Reputation for Quality*

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Abstract

We propose a new model of firm reputation that interprets reputation directly as the market belief about product quality. Quality is persistent and is determined endogenously by the firm's past investments. We analyse how investment incentives depend on the firm's reputation and derive implications for reputational dynamics.

We consider three types of consumer learning. When consumers learn about quality through good news, investment incentives are decreasing in reputation, leading to a unique work-shirk equilibrium and convergent dynamics. When consumers learn through bad news, investment incentives are increasing in reputation, leading to a continuum of shirk-work equilibria and divergent dynamics. Finally, when consumers learn through Brownian news and the cost of investment is low, incentives are hump-shaped but a work-shirk equilibrium exists and is essentially unique.

1 Introduction

In most industries firms can invest into the quality of their products through human capital investment, research and development, and organisational change. While imperfect monitoring by customers gives rise to a moral hazard problem, the firm can share in the created value by building a reputation for quality, justifying premium prices. This paper analyses the incentives for investment in such a market, characterising how these incentives are determined by the current reputation of the firm and the market information structure.

Our key innovation is to model reputation directly as the market belief about the firm's endogenous product quality. As quality is determined by past investments, it is persistent and can

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serve as a Markovian state variable. This is in contrast to repeated games models, which do not have a state variable, and reputation models in which the state variable is exogenous. As a consequence, reputational dynamics in our model are endogenously driven by reputational incentives, rather than trailing exogenous shocks.

The model captures key features of many important industries. In labour markets such as those for academics, artists and advertising executives, agents spend much of their time investing in skills and perfecting their trade. Their reputation and future compensation, however, depends heavily on their best paper, performance or campaign. In the computer industry, component manufacturers invest heavily into research and development while customers are only able to observe the performance of the entire computer. Customers therefore often learn about the quality of the product through newsworthy incidences, such as Dell's 2006 recall of 4 million Sony lithium-ion batteries.[1] In the car industry, firms devote considerable resources to improving quality standards through organisational change and new production processes. Since these investments are not observable, customers only learn about the true quality slowly, through consumer reports and the media.[2]

In the model, illustrated in figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a function of the firm's past investments. The quality then determines future prices through imperfect market learning: a high quality product generally leads to a higher consumer utility than a low quality product, but learning is obstructed by noise. At each point in time, consumers' willingness to pay is determined by the market belief that the quality is high, x_t, which we call the *reputation* of the firm. This reputation changes over time as a function of (a) the equilibrium beliefs of the firm's investments, and (b) market learning about the product quality. Our model nests three types of market information structures that have received attention in the literature on imperfect monitoring:

1. In the *good news* case the product usually generates constant utility. However, at random times a high-quality product enjoys a breakthrough, revealing its high quality. Such good news may occur in academia when a paper becomes famous, in the bio-tech industry when a trial succeeds, and for actors when they win an Oscar.

2. In the *bad news* case the product usually generates constant utility. However, at random times the low quality product suffers a breakdown, revealing its low quality. Such bad news may occur in the computer industry when batteries explode, for borrowers when the default on a loan, and for doctors when they are sued for medical malpractice.

3. In the *Brownian news* case a high-quality product generates a higher mean utility than a low-quality product, but customers learn slowly because of a normally distributed random

[1] See "Dell to Recall 4m Laptop Batteries", Financial Times, 15th August 2006.
[2] See "Detroit Carmakers on a Journey to Recover Reputation", Financial Times, 24th December 2008.

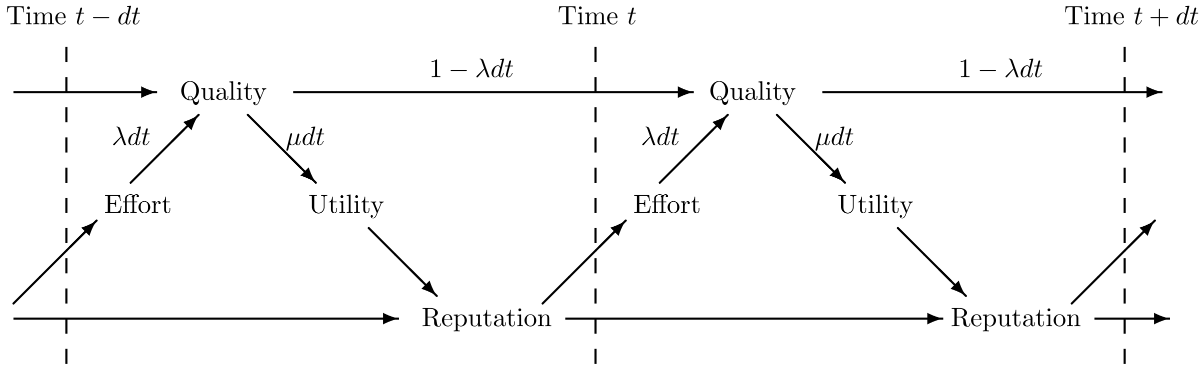

Figure 1: **Timeline.**

error. As a result, beliefs changes continuously over time. Such continuous updating occurs as drivers learn about the build-quality of a car, as clients learn about the skills of a consultancy, and as callers learn about the customer service of a telephone service provider.

In a Markovian equilibrium the firm's value is a function of its product quality and its reputation. As illustrated in figure 1, both quality and reputation move slowly and therefore can be interpreted as assets, which the firm builds up at times, and which it depletes at other times. Reputation is valuable because it determines the firm's revenue. Quality in turn is valuable because a high quality product yields higher expected utility to customers, increasing the firm's future reputation. Crucially, as quality is persistent, this reputational payoff does not take the form of an immediate one-off reputational boost but it accrues to the firm as a stream of future *reputational dividends.* Theorem 1 formalizes this idea by writing the asset value of quality, i.e. the difference in the value between a high quality firm and a low quality firm, as the net present value of its future reputational dividends. This formula is important because it is precisely this value of quality which determines the investment incentives of the firm.

To analyse these incentives further, we have to take a stance on the market information structure and do so by focusing on the three cases above. These cases are analytically tractable and we discuss in Section 7.1 how their insights qualitatively carry over to other market information structures.

In the *good news case* (Section 4) equilibria are *work-shirk*: The firm works if and only if its reputation lies below a cutoff x^*. Intuitively, the reputational dividend consists of the possibility of a product breakthrough, revealing the firm's high quality and boosting its reputation to 1. Since the benefit of such a reputational boost decreases in the firm's reputation, so do investment incentives. The form of the equilibrium implies that reputational dynamics converge to a cycle: A firm with low reputation works, eventually jumps to reputation 1 where it starts shirking; the firm's reputation then drifts down until it hits the cutoff and starts working again.

The *bad news case* (Section 5) is in many ways the opposite to the good news case. Equilibria are *shirk-work*: The firm works if and only if its reputation lies above a cutoff x^*. Intuitively, the reputational dividend is insurance against a product breakdown, revealing the firm's low quality and destroying its reputation. Since the benefit of such insurance increases in the firm's reputation, so do investment incentives. The form of the equilibrium implies that reputational dynamics diverge: A firm with reputation below the cutoff shirks forever, causing its reputation to fall to 0; a firm with reputation above the cutoff works forever, causing its reputation to approach 1.

Our analysis of the *Brownian news case* (Section 6) indicates that the good news results are more robust than those for bad news. When effort is sufficiently cheap, equilibrium is essentially unique and work-shirk. In contrast, there is never a shirk-work equilibrium. This asymmetry hinges on the reputational drift due to equilibrium beliefs. When $x \approx 0$ and $x \approx 1$, market learning is slow and the reputational dividend is small. At the top, work is not sustainable: if the firm is believed to working, the firm's reputation stays high and the reputational dividend stays small, undermining the incentive to invest. At the bottom, work may be sustainable: if the firm is believed to work, the firm's reputation drifts up and reputational dividends increase, sustaining the incentive to invest. Crucially, a firm exerts effort at $x \approx 0$ not because of current reputational dividends, but because of those in the future. This argument is self-fulfilling: the firm works at low reputations because it is believed to work. This suggests another, *shirk-work-shirk*, type of equilibrium where a firm with a low reputation is trapped in a shirk-hole in which market learning is too slow to incentivise effort. While such an equilibrium may exist, Theorem 4 shows that it disappears for small costs.

We can link our results to models in which quality is chosen in every period (e.g. Klein and Leffler (1981), Mailath and Samuelson (2001)) by taking the obsolescence rate of quality λ to infinity. With complete information, an increase in λ front-loads the returns to investment and increases investment incentives. With incomplete information, there is a countervailing effect: For large values of λ, equilibrium beliefs dominate market learning in determining reputational dynamics. In the good news and Brownian news cases, work-shirk profiles with positive effort cannot be supported when λ is high since the distribution of reputations degenerates to a peak at the work-shirk cutoff and expected reputational dividends vanish. On the other hand, for high λ, pure shirking is an equilibrium. In the bad news case, to the contrary, investment incentives increase in the obsolescence rate and any shirk-work profile is sustainable as an equilibrium.

1.1 Theoretical Literature

Our paper forms a bridge between classical models of reputation with exogenous types, and models of repeated games. In contrast to the repeated games literature, we suppose there is a state variable which links the periods. In contrast to reputation models, we suppose the state variable is the quality of the firm's product rather than some exogenous ability type of the firm (see figure 2). As a consequence, long-term dynamics are driven by reputational investment incentives rather than by exogenous type changes.

In their reputation paper, Mailath and Samuelson (2001) consider a firm that sells a good of unknown quality. There are two types of firms: a competent firm who can choose high or low effort, and an inept firm who can only choose low effort. The actual product quality is then a noisy function of the firm's effort. From the consumer's perspective, utility is determined by the probability the firm is competent (the firm's reputation) multiplied by the probability that a competent firm exerts effort.

Mailath and Samuelson derive a striking result: there is a unique Markov perfect equilibrium in pure strategies in which the competent firm always chooses low effort. When the reputation is close to 1, it is impossible to sustain high effort for the same reason as in our paper. Effort then unravels from the top: If the firm is known to be shirking when its reputation passes some cutoff, it has no incentive to exert effort just below this cutoff since success would mean an increase in reputation and an immediate collapse in the price. In contrast, in our paper, product quality is persistent. Thus, the price drifts down continuously when the firm starts to shirk, and unravelling is prevented.

Holmström (1999) examines a signal-jamming model where an agent of unknown ability can exert effort to confuse the learning of her employer. When the agent's type is constant, the employer gradually learns the agent's ability, and effort declines over time. When the agent's type exogenously changes over time, some effort level is sustained in the stationary equilibrium.

There is a wider literature on reputation models with moral hazard and fixed types, surveyed in Bar-Isaac and Tadelis (2008). A number of these papers examine how a firm's incentives to exert effort vary over its lifecycle. First, incentives are low towards the end of the firm's life (Kreps et al. (1982), Diamond (1989)). Second, incentives are low when updating is slow (Benabou and Laroque (1992), Mailath and Samuelson (2001)). Third, when reputation can be lost with one piece of bad news, incentives increase in the level of reputation (Diamond (1989)). Together these papers help explain how demand varies across firms and over time (Foster, Haltiwanger, and Syverson (2008)).[3]

When compared to the repeated games literature (e.g. Fudenberg, Kreps, and Maskin (1990)), our model has an evolving state variable. Nevertheless, the way that incentives for effort are determined by the signal structure in our paper echoes a similar theme in the repeated games literature. Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007a) consider a repeated prisoners' dilemma game with imperfect public monitoring. They find that first-best is attainable as players become patient if the public signal indicates defection (∼bad news), but is not attainable if the public signal indicates cooperation (∼good news) or is generated by Brownian motion.[4]

[3] Industry dynamics have been analysed with complete information models with exogenous firm types (Jovanovic (1982), Hopenhayn (1992)) or endogenous capital accumulation (Ericson and Pakes (1995)). The difference between these two approaches is analogous to the distinction between our paper and classical reputation papers.

[4] Also see Faingold and Sannikov (2007) on games with long-run and short-run players, and Sannikov and Skrzy-

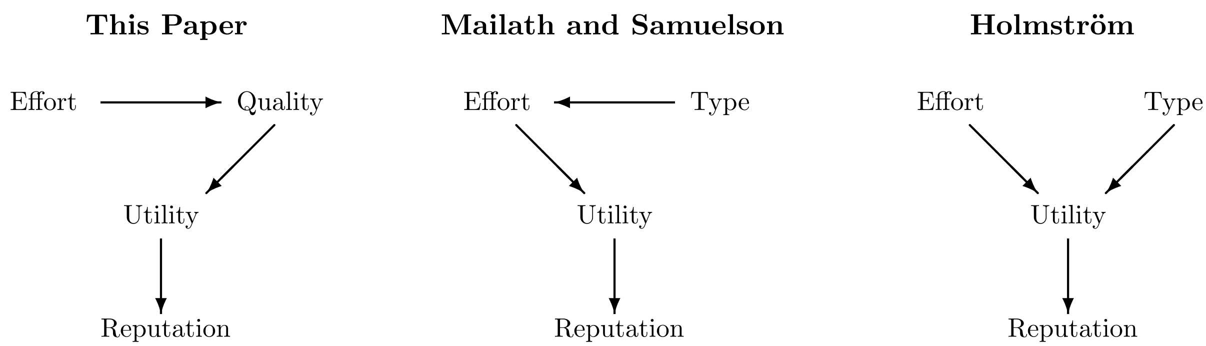

Figure 2: **Relation to Literature.** This figure shows the relationships between this paper, Mailath and Samuelson (2001) and Holmström (1999).

Finally, our paper is related to contract design with persistent effort. Fernandes and Phelan (2000) suppose that an agent's output depends today on effort both today and yesterday, and derive a recursive formulation to solve for the principal's optimal contract. Jarque (2008) shows that the problem is much simpler when output depends on the geometric sum of past efforts and the cost of effort is linear. Unlike these papers, our consumers simply react to the firm's actions, rather than designing contingent contracts.

1.2 Empirical Literature

There are a number of empirical papers examining the importance of reputation in internet auctions (eBay). Resnick, Zeckhauser, Swanson, and Lockwood (2006) find that a new seller obtains significantly lower prices than a seller with a good feedback score. Cabral and Hortaçsu (2009) similarly find that a seller with negative feedback obtains significantly lower prices. More interesting, Cabral and Hortaçsu (2009) look at the seller's reactions, showing that a seller who receives one negative feedback is more likely to obtain a second negative feedback, and is more likely to exit. This suggests that either underlying quality is correlated over time, or a seller who receives negative feedback exerts less effort, as in our bad news case.

Studies have also examined the role of reputation in other markets. In the airline industry, a crash reduces the stock market value of the airline and manufacturer in question, reduces demand for all aviation, but increases the value of firms who compete directly with the crashed airline (Chalk (1987), Borenstein and Zimmerman (1988), Bosch, Eckard, and Singal (1998)). In the restaurant market, the introduction of grade cards increased investments in hygiene, and had the biggest effect on non-chain restaurants (Jin and Leslie (2003, 2009)). In the vehicle emission testing market, garages with higher pass rates can demand higher prices (Hubbard (1998, 2002)). In all of these cases, firms make investments that affect the quality of the product, and hence their reputation. While these studies demonstrate the importance in maintaining a reputation, there is

pacz (2007b) on simultaneous move games with Brownian and Poisson news.

little evidence on the effect of reputation on the firm's investment incentives, as examined in this paper.

