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REPRODUCING KERNELS FOR q-JACOBI POLYNOMIALS[1]

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ABSTRACT. We derive a family of reproducing kernels for the q-Jacobi polynomials $\Phi_n^{(\alpha,\beta)}(x) = {}_2\Phi_1(q^{-n}, q^{n-1+\beta}; q^\alpha; q, qx)$. This is achieved by proving that the polynomials $\Phi_n^{(\alpha,\beta)}(x)$ satisfy a discrete Fredholm integral equation of the second kind with a positive symmetric kernel, then applying Mercer's theorem.

1. **Introduction.** The purpose of the present note is to construct a family of reproducing kernels or bilinear formulas

$$\sum_{n=0}^{\infty} \theta_n^{(j)} \Phi_n^{(\alpha,\beta)}(x) \Phi_n^{(\alpha,\beta)}(y) = K^{(j)}(x, y) \tag{1.1}$$

for the q-Jacobi polynomials $\{\Phi_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$. For definitions and notations, see §2. These reproducing kernels are obtained by finding a linear integral operator that maps a q-Jacobi polynomial to another q-Jacobi polynomial of the same degree but with different parameters. This integral operator is the q-fractional integral $L^{(\alpha,\eta)}$ of (2.9). The results obtained below are q-analogues of Ismail's results in **[5]**. Actually Al-Salam and Ismail **[1]** used certain discrete transforms, that map the Charlier and Meixner polynomials to Laguerre polynomials, to derive several bilinear formulas for the Charlier and Meixner polynomials. Later Ismail **[5]** modified these transforms and obtained similar formulas for the Hahn polynomials. Related results were also obtained by Rahman **[6]**, **[7]** by using a completely different approach.

In the next section we define the q-Jacobi polynomials and a q-fractional integral operator. §3 contains our main results, the family of bilinear formulas (3.13). The last section, §4, is devoted to proving the square integrability of the kernel $K(q^r, q^s; \alpha, \beta, \eta; q)$ of (3.6) with respect to the measure $\mu(x, y)$ of (3.9) and (3.10).

For an excellent survey of the theory of reproducing kernels we refer the interested reader to Hille **[4]**.

2. **Preliminaries.** Throughout this work we shall always assume that $0 < q < 1$. The symbol $(a; q)_\infty$ shall stand for the convergent infinite product

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$\Pi_0^\infty(1 - aq^n)$. By $(a; q)_\alpha$, or equivalently $[1 - a]_\alpha$ we mean

$$(a; q)_\alpha = [1 - a]_\alpha = (a; q)_\infty / (aq^\alpha; q)_\infty,$$

so that, in particular, we have $(a; q)_0 = 1$, $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ for $n = 1, 2, 3, \cdots$.

The generalized basic or q-hypergeometric function ${}_r\Phi_s$ is defined by

$$\begin{aligned} &{}_r\Phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) \\ &= {}_r\Phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r; & \\ & z \\ b_1, b_2, \ldots, b_s; & \end{matrix}\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(b_1; q)_n (b_2; q)_n \cdots (b_s; q)_n} \frac{z^n}{(q; q)_n}. \end{aligned} \tag{2.1}$$

In particular, we have the q-analogue of the binomial theorem **[8**, p. 92]

$$ {}_1\Phi_0(a; -; q, z) = (az; q)_\infty / (z; q)_\infty \tag{2.2}$$

valid for $|z| < 1$.

Let us recall the q-integral

$$\int_0^x f(t)\, d(t; q) = x(1 - q) \sum_{n=0}^{\infty} q^n f(xq^n). \tag{2.3}$$

This note is concerned with what we call q-Jacobi polynomials as defined by

$$\Phi_n^{(\alpha,\beta)}(x) = {}_2\Phi_1(q^{-n}, q^{n+\beta-1}; q^\alpha; q; qx). \tag{2.4}$$

However we should mention here that Andrews and Askey **[2]** studied a more general set of polynomials, namely, ${}_3\Phi_2(q^{-n}, q^{n+\alpha+\beta+1}, x; q^{\alpha+1}, -q^{d+1}; q, q)$, which they also refer to as the q-Jacobi polynomial. They refer to (2.4) as the ${}_2\Phi_1$-Jacobi polynomials.

