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RESEARCH INTO ADVANCED CONCEPTS OF MICROWAVE POWER AMPLIFICATION AND GENERATION UTILIZING LINEAR BEAM DEVICES

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ABSTRACT

This is an interim report which summarizes work during the past six months on some aspects of symmetry analysis applied to the interaction between electron beams and electromagnetic fields. Symmetry analysis is combined with coupled mode theory to explore the space charge waves and their associated phase constants for a coupled system of symmetrically located, parallel, electron beams. Both confined flow and Brillouin focused electron beams are considered.

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I. INTRODUCTION

The objective of this research program is to explore theoretically some aspects of the interaction between a drifting stream of electrons and electromagnetic fields; particular emphasis being given to the possible generation and amplification of millimeter waves. This interim report discusses an application of symmetry analysis to the study of the interaction between electron beams and electromagnetic fields. A system of parallel, uniform electron beams located symmetrically around the circumference of a circle is analyzed. Symmetry analysis enables the number and possible degeneracy of the space charge wave modes of the coupled system to be specified exactly. It also provides a qualitative estimate of the variation of the coupled system's phase constants with the degree of coupling. Combining coupled mode theory with symmetry analysis enables a quantitative estimate of the coupling factors to be made. Expressions are developed for both confined flow and Brillouin focused electron beams.

II. SYMMETRY ANALYSIS OF SPACE CHARGE WAVES ON COUPLED ELECTRON BEAMS

A. Introduction

Symmetry analysis has a number of potential applications to the study of the interaction between electron beams and electromagnetic waves. As an example of one application of symmetry analysis to electron beam devices, the space charge waves for a system of symmetrically located, parallel, coupled electron beams will be discussed. The analysis illustrates how the lowest order space charge wave pair on an isolated electron beam can be used to approximate the space charge waves on a symmetric system of electron beams, and how the resulting space charge wave reduction factors can be determined to good accuracy. In the course of this symmetry analysis, some aspects of group theory are combined with coupled mode theory.

The electron beam system considered has a number of identical, uniform, circular electron beams of radius b, located symmetrically around the circumference of a circle of radius a ($a >> b$). These beams drift parallel to the z axis with a d-c velocity u_o. Initially, confined flow electron beams (infinite magnetic field, no transverse electron motion) are considered; later, electron beams with a finite focusing magnetic field (including the case of Brillouin focusing) will be considered briefly. As a particular example, a system of four electron beams will be analyzed (see Figure 1). However,