2 Model

Basics: Time $t \in [0, \infty)$ is continuous and infinite. The common interest rate is $r \in (0, \infty)$.

Firm and Consumers: There is one firm and a continuum of consumers. At any point in time t the firm's product can have high or low quality, $\theta_t \in \{L = 0, H = \mu\}$, where $\mu > 0$. The expected instantaneous value of the product to the consumer equals the increment of a stochastic process $dZ_t = dZ_t(\theta_t, \varepsilon_t)$ with expected value $E[dZ_t] = \theta_t dt$ and stochastic component ε_t that is independent over time. We will often focus on three functional specifications:

(a) *Good news*: $dZ_t = 0$ almost always but a good product has a *breakthrough* with arrival rate μ generating consumer utility of $dZ_t = 1$

(b) *Bad news*: $dZ_t = \mu dt$ almost always but a bad product has a *breakdown* with arrival rate μ generating consumer disutility of $dZ_t = -1$

(c) *Brownian news*: $dZ_t = \theta_t dt + dW_t$

Strategies: At time t the firm chooses effort $\eta_t \in [0, 1]$ at cost $c\eta_t dt$. Product quality θ_t is a function of past effort $(\eta_s)_{0 \leq s \leq t}$ via a Poisson process with arrival rate λ (independent of ε_t) that models quality obsolescence. Absent a shock, quality is constant: $\theta_{t+dt} = \theta_t$, while at a shock, previous quality becomes obsolescent and quality is determined by the level of investment: $Pr(\theta_{t+dt} = \mu) = \eta_t$. This implies $\Pr(\theta_t = H) = \int_0^t \lambda e^{\lambda(s-t)} \eta_s ds + e^{-\lambda t} \Pr(\theta_0 = H)$.[5]

Information: Realized consumer utility dZ_t is public information, while actual product quality θ_t is observed only by the firm.[6] The market belief about product quality $x_t = \Pr(\theta_t = \mu)$ at time t is called the firm's reputation.

Reputation Updating: The reputation increment $dx_t = x_{t+dt} - x_t$ is governed by realized consumer utility dZ_t and believed effort $\widetilde{\eta}_t$. As these are independent, dx_t can be decomposed additively:

$$dx_t = \lambda(\widetilde{\eta}_t - x_t)dt + x_t(1 - x_t)\frac{\Pr(dZ_t|H) - \Pr(dZ_t|L)}{x_t \Pr(dZ_t|H) + (1 - x_t)\Pr(dZ_t|L)}. \tag{2.1}$$

[5]This formulation provides a simple way to allow the firm's type to depend on its past investments. One can view effort as the choice of absorptive capacity, determining the ability of a firm to recognise new external information and apply it to commercial ends (Cohen and Levinthal (1990)). Equivalently, one could assume the firm first sees the new technology arrive and chooses whether to adopt it at cost $k = rc/\lambda$.

[6]The assumption that the firm knows the quality of its product is irrelevant since the cost is independent of θ.

We denote by $d_\theta x_t$ the increments conditional on quality θ.[7]

Profit and Consumer Surplus: The firm and consumers are risk-neutral. At time t the firm sets price equal to the expected value μx_t. While consumers get utility 0 in expectation, the firm's instantaneous profit is $(\mu x_t - c\eta_t)dt$ and its discounted present value is thus given by:

$$V_\theta\left(x;\eta,\widetilde{\eta}\right) := r\int_{t=0}^{\infty} e^{-rt}\mathbb{E}_{\theta_0=\theta,x_0=x,\eta,\widetilde{\eta}}\left[\mu x_t - c\eta_t\right]dt. \tag{2.2}$$

Markov-Perfect-Equilibrium: We assume Markovian beliefs $\widetilde{\eta} = \widetilde{\eta}\left(x\right)$ and show below that optimal effort $\eta = \eta\left(x\right)$ is independent of history and current product quality θ. A Markov-Perfect-Equilibrium $\langle\eta,\widetilde{\eta}\rangle$ consists of a Markovian effort function $\eta : [0,1] \to [0,1]$ for the firm and Markovian market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that 1) $\eta \in \eta^*\left(\widetilde{\eta}\right) := \arg\max_\eta \left\{V_\theta\left(x;\eta,\widetilde{\eta}\right)\right\}$ maximizes firm value $V_\theta\left(x;\eta,\widetilde{\eta}\right)$, given x and $\widetilde{\eta}$, and 2) market beliefs are correct: $\widetilde{\eta} = \eta$. In a Markovian equilibrium η, we will write the firm's value as a function of its quality and its reputation: $V_\theta(x)$.

2.1 Optimal Investment Choice

In principle, the firm's effort choice η as well as market beliefs $\widetilde{\eta}$ could depend on the entire public history $dZ^t = (dZ_s)_{0\le s<t}$, as well as the private history $\theta^t = (\theta_s)_{0\le s<t}$ and time t. We assume that market beliefs $\widetilde{\eta}$ are Markovian because we think of the continuum of consumers as sharing their experience in an imperfect way, e.g. through consumer reports. For Markovian beliefs $\widetilde{\eta}$, all payoff relevant parameters at time t depend on the history only via the current product quality θ_t and the firm's reputation x_t. Thus, the optimal effort choice of the firm only needs to depend on these two parameters.

The benefit of effort in $[t;t+dt]$ is the probability of a technology shock hitting, λdt, times the difference in value functions $\Delta\left(x\right) := V_H\left(x\right) - V_L\left(x\right)$, which we call the *asset value of quality*. The marginal cost of investment is rc, and thus optimal effort $\eta\left(x\right)$ is given by

$$\eta\left(x\right) = \begin{cases} 1 & \text{if } rc < \lambda\Delta\left(x\right) \\ 0 & \text{if } rc > \lambda\Delta\left(x\right) \end{cases}. \tag{2.3}$$

Optimal effort is therefore independent of the current quality of the firm θ_t.

Lemma 1 summarizes this discussion:

[7] The reason for modelling time as continuous is purely pragmatic. If time were measured in discrete periods, the updating equation (2.1) would be complicated by a dt^2 term because in every period market learning would already take the equilibrium effort decision into account.

Lemma 1 *For Markovian beliefs $\widetilde{\eta}(x)$ there is an optimal Markovian effort function $\eta(x)$ that depends solely on the firm's reputation but not on its product quality. Additionally, $\eta(x)$ satisfies equation (2.3).*

Equation (2.3) makes the model more tractable and is the reason that we assume the cost of effort to be independent of product quality and past effort. An implication of equation (2.3) is that our results are not driven by the asymmetric information about product quality θ, but solely by the unobserved investment η into future quality. We analyse asymmetric costs in Section **??**.

2.2 Cutoff Equilibria and Reputational Dynamics

We call an equilibrium *work-shirk*, if there exists a cutoff x^* such that a firm with low reputation $x < x^*$ exerts effort, $\eta(x) = 1$, whereas a firm with a high reputation $x > x^*$ does not, $\eta(x) = 0$. The opposite case, where low reputations shirk and high reputations work, is called a *shirk-work* equilibrium.

Reputational dynamics of work-shirk equilibria are fundamentally different from those of shirk-work equilibria. Net of market learning, the dynamics $dx = \lambda(\widetilde{\eta}_t - x_t)dt$ are convergent in a work-shirk equilibrium, i.e. $dx > 0$ for $x < x^*$ and $dx < 0$ for $x > x^*$, but divergent in a shirk-work equilibrium.

A set of reputations $S \subseteq [0, 1]$ is called a *shirk-hole* if the firm shirks $\eta(x) = 0$ for all $x \in S$ and S is closed under reputational dynamics in that $\Pr(x_t \in S) = 1$ if $x_0 \in S$. For future use, define $\Delta_{x^*}(x)$ to be the asset value of quality for a firm with reputation x when both actual effort η and believed effort $\widetilde{\eta}$ are work-shirk with cutoff x^*.

3 General Results

In this section we prepare the ground-work for the analysis of the good, bad and Brownian news cases. In Section 3.1 we derive the welfare maximising effort. As noted above, the firm's value $V_\theta(x)$ is a function of it's reputation and quality. In Section 3.2 we show that an increase in reputation increases the price today and tomorrow, and therefore increases $V_\theta(x)$. In Section 3.3 we show that a higher quality derives its value indirectly, through its effect on future reputation. In particular, Theorem 1 interprets the asset value of quality as the net present value of future reputational dividends. Finally, Section 3.4 derives some general results about the firm's effort choice.

3.1 Welfare

Suppose product quality is publicly observed. Then the benefit of exerting effort equals the obsolescence rate λ times the price differential μ divided by the effective discount rate $r + \lambda$. Thus the first-best effort choice is given by:

$$\eta = \begin{cases} 1 & \text{if } c < \frac{\lambda}{r+\lambda}\mu \\ 0 & \text{if } c > \frac{\lambda}{r+\lambda}\mu \end{cases} . \tag{3.1}$$

There is no equilibrium with positive effort if $c > \frac{\lambda}{r+\lambda}\mu$. In this case, welfare is negative and the firm makes negative profits as consumers receive zero utility in equilibrium. The firm therefore prefers to shirk at all levels of reputation, thereby guaranteeing itself a non-negative payoff.

We thus restrict attention in the paper to the case $c < \frac{\lambda}{r+\lambda}\mu$.

3.2 Value of Reputation

Lemma 2 shows that, when the firm is choosing its effort optimally, the value function is increasing in reputation.[8] To prove the lemma, we need to rule out the possibility that a firm with a higher initial reputation may shirk, lose its product quality, and fall behind a firm with a lower initial reputation. We do this by observing that a firm always has the option to mimic a firm with a slightly lower reputation.

Lemma 2 *Given an optimal response to market beliefs $\eta^*(\widetilde{\eta})$, the value function of the firm $V_\theta(x;\eta^*(\widetilde{\eta}),\widetilde{\eta})$ is strictly increasing in its reputation x and increasing in market beliefs $\widetilde{\eta}$.*

Proof. Fix θ, $(x', \widetilde{\eta}')$ and $(x'', \widetilde{\eta}'') \geq (x', \widetilde{\eta}')$, i.e. $x'' \geq x'$ and $\widetilde{\eta}''(x) \geq \widetilde{\eta}'(x)$ for all x. Write the best response $\eta^*(\widetilde{\eta}')$ to the Markovian beliefs $\widetilde{\eta}'$ in a non-Markovian way as a function of the public history $\overline{\eta}(dZ^t) = \eta\left(x\left(dZ^t, \widetilde{\eta}', x'\right)\right)$. For any realization of the random processes, denote by $(x''_t, \theta''_t, \eta''_t, dZ''_t)$ the trajectory of reputation, quality, effort and utility given effort $\overline{\eta}(dZ^t)$, initial reputation x'' and market beliefs $\widetilde{\eta}''$, and by $(x'_t, \theta'_t, \eta'_t, dZ'_t)$ the corresponding trajectory given $\overline{\eta}, x', \widetilde{\eta}'$.

By construction, effort $\eta'_t = \eta''_t$, quality $\theta'_t = \theta''_t$, and utility $dZ'_t = dZ''_t$ will coincide for all times t. Reputation on the other hand may start at different levels $x'' \geq x'$ and because the updating equation (2.1) implies that $x_{t+dt}(x_t, \widetilde{\eta}_t, dZ_t)$ is increasing in x_t and $\widetilde{\eta}_t$, we get $x''_t \geq x'_t$.

Thus, by mimicking the effort of the firm with lower initial reputation x', the firm with a strictly higher initial reputation x'' can secure itself a strictly higher value. By Lemma 1 there must be a Markov strategy that is at least as good as this mimicking strategy. □

Lemma 2 implies that across equilibria η, η', with $\eta'(x) \geq \eta(x)$ for all x, the firm's value is increasing in effort $V_\theta(x;\eta',\eta') \geq V_\theta(x;\eta,\eta)$.

[8] While it is unclear whether these monotonicity results hold for non-equilibrium $\langle \eta, \widetilde{\eta} \rangle$, Lemma 6 in Appendix C.1 extends them to work-shirk effort functions where the cutoff type x^* is indifferent between working and shirking.

3.3 Value of Quality

As shown in Section 2.1, investment incentives are driven by the asset value of quality $\Delta(x)$. To analyse this value, we can decompose it into (a) the immediate benefit of a positive market signal, called the *reputational dividend*, and (b) the continuation benefit of a high quality product:

$$\begin{aligned} \Delta(x) &= (1-rdt)(1-\lambda dt)\mathbb{E}_x\left[V_H(x+d_Hx)-V_L(x+d_Lx)\right] \\ &= (1-rdt-\lambda dt)\mathbb{E}_x\left[(V_H(x+d_Hx)-V_H(x+d_Lx))+\Delta(x+d_Lx)\right]. \end{aligned} \tag{3.2}$$

The first line uses the principle of dynamic programming, while the second adds and subtracts $V_H(x+d_Lx)$. Integrating up yields equation (3.3) in Theorem 1, which expresses the asset value of quality as the discounted sum of future reputational dividends. Equation (3.4) follows from the alternative decomposition of (3.2) when we add and subtract $V_L(x+d_Hx)$ instead of $V_H(x+d_Lx)$. These expressions serve as a work-horses throughout the paper.

Theorem 1 *Fix any Markovian beliefs $\widetilde{\eta}$ and a Markovian best response $\eta^*(\widetilde{\eta})$. Then two closed-form expressions for the value of quality $\Delta(x)$ are given by*

$$\Delta(x) = \int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}[D_H(x_t)]dt \tag{3.3}$$

$$= \int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=H}[D_L(x)]dt \tag{3.4}$$

where $\theta^t = L$ is short for $\theta_s = L$ for all $s \in [0;t]$, and the reputational dividend $D_\theta(x)$ is defined by

$$D_\theta(x) := \mathbb{E}[V_\theta(x+d_Hx)-V_\theta(x+d_Lx)]/dt.$$

Proof. To integrate up (3.2), fix x and set $\psi(t) := \mathbb{E}_{x_0=x,\theta^t=L}\left[\Delta(x_t)\right]$. Up to terms of order $o(dt)$ we have

$$\begin{aligned} -d\left(\psi(t)e^{-(r+\lambda)t}\right) &= -e^{-(r+\lambda)t}\left(\psi(t+dt)-\psi(t)-(r+\lambda)dt\psi(t)\right) \\ &= e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}\left[-\mathbb{E}_{x_t}\left[\Delta(x_t+d_Lx_t)\right]+(1+(r+\lambda)dt)\Delta(x_t)\right] \\ &= e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}\left[D_H(x_t)\right]dt \end{aligned}$$

and (3.3) follows. □

Corollary 1 *Fix any Markovian beliefs $\widetilde{\eta}$ and a Markovian best response $\eta^*(\widetilde{\eta})$. For a given reputation x, a high-quality firm has a higher value than a low-quality firm, i.e. $V_H(x) \geq V_L(x)$.*

Proof. By the updating equation (2.1) we have $d_Hx \geq d_Lx$, by Lemma 2 we get $D_\theta(x) = V_\theta(x_t+d_Hx_t)-V_\theta(x_t+d_Lx_t) \geq 0$. Finally by Theorem 1 we get $\Delta(x) = V_H(x)-V_L(x) \geq 0$. □

3.4 Investment Levels

Lemma 3 (Some Effort Somewhere) *For sufficiently low costs c, pure shirking, i.e.* $\eta(x) = 0$ *for all x, is not an equilibrium.*

Proof. Fixing $\eta(x) = \widetilde{\eta}(x) = 0$, value functions and Δ do not depend on c, and $\Delta(x) > 0$ for some x. Thus, as the cost of effort vanish $c \to 0$, its benefits $\lambda\Delta(x)$ are constant and thus bounded away from 0, contradicting the equilibrium condition. □

This result is in contrast to reputational models with inept firms where shirking is always an equilibrium, even if costs of working are 0. As discussed in the introduction the critical difference in this model is that expected quality and price still depend on the firm's reputation - and in particular are greater than 0 - even when the firm is believed to be shirking.