The q-Jacobi polynomials satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \Phi_n^{(\alpha,\beta)}(x) \Phi_m^{(\alpha,\beta)}(x)\, d\Psi(x; \alpha, \beta) = F_n(\alpha,\beta)\delta_{nm} \tag{2.5}$$

where

$$F_n(\alpha, \beta) = q^{\alpha n} \frac{(q; q)_n (q^{\beta-\alpha}; q)_n (q^{n+\beta-1}; q)_n}{(q^\alpha; q)_n (q^\beta; q)_{2n}} \tag{2.6}$$

and $\Psi(x; \alpha, \beta)$ is the step function with the jumps

$$d\Psi(x; \alpha, \beta) = \frac{(q^\alpha; q)_\infty (q^{\beta-\alpha}; q)_k}{(q^\beta; q)_\infty (q; q)_k} q^{\alpha k} \tag{2.7}$$

at $x = q^k$ $(k = 0, 1, 2, 3, \ldots)$, $\beta > \alpha > 0$.

(2.5) may also be written as a q-integral

$$\int_0^1 [1-qt]_{\beta-\alpha-1} t^{\alpha-1}\Phi_n^{(\alpha,\beta)}(t)\Phi_m^{(\alpha,\beta)}(t)\,d(t;q) \tag{2.8}$$
$$= \frac{(q;q)_\infty (q^\beta;q)_\infty}{(q^{\beta-\alpha};q)_\infty (q^\alpha;q)_\infty} F_n(\alpha,\beta)\delta_{nm}$$

We shall also make use of the fractional operator $\mathfrak{L}^{(\alpha,\eta)}$ for $\eta > 0$,

$$\mathfrak{L}^{(\alpha,\eta)} f(x) = \int_0^1 f(xt)\,d\Psi(t;\alpha,\eta+\alpha) \tag{2.9}$$
$$= \frac{(q^\alpha;q)_\infty (q^\eta;q)_\infty}{(q^{\alpha+\eta};q)_\infty (q;q)_\infty (1-q)} \cdot \int_0^1 [1-qt]_{\eta-1} t^{\alpha-1} f(xt)\,d(t;q).$$

In particular, using (2.2), we can show that

$$\mathfrak{L}^{(\alpha,\eta)}\{x^n\} = \frac{(q^\alpha;q)_n}{(q^{\alpha+\eta};q)_n} x^n \qquad (n = 0, 1, 2, 3, \cdots) \tag{2.10}$$

so that

$$\mathfrak{L}^{(\alpha,\eta)}\Phi_n^{(\alpha,\beta)}(x) = \Phi_n^{(\alpha+\eta,\beta)}(x). \tag{2.11}$$

More explicitly, (2.11) can also be written as

$$\sum_{k=0}^\infty q^{\alpha k} \frac{(q^\alpha;q)_\infty (q^\eta;q)_k}{(q^{\eta+\alpha};q)_\infty (q;q)_k} \Phi_n^{(\alpha,\beta)}(xq^k) = \Phi_n^{(\alpha+\eta,\beta)}(x). \tag{2.12}$$

3. **Reproducing kernels.** (2.11) shows that the q-fractional integral operator $\mathfrak{L}^{(\alpha,\beta)}$ maps a q-Jacobi polynomial to a q-Jacobi polynomial. Hence we have

$$F_n(\alpha+\eta,\beta)\delta_{nm} \tag{3.1}$$
$$= \int_{-\infty}^\infty \{\mathfrak{L}^{(\alpha,\eta)}\Phi_n^{(\alpha,\beta)}(x)\}\{\mathfrak{L}^{(\alpha,\eta)}\Phi_n^{(\alpha,\beta)}(x)\}\,d\Psi(x;\alpha+\eta,\beta).$$