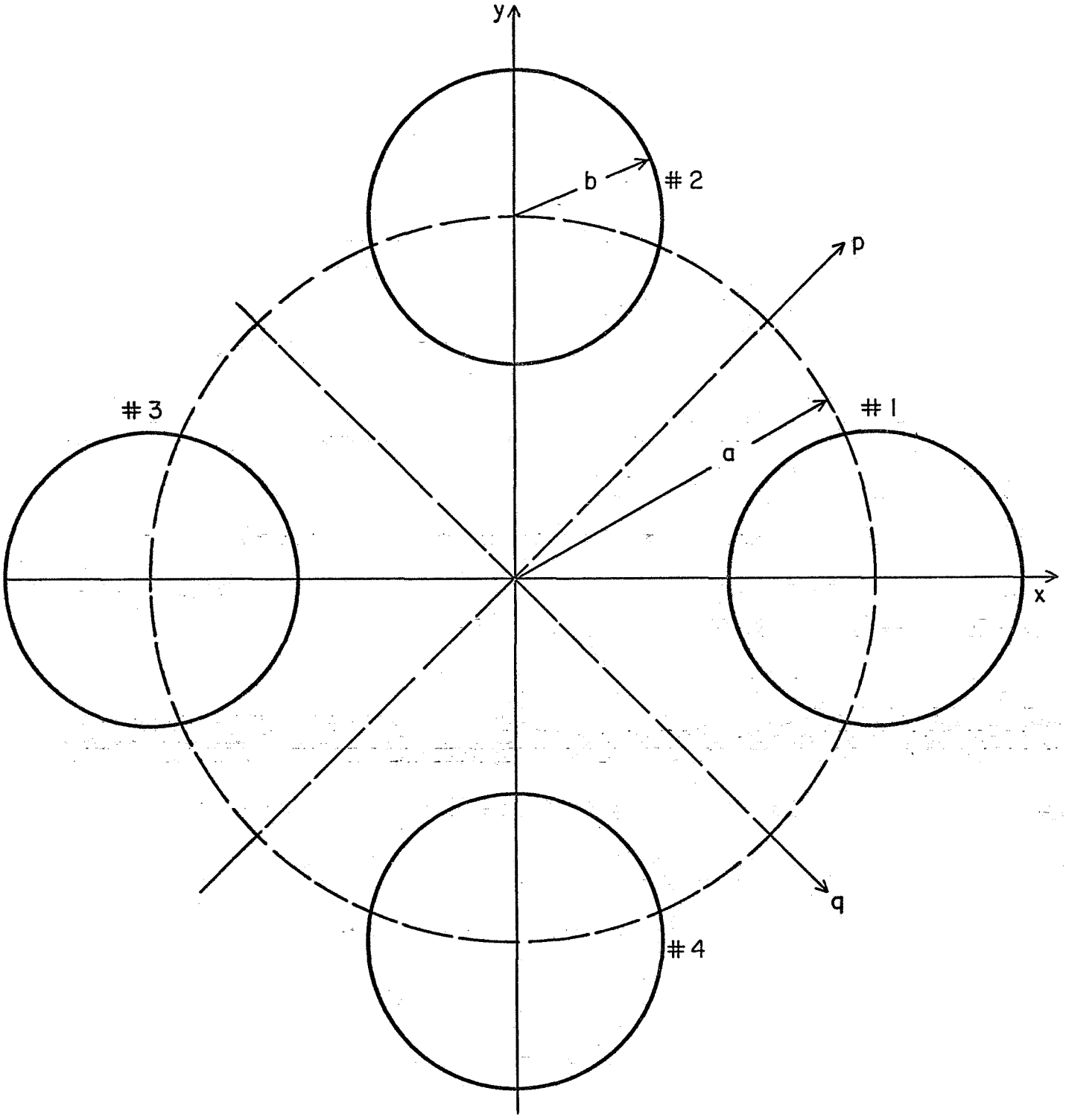

FIGURE 1. Four Coupled Electron Beams of Radius b Symmetrically Located on a Circle of Radius a.

the method is applicable to any number of electron beams.

Coupled mode theory assumes that the modes of the coupled system are well approximated by a linear combination of the modes which propagate on the various electron beams in the absence of coupling. It assumes, then, that the uncoupled modes are perturbed only a small amount, and hence, that the coupling is weak.

For the coupled electron beam system, the eigenvector equation

$$HA = \beta A \tag{1}$$

must be satisfied. Here H is the matrix of coupling coefficients relative to a basis which is taken to be the vector fields of the uncoupled electron beams, A is an eigenvector of H giving the coordinates of a mode of the coupled electron beam system relative to the chosen basis, and β is the phase constant (an eigenvalue) of the mode of the coupled electron beam system with coordinates given by A.

If the electron beams are symmetrically located, then a particular group of symmetry operations will map the system onto itself. The matrix H must commute with the representation D(S) (formed from the basis functions) associated with each symmetry operation S of the symmetry group characteristic of the system,

$$D(S)\ H = H\ D(S). \tag{2}$$

Symmetry, therefore, restricts the form that H can take.

B. Space Charge Waves on a Confined Flow Electron Beam

The solutions for the lowest order, axially symmetric, space charge wave pair on a confined flow, circular electron beam are well known.[1,2] Assuming a time and z dependence of $\exp(j\omega t - j\beta z)$, all the field components, the current density, and the velocity can be expressed in terms of the axial electric field E_z which must satisfy the equations

$$(\nabla_T^2 + \xi^2)E_z = 0, \qquad 0 < r < b, \tag{3a}$$

$$(\nabla_T^2 - \gamma^2)E_z = 0, \qquad r > a, \tag{3b}$$

$$\gamma^2 = \beta^2 - k^2, \tag{3c}$$

$$\xi^2 = \gamma^2 \left[\frac{\beta_p^2}{(\beta_e - \beta)^2} - 1\right], \tag{3d}$$

$$\beta_e = \omega/u_o, \qquad \beta_p = \omega_p/u_o, \tag{3e}$$

where ω_p is the plasma frequency for the electron beam.