Lemma 4 (No Effort at the Top) *If no market signal* dZ_t *perfectly reveals low quality, i.e. if* $\frac{\Pr(dZ_t|L)}{\Pr(dZ_t|H)} < \infty$ *for all* dZ_t*, then a firm with a perfect reputation* $x = 1$ *must be shirking with positive probability:* $\eta(1) < 1$.

Proof. Assume to the contrary that $\eta(1) = 1$ in equilibrium. By the assumption that $\frac{\Pr(dZ_t|L)}{\Pr(dZ_t|H)} < \infty$, a firm with perfect reputation $x = 1$ can never lose its reputation, irrespectively of the consumer utility dZ_t it generates. Thus $d_H x_t = d_L x_t = 0$ and by Theorem 1 the asset value of quality $\Delta(x)$ equals 0. Thus, $\eta(1) = 1$ cannot be sustained in equilibrium. □

The proof actually shows a slightly stronger statement: In no equilibrium the firm can be working for all reputation levels x in an interval of arbitrary high reputations $(1 - \varepsilon, 1)$.

4 Good News

Assume that consumers learn about quality θ_t from infrequent product *breakthroughs* that reveal $\theta = H$ with arrival rate μ. Absent a breakthrough, updating evolves deterministically according to:

$$\frac{dx}{dt} = \lambda(\eta(x) - x) - \mu x(1 - x). \tag{4.1}$$

We define x_t as the deterministic solution of the ODE (4.1) with initial value x_0.

The reputational dividend, which is the value of having a high quality in the next instant, equals the value of increasing the reputation from x to 1 times the probability of a breakthrough:

$$D_H(x) = V_H(x + d_H x) - V_H(x + d_L x) = \mu(V_H(1) - V_H(x))dt$$

Conditioning on the firm always being low quality, in order to prevent a breakthrough, the time path of reputation is given by (4.1). Using equation (3.3), the asset value of quality is:

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu[V_H(1) - V_H(x_t)]dt \tag{4.2}$$

The reputational dividend $V_H(1) - V_H(x_t)$ is decreasing in x_t, so that $\Delta(x_0)$ is decreasing in x_0. It follows that any equilibrium is work-shirk. Intuitively, when a breakthrough occurs the firm's reputation immediately jumps to 1. Since this jump is larger for a low-reputation firm, incentives to invest decrease in reputation and the equilibrium is work-shirk.

The form of the equilibrium implies that the reputational dynamics converge to a cycle. Absent a breakthrough, the firm's reputation converges to a stationary point $\hat{x} = \min\{\lambda/\mu, x^*\}$ where the firm works with positive probability. When a breakthrough occurs, the firm's reputation jumps to 1. The firm is then believed to be shirking, so its reputation drifts down to $\hat{x}$, absent another breakthrough. In the long-run, the firm's reputation therefore cycles over the range $[\hat{x}, 1]$.

Theorem 2 *In the good news case*

(a) Every equilibrium is work-shirk.

(b) Reputational dynamics converge to a non-trivial cycle.

(c) If $\lambda \geq \mu$, the equilibrium is unique.

(d) For sufficiently high λ, zero-effort is the only equilibrium.

Proof. Part (a). Reputation x_t follows (4.1), so an increase in x_0 raises x_t at each point in time. Lemma 2 says that $V_H(x)$ is strictly increasing in x, so equation (4.2) implies that $\Delta(x_0)$ is decreasing in x_0. Part (b) follows from (a).

Part (c). We will show that $\Delta_{x^*}(x^*)$ is decreasing in x^*. Under the assumption $\lambda \geq \mu$, the reputational dynamics x_t starting at cutoff $x_0 = x^*$ are stuck at x^*. Thus, by equation (4.2) the value of quality is the discounted present value of an reputational boost from x^* to 1: $\Delta_{x^*}(x^*) = \frac{\mu}{r+\lambda}\left(V_{H,x^*}(1) - V_{H,x^*}(x^*)\right)$. In Appendix A.1 we show that the latter term is given by

$$V_{H,x^*}(1) - V_{H,x^*}(x^*) = \int_{t=0}^\infty e^{-rt} r\mu(x_t - x^*)\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t}\right]dt, \tag{4.3}$$

where $x_0 = 1$. Note that, in this expression, the terms in the brackets capture the possibilities of λ and μ shocks while x_t descends from 1 to x^*. Equation (4.3) is decreasing in the cutoff, so the equilibrium is unique.

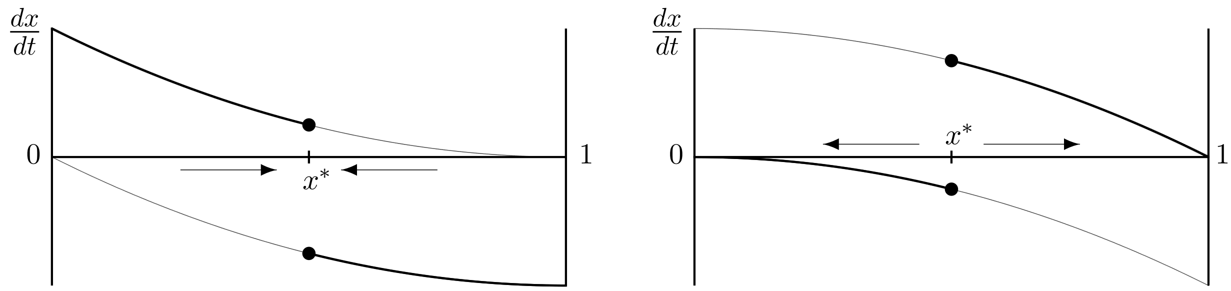

Figure 3: **Dynamics in Good News (left) and Bad News (right).** This picture illustrates how the reputational drift dx/dt, absent a breakthrough, changes with the reputation of the firm, x. These pictures assume $\lambda = \mu$ and $x^* = 1/2$. The dark line shows the drift when beliefs are correct.

Part (d). Let $x^* = 0$. Using part (a) and $x_t \leq e^{-\lambda t}$, we get

$$\lambda \Delta_0(0) \leq \frac{\lambda}{r+\lambda} \mu \int_{t=0}^{\infty} e^{-(r+\lambda)t} r\mu \, dt = \frac{\lambda}{r+\lambda} \frac{r\mu^2}{r+\lambda}$$

When λ is sufficiently high $\lambda \Delta_0(0) < rc$, and the unique equilibrium exhibits zero effort. $\square$

Suppose that $\lambda \geq \mu$, so the firm's reputation drifts up whenever it is known to be working (see figure 3). In this case, the dynamics are stationary at $\hat{x} = x^*$, at which point the firm chooses to work with probability $\eta(x^*) = x^* \left(1 + \frac{\mu}{\lambda}(1 - x^*)\right)$ so as to keep its reputation constant.

Theorem 2(c) shows that the equilibrium is unique. To understand this result, suppose the market believes the cutoff is $\tilde{x}$, and denote the firm's best response by $x^*(\tilde{x})$. An increase in $\tilde{x}$ means the firm's reputation will not drift down as far, absent a breakthrough. This change benefits low-quality firms more than high-quality firms, reducing $\Delta(x)$. As a result, $x^*(\tilde{x})$ is decreasing in $\tilde{x}$ and there is a unique fixed point where $x^*(\tilde{x}) = \tilde{x}$.

Theorem 2(d) says that, as technology shocks become more frequent the incentives to exert effort disappear. Intuitively, when λ is high, equilibrium beliefs move the reputation towards x^* very quickly. Hence a breakthrough, which raises the reputation to 1, is quickly depreciated. This means there is little value to being a high-quality firm and little value to investing. Theorem 2(d) implies that investment is non-monotonic in the frequency of technology shocks. When shocks are too frequent, equilibrium beliefs drive reputation and there is no incentive to invest in actual quality. When shocks are too infrequent, investment takes too long to pay off and, again, there is little incentive to invest in quality. For intermediate frequencies, however, effort can be supported.

5 Bad News

Assume that x_t is generated by *breakdowns* that reveal $\theta = L$ with arrival rate μ. Absent a breakdown, updating evolves deterministically according to:

$$\frac{dx}{dt} = \lambda(\eta(x) - x) + \mu x(1 - x). \tag{5.1}$$

We define x_t as the solution of the ODE (5.1) with initial value x_0.

The reputational dividend, which is the value of having a high quality in the next instant, equals the value of not losing one's reputation times the probability of a breakdown:

$$D_L(x) = V_L(x_t + d_H x_t) - V_L(x_t + d_L x_t) = \mu(V_L(x_t) - V_L(0))dt.$$

Conditioning on the firm always being high quality, in order to prevent a breakdown, the time path of reputation is given by (5.1). Using equation (3.4), the asset value of quality is:

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu[V_L(x_t) - V_L(0)]dt. \tag{5.2}$$

The jump size $V_L(x_t) - V_L(0)$ is increasing in x_t, so that $\Delta(x)$ is increasing in x. It follows that any equilibrium is shirk-work. Intuitively, when a breakdown occurs a firm immediately jumps to reputation $x = 0$. Since this jump is larger for a high-reputation firm, incentives to invest increase in reputation and the equilibrium is shirk-work.[9]

The form of the equilibrium implies that the reputational dynamics diverge. Suppose we are in an equilibrium with $x^* \in (0, 1)$, so firms below x^* shirk while firms above x^* work. A firm that starts with reputation above x^* converges to reputation $x = 1$, absent a breakdown. If the firm is hit by such a breakdown while its product quality is still low, it gets stuck in a shirk-hole with reputation $x = 0$. A firm with reputation below x^* initially shirks and may have either rising or falling reputation, depending on parameters. In either case, its reputation will either end up at $x = 0$ or $x = 1$.

To allow for positive effort in some equilibrium we impose the following assumption:

$$\frac{\lambda}{r + \lambda + \mu}\mu(\mu - c) > rc \tag{PE}$$

Theorem 3 *Assume (PE) holds. In the bad news case*

[9]For example, when writing about the explosion of Sony's batteries in Dell's laptops the Financial Times wrote that "The withdrawal comes at a sensitive time for [Dell], which has been fighting broader perceptions of poor customer service and slowing sales growth. [However] it could have a deeper impact on Sony, given the Japanese company's reputation for quality in the consumer electronics industry"(15th August 2006). This illustrates the point that the jump in reputation is larger for higher reputation firms.

(a) *Every equilibrium is shirk-work.*

(b) *If* $x^* \in (0,1)$ *then reputational dynamics diverge to* 0 *or* 1.

(c) *If* $\lambda \geq \mu$ *there is a non-empty interval* $[a,b]$ *such that every cutoff* $x^* \in [a,b]$ *defines an equilibrium.*

(d) *If* λ *is sufficiently high, every* $x^* \in (0,1]$ *defines an equilibrium.*

Proof. Part (a). Reputation x_t follows (5.1), so an increase in x_0 raises x_t at each point in time. Lemma 2 says that $V_L(x)$ is strictly increasing in x, so equation (5.2) implies that $\Delta(x_0)$ is increasing in x_0. Part (b) follows from (a).

Part (c). If $\lambda \geq \mu$, the dynamics are divergent at x^*: if $x_0 = x_t - \epsilon$, then $\lim x_t = 0$; if $x_0 = x_t + \epsilon$, then $\lim x_t = 1$. Thus, to define value functions and Δ at the cutoff x^* we need to specify whether or not x^* works. Denote by $\Delta^-_{x^*}(x)$ (resp. $\Delta^+_{x^*}(x)$) the value of quality at x when x^* is believed to be shirking (resp. working). At $x^* \in (0,1)$ we have $\Delta^-_{x^*}(x^*) = \lim_{x \nearrow x^*} \lambda \Delta_{x^*}(x)$ and $\Delta^+_{x^*}(x^*) = \lim_{x \searrow x^*} \lambda \Delta_{x^*}(x)$. Lemma 2 says that $V_L(x_t)$ is strictly increasing in x_t, so (5.2) implies that

$$\Delta^-_{x^*}(x^*) < \Delta^+_{x^*}(x^*). \tag{5.3}$$

A cutoff $x^* \in (0,1]$ then defines a shirk-work equilibrium iff[10]

$$\lambda \Delta^-_{x^*}(x^*) \leq rc \leq \lambda \Delta^+_{x^*}(x^*). \tag{5.4}$$

Equation (5.2) implies that $\Delta^+_{x^*}(x^*)$ and $\Delta^-_{x^*}(x^*)$ are increasing and continuous in x^*. For the lower bound, observe that $\lambda \Delta^-_0(0) = 0$, because a firm with no reputation that is believed to be shirking is stuck at 0 forever. For the upper bound, $\lambda \Delta^+_1(1) = \frac{\lambda}{r+\lambda+\mu}\mu(\mu - c)$ because $V_{L,1}(1) = \frac{r+\lambda}{r+\lambda+\mu}(\mu - c)$, $V_{L,1}(0) = 0$, and $\lambda \Delta^+_1(1) = \frac{\lambda\mu}{r+\lambda}(V_{L,1}(1) - V_{L,1}(0))$. Under assumption (PE) equation (5.4) therefore defines a non-empty interval of cutoffs, $[a,b]$.

Part (d). Pick any $x^* > 0$. First, suppose $x_0 > x^*$ and observe that $x_t \geq 1 - e^{-\lambda t}$. In Appendix B.1 we derive the following formula for the value of quality:

$$\Delta_{x^*}(x) = \frac{\mu}{\lambda + \mu} \int_{t=0}^{\infty} e^{-rt} r(\mu x_t - c)(1 - e^{-(\lambda+\mu)t})\, dt \tag{5.5}$$

[10] The case $x^* = 0$ is more subtle because there are two qualitatively different effort-profiles with cutoff $x^* = 0$.