Substituting from (2.12) we get

$$F_n(\alpha+\eta;\beta)\delta_{mn} = \left[\frac{(q^\alpha;q)_\infty}{(q^{\alpha+\eta};q)_\infty}\right]^2 \int_{-\infty}^\infty \sum_{k,j=0}^\infty \frac{(q^\eta;q)_j (q^\eta;q)_k}{(q;q)_j (q;q)_k} \tag{3.2}$$
$$\cdot\, q^{\alpha(k+j)}\Phi_n^{(\alpha,\beta)}(xq^k)\Phi_m^{(\alpha,\beta)}(xq^j)\,d\Psi(x;\alpha+\eta,\beta).$$

Substituting for $d\Psi(x;\ \alpha+\eta,\beta)$ in (3.2) from (2.7) we get, after some simplification,

$$F_n(\alpha+\eta;\beta)\delta_{mn} = (q^\alpha;q)_\infty (q^\alpha;q)_\infty / (q^\beta;q)_\infty (q^{\alpha+\eta};q)_\infty \tag{3.3}$$
$$\cdot \sum_{r,s=0}^\infty \Phi_n^{(\alpha,\beta)}(q^r)\Phi_m^{(\alpha,\beta)}(q^s)$$
$$\cdot\, q^{\alpha(r+s)} \frac{(q^\eta;q)_r (q^\eta;q)_s}{(q;q)_r (q;q)_s}$$
$$\cdot\, {}_3\Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta};\ q^{1-\eta-r}, q^{1-\eta-s};\ q^{2-\eta-\alpha}).$$

Now using (2.6) and (3.3) we obtain

$$F_n(\alpha;\beta)\delta_{mn} = \frac{(q^\alpha;q)_\infty (q^{\beta-\alpha};q)_n (q^{\alpha+\eta};q)_n}{(q^{\alpha+\eta};q)_\infty (q^{\beta-\alpha-\eta};q)_n (q^\alpha;q)_n}$$

$$(3.4)\qquad \cdot \sum_{r=0}^{\infty} \Phi_n^{(\alpha,\beta)}(q^r) \frac{(q^\eta;q)_r}{(q^{\beta-\alpha};q)_r} \sum_{s=0}^{\infty} \Phi_m^{(\alpha,\beta)}(q^s) \frac{(q^\eta;q)_s}{(q;q)_s} q^{\alpha s}$$
$$\cdot {}_3\Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta}; q^{1-\eta-r}, q^{1-\eta-s}; q, q^{2-\eta-\alpha}) d\Psi(q^r;\alpha,\beta).$$

Combining (3.4) and (2.5) with the uniqueness of the orthogonal polynomials we get

$$\Phi_n^{(\alpha,\beta)}(q^r) = \frac{(q^\alpha;q)_\infty (q^{\beta-\alpha};q)_n (q^{\alpha+\eta};q)_n}{(q^{\alpha+\eta};q)_\infty (q^{\beta-\alpha-\eta};q)_n (q^\alpha;q)_n} \frac{(q^\eta,q)_r}{(q^{\beta-\alpha};q)_r} q^{-m\eta}$$

$$(3.5)\qquad \left\{ \sum_{s=0}^{\infty} \Phi_n^{(\alpha,\beta)}(q^s) q^{\alpha s} \frac{(q^\eta;q)_s}{(q;q)_s} \right.$$
$$\left. \cdot {}_3\Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta}; q^{1-\eta-r}, q^{1-\eta-s}; q, q^{2-\eta-\alpha}) \right\}.$$

