\int_H |\langle f,\mu(h)\phi\rangle|^2\,dh &= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} f(x)t^{1/2}e^{\pi i\langle B_yx,x\rangle}\overline{\phi(t^{1/2}S_{-\ell/2}x)}\,dx\right|^2 t^{-3}dydtd\ell \\
&= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} f(x)t^{1/2}e^{2\pi i\langle y,\Psi(x)\rangle}\overline{\phi(t^{1/2}S_{-\ell/2}x)}\,dx\right|^2 t^{-3}dydtd\ell, \qquad (4.12)
\end{aligned}$$

where we have used property (d) of Ψ stated in Proposition 4.1.

We take $f \in L^2(\mathbb{R}^2)$ vanishing outside a ring and apply Lemma 4.4 to the right-hand side of (4.12). The computation of $|h(x)+h(-x)|^2$, for the function $h(x) = f(x)t^{1/2}\overline{\phi(t^{1/2}S_{-\ell/2}x)}$ is immediate and we thus obtain

$$\begin{aligned}\int_H |\langle f, \mu(h)\phi\rangle|^2\, dh = \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2} \big\{ &|f(x)|^2 t|\phi(t^{1/2}S_{-\ell/2}x)|^2 \\ &+ |f(-x)|^2 t|\phi(-t^{1/2}S_{-\ell/2}x)|^2 \\ &+ 2\mathcal{R}e\, f(x)\overline{f(-x)}t\phi(-t^{1/2}S_{-\ell/2}x)\overline{\phi(t^{1/2}S_{-\ell/2}x)}\big\} \frac{dx}{x_2^2}t^{-3}dtd\ell.\end{aligned}$$

First, we consider f with $f(x_1, x_2) = 0$ for $x_2 < 0$. We perform the change of variables $(t, \ell) \mapsto y = (y_1, y_2)$, given by $t^{1/2}S_{-\ell/2}x = y$. Hence $dtd\ell = 4x_2^{-2}dy$ and

$$\begin{aligned}\int_H |\langle f, \mu(h)\phi\rangle|^2\, dh &= \int_{\mathbb{R}^2_+} |f(x)|^2 \left(\int_{\mathbb{R}^2_+} |\phi(y)|^2 \frac{4}{y_2^4}\, dy \right) dx \\ &= \|f\|_2^2 \left(\int_{\mathbb{R}^2_+} |\phi(y)|^2 \frac{4}{y_2^4}\, dy \right).\end{aligned}$$

If $f(x_1, x_2) = 0$ for $x_2 > 0$, then arguing in a similar way we obtain

$$\begin{aligned}\int_H |\langle f, \mu(h)\phi\rangle|^2\, dh &= \int_{\mathbb{R}^2_+} |f(x)|^2 \left(\int_{\mathbb{R}^2_+} |\phi(-y)|^2 \frac{4}{y_2^4}\, dy \right) dx \\ &= \|f\|_2^2 \left(\int_{\mathbb{R}^2_+} |\phi(-y)|^2 \frac{4}{y_2^4}\, dy \right).\end{aligned}$$

Therefore, if the window function ϕ fulfills (4.10), then (4.9) holds for every f which vanishes either for $x_2 < 0$ or for $x_2 > 0$. Take now a bounded function f with support in some annulus $c < \|x\| < C$. Then

$$2\mathcal{R}e\, f(x)\overline{f(-x)}t\phi(-t^{1/2}S_{-\ell/2}x)\overline{\phi(t^{1/2}S_{-\ell/2}x)}\frac{1}{x_2^2}$$

is integrable with respect to the measure $dx_1dx_2t^{-3}dtd\ell$ and its integral is easily seen to vanish. Finally,

$$\begin{aligned}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} & 2\mathcal{R}e\, f(x)\overline{f(-x)}t\phi(-t^{1/2}S_{-\ell/2}x)\overline{\phi(t^{1/2}S_{-\ell/2}x)}\frac{1}{x_2^2}dx_1dx_2t^{-3}dtd\ell \\ &= 4\int_{\mathbb{R}^2_+} f(x)\overline{f(-x)}dx \int_{\mathbb{R}^2_+} \overline{\phi(y)}\phi(-y)\frac{dy}{y_2^4},\end{aligned}$$

and (4.11) follows.