If $x^* = 0$ is believed to be working, there is no shirk-hole, and the necessary and sufficient condition for equilibrium is that this is in the firm's best interest, i.e. $rc \leq \lambda \Delta^+_0(0)$.

However, if $x^* = 0$ is believed to be shirking, then $x^* = 0$ is a shirk-hole and the equilibrium condition is $rc \leq \lambda \Delta^-_0(x)$ for all $x > 0$ (and $rc \geq \lambda \Delta^-_0(0)$ which is automatically satisfied because $\lambda \Delta^-_0(0) = 0$). This condition is weaker than (5.4) as $\Delta^+_0(0) < \lim_{x \to 0} \Delta^-_0(x)$: The existence of a shirk-hole increases the value of quality. Formally, this is reflected in equation (5.2) in that $V_L(0) = 0$ if reputation 0 shirks, but $V_L(0) > 0$ if reputation 0 works.

Taking limits,

$$\lim_{\lambda\to\infty} \lambda\Delta_{x^*}(x) \geq \lim_{\lambda\to\infty} \frac{\lambda\mu}{\lambda+\mu} r \int_{t=0}^{\infty} e^{-rt}(\mu(1-e^{-\lambda t})-c)(1-e^{-(\lambda+\mu)t})\,dt = \mu(\mu-c)$$

where the final equality uses the fact that the integral converges to $(\mu-c)/r$. Assumption (PE) implies that $\mu(\mu-c) > rc$. Hence for sufficiently large λ, working is optimal for all $x > x^*$ and any x^*.

Next suppose $x_0 < x^*$ and observe that $x_t \leq e^{-(\lambda-\mu)t}$. In Appendix B.1, we derive the following formula for the value of quality:

$$\Delta_{x^*}(x) = \frac{\mu}{\lambda-\mu} \int_{t=0}^{\infty} e^{-rt} r\mu x_t (e^{-\mu t} - e^{-\lambda t})\,dt \tag{5.6}$$

Taking limits,

$$\lim_{\lambda\to\infty} \lambda\Delta_{x^*}(x) \leq \lim_{\lambda\to\infty} \frac{\lambda\mu}{\lambda-\mu} \int_{t=0}^{\infty} e^{-rt} r\mu e^{-(\lambda-\mu)t}\,dt = 0$$

Hence for sufficiently large λ, shirking is optimal for all $x < x^*$ and any x^*. $\square$

Suppose $\lambda \geq \mu$, so that whenever the firm is known to be shirking its reputation drifts down (see figure 3). In this case, the region below x^* is a shirk-hole: when a firm's reputation is below the cutoff, it is certain to see its reputation decrease because of the unfavourable equilibrium beliefs. Such a firm always shirks, eventually giving rise to a low quality product and a product breakdown destroying whatever is left of its reputation. When a firm's reputation is above the cutoff, favourable market beliefs contribute to an increasing reputation and the firm invests to insure itself against a product breakdown. At the cutoff, the firm works when it is believed to be working and shirks whenever it is believed to be shirking.[11]

Theorem 3(c) shows that there is an interval of equilibrium cutoffs satisfying (5.4). The multiplicity is driven by a discontinuity in the value function at x^*, caused by the divergent reputational dynamics. Intuitively, the market's beliefs become self-fulfilling. If the market believes the firm is shirking, it's reputation falls, undermining any incentive to invest. Conversely, if the market believes the firm is working, its reputation rises, causing the firm to invest in order to protect its appreciating reputation.

When $\lambda < \mu$ the dynamics have additional interesting features: Define $\widehat{x} = 1 - \frac{\lambda}{\mu} \in (0,1)$ to be the stationary point in the dynamics when the firm is believed to be shirking. There are two types of equilibria:

1. *Trapped equilibria.* When $\widehat{x} < x^*$, a firm with reputation $x \in (0, x^*)$ finds its reputation converging to $\widehat{x}$, and remains stuck in a shirk-hole. At some point is suffers a breakdown

[11] The divergent dynamics imply that there will be path dependence in reputations. This is consistent with the existence of credit traps in financial markets, and may help explain why political scandals have such dramatic effects on politicians careers (Diermeier, Keane, and Merlo (2005)).

and remains at $x = 0$ thereafter. Since the dynamics are divergent at x^* the value function is discontinuous, and there is an interval of such equilibria.

2. *Permeable equilibria.* When $\widehat{x} > x^*$, a firm with reputation $x \in (0, x^*)$ finds it's reputation increasing. If x_t passes x^* before a breakdown hits, the firm starts to work and it's reputation may converge to one. Since the value functions are continuous at a permeable cutoff x^*, there is at most one permeable equilibrium.

Finally, Theorem 3(d) shows that as technology shocks become more frequent, then any cutoff can be an equilibrium. Intuitively, a firm that starts below the cutoff finds it's reputation falling to zero instantly and gives up, while a firm above the cutoff finds it's reputation rising to one instantly and works to stay there. While outside the model, this multiplicity creates an incentive for firms to invest in marketing in order to shape consumers expectations.

6 Brownian News

We now assume that consumers learn about quality θ_t gradually, through the evolution of a Brownian motion with state-dependent drift, $dZ_t = \theta_t dt + dW_t$. Updating evolves according to

$$\begin{aligned} d_H x &= \lambda(\eta(x) - x)dt + \mu^2 x(1-x)^2 dt + \mu x(1-x)dW \\ d_L x &= \lambda(\eta(x) - x)dt - \mu^2 x^2(1-x)dt + \mu x(1-x)dW. \end{aligned} \tag{6.1}$$

To calculate the value of quality we apply Itô's formula to get:

$$\begin{aligned} \mathbb{E}_x[V_\theta(x + d_H x)] &= V_\theta(x) + \mu^2 x(1-x)^2 V'_\theta(x)dt + \frac{(\mu x(1-x))^2}{2} V''_\theta(x)dt \\ \mathbb{E}_x[V_\theta(x + d_L x)] &= V_\theta(x) - \mu^2 x(1-x)^2 V'_\theta(x)dt + \frac{(\mu x(1-x))^2}{2} V''_\theta(x)dt. \end{aligned}$$

The reputational dividend is thus:

$$D_H(x) = \mathbb{E}_x[V_H(x + d_H x) - V_H(x + d_L x)]/dt = \mu^2 x(1-x)V'_H(x). \tag{6.2}$$

$D_H(x)$ declines to zero in either tail as reputational updating becomes very slow. Using equation (3.3), the value of quality reduces to

$$\Delta(x) = \int_0^\infty e^{-(r+\lambda)t} \mathbb{E}_{x_0=x,\theta^t=L}[\mu^2 x_t(1-x_t)V'_H(x_t)]dt. \tag{6.3}$$

Theorem 4 shows that, for small costs, there exists an equilibrium whereby the firm works when its reputation is below some cutoff, x^*. In such an equilibrium, the reputation of a firm below the cutoff tends to rise, whereas the reputation of a firm above the cutoff tends to fall, leading to cyclical dynamics. Theorem 4 also shows that, for small costs, this equilibrium is essentially unique in that (i) the work-shirk cutoff is unique; and (ii) the work region in the work-shirk equilibrium is arbitrarily close to the work-region in any other equilibrium. Denote the work region in an arbitrary equilibrium by $R_\eta = \{x : \eta(x) = 1\}$ and let R^* be the work-region in the work-shirk equilibrium. Let h be the Hausdorff metric.

Theorem 4 *There is c^* such that for all $c \in (0, c^*)$:*

(a) There exists $x^* \in (0,1)$ such that work-shirk with cutoff x^* is an equilibrium.

(b) Reputational dynamics converge to a non-trivial cycle.

(c) Equilibrium is essentially unique: (i) The work-shirk cutoff x^* is uniquely determined; and (ii) $h(R_\eta, R^*) < \epsilon$, for any equilibrium η and any $\epsilon > 0$.

(d) For fixed c and high enough λ, there is no work-shirk equilibrium, but zero-effort is an equilibrium.

Proof. See Appendix C. □

Section 6.1 discusses the existence of work-shirk equilibria. Section 6.2 considers other forms of equilibria. Section 6.3 discusses the effect of a high obsolescence rate ($\lambda \to \infty$).

6.1 Work-Shirk Equilibria

When costs are low, there is a work-shirk equilibrium but no shirk-work equilibrium (Lemma 4). This asymmetry is illustrated in the left panel of figure 4. Intuitively, when the firm is believed to be working, the asset value of quality is zero at $x = 1$ since current dividends are zero and, as the firm's reputation stays at $x = 1$, future dividends are zero. In contrast, the asset value of quality is positive at $x = 0$ since the market's belief that the firm is working causes its reputation to drift into the interior of $(0,1)$, enabling it to collect high dividends in the future. In other words, the firm wishes to invest at $x = 0$ not because immediate reputational dividends, which are close to 0, but because of future dividends, when the firm's reputation is sensitive to true quality.

Figure 4 is almost a proof of Theorem 4(a). Elementary calculations show that:

$$\Delta_1(x) \begin{cases} > 0 & \text{for } x < 1 \\ = 0 & \text{for } x = 1 \end{cases} \quad \text{and } \Delta_1'(1) < 0.$$

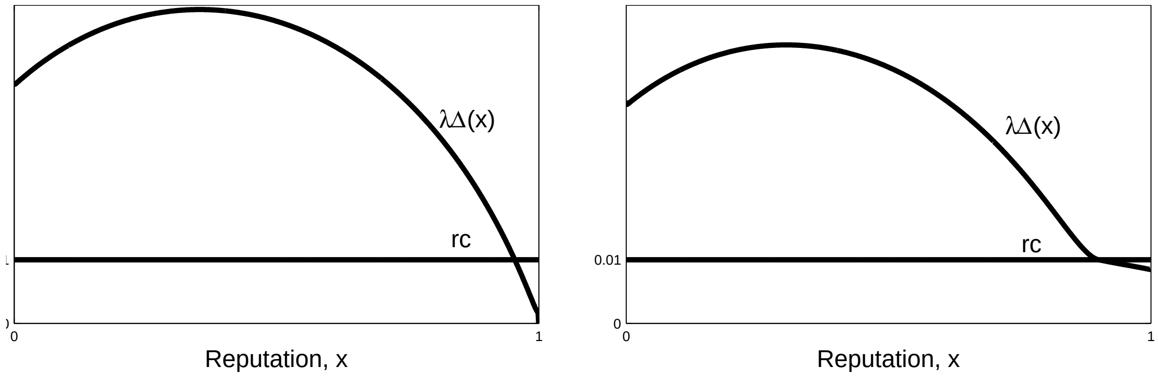

Figure 4: **Asset Value of Quality under Full Effort (left) and in Work-Shirk Equilibrium (right).** This figure assumes that $\mu = 1$, $\lambda = 1$, $r = 1$ and $c = 0.01$. In the work-shirk equilibrium, the resulting cutoff is is $x^* = 0.900$.

and thus for small c there exists x^* such that

$$\lambda\Delta_1(x) \begin{cases} > rc & \text{for } x < x^* \qquad \text{(Low reputations work)} \\ = rc & \text{for } x = x^* \qquad (x^* \text{ indifferent}) \\ < rc & \text{for } x > x^* \qquad \text{(High reputations shirk).} \end{cases} \tag{6.4}$$

To prove part (a) we just need to replace Δ_1 on the LHS with Δ_{x^*}. The problem with this simple argument is that it implicitly assumes continuity of Δ'_{x^*} as $x^* \to 1$. However, it is straightforward to show that $\lim_{x^* \to 1} \Delta'_{x^*}(1) = 0 > \Delta'_1(1)$. As a result, it could be that $\Delta_{x^*}(x)$ is increasing in x for $x > x^*$, contradicting the last condition in (6.4).

To understand this complication, consider the marginal value of reputation $V'_\theta(x)$ to a firm with reputation $x \in [x^*, 1]$ where $x^* \approx 1$. A reputational increment dx is valuable to the firm only as long as $x_t|_{x_0=x+dx} > x^*$: As soon as $x_t|_{x_0=x+dx} = x^*$ the increment $x_t|_{x_0=x+dx} - x_t|_{x_0=x}$ vanishes because of the difference of drift $\mathbb{E}[dx]$ to the left of x^* and to the right of x^*. As a consequence, $V'_\theta(x)$ and $D_\theta(x)$ may be minimized at the cutoff x^*, and one may be concerned that $\Delta_{x^*}(x)$ is also minimised at x^*.

To overcome this complication and show that $\lambda\Delta_{x^*}(x^*) > \lambda\Delta_{x^*}(x)$ for $x \in (x^*, 1]$ we need a better understanding of the reputational dynamics dx and the marginal values $V'_{\theta,x^*}(x)$ for $x, x^* \approx 1$. Fortunately, the dynamics of $(1-x)$ approximate a geometric Brownian motion which is reflected at $(1-x^*)$ by the large relative difference in the drift terms. For the high quality firm,

$$\begin{aligned} d_H(1-x) &= -\lambda\,(\eta - x)\,dt - \mu^2 x\,(1-x)^2\,dt + \mu x\,(1-x)\,dW \\ &\approx \begin{cases} -\lambda\,(1-x)\,dt - \mu\,(1-x)\,dW & \text{for } x < x^* \\ \lambda x dt & \text{for } x > x^* \end{cases} \end{aligned}$$

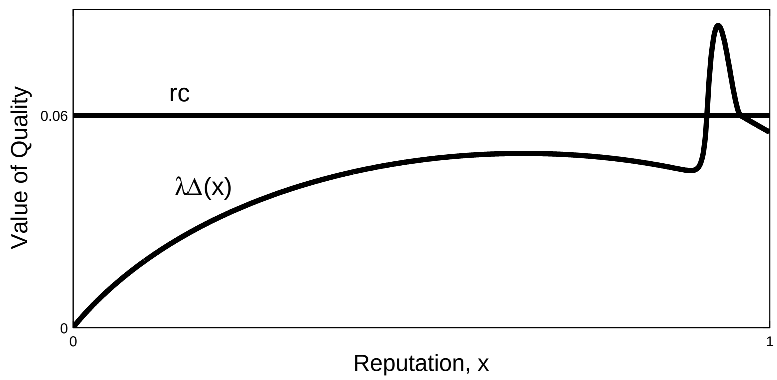

Figure 5: **Shirk-Work-Shirk Equilibrium.** This figure illustrates the asset value of quality in a work-shirk equilibrium, $\Delta_{x^*}(x)$. The straight line equals rc/λ. This figure assumes that $\mu = 1$, $\lambda = 1$, $r = 1$ and $c = 0.06$. The resulting cutoffs are $\underline{x} = 0.910$ and $\overline{x} = 0.958$.

and likewise for $d_L(1 - x)$.