Clearly (3.5) is an integral equation satisfied by $\Phi_n^{(\alpha,\beta)}(x)$. The kernel in (3.5) is not symmetric but can be symmetrized by rewriting (3.5) as

$$q^{\alpha r/2} \left\{ (q^{\beta-\alpha};q)_r (q^\alpha;q)_\infty / (q;q)_r (q^\beta;q)_\infty \right\}^{1/2} \Phi_n^{(\alpha,\beta)}(q^r)$$
$$= \lambda_n(\alpha,\beta,\eta,q) \sum_{s=0}^{\infty} q^{\alpha s/2} \left\{ \frac{(q^{\beta-\alpha};q)_s (q^\alpha;q)_\infty}{(q;q)_s (q^\beta,q)_\infty} \right\}^{1/2}$$
$$\Phi_n^{(\alpha,\beta)}(q^s) K(q^s, q^r; \alpha, \beta, \eta, q).$$

where $K(q^s, q^r; \alpha, \beta, \eta, q)$ is the symmetric kernel

$$K(q^s, q^r; \alpha, \beta, \eta; q) = q^{(r+s)\alpha/2} (q^\eta;q)_r (q^\eta;q)_s$$
$$(3.6)\qquad \cdot \left\{ (q^{\beta-\alpha};q)_r (q^{\beta-\alpha};q)_s (q;q)_r (q;q)_s \right\}^{-1/2}$$
$$\cdot {}_3\Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta}; q^{1-\eta-r}, q^{1-\eta-s}; q, q^{2-\eta-\alpha}),$$

and λ_n are the eigenvalues

$$(3.7)\qquad \lambda_n(\alpha,\beta,\eta;q) = \frac{(q^{\alpha+\eta};q)_\infty (q^{\beta-\alpha};q)_n q^{-m\eta}}{(q^{\alpha+\eta+n};q)_\infty (q^{\beta-\alpha-\eta};q)_n} \qquad (n = 0, 1, 2, \ldots).$$

The completeness of the system of orthogonal polynomials $\{\Phi_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ follows from Weierstrass' approximation theorem. Therefore, the eigenvalues given by (3.7) are all the eigenvalues of the integral equation

$$(3.8)\qquad f(q^r) = \lambda \sum_{s=0}^{\infty} K(q^r, q^s; \alpha, \beta, \eta; q) f(q^s), \qquad r = 0, 1, 2, \ldots.$$

The eigenvalues $\lambda_n(\alpha,\beta,\eta;q)$ are positive and increasing to ∞ for $\alpha > 0$,

$\beta > \alpha + \eta$ and $\eta > 0$, and the symmetric kernel $K(q^r, q^s;\ \alpha, \beta, \eta;\ q)$ belongs to $L^2(d\mu)$ where

$$\mu(x, y) = \nu(x)\nu(y), \tag{3.9}$$

and

$$d\nu(x) = \begin{cases} 1 & \text{if } x = q^k, k = 0, 1, \ldots, \\ 0 & \text{otherwise;} \end{cases} \tag{3.10}$$

see §4. Now we are in a position to apply the extension of Mercer's theorem[2] (see Tricomi [9, p. 125]) to the discrete integral equation (3.8) and derive the bilinear formula

$$\sum_{n=0}^{\infty} \frac{f_n(q^r; \alpha, \beta) f_n(q^s; \alpha, \beta)}{\lambda_n(\alpha, \beta, \eta; q)} = K(q^r, q^s; \alpha, \beta, \eta; q) \tag{3.11}$$

with

$$f_n(q^r; \alpha, \beta) = q^{\alpha r/2} \left\{ \frac{(q^{\beta-\alpha}; q)_r (q^{\alpha}; q)_{\infty}}{(q; q)_r (q^{\beta}; q)_{\infty}} \right\}^{1/2} \Phi_n^{(\alpha,\beta)}(q^r). \tag{3.12}$$

