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GOLDEN OLDIE EDITORIAL

Editorial note to: Roger Penrose, Conformal treatment of infinity

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In the late 1950's F. A. E. Pirani, A. Trautman, R. K. Sachs, H. Bondi, E. T. Newman, R. Penrose and others began exploring possibilities to establish in Einstein's non-linear theory a notion of gravitational radiation which is covariant and not based on approximations. The proposal they finally came up with relies, however, on an idealization. The solutions to Einstein's field equations were assumed to be *asymptotically flat at null infinity* in the sense that they admit distinguished coordinate systems which include a null coordinate whose level hypersurfaces open up in the future so that the generating null geodesics are future complete and the curvature decays to zero in the infinite future. Apart from subtle questions about the precise decay conditions, there was a general agreement that one had arrived at the right concept. A survey of this development and of the results supporting this view is given by R. K. Sachs in the same volume which contains the article reprinted here [1].

Shortly after this development Roger Penrose put forward a remarkable idea. Seeking a formulation that emphasizes the role of the null cone or, equivalently, the conformal structure, which are the basic geometric elements underlying the earlier considerations, he proposes to characterize the required fall-off behaviour of the fields at null infinity by the condition that the conformal structure of the space–time admits an extension across null infinity of a certain smoothness. This provides a *geometric notion*

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of asymptotic flatness at null infinity which allows null infinity to be represented by a *conformal boundary*, a hypersurface of some smoothness in the extended space–time.

An outline of the new idea with applications to the radiation problem was first given in the article [2]. It is further expounded in the article reprinted here (‘the article’), in which the author also shows that various known solutions to Einstein’s field equations admit such extensions and indicates that the conformal techniques are of a much wider applicability than considered in [2], including solutions with cosmological constant $\lambda \neq 0$ and, going beyond the theme indicated in the title, certain types of big bang singularities. Various topics of this article are then extended and discussed with detailed arguments in the article [3].

Of all the achievements of this work the most conspicuous one is its bird’s-eye view on the global causal structure of space–time. This is exemplified clearly by the discussion of the dependence of the causal properties of the conformal boundary on the sign of the cosmological constant λ and its relations to the existence of particle and event horizons, notions which gave rise to much confusion in the early days of relativity. All this is today’s standard but at the time it must have come as a revelation.

The article provides, of course, also a new view on the radiation problem. Concepts and formulae expressed in terms of complicated limits in the earlier analysis simplify considerably in the new setting. Incoming and outgoing gravitational radiation fields, total energy–momentum and its loss by radiation, and the Bondi–Metzner–Sachs asymptotic symmetry group are discussed on the conformal boundary in terms of local differential geometry. The ‘peeling off’ property of the conformal Weyl tensor at null infinity, emphasized earlier by Sachs, is obtained as a consequence of the assumptions. This is an important and subtle point which is intimately related to the precise smoothness requirement on the conformal extension.

As a consequence of this elegant, geometric and concrete treatment of the radiation problem there developed in the following years a large body of literature which discusses in detail diverse aspects of the new setting and the concepts coming along with it, which studies the setting or weakended versions of it under symmetry assumptions, and which analyses for various explicit solutions to what an extent they admit the required conformal extension. References to a large part of this work can be found in [4] and [5].

As stressed in the article, analysing the asymptotic behaviour of fields that satisfy on a given space–time conformally covariant equations is simplified immensely if the space–time admits a conformal extension of sufficient smoothness. In this case the assumptions on the space–time, on the equation governing the fields, and on the initial data for the fields on suitable hypersurfaces are independent of each other and can be arranged as convenient. If the space–time is supposed to satisfy Einstein’s field equations, the situation is, however, much more subtle and there remains a problem.

It is emphasized in the article that the Bianchi equation, satisfied by the conformal Weyl tensor in the vacuum case for any sign of λ, does show a certain conformal covariance while the Einstein equations do not possess this feature. Moreover, in the case of Cauchy data which are *asymptotically flat in the sense of the standard Cauchy problem* the conformal scaling behaviour of the various fields and the structure of the initial data at space-like infinity are tied up with each other in a delicate way. A priori, it is unclear how the notion of asymptotic flatness of Cauchy data at space-like infinity relates to the geometric notion of asymptotic flatness at null infinity inherent in Penrose's conditions. This relation is mediated by the field equations and its understanding requires global control on the solutions and, on the top of it, precise results of their asymptotic behaviour.

Today quite a few general results on the asymptotic behaviour of solutions are available. In particular, we know now that there exist large classes of vacuum solutions which arise from asymptotically flat data (in the Cauchy sense) or from data on compact initial slices with cosmological constant $\lambda > 0$ (assuming $sign(\mathrm{g}) = (-, +, +, +)$) which develop into solutions satisfying Penrose's conditions [6], [7], [8]. There exist even more such solutions if condition (c) of Lecture II is dropped, including solutions with $\lambda < 0$ [9]. It is also known, however, that there exist large sets of asymptotically flat data which develop into solutions that still admit a notion of radiation field but do not show Sachs' peeling behaviour and thus cannot admit conformal extensions at null infinity with the smoothness as required in the article [10], [11].

It is not understood yet which data exactly evolve into solutions that admit geometric asymptotic flatness at null infinity [11]. Obtaining a precise characterization of this subset in the set of 'all' asymptotically flat Cauchy data is of considerable interest. The analysis will provide detailed insights into the evolution process defined by Einstein's field equations and it will yield sharp results on the asymptotic behaviour of the fields. When this problem is understood we can decide whether the geometric notion of asymptotic flatness allows us to model all physical processes of interest in the radiation problem or whether it is too restrictive and requires subtle generalizations.

Whatever the final answer will be, the pictures and methods put forward in the article changed our views on space–time in a fundamental way and they have been and will remain to be of importance for our understanding and interpretation of their global structure and their asymptotic properties.

A brief biography of Sir Roger Penrose was printed together with another Golden Oldie in Gen. Relativ. Gravit. **34**, 1135 (2002), doi:10.1023/A:1016534604511.

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