This has two implications. First, while the dividend may be minimised at x^*, the value of quality at the cutoff $\Delta_{x^*}(x^*)$ is largely determined by the dividends at $x < x^*$. Second, the marginal value of reputation and the dividend at $x > x^*$ are small in relation to those at $x < x^*$. This is because a reputational increment essentially disappears when $x_t = x^*$ and this happens much sooner for initial reputations $x_0 > x^*$ than for $x_0 < x^*$. Hence for $x > x^*$, $\Delta_{x^*}(x)$ is an average of low dividends while $x_t > x^*$, and a continuation value $\Delta_{x^*}(x^*)$ when x_t hits x^*. This average comes to less than $\Delta_{x^*}(x^*)$, as required.

6.2 Other Equilibria

Given that reputational updating is slow at $x \approx 0$ and $x \approx 1$, one may expect there to exist *shirk-work-shirk* equilibria, where a firm works when its reputation is between two cutoffs, $x \in [\underline{x}, \overline{x}]$ and shirks elsewhere. A firm with a low reputation is thus trapped in a lower shirk-hole in which market learning is too slow to incentivise effort, while a firm above $\underline{x}$ experiences convergent dynamics around $\overline{x}$.

Simulations indicate that such equilibria may exist for certain parameter ranges (figure 5). Investment incentives in this equilibrium (with $c = 0.06$) are much higher than in the above work-shirk equilibrium (with $c = 0.01$). This is because a firm at the work-shirk cutoff has more to lose when a sequence of bad utility draws can push its reputation into a shirk-region, where it may be stuck forever.

For low costs, Theorem 4(c) shows that shirk-work-shirk equilibria disappear because a firm at the shirk-work cutoff $\underline{x}$ strictly prefers to work. Intuitively, reputational dividends and the value of quality are uniformly bounded below on any interval $[\varepsilon; 1 - \varepsilon]$ so, when costs are small,

all intermediate reputations prefer to work. At the lower cutoff, $\underline{x}$, working is then very profitable since it can make the difference between falling into the shirk-hole and sinking to $x = 0$, and climbing into the work region and rising to $x = \overline{x}$.

While no shirk-work-shirk equilibria exist for low costs, there may be other equilibria. However, these extra equilibria are qualitatively similar to the work-shirk equilibrium and can be characterised by $\{\underline{x}, \overline{x}\}$ with $\eta(x) = 1$ for $x < \underline{x}$ and $\eta(x) = 0$ for $x > \overline{x}$.[12] These equilibria all involve work on $[0, 1-\epsilon]$, so they converge to the work-shirk equilibrium in the Hausdorff metric as $c \to 0$. Moreover, the work-shirk transitions act like reflection barriers implying that $\underline{x} \approx x^*$, so these extra equilibria entail as least as much work as the work-shirk equilibrium.

6.3 High Obsolescence Rate

Theorem 4(d) states that, for sufficiently high λ, the work-shirk equilibria disappear and zero-effort is an equilibrium. The intuition is the same as in the good news case: The high drift towards a work-shirk cutoff $x^* \in (0, 1)$ drives the marginal value of reputation to 0, and with it the reputational dividend and the value of quality. Zero-effort is an equilibrium because the drift towards 0 ensures low values of x_t, and with this a diminishing reputational dividend $D_\theta(x) = \mu^2 x (1-x) V'_\theta(x)$.

We conjecture that zero-effort is the unique equilibrium as $\lambda \to \infty$, as suggested by our numerical simulations. For example, with a shirk-work-shirk profile, the need to incentivise effort at $x = \overline{x}$ implies that the size of this region, $\overline{x} - \underline{x}$, must become small so that having high-quality affects the probability of falling into the shirk region. However, as $\overline{x} - \underline{x}$ becomes sufficiently small, the probability of entering the shirk region grows, and $V(\overline{x})$ and $\Delta(\overline{x})$ converge to 0.

7 Extensions

7.1 Generalized Poisson-Learning

We focused above on good, bad and Brownian news for the sake of clean analytical results. We now discuss how the qualitative insights can shed light on market learning via more general Poisson processes.

7.1.1 Good and Bad News

Assume that the product can enjoy both breakthroughs revealing high quality with intensity μ_H, and breakdowns revealing low quality with intensity μ_L. In this case the reputational dividend is

[12] For example, given the parameters in figure 4, there is another equilibrium with working on $[0, 0.900]$, shirking on $[0.900, 0.944]$, working on $[0.944, 0.9605]$ and shirking on $[0.9605, 1]$.

given by:

$$\begin{aligned} D_\theta(x) &= \mu_H\left(V_\theta(1)-V_\theta(x)\right)+\mu_L\left(V_\theta(x)-V_\theta(0)\right) \\ &= \left(\mu_L-\mu_H\right)V_\theta(x)+\mu_H V_\theta(1)-\mu_L V_\theta(0). \end{aligned}$$

Assume further that breakdowns are more likely than breakthroughs: $\mu_L > \mu_H$. This case is similar to the pure bad news case of section 5:[13] Reputational dividend and value of quality are increasing in reputation, and any equilibrium must be shirk-work.

However, parts (b), (c) and (d) of Theorem 3 no longer hold true: Equilibrium dynamics are similar, but now a shirking low-reputation firm with a high product quality may escape the shirk-hole with a product breakthrough. This renders the value of quality at $x = 0$ strictly positive, even under adverse market beliefs. Thus, proper shirk-work equilibria, where $x = 0$ shirks and $x = 1$ works, fail to exist for low c. While incentives can thus be too high for any proper shirk-work equilibrium, they can at the same time be too low for a full-work equilibrium, e.g. for high values of λ, and equilibrium may fail to exist.[14]

7.1.2 Generalized Bad News

We now modify the bad news case by allowing for occasional failures of high-quality products with intensity $\mu_H < \mu_L$, as is common in the literature, e.g. Abreu, Milgrom, and Pearce (1991). While a product breakdown still causes a discrete hit to the firm's reputation it does not destroy it completely. Lemma 4 states that in equilibrium a firm with a perfect reputation cannot believed to be working. However, the insights of the bad news section are robust to this variation in that, for low costs c, we can still show the existence of shirk-work-shirk, and work-shirk equilibria (rather than work-shirk and full work equilibria in the case of pure bad news).[15]

The shirk-work-shirk equilibrium captures the idea that a product breakdown can put a firm in the "hot-seat" where one more breakdown would finish the firm off, by pushing it into a shirk-hole.

[13] The alternative case, with $\mu_H > \mu_L$, is similar to the pure good news case of section 4. Finally, when quality is discovered with equal intensity, $\mu_H = \mu_L$, incentives are flat and either full-effort or zero-effort is an equilibrium.

[14] This argument does not yet prove that for certain parameter ranges equuilibrium does not exist: For even if incentives are too high for a proper shirk-work equilibrium (and too low for a full-effort equilibrium), they need not be too high for a zero-effort equilibrium.

[15] To prove this we need to slightly adopt the equilibrium existence proof for Brownian learning in Appendix C:

For the work-shirk equilibrium, the only remarkable difference is that an extra term appears in equation (C.4) as the discontinuous reputation dynamics may now jump over the cutoff. However, this term can be matched by an equivalent term - for the firm with initial reputation ℓ^* when it suffers a breakdown - which outweighs the first term by Lemma 9.

The shirk-work-shirk equilibrium on the other hand is in contrast to the Brownian case, in particular to Lemma 12. The key difference is that value functions are now discontinuous, incentivising effort above the cutoff but not in the shirk-hole just below. To construct the shirk-work-shirk equilibrium, we can first choose the lower, shirk-work cutoff low enough so as to discourage work in the shirk-hole, and then reapply the arguments in Appendix C to prove existence of the upper, work-shirk cutoff with the required properties.

In such an equilibrium a firm that fails once will try hard but a firm that fails repeatedly gives up.

7.2 Asymmetric Costs

Throughout this paper we have assumed that firms of different types deliver different utilities to customers but have the same cost of investment. One can extend the model to allow for asymmetric investment costs.

To be precise, suppose the low- and high-quality firms have costs $c_L \geq c_H$. Thus, it is easier for a high-quality firm to stay on the frontier than for a low-quality firm to catch up. Reputational updating is given by equation (2.1). In particular, when

$$rc_L > \lambda\Delta(x) \geq rc_H$$

the drift term disappears and reputation evolves according to

$$dx_t = x_t(1-x_t)\frac{\Pr(dZ_t|H) - \Pr(dZ_t|L)}{x_t \Pr(dZ_t|H) + (1-x_t)\Pr(dZ_t|L)}.$$

In the good news model, equilibria are still work-shirk, with a low quality firm's cutoff x_L^* lying below a high quality firm's cutoff, x_H^*. Using equation (4.1) the reputation in the intermediate range evolves according to $dx/dt = -\mu x(1-x)$. Assuming $\lambda \geq \mu$, it follows that, as in the symmetric case, the firm's reputation ultimately cycles over $[x_L^*, 1]$.

In the bad news model, equilibria are still shirk-work, with a low quality firm's cutoff x_L^* lying above a high quality firm's cutoff, x_H^*. Using equation (5.1) the reputation in the intermediate range evolves according to $dx/dt = \mu x(1-x)$. It follows that, as in the symmetric case, the firm's reputation converges to 0 or 1.

Finally, in the Brownian model, a work-shirk equilibrium consists of cutoffs for the high- and low-quality firms, $x_L^* < x_H^*$. Using equation (6.1) the reputation in the intermediate range evolves according to $d_H x = \mu^2 x(1-x)^2 dt + \mu x(1-x)dW$ and $d_L x = -\mu^2 x^2(1-x)dt + \mu x(1-x)dW$. We conjecture that one can then extend Theorem 4 to show that a work-shirk equilibrium exists when c_L and c_H are sufficiently small. In such an equilibrium, the reputation of a high- and low-quality firm tends to converge to x_H^* and x_L^* respectively.

8 Conclusion

This paper develops a new model of reputation, where the firm invests in its quality, and the quality is imperfectly observed by the market. As customers experience the product, the firm's reputation evolves. This evolution, in turn, affects the incentives of the firm to invest in quality. The model forms a bridge between repeated games and classical models of reputation. In contrast to repeated games, different firms may have different capabilities. In contrast to classical models of reputation, firm's capabilities are a function of past decisions and are therefore endogenous. This model seems realistic: The current state of General Motors is a function of its past hiring policies, investment decisions and reorganisations, all of which are endogenous.

Our results highlight the role of the market information structure in determining reputational incentives: When the market learns through good news, there is a unique work-shirk equilibrium and convergent dynamics. When the market learns through bad news, there is a continuum of shirk-work equilibria and divergent dynamics. The results for Brownian news looks like those for good news: for low costs there is a work-shirk equilibrium, but no shirk-work equilibrium.

The model can be extended in many ways. Within the current framework, one would like to study more general Poisson processes. More generally, one could consider multiple quality levels or allow for quality ladders. Finally, it would be interesting to analyse a model with multiple firms, or models in which firms could enter and exit.

A Good News

A.1 Derivation of Equation (4.3) in Proof of Theorem 2

For a low-quality firm, the value at reputation x^* is given by:

$$V_L(x^*) = \mu x^*.$$

For a high-quality firm, the value at reputation $x^* \in [x^*, 1]$ is given by the differential equation:

$$rV_H(x) = r\mu x + \frac{d}{dt}V_H(x) - \lambda[V_H(x) - V_L(x)] + \mu[V_H(1) - V_H(x)].$$

The firm's value at the cutoff x^* is therefore:

$$\begin{aligned} V_H(x^*) &= \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t}[r\mu x^* + \lambda V_L(x^*) + \mu V_H(1)]\,dt \\ &= \frac{r+\lambda}{r+\lambda+\mu}\mu x^* + \frac{\mu}{r+\lambda+\mu}V_H(1), \end{aligned} \tag{A.1}$$

and the value of the jump in reputation is given by:

$$V_H(1) - V_H(x^*) = \frac{r+\lambda}{r+\lambda+\mu}[V_H(1) - \mu x^*].$$

We now wish to evaluate $V_H(1) - \mu x^*$. As in equation (A.1),

$$V_H(1) - \mu x^* = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t}[r\mu(x_t - x^*) + \lambda(V_L(x_t) - \mu x^*) + \mu(V_H(1) - \mu x^*)]\,dt \tag{A.2}$$

Evaluating the second term on the right hand side,

$$\begin{aligned} \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t}\lambda(V_L(x_t) - \mu x^*)\,dt &= \lambda \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t}\left[\int_{s=t}^{\infty} e^{-r(s-t)}r\mu(x_t - x^*)\,dt\right] \\ &= \frac{\lambda}{\mu+\lambda}\int_{s=0}^{\infty} e^{-rs}r\mu(x_s - x^*)[1 - e^{-(\mu+\lambda)s}]\,dt \end{aligned}$$

Plugging back into (A.2),

$$\frac{r+\lambda}{r+\mu+\lambda}(V_H(1) - \mu x^*) = \int_{t=0}^{\infty} e^{-rt}r\mu(x_t - x^*)\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t}\right]\,dt$$

as desired.

B Bad News

B.1 Derivation of Equations (5.5) and (5.6) in Proof of Theorem 3

Since $\lambda \geq \mu$, $V_L(0) = 0$. Equation (5.2) implies that:

$$\Delta_{x^*}(x) = \int_{t=0}^{\infty} e^{-(r+\lambda)t} \mu V_L(x_t) dt. \tag{B.1}$$

Suppose $x > x^*$. Then the value of a low-quality firm is given by:

$$V_L(x) = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} [r(\mu x_t - c) + \lambda V_H(x_t) + \mu \cdot 0] \, dt. \tag{B.2}$$

The second term here is:

$$\begin{aligned} \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda V_H(x_t) \, dt &= \lambda \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \left[\int_{s=t}^{\infty} e^{-r(s-t)} r(\mu x_s - c) \, ds \right] dt \\ &= \frac{\lambda}{\mu + \lambda} \int_{s=0}^{\infty} e^{-rs} r(\mu x_s - c)[1 - e^{-(\lambda+\mu)s}] \, ds. \end{aligned}$$

Plugging back into (B.2):

$$V_L(x) = \int_{t=0}^{\infty} e^{-rt} r(\mu x_t - c) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t} \right] dt.$$

Using equation (B.1):

$$\begin{aligned} \Delta_{x^*}(x) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t} \mu \left[\int_{s=t}^{\infty} e^{-r(s-t)} r(\mu x_s - c) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)(s-t)} \right] ds \right] dt \\ &= \frac{\mu}{\lambda + \mu} \int_{s=0}^{\infty} e^{-rs} r(\mu x_s - c)(1 - e^{-(\lambda+\mu)s}) \, ds, \end{aligned}$$

which gives us equation (5.5).

Next, suppose $x < x^*$. A low-quality firm's value function is given by $V_L(x) = \int_{t=0}^{\infty} e^{-(r+\mu)t} r\mu x_t \, dt$. Using equation (B.1):

$$\begin{aligned} \Delta_{x^*}(x) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t} \mu \left[\int_{s=t}^{\infty} e^{-(r+\mu)(s-t)} r\mu x_s \, ds \right] dt \\ &= \frac{\mu}{\lambda - \mu} \int_{s=0}^{\infty} e^{-rs} r\mu x_s (e^{-\mu s} - e^{-\lambda s}) \, ds, \end{aligned}$$

which gives us equation (5.6).