The reproducing kernel (3.6) can be iterated to give

$$\begin{aligned} &\sum_{n=0}^{\infty} \{\lambda_n(\alpha, \beta, \eta; q)\}^{-j} f_n(q^r; \alpha, \beta) f_n(q^s; \alpha, \beta) \\ &\qquad = K^{(j)}(q^r, q^s; \alpha, \beta, \eta; q), \qquad j = 1, 2, \ldots, \end{aligned} \tag{3.13}$$

and the iterated kernels $K^{(j)}$ are defined inductively by

$$K^{(1)}(q^r, q^w; \alpha, \beta, \eta; q) = K(q^r, q^s; \alpha, \beta; \eta; q), \tag{3.14}$$

$$\begin{aligned} &K^{(j+1)}(q^r, q^s; \alpha, \beta, \eta; q) \\ &\qquad = \sum_{l=0}^{\infty} K^{(j)}(q^r, q^l; \alpha, \beta, \eta; q) K(q^l, q^s; \alpha, \beta, \eta; q). \end{aligned} \tag{3.15}$$

One can easily take special cases of the above bilinear formulas and derive similar formulas for the q-Laguerre and q-Legendre polynomials.

4. **The square integrability of** $K(q^r, q^s;\ \alpha, \beta, \eta;\ q)$. In this section we shall prove that the kernel $K(q^r, q^s;\ \alpha, \beta, \eta;\ q)$ of (3.6) belongs to $L^2(d\mu)$, μ being the measure defined by (3.4) and (3.10). Let $\|K\|_\mu$ be the $L^2(d\mu)$ norm of the above mentioned kernel. We now prove that $\|K\|_\mu$ is finite.

From (3.5) we have, for $\alpha > 0$, $\beta > \alpha + \eta$, $\eta > 0$,

$$\|K\|_\mu^2 = \sum_{r,s=0}^{\infty} \frac{(q^{\eta}; q)_r (q^{\eta}; q)_s q^{\alpha(r+s)}}{(q^{\beta-\alpha}; q)_r (q^{\beta-\alpha}; q)_s (q; q)_r (q; q)_s} \cdot \left\{ {}_3\Phi_2(q^{-r}, q^{-s}, q^{\beta-\alpha-\eta}; q^{1-\eta-r}, q^{1-\eta-s}; q, q^{2-\eta-\alpha}) \right\}^2.$$

[2]The extension is straightforward and we believe it is known.

Applying the Cauchy-Schwarz inequality on the $\{{}_3\Phi_2\}^2$ we get

$$\|K\|_\mu^2 \leqslant \sum_{r,s=0}^{\infty} \frac{(q^\eta; q)_r^2 (q^\eta; q)_s^2 q^{\alpha(r+s)}}{(q^{\beta-\alpha}; q)_r (q^{\beta-\alpha}; q)_s (q; q)_r (q; q)_s}$$

$$\cdot \sum_k rs \frac{(q^{-r}; q)_k^2 (q^{-s}; q)_k^2 (q^{\beta-\alpha-\eta}; q)_k^2}{(q; q)_k^2 (q^{1-\eta-r}; q)_k^2 (q^{1-\eta-s}; q)_k^2} q^{2(2-\eta-\alpha)k}.$$

Interchanging summations and using the identity

$$(q^{a-r}; q)_k = (-1)^k q^{k(a-r)+k(k-1)/2} \frac{(q^{1-a}; q)_r}{(q^{1-a}; q)_{r-k}},$$

we get

$$\|K\|_\mu^2 \leqslant \sum_{k=0}^{\infty} \frac{(q^{\beta-\alpha-\eta}; q)_k^2 q^{2\eta k}}{(q^{\beta-\alpha}; q)_k} \left\{ \sum_r \frac{(r+k) q^{\alpha r} (q^{1+r}; q)_r (q^\eta; q)_r^2}{(q^{\beta-\alpha+k}; q)_r (q; q)_r^2} \right\}^2.$$

The inside summation on the right-hand side of the above inequality behaves for large values of k as $(A + Bk)^2$ where A and B are constants. Hence the right-hand side is convergent (by the ratio test) under the stated conditions. This proves our assertion.

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