Conversely, if conditions (4.10) and (4.11) are satisfied and if f is a function as in the assumptions of Lemma 4.4, then all of the terms of (4.12) are integrable and the reproducing formula (4.9) is true for all

$f \in L^2(\mathbb{R}^2)$. To see this, take $f \in L^2(\mathbb{R}^2)$ and let f_n be a sequence of functions as in Lemma 4.4 which tends to f in the L^2-norm. The sequence $F(f_n) = \langle f_n, \mu(h)\phi\rangle$ is a Cauchy sequence in $L^2(H, dh)$ which tends pointwise to $F(f) = \langle f, \mu(h)\phi\rangle$. Since (4.9) holds for all f_n, it follows that it also holds for f. □

Example. We finish by giving an example of admissible wavelets. Take a Meyer wavelet ψ (see, e.g., [11]). It is the Schwartz function defined by

$$\hat{\psi}(x) = e^{-ix/2} \sin(\omega(x))$$

where $\omega \in C_c^\infty(\mathbb{R})$ is an even function with support in $\{x : \frac{2\pi}{3} < |x| < \frac{8\pi}{3}\}$. The function $g(x) = \hat{\psi}(x)$ satisfies:

$$c_g := \int_0^{+\infty} \frac{|g(x)|^2}{x^4}\, dx = \int_0^{+\infty} \frac{|g(-x)|^2}{x^4}\, dx < +\infty.$$

Next we consider an asymmetric step function f and observe that

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |f(-x)|^2 = 2.$$

The admissible wavelet ϕ is defined by:

$$\phi(x_1, x_2) = f(x_1)g(x_2)$$

It follows that:

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_0^{+\infty} |\phi(x)|^2\, \frac{dx}{x_2^4} &= \int_{-\infty}^{+\infty}\int_0^{+\infty} |f(x_1)g(x_2)|^2\, \frac{dx_2}{x_2^4} dx_1 \\
&= \left(\int_0^{+\infty} \frac{|g(x_2)|^2}{x_2^4}\, dx_2\right)\left(\int_{-\infty}^{+\infty} |f(x_1)|^2\, dx_1\right) \\
&= c_g \cdot 2
\end{aligned}$$

and similarly for the integral in $-x$. Finally

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^{+\infty} \phi(x)\overline{\phi(-x)}\, \frac{dx}{x_2^4} \\
&\qquad = \int_{-\infty}^{+\infty}\int_0^{+\infty} f(x_1)g(x_2)\overline{f(-x_1)g(-x_2)}\, \frac{dx_2}{x_2^4} dx_1 \\
&\qquad = \left(\int_0^{+\infty} \frac{g(x_2)\overline{g(-x_2)}}{x_2^4}\, dx_2\right)\left(\int_{-\infty}^{+\infty} f(x_1)f(-x_1)\, dx_1\right)
\end{aligned}$$

and the latter integral vanishes due to the asymmetry of f.

References

[1] S. T. Ali, J. P . Antoine, J. P .Gazeau, *Coherent states, wavelets and their generalizations*, Springer-Verlag, New York, 2000.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[3] D. Bernier and K. F. Taylor. Wavelets from square-integrable representations. *SIAM J. Math. Anal.*, 27(2):594–608, 1996.

[4] E. Cordero, F. De Mari, K. Nowak and A. Tabacco *Analytic features of reproducing groups for the metaplectic representation*, preprint, 2005.

[5] F. De Mari and K. Nowak. Analysis of the affine transformations of the time-frequency plane. *Bull. Austral. Math. Soc.*, 63(2):195–218, 2001.

[6] J. Dixmier. *Les C^*-Algèbres et leurs représentations*, Gauthier-Villars Éditeur, Paris, 1969.

[7] M. N. Do and M. Vetterli. New tight frames of curvelets and optimal representations of objects with piecewise $C\,\mathrm{span}\,2$ singularities. *IEEE Trans. Signal Process.*, to appear.

[8] G. B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, 1989.

[9] K. Gröchenig. *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[10] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978.

[11] E. Hernández, G. L. Weiss. *A First Course on Wavelets*, CRC Press, Boca Raton, 1996.

[12] A. W. Knapp, *Lie Groups Beyond an Introduction*, Secon edition, Birkhäuser, Boston, 2002.

[13] R. S. Laugesen, N. Weaver, G. L. Weiss, and E. N. Wilson. A characterization of the higher dimensional groups associated with continuous wavelets. *J. Geom. Anal.*, 12(1):89–102, 2002.

[14] J. Williamson, On an algebraic problem, concerning the normal forms of linear dynamical systems, *Amer. J. of Math.* 58 (1): 141–163, 1936.

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy
E-mail address: cordero@dm.unito.it

DIPTEM, Piazzale J. F. Kennedy, Pad. D., 16129 Genova, Italy
E-mail address: demari@dima.unige.it

Department of Mathematics and Computer Science, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104-2875
E-mail address: nowak.mcs.drexel.edu

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy
E-mail address: anita.tabacco@polito.it