C Brownian Motion: Proof of Theorem 4

We will now show that for sufficiently small c there exists a cutoff x^* such that:

(a) Cutoff is indifferent: $\lambda\Delta_{x^*}(x^*) = rc$,

(b) High reputations shirk: $\lambda\Delta_{x^*}(x) < rc$ for $x > x^*$,

(c) Low reputations work: $\lambda\Delta_{x^*}(x) > rc$ for $x < x^*$.

The proof is structured as follows. In Section C.1 we show how to write the marginal value of reputation $V_\theta'(x)$ as the function of the "durability" of incremental reputation. In Section C.2 we perform a change of variables, replacing the probability of high quality $x = \Pr(\theta = H)$ with the log-likelihood ratio $\ell = \log \frac{x}{1-x}$.

Lemma 7 proves part (a): For every small c there exists a cutoff x^* close to 1 such that the cutoff is indifferent: $\lambda\Delta_{x^*}(x^*) = rc$. Lemmas 8 and 9 prove part (b) and (c): Suppose x^* is high and $\lambda\Delta_{x^*}(x^*) = rc$, as guaranteed by Lemma 7. Then $\lambda\widehat{\Delta}_{x^*}(x) < rc$ for $x > x^*$ and $\lambda\widehat{\Delta}_{x^*}(x) > rc$ for $x < x^*$.

Section C.6 shows the essential uniqueness of the work-shirk equilibrium. Lemma 10 proves uniqueness of the cutoff x^* satisfying $\lambda\Delta_{x^*}(x^*) = rc$. Lemmas 11 and 12 show that any other equilibria must also display $\eta = 1$ on $[0; 1-\varepsilon]$, while Lemma 4 ensures $\eta = 0$ on some $[0; 1-\varepsilon']$.

Finally, Lemma 13 shows that work-shirk equilibria disappear as $\lambda \to \infty$, while zero-effort becomes an equilibrium.

C.1 Value of Quality and Reputation: Reprise

Subsection 3.3 showed that both reputation and quality have value in equilibrium. Here we derive non-equilibrium variations of those results and provide an explicit formula for the marginal value of reputation (equation C.1). Like in the proof of Lemma 2 we will utilize non-Markovian effort functions $\overline{\eta}\left(dZ^t\right)$ to do so.

Lemma 5 *For any effort and belief function $\langle \eta, \widetilde{\eta}\rangle$, and any (non-Markovian) alternative effort function $\overline{\eta}\left(dZ^t\right)$ the firm's value equals:*

$$V_\theta(x) = \mathbb{E}_{x_0=x,\theta_0=\theta,\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(r\left(\mu x_t - c\overline{\eta}\left(dZ^t\right)\right) + \left(\eta(x_t) - \overline{\eta}\left(dZ^t\right)\right)\left(\lambda\Delta(x_t) - rc\right)\right)dt\right]$$

Proof. Fix θ_0, x_0 and $\widetilde{\eta}$. Consider first a "one shot deviation" from η, i.e. an alternative effort function $\overline{\eta}$ that differs from η only for $t \in [0, dt]$, say $\overline{\eta} = 1$ while $\eta = 0$. A firm that exerts effort according to η but whose quality θ_{dt} is governed by $\overline{\eta}$ gains $\lambda\Delta(x_0)dt$. Thus, the firm's actual value is the value under the more favorable process $\overline{\eta}$, minus the fair value of the quality subsidy:

$$V_\theta(x) = \mathbb{E}_{x_0=x,\theta_0=\theta,\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt} r\left(\mu x_t - c\eta(x_t)\right)dt\right] - \lambda\Delta(x_0)dt.$$

For "multi-period" deviations, we accumulate a term $\left(\eta(x_t) - \overline{\eta}\left(dZ^t\right)\right)\lambda\Delta(x_t)dt$ whenever $\eta(x_t) \neq \overline{\eta}\left(dZ^t\right)$. Thus, in general we have

$$\begin{aligned} V_\theta(x) &= \mathbb{E}_{x_0=x,\theta_0=\theta,\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(r\left(\mu x_t - c\eta(x_t)\right) + \left(\eta(x_t) - \overline{\eta}\left(dZ^t\right)\right)\lambda\Delta(x_t)\right)dt\right] \\ &= \mathbb{E}_{x_0=x,\theta_0=\theta,\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(r\left(\mu x_t - c\overline{\eta}\left(dZ^t\right)\right) + \left(\eta(x_t) - \overline{\eta}\left(dZ^t\right)\right)\left(\lambda\Delta(x_t) - rc\right)\right)dt\right] \end{aligned}$$

as required. $\square$

This lemma is of a certain intrinsic interest in formalizing one more way to tweak value functions by trading off changes in current payoffs $\lambda\Delta(x_t) - rc$ against the evolution of the state variables. The real reason for deriving it, however, is to apply it to work-shirk effort profiles in situations where $\lambda\Delta(x_t) - rc$ is either small or we know its sign.

To state and prove the next lemma, we need to write the firm's reputation at time t as a function of its initial reputation x_0, realized utilities dZ^t and the Markovian beliefs $\widetilde{\eta}$: $x_t = x_t\left(x_0, dZ^t, \widetilde{\eta}\right)$.

Lemma 6 *(a) In a work-shirk profile with cutoff x^* in which x^* weakly prefers to shirk, i.e. $\lambda\Delta_{x^*}(x^*) \leq rc$, the marginal value of reputation is strictly positive:*

$$V'_\theta(x) > 0 \text{ for all } x \in [0;1].$$

(b) In a work-shirk profile with cutoff x^* in which x^* is indifferent, i.e. $\lambda\Delta_{x^*}(x^*) = rc$, the marginal value is given by:

$$V'_\theta(x) = r\mu\int e^{-rt}\mathbb{E}_{\theta_0=\theta}\left[\frac{\partial x_t}{\partial x}(x, dZ^t, \widetilde{\eta})\right]dt > 0 \text{ for all } x \in [0;1]. \tag{C.1}$$

(c) In a work-shirk profile with cutoff x^*, the value of quality at x^* is strictly positive:

$$\Delta_{x^*}(x^*) > 0.$$

Proof. For (a) and (b) let $\overline{\eta}\left(dZ^t\right) = \eta(x_t(x+dx, dZ^t, \widetilde{\eta}))$ be the non-Markovian strategy of a firm with initial reputation x that mimicks the effort of a firm starting with initial reputation $x + dx$.

We decompose the incremental value of reputation as follows:

$$V_{\theta,\eta}(x+dx) - V_{\theta,\eta}(x) = \left[V_{\theta,\eta}(x+dx) - V_{\theta,\overline{\eta}}(x)\right] + \left[V_{\theta,\overline{\eta}}(x) - V_{\theta,\eta}(x)\right]$$

The first term is the reputational advantage under mimicked effort. It is determined by the partial derivative of future reputation with respect to current reputation:

$$V_{\theta,\eta}(x+dx) - V_{\theta,\overline{\eta}}(x) = r\mu \int e^{-rt}\mathbb{E}_{\theta_0=\theta,\overline{\eta}}\left[x_t(x+dx, dZ^t, \widetilde{\eta}) - x_t\left(x, dZ^t, \widetilde{\eta}\right)\right] dt$$

This term is always positive. Taking the limit as $dt \to 0$ gives rise to equation C.1, proving (b).

The second term is the net value of shirking whenever $\overline{\eta}\left(dZ^t\right) = 0 < 1 = \eta(x_t)$. It is given by

$$\begin{aligned}
V_{\theta,\overline{\eta}}(x) - V_{\theta,\eta}(x) &= \mathbb{E}_{x_0=x,\theta_0=\theta,\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\eta(x_t) - \overline{\eta}\left(dZ^t\right)\right)(rc - \lambda\Delta(x_t))\, dt\right] \\
&= \mathbb{E}_{\theta_0=\theta,\overline{\eta}}\left[\int_0^\infty e^{-rt}\left(\eta(x_t(x, dZ^t, \widetilde{\eta})) - \eta(x_t(x+dx, dZ^t, \widetilde{\eta}))\right)\left(rc - \lambda\Delta(x_t(x, dZ^t, \widetilde{\eta}))\right) dt\right] \\
&= \mathbb{E}_{\theta_0=\theta,\overline{\eta}}\left[\int_{x_t(x,dZ^t,\widetilde{\eta})<x^*<x_t(x+dx,dZ^t,\widetilde{\eta})} e^{-rt}\left(rc - \lambda\Delta(x_t(x, dZ^t, \widetilde{\eta}))\right) dt\right]
\end{aligned}$$

The first line applies Lemma 5. The second line applies the definition of $\overline{\eta}\left(dZ^t\right)$. The third line utilizes that $x_t(x, dZ^t, \widetilde{\eta}) < x_t(x+dx, dZ^t, \widetilde{\eta})$ and the effort functions disagree iff these trajectories are on opposite sides of x^*.

When $\lambda\Delta_{x^*}(x^*) \leq rc$, this term is positive, proving part (a): Reducing effort on the margin is profitable if its cost exceed its benefits. When $\lambda\Delta_{x^*}(x^*) = rc$, this term is of order dx^2 because both $\int e^{-rt}\Pr_{\theta_0=\theta,\overline{\eta}}\left[x_t(x, dZ^t, \widetilde{\eta}) < x^* < x_t(x+dx, dZ^t, \widetilde{\eta})\right] dt$ and $rc - \lambda\Delta(x_t(x, dZ^t, \widetilde{\eta}))$ are of order dx, proving part (b).

(c) If the value of quality at the cutoff was non-positive $\Delta_{x^*}(x^*) \leq 0$, the marginal value of reputation is positive $V_H'(x) > 0$ by part (a) and so is the reputational dividend $D_H(x) = \mu^2 x(1-x) V_H'(x) > 0$. But then, also the value of quality would be positive $\Delta_{x^*}(x^*) > 0$.

the premise in part (a) would be satisfied, yielding $V_H'(x) > 0$ for all x. Thus (6.3) implies that $\Delta_{x^*}(x^*) > 0$, yielding a contradiction. $\square$

Part (b) has the flavor of the envelope theorem: when the firm's first-order condition holds at the cutoff, then a change in the initial reputation only affects its payoff through the reputational evolution. Intuitively, a firm with a lower initial reputation works more, leading to a gain of $\Delta(x)$ when a technology shock hits; however, this is exactly offset by the extra cost born by the firm. The marginal value of reputation $V_\theta'(x)$ is thus determined solely by the "durability" of the reputational increment $\frac{\partial x_t}{\partial x_0}$.

Remark: When the reputational evolution hits the cutoff at time T, $x_T = x^*$, the difference in the drift terms $\mathbb{E}\left[d_\theta(x-\varepsilon)\right] - \mathbb{E}\left[d_\theta(x+\varepsilon)\right] = \lambda dt$ diminishes any reputational increment, i.e. $\frac{\partial x_t}{\partial x_0}$ decreases at precisely this rate. When the cutoff x^* is "reflecting" because either $x^* \approx 1$ or $\lambda \gg 0$, the reputational increment dx_t approximately disappears at T and we can restrict the integral in (C.1) to times $t \leq T$.

C.2 Updating Log-likelihood Ratios

Define $\ell(x) = \log(x/(1-x)) \in \mathbb{R} \cup \{-\infty, \infty\}$. Note that $x(\ell) = \frac{e^\ell}{1+e^\ell}$ and $\frac{d\ell}{dx} = \frac{\left(1+e^\ell\right)^2}{e^\ell} = \frac{1}{x(1-x)}$. According to Bayes rule, equilibrium beliefs $\widetilde{\eta}$ affect ℓ via

$$\frac{d\ell}{dt} = \frac{d\ell}{dx}\lambda\left(\widetilde{\eta} - x\right) = \lambda \frac{\left(1+e^\ell\right)^2}{e^\ell}\left(\widetilde{\eta} - \frac{e^\ell}{1+e^\ell}\right) = \begin{cases} \lambda\left(1+e^{-\ell}\right) & \text{for } \widetilde{\eta} = 1, \\ -\lambda\left(1+e^{\ell}\right) & \text{for } \widetilde{\eta} = 0. \end{cases} \tag{C.2}$$

Market learning $dZ = \theta dt + dW$ affects ℓ via [16]

$$d_\theta \ell = \frac{d\ell}{dx} d_\theta x + \frac{1}{2}\frac{d^2\ell}{dx^2} Var\left(d_\theta x\right) dt = \begin{cases} \frac{\mu^2}{2} + \mu dW & \text{for } \theta = H, \\ -\frac{\mu^2}{2} + \mu dW & \text{for } \theta = L. \end{cases}$$

Thus, in a work-shirk profile with cutoff $\ell^* \gg 0$, high reputations $\ell_t \gg 0$ approximately follow a Brownian motion with drift $\lambda \pm \frac{\mu^2}{2}$ reflected at ℓ^*:

$$d_\theta \ell \approx \begin{cases} \left(\lambda \pm \frac{\mu^2}{2}\right) dt + \mu dW & \text{for } \ell < \ell^*, \\ -\infty & \text{for } \ell > \ell^*. \end{cases}$$

Finally, we can write NPVs $\widehat{V}_\theta(\ell) := V_\theta\left(\frac{e^\ell}{1+e^\ell}\right)$, value of quality $\widehat{\Delta}(\ell) := \Delta\left(\frac{e^\ell}{1+e^\ell}\right)$, and effort $\widehat{\eta}(\ell) := \eta\left(\frac{e^\ell}{1+e^\ell}\right)$ as functions of ℓ and obtain:

$$\widehat{\Delta}(\ell) = \int_0^\infty e^{-(r+\lambda)t} \mathbb{E}_{\ell_0=\ell, \theta^t = L}\left[\widehat{D}(\ell_t)\right] dt, \tag{C.3}$$

where the dividend is

$$\widehat{D}(\ell) = \mu^2 \widehat{V}_H'(\ell_t).$$

C.3 Indifference of Cutoff

We now show that for small costs there exists a high cutoff who satisfies the indifference condition. Since $\widehat{\Delta}$ and $\widehat{V}$ depend on c, we subscript them with c where useful.

Lemma 7 *For every $\ell \in \mathbb{R}$ there exists $c > 0$ such that for all $c^* < c$ there exists $\ell^* > \ell$ such that $rc^* = \lambda \widehat{\Delta}_{\ell^*, c^*}(\ell^*)$.*

[16] Remember $\frac{d\ell}{dx} = \frac{1}{x(1-x)}$ and $\frac{d^2\ell}{dx^2} = \frac{2x-1}{x^2(1-x)^2}$. For $\theta = H$ we have $d_H x = \mu^2 x(1-x)^2 dt + \mu x(1-x) dW$ and $Var\left(d_H x\right) = \mu^2 x^2 (1-x)^2$ so that:

$$d_H \ell = \frac{d\ell}{dx} d_H x + \frac{1}{2}\frac{d^2\ell}{dx^2} Var\left(d_\theta x\right) dt = \mu^2(1-x)dt + \mu dW + \frac{1}{2}\mu^2(2x-1) = \frac{\mu^2}{2} + \mu dW.$$

Proof. Fix a cutoff $\ell \in \mathbb{R}$ and consider $\widehat{\Delta}_{\ell,c}(\ell)$ as a function of $c \in [0, \mu\lambda/(r+\lambda)]$. By Lemma 6(c) we have $\widehat{\Delta}_{\ell,c}(\ell) > 0$ for all c. Since $\widehat{\Delta}_{\ell,c}(\ell)$ is continuous in c, it takes on its minimum $\widehat{\Delta}_{\ell,c'}(\ell) > 0$ at some c'.

Define c by $rc = \lambda\widehat{\Delta}_{\ell,c'}(\ell)$ and fix $c^* \in (0, c)$. Using the definitions of c' and c^*,

$$\lambda\widehat{\Delta}_{\ell,c^*}(\ell) \geq \lambda\widehat{\Delta}_{\ell,c'}(\ell) > rc^*,$$

so the firm prefers to work. On the other hand, at $\ell = \infty$:

$$rc^* > r\widehat{\Delta}_{\infty,c^*}(\infty) = 0,$$

so the firm prefers to shirk. By continuity of $\widehat{\Delta}_{\ell,c^*}(\ell)$ as a function of $\ell \in \mathbb{R} \cup \{-\infty, \infty\}$, there exists $\ell^* \in (\ell, \infty)$ with $rc^* = \lambda\widehat{\Delta}_{\ell^*,c^*}(\ell^*)$. $\square$

The daunting array of quantifiers in the statement of this lemma guarantees that we can assume ℓ^* with $rc^* = \lambda\widehat{\Delta}_{\ell^*,c^*}(\ell^*)$ as large as necessary in the upcoming arguments.

C.4 High Reputations Shirk

Lemma 8 shows that firms with high reputations shirk. In proving this result we show that the dividend of quality $\widehat{D}(\ell)$ is much greater below ℓ^* than above ℓ^*. Intuitively, incremental reputation above ℓ^* is less "durable" because it disappears when ℓ_t hits ℓ^* and reputational updating $\frac{d\ell}{dt}$ decelerates from $-\lambda\left(1+e^{\ell}\right) \approx -\infty$ to $\lambda\left(1+e^{-\ell}\right) \approx \lambda$.

Lemma 8 *Suppose ℓ^* is large and $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = rc$. Then $\lambda\widehat{\Delta}_{\ell^*}(\ell) < rc$ for all $\ell > \ell^*$.*

Proof. Fix $\ell > \ell^*$. Suppose ℓ_t hits the cutoff ℓ^* for the first time at $t = T$. We can then write,

$$\begin{aligned}
\widehat{\Delta}_{\ell^*}(\ell) - \widehat{\Delta}_{\ell^*}(\ell^*) &= \mathbb{E}\left[\int_0^T e^{-(r+\lambda)t}\mathbb{E}_{\ell_0=\ell,\theta^t=L}\left[\mu^2\widehat{V}_H'(\ell_t)\right]dt + e^{-(r+\lambda)T}\widehat{\Delta}_{\ell^*}(\ell^*)\right] - \widehat{\Delta}_{\ell^*}(\ell^*) \\
&= \int_0^T e^{-(r+\lambda)t}\left(\mathbb{E}_{\ell_0=\ell,\theta^t=L}\left[\mu^2\widehat{V}_H'(\ell_t)\right] - (r+\lambda)\widehat{\Delta}_{\ell^*}(\ell^*)\right)dt \qquad \text{(C.4)}
\end{aligned}$$

We wish to show that (C.4) is negative, and do so using two claims.

Claim 1. Fix $\alpha > \beta > 0$ and sufficiently high ℓ^*. Then there exists a $\gamma > 0$ such that the discounted probability that the future reputation ℓ_t is between $\ell^* - \alpha$ and $\ell^* - \beta$,

$$(r+\lambda)\int_0^\infty e^{-(r+\lambda)t} \Pr_{\ell_t=\ell^*,\theta^t=L}\left[\ell_t \in [\ell^*-\alpha; \ell^*-\beta]\right]dt \qquad \text{(C.5)}$$

is bounded below by γ independently of ℓ^*.

Proof. Remember that the reputational dynamics ℓ_t for high values of ℓ^* are approximated by a reflected Brownian motion with finite drift:

$$d\ell^L \approx \begin{cases} \left(\lambda - \frac{\mu^2}{2}\right) dt + \mu dW & \text{for } \ell < \ell^* \\ -\infty & \text{for } \ell > \ell^* \end{cases} \tag{C.6}$$

The process ℓ_t therefore has a positive probability of lying in $[\ell^* - \alpha; \ell^* - \beta]$ at any time t, so (C.5) can be bounded below by $\varepsilon > 0$, for all sufficiently high ℓ^*.

Claim 2. Fix $\alpha > \beta > 0$ and $M > 0$. Suppose ℓ^* is sufficiently high and $rc = \lambda \widehat{\Delta}_{\ell^*,c}(\ell^*)$. Then

$$\widehat{V}'_{H,\ell^*}(\ell^* - \gamma) > M\widehat{V}'_{H,\ell^*}(\ell^* + \delta) \text{ for all } \gamma \in [\beta; \alpha] \text{ and all } \delta > 0.$$

Proof. Let $\gamma \in [\beta; \alpha]$. When ℓ^* is sufficiently high, the reputational dynamics are given by (C.6). The expected drift is bounded above, and the expected time T before the reputational dynamic starting at $\ell_0 = \ell^* - \gamma$ reaches the cutoff $\ell_T|_{\ell_0 = \ell^* - \gamma} = \ell^*$, is bounded below independently of ℓ^*. Thus, $\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell')\right]$ for $\ell' \in [\ell^* - \alpha; \ell^* - \beta]$ is bounded away from 0 as $\ell^* \to \infty$.

Next, consider $\ell^* + \delta$. The expected time T before $\ell_T|_{\ell_0 = \ell^* + \delta} = \ell^*$ uniformly converges to 0. This is easier to see for the posterior x_t than for the log-likelihood-ratio ℓ, as $\mathbb{E}\left[\frac{d_H x}{dt}\right] = -\lambda x$ is bounded away from 0 while $1 - x^*$ converges to 0. Thus, $\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* + \delta)\right]$ for $\delta > 0$ converges to 0 as $\ell^* \to \infty$.

For large values of ℓ^* we can ignore in $\widehat{V}'_{H,\ell^*}(\ell) \geq r\mu \int e^{-rt}\mathbb{E}\left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{d\ell_t}{d\ell_0}(\ell)\right] dt$ all terms where ℓ_t and ℓ are on different sides of ℓ^*. Since $e^\ell/(1 + e^\ell)^2$ is decreasing in $\ell > 0$ we get bounds $e^{\ell_t(\ell^* - \gamma)}/(1 + e^{\ell_t(\ell^* - \gamma)})^2 \geq e^{\ell^*}/(1 + e^{\ell^*})^2 \geq e^{\ell_t(\ell^* + \delta)}/\left(1 + e^{\ell_t(\ell^* + \delta)}\right)^2$ and equation (C.1) implies

$$\frac{\widehat{V}'_{H,\ell^*}(\ell^* - \gamma)}{\widehat{V}'_{H,\ell^*}(\ell^* + \delta)} \geq \frac{r\mu\left[\frac{e^{\ell^*}}{(1+e^{\ell^*})^2}\right]\int e^{-rt}\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* - \gamma)\right] dt}{r\mu\left[\frac{e^{\ell^*}}{(1+e^{\ell^*})^2}\right]\int e^{-rt}\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* + \delta)\right] dt} \geq \frac{\int e^{-rt}\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* - \gamma)\right] dt}{\int e^{-rt}\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* + \delta)\right] dt}$$

Therefore, $\widehat{V}'_{H,\ell^*}(\ell^* - \gamma)/\widehat{V}'_{H,\ell^*}(\ell^* + \delta)$ diverges to ∞ as $\ell^* \to \infty$, uniformly over all $\gamma \in [\beta; \alpha]$ and $\delta > 0$.

Proof of Lemma. We now show that the right hand side of equation (C.4) is negative. Let $\ell > \ell^* \gg 0$. Fix $\alpha > \beta > 0$ and $\varepsilon > 0$, and choose $M = \frac{1}{\varepsilon}$. Equation (6.3) implies that

$$\begin{aligned}
(r+\lambda)\widehat{\Delta}_{\ell^*}(\ell^*) &= (r+\lambda)\int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{\ell_0=\ell^*,\theta^t=L}\left[\mu^2\widehat{V}_H'(\ell_t)\right]dt \\
&\geq (r+\lambda)\int_0^\infty e^{-(r+\lambda)t}\begin{pmatrix} \Pr_{\ell_0=\ell^*,\theta^t=L}(\ell_t\in[\ell^*-\beta;\ell^*-\alpha]) \\ *\min_{\ell'\in[\ell^*-\beta;\ell^*-\alpha]}\left\{\mu^2\widehat{V}_H'(\ell')\right\}\end{pmatrix}dt \\
&\geq \varepsilon M \max_{\ell\in[\ell^*,\infty)}\left\{\mu^2\widehat{V}_H'(\ell)\right\} \\
&= \max_{\ell\in[\ell^*,\infty)}\left\{\mu^2\widehat{V}_H'(\ell)\right\}
\end{aligned}$$

where the second inequality uses Claims 1 and 2. Hence (C.4) is negative, as required. □

C.5 Low Reputations Work

Lemma 9 shows that firms with low reputations work. For reputations $\ell \in \left[\overline{\ell}, \ell^*\right]$ for some $\overline{\ell}$ defined below, the optimality of working follows directly by showing that $\widehat{V}_\theta'(\ell)$ and $\widehat{\Delta}'(\ell)$ are decreasing on $\left[\overline{\ell}, \ell^*\right]$. For reputations $\ell < \overline{\ell}$ the result follows from the closeness of $\widehat{\Delta}_{\ell^*}(\cdot)$ and $\widehat{\Delta}_\infty(\cdot)$.

Lemma 9 *Assume ℓ^* is large, costs c are small and $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = rc$. Then $\lambda\widehat{\Delta}_{\ell^*}(\ell) > rc$ for all $\ell < \ell^*$.*

Proof. *Claim 1.* There exists $\underline{\ell}$ sufficiently large such that $\widehat{V}_{H,\ell^*}'(\cdot)$ is strictly decreasing on $[\underline{\ell}, \ell^*]$ for any $\ell^* > \underline{\ell}$.

Proof. For $0 \ll \ell_t < \ell^*$, the reputational dynamics ℓ_t are approximately a Brownian motion (C.6). Let $\underline{\ell}$ be sufficiently high and consider $\ell_0 \in [\underline{\ell}, \ell^*]$. As long as $\ell_t < \ell^*$ we have $\frac{\partial \ell_t}{\partial \ell_0} \approx 1$. When the trajectory hits ℓ^* then $\frac{\partial \ell_t}{\partial \ell_0} \approx 0$ since ℓ^* is reflecting. Using equation (C.1)

$$\widehat{V}_H'(\ell) = r\mu\int_0^\infty e^{-rt}\mathbb{E}_{\ell_0}\left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}\right]dt \approx r\mu\int_0^\infty e^{-rt}\mathbb{E}_{\ell_0}\left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\mathbf{1}_{t<T(\ell_0)}\right]dt \qquad \text{(C.7)}$$

where $T(\ell_0)$ is the time ℓ_t first hits ℓ^*. Since $e^{\ell_t}/(1+e^{\ell_t})^2$ is strictly decreasing for $\ell_t > 0$, and $T(\ell_0)$ is decreasing in ℓ_0, equation (C.7) is strictly decreasing in ℓ on $[\underline{\ell}, \ell^*]$.

Claim 2. There exists $\overline{\ell}$ sufficiently large such that $\widehat{\Delta}_{\ell^*}(\cdot)$ is strictly decreasing on $\left[\overline{\ell}, \ell^*\right]$ for any $\ell^* > \overline{\ell}$.

Proof. Pick $\underline{\ell}$ as in Claim 1. Since the reputational dynamics are determined by reflected Brownian motion with finite drift (C.6), we can pick $\overline{\ell} \gg \underline{\ell}$ such that for any $\ell^* > \overline{\ell}$, $\Pr_{\ell_0\in[\overline{\ell},\ell^*]}(\ell_t \in [\underline{\ell}, \ell^*]) \approx 1$. Claim 1 says that $\widehat{V}_{H,\ell^*}'(\cdot)$ is strictly decreasing on $[\underline{\ell}, \ell^*]$. Equation (C.3) says that

$\widehat{\Delta}_{\ell^*}(\ell)$ is the integral over $\widehat{V}'_{H,\ell^*}(\ell_t)$, yielding the result.

Claim 3. Assume that $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = rc$ and fix any $\bar{\ell}$. Then $\widehat{\Delta}_{\ell^*}(\cdot)$ converges to $\widehat{\Delta}_\infty(\cdot)$ uniformly on $[-\infty, \bar{\ell}]$ as $\ell^* \to \infty$.

Proof. As $\ell^* \to \infty$, $\widehat{\Delta}_{\ell^*}(\ell)$ converges pointwise to $\widehat{\Delta}_\infty(\ell)$ for all ℓ. Let $\ell^* \gg \bar{\ell}$. For any $\ell < \bar{\ell}$, equations (C.1) and (C.3) imply that, $\widehat{V}'_{\theta,\ell^*}(\ell)$, and thus $\widehat{\Delta}_{\ell^*}(\ell)$, depend on ℓ^* only on trajectories ℓ_t that reach ℓ^*. The future discounted probability of these trajectories converges to 0 as $\ell^* \to \infty$, so the convergence is uniform for $\ell < \bar{\ell}$.

Proof of Lemma. Choose $0 \ll \bar{\ell} \ll \ell^*$. Claim 2 implies that $\widehat{\Delta}_{\ell^*}(\ell)$ is strictly decreasing in ℓ for $\ell \in [\bar{\ell}, \ell^*)$. Since $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = rc$, we have

$$\lambda\widehat{\Delta}_{\ell^*}(\ell^*) > rc \quad \text{for } \ell \in [\bar{\ell}, \ell^*).$$

The function $\widehat{\Delta}_\infty(\cdot)$ is bounded away from 0 on $[-\infty, \bar{\ell}]$.[17] Hence Claim 3 implies that $\widehat{\Delta}_{\ell^*}(\ell)$ is bounded away from zero. For small costs c, we get

$$\lambda\widehat{\Delta}_{\ell^*}(\ell^*) > rc \quad \text{for } \ell \in [-\infty, \bar{\ell}],$$

as required. □

C.6 Essential Uniqueness

Lemma 10 shows that, when costs are low, there is at most one work-shirk equilibrium. Intuitively, the cutoff ℓ^* must be close to ∞ because firms with intermediate reputations will prefer to work. The proof then invokes a property of the Brownian reputation dynamics to conclude that the value of quality $\widehat{\Delta}_{\ell^*}(\ell^*)$ is decreasing for high ℓ^*, and that equilibrium is unique.

Lemma 10 *For sufficiently small c, there is at most one equilibrium cutoff ℓ^*.*

Proof. *Claim 1.* For every $\bar{\ell} < \infty$ and sufficiently low $c = c\left(\bar{\ell}\right)$, there is no work-shirk equilibrium with $\ell^* < \bar{\ell}$.

Proof. The value of quality of a firm with a perfect reputation $\widehat{\Delta}_{\ell^*}(\infty)$ is bounded below uniformly in cost of effort c and "not-too-high" cutoffs $\ell^* < \bar{\ell}$. Thus, for small c the firm would prefer to work: $rc < \lambda\widehat{\Delta}_{\ell^*}(\infty)$.

Claim 2. There exists $\bar{\ell} < \infty$, such that $\widehat{\Delta}_{\ell^*}(\ell^*)$ is strictly decreasing in ℓ^* for $\ell^* \in [\bar{\ell}, \infty]$.

[17] This also follows from Lemma 12.

Proof. For $0 \ll \ell_t < \ell^*$, the reputational dynamics ℓ_t are approximately a Brownian motion, reflected at ℓ^*. Moreover, the dynamics $d\ell$ are approximately identical for a firm with reputation $\ell_t + \epsilon$ and cutoff $\ell^* + \epsilon$. Therefore when $\bar{\ell}$ is sufficiently high and $\ell^* > \bar{\ell}$

$$\begin{aligned}\widehat{V}'_{H,\ell^*}(\ell_0) &= r\mu \int e^{-rt}\mathbb{E}_{\ell_0}\left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}(\ell_0;\ell^*)\right] dt \\ &> r\mu \int e^{-rt}\mathbb{E}_{\ell_0+\epsilon}\left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial \ell_t}{\partial \ell_0}(\ell_0+\epsilon;\ell^*+\epsilon)\right] dt \\ &= \widehat{V}'_{H,\ell^*+\epsilon}(\ell_0+\epsilon)\end{aligned}$$

where the inequality follows from the fact that $e^{\ell_t}/(1+e^{\ell_t})^2$ is decreasing in $\ell_t > 0$. Using equation (C.3), we conclude that

$$\widehat{\Delta}_{\ell^*}(\ell^*) > \widehat{\Delta}_{\ell^*+\epsilon}(\ell^*+\epsilon),$$

as required.

Proof of Lemma. When costs are small there is no equilibrium with $\ell < \bar{\ell}$. Suppose there are two equilibria with cutoffs $\ell_1^*, \ell_2^* \geq \bar{\ell}$. Then $\lambda\widehat{\Delta}_{\ell_1^*}(\ell_1^*) = \lambda\widehat{\Delta}_{\ell_1^*}(\ell_2^*)$, which contradicts Claim 1. $\square$

The following two Lemmas show that low reputations work, while Lemma 4 ensures that high reputations shirk.

Lemma 11 *Fix $\varepsilon > 0$. The there exists c' such that for all $c < c'$ and $x \in [\varepsilon; 1-\varepsilon]$,*

$$\lambda\Delta_{\eta,c}(x) > rc'. \tag{C.8}$$

Proof. We bound the left hand side of (C.8) below, uniformly across c, η, x. To do this define the discounted occupancy time by

$$\Phi_t(x|x_0) = \int e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}[\mathbf{1}_{x_t\leq x}]dt$$

and let $\phi_t(x|x_0) = \Phi_t'(x|x_0)$ be the discounted density of x_t. On $[\varepsilon; 1-\varepsilon]$ the density $\phi_t(x|x_0)$ is bounded below by $\underline{\phi}_t$ uniformly across (x_0, η) because the drift of the Brownian motion is bounded above. We then have,

$$\begin{aligned}
\lambda\Delta_{\eta,c}(x) &= \lambda\mu^2\int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}\left[x_t(1-x_t)V_H'(x_t)\right]dt \\
&\geq \lambda\mu^2\varepsilon(1-\varepsilon)\int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}\left[V_H'(x_t)\,\mathbf{1}_{x_t\in[\varepsilon;1-\varepsilon]}\right]dt \\
&= \lambda\mu^2\varepsilon(1-\varepsilon)\int_0^\infty\left[\int_\epsilon^{1-\epsilon}V_H'(x)\phi_t(x|x_0)\,dx\right]dt \\
&\geq \lambda\mu^2\varepsilon(1-\varepsilon)\int_0^\infty \underline{\phi}_t[V_H(1-\epsilon)-V_H(\epsilon)]\,dt \\
&\geq \text{const.}
\end{aligned}$$

for some positive constant, as required. □

Lemma 12 *Fix $\varepsilon > 0$. For sufficiently low costs, a firm with reputation below $1-\varepsilon$ works, i.e. $\lambda\Delta_{\eta,c}(x) > rc$.*

Proof. Fix $\varepsilon > 0$. From Lemma 11 we know that (C.8) holds on $[\varepsilon, 1-\varepsilon]$. By contradiction, assume that there is a shirk-region in $[0;\varepsilon]$ and let the highest shirk-work cutoff be $x^* < \varepsilon$. Since dividends are positive, Theorem 1 implies that

$$\Delta_{\eta,c}(x^*) \geq E[e^{(r+\lambda)T}]\Delta_{\eta,c}(\varepsilon)$$

where T is the time that a process starting at x^* hits ϵ. Since the firm is working for $x \in [x^*, \varepsilon]$, $E[e^{(r+\lambda)T}]$ and therefore $\Delta_{\eta,c}(x^*)/\Delta_{\eta,c}(\varepsilon)$ are bounded away from 0, independently of ε.

Let the cost be sufficiently low so that $c < c'\Delta_{\eta,c}(x^*)/\Delta_{\eta,c}(\varepsilon)$. Then we have

$$rc < rc'\frac{\Delta_{\eta,c}(x^*)}{\Delta_{\eta,c}(\varepsilon)} < \lambda\Delta_{\eta,c}(x^*),$$

where the second inequality uses (C.8) at $x = \varepsilon$. This contradicts the assumption that x^* is indifferent between working and shirking, as required. □

C.7 Zero-effort as $\lambda \to \infty$

Lemma 13 *For sufficiently high λ:*
(a) Pure shirking is an equilibrium.
(b) There is no work-shirk equilibrium with $x^* \in (0,1]$.

Proof. We will show that the dividend $\widehat{V}_\theta'(\ell)$ approaches 0 uniformly over all reputations and all work-shirk effort profiles. That is, $\lim_{\lambda\to\infty}\sup_{\ell^*\in[-\infty,\infty],\ell\in\mathbb{R}}\widehat{V}_{\theta,\ell^*}'(\ell) = 0$, where $\ell^* = \pm\infty$ captures the pure shirk (resp. work) profile. Equation (C.3) then implies that $\lim_{\lambda\to\infty}\sup_{\ell^*\in[-\infty,\infty],\ell\in\mathbb{R}}\lambda\widehat{\Delta}_{\ell^*}(\ell)=0$.

To do so, fix ϵ and let $\ell^{**}>0$ solve $e^{\ell^{**}}/\left(1+e^{\ell^{**}}\right)^2=\epsilon$. First, consider a cutoff ℓ^* in the tail, i.e. $|\ell^*|>\ell^{**}$. For any two processes ℓ_t and ℓ_t', equation (C.2) implies that in a work-shirk equilibrium, $|\ell_t'-\ell_t|$ decreases over time for any realisation of W_t. Hence $\partial\ell_t/\partial\ell_0\leq 1$. Using equation (C.1) we get:

$$\begin{aligned}
\lim_{\lambda\to\infty}\sup_{|\ell^*|>\ell^{**},\ell_0\in\mathbb{R}}\widehat{V}_{\theta,\ell^*}'(\ell_0) &= \lim_{\lambda\to\infty}\sup_{|\ell^*|>\ell^{**},\ell_0}\mathbb{E}\left[r\int e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial\ell_t}{\partial\ell_0}dt\right]\\
&\leq \lim_{\lambda\to\infty}\sup_{|\ell^*|>\ell^{**},\ell_0}\mathbb{E}\left[r\int e^{-rt}\left(\frac{\Pr(|\ell_t|<\ell^{**})}{4}+\frac{\Pr(|\ell_t|\geq\ell^{**})e^{\ell^{**}}}{\left(1+e^{\ell^{**}}\right)^2}\right)dt\right]\\
&\leq\varepsilon.
\end{aligned}$$

The second line uses $e^{\ell}/(1+e^{\ell})^2\leq 1/4$ and $\partial\ell_t/\partial\ell_0\leq 1$. The first term under the integral vanishes because $\lim_{\lambda\to\infty}\sup_{\ell_0}\Pr\left(|\ell_t|<\ell^{**}\right)=0$ for all t. The second term is bounded by ε.

Next, suppose that $|\ell^*|\leq\ell^{**}$. Suppose the process ℓ_t hits ℓ^* at time T. Using equation (C.1) we get:

$$\begin{aligned}
\lim_{\lambda\to\infty}\sup_{|\ell^*|<\ell^{**},\ell_0\in\mathbb{R}}\widehat{V}_\theta'(\ell_0) &= \lim_{\lambda\to\infty}\sup_{|\ell^*|<\ell^{**},\ell_0}\mathbb{E}\left[r\int e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial\ell_t}{\partial\ell_0}dt\right]\\
&\leq \lim_{\lambda\to\infty}\sup_{|\ell^*|<\ell^{**},\ell_0}\mathbb{E}\left[\frac{r}{4}\int_{t=0}^{T}e^{-rt}dt+\frac{r}{4}\int_{t=T}^{\infty}e^{-rt}\frac{\partial\ell_t}{\partial\ell_0}dt\right]\\
&=0.
\end{aligned}$$

The second line again uses $e^{\ell}/(1+e^{\ell})^2\leq 1/4$ and $\partial\ell_t/\partial\ell_0\leq 1$. The first integral vanishes because $\sup_{\ell_0,|\ell^*|<\ell^{**}}\mathbb{E}[T]\to 0$ as $\lambda\to\infty$. The second integral vanishes because reputational increments disappear at an absorbing boundary, i.e. $E[\partial\ell_t/\partial\ell_0(\ell^*)|t\geq T]\to 0$ as $\lambda\to\infty$. $\square$

References

ABREU, D., P. MILGROM, AND D. PEARCE (1991): "Information and Timing in Repeated Partnerships Information and Timing in Repeated Partnerships," *Econometrica*, 59(6), 1713–1733.

BAR-ISAAC, H., AND S. TADELIS (2008): "Seller Reputation," *Foundations and Trends in Microeconomics*, 4(4), 273–351.

BENABOU, R., AND G. LAROQUE (1992): "Using Priviledged Information to Manipulate Markets: Insiders, Gurus and Credibility," *Quarterly Journal of Economics*, 107(3), 921–958.

BORENSTEIN, S., AND M. B. ZIMMERMAN (1988): "Market Incentives for Safe Commercial Airline Operation," *American Economic Review*, 75(5), 913–935.

BOSCH, J.-C., E. W. ECKARD, AND V. SINGAL (1998): "The Competitive Impact of Air Crashes: Stock Market Evidence The Competitive Impact of Air Crashes: Stock Market Evidence," *Journal of Law and Economics*, 41(2), 503–519.

CABRAL, L., AND A. HORTAÇSU (2009): "The Dynamics of Seller Reputation: Theory and Evidence from eBay," forthcoming.

CHALK, A. J. (1987): "Market Forces and Commercial Aircraft Safety Market Forces and Commercial Aircraft Safety," 36(1), 61–81.

COHEN, W. M., AND D. A. LEVINTHAL (1990): "Absorptive Capacity: A New Perspective on Learning and Innovation," *Administrative Science Quarterly*, 35(1), 128–152.

DIAMOND, D. W. (1989): "Reputation Acquisition in Debt Markets," *Journal of Political Economy*, 97(4), 828–862.

DIERMEIER, D., M. KEANE, AND A. MERLO (2005): "A Political Economy Model of Congressional Careers," *American Economic Review*, 95(1), 347–373.

ERICSON, R., AND A. PAKES (1995): "Markov-Perfect Industry Dynamics: A Framework for Empirical Work," *Review of Economic Studies*, 62(1), 53–82.

FAINGOLD, E., AND Y. SANNIKOV (2007): "Reputation Effects and Degenerate Equilibria in Continuous-Time Games," Working paper, Yale University.

FERNANDES, A., AND C. PHELAN (2000): "A Recursive Formulation for Repeated Agency with History Dependence," *Journal of Economic Theory*, 91(2), 223–247.

FOSTER, L., J. HALTIWANGER, AND C. SYVERSON (2008): "Reallocation, Firm Turnover and Efficiency: Selection on Productiviy or Profitability?," *American Economic Review*, 98(1), 394–425.

Fudenberg, D., D. Kreps, and E. Maskin (1990): "Repeated Games with Long-Run and Short-Run Players," *Review of Economic Studies*, 57(4), 555–573.

Holmström, B. (1999): "Managerial Incentive Problems: A Dynamic Perspective," *Review of Economic Studies*, 66(1), 169–182.

Hopenhayn, H. A. (1992): "Entry, Exit, and firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127–1150.

Hubbard, T. N. (1998): "An Empirical Examination of Moral Hazard in the Vehicle Inspection Market," *RAND Journal of Economics*, 29(2), 406–426.

——— (2002): "How Do Consumers Motivate Experts? Reputational Incentives in an Auto Repair Market," *Journal of Law and Economics*, 45(2), 437–468.

Jarque, A. (2008): "Repeated Moral Hazard with Effort Persistence," Working paper, Federal Reserve Bank of Richmond.

Jin, G. Z., and P. Leslie (2003): "The Effect of Information on Product Quality: Evidence from Restaurant Hygiene Grade Cards," *Quarterly Journal of Economics*, 118(2), 409–451.

——— (2009): "Reputational Incentives for Restaurant Hygiene," *AEJ: Microeconomics*, 1(1), 237–67.

Jovanovic, B. (1982): "Selection and the Evolution of Industry," *Econometrica*, 50(3), 649–670.

Klein, B., and K. B. Leffler (1981): "The Role of Market Forces in Assuring Contractual Performance," *Journal of Political Economy*, 89(4), 615–641.

Kreps, D., P. Milgrom, J. Roberts, and R. Wilson (1982): "Rational Cooperation in the Finitely Repeated Prisoners' Dilemma," *Journal of Economic Theory*, 27(2), 245–252.

Mailath, G., and L. Samuelson (2001): "Who Wants a Good Reputation," *Review of Economic Studies*, 68(2), 415–441.

Resnick, P., R. Zeckhauser, J. Swanson, and K. Lockwood (2006): "The value of reputation on eBay: A controlled experiment," *Experimental Economics*, 9(2), 79–101.

Sannikov, Y., and A. Skrzypacz (2007a): "Impossibility of Collusion under Imperfect Monitoring with Flexible Production," *American Economic Review*, 97(5), 1794–1823.

——— (2007b): "The Role of Information in Games with Frequent Actions," Working paper, Princeton University.