The axially symmetric solution to these equations is

$$E_z = E_o J_o(\xi r)e^{j\omega t - j\beta z}, \qquad 0 < r < b, \tag{4a}$$

$$E_z = E_o \frac{J_o(\xi b)}{K_o(\gamma b)} K_o(\gamma r)e^{j\omega t - j\beta z}, \qquad r > b. \tag{4b}$$

The phase constant β is determined from the characteristic equation

$$(\xi b)\frac{J_1(\xi b)}{J_o(\xi b)} = (\gamma b)\frac{K_1(\gamma b)}{K_o(\gamma b)}. \tag{5}$$

In these equations, J_o and J_1 are Bessel functions of the

first kind, and K_o and K_1 are modified Bessel functions of the second kind. For each set of parameters, b, u_o, ω, and ω_p, there are two lowest order solutions for β, corresponding to the fast and slow space charge waves. The phase constant β can be written as

$$\beta = \beta_e \pm \beta_q = \beta_e \pm R\beta_p, \tag{6}$$

where R is the plasma frequency reduction factor ($R < 1$). Branch and Mihran[3] have published curves of R versus $\beta_e b$ for a variety of geometries and parameter values.

A planar, confined flow electron beam, that is, a confined flow electron beam with infinite radius, has $R = 1$. For all confined flow electron beams with a finite radius, R will be in the range from 0 to 1. From (6), $R = \pm(\beta - \beta_e)/\beta_p$ for slow and fast space charge waves, respectively. Thus, R is a measure of the deviation of the phase constant from β_e, measured relative to the magnitude of β_p. There is another physical interpretation of the plasma frequency reduction factor R that can be made. If the a-c axial current density J_z inside the electron beam is calculated, one finds that

$$- j\omega\varepsilon_o \frac{E_z}{J_z} = R^2 \tag{7}$$

for a confined flow electron beam. Thus, R is a measure of the square root of the ratio of the axial electric field to the axial current density. This result will be used later to determine the perturbed phase constant in a multiple electron beam system.

It is assumed here that the magnitude of β_q is appreciable compared to β_e. In other words, the phase velocities of the slow and fast space charge waves,

$$v = \frac{\omega}{(\beta_e \pm \beta_q)} , \tag{8}$$

differ appreciably. As a consequence of this assumption, in the multiple electron beam system the fast space charge waves of the various electron beams will couple together, as will the slow space charge waves. However, there will be no significant coupling between fast and slow space charge waves. This simplifies the analysis of the coupled system.

C. Symmetry of the Multiple Confined Flow Electron Beams

A multiple electron beam system with n confined flow electron beams symmetrically located around the circumference of a circle of radius a has an associated symmetry group designated by C_{nv} (for an introduction to symmetry analysis refer to McWeeny[4]). Thus the four-beam system of Figure 1 belongs to the symmetry group C_{4v}. This group contains the following eight symmetry operations; identity (E), rotation by +90° about the z axis (C_4), rotation by 180° about the z axis (C_2), rotation by -90° about the z axis ($\bar{C}_4$), reflection in the xz plane (σ_y), reflection in the yz plane (σ_x), reflection in the qz plane (σ_p), and reflection in the pz plane (σ_q). Each of these symmetry operations produces a rearrangement of the electron beam system which is indistinguishable from the original arrangement.

The axial electric field of the four-beam coupled system can be expressed in terms of the axial electric fields of the isolated electron beams. That is, the E_{z1}, E_{z2}, E_{z3}, and E_{z4} for the isolated electron beams (given by equations of the form of (4a,b)) are used as a basis for the coupled system. It is necessary to construct a matrix representation of C_{4v} relating to this basis. This is done by determining which of the five irreducible representations of C_{4v} are contained in the representation based on the E_z (refer to McWeeny[4] for a discussion of representations of symmetry groups). The five irreducible representations of C_{4v} are given in the Appendix; there are four one-dimensional and one two-dimensional representations. It is easiest to work with the characters, that is, the traces, of the matrix representations.

The character of one of the matrix representations of C_{4v} is determined by counting the number of basis functions transformed into themselves by the associated symmetry operation. Or, in what amounts to the same procedure, by counting the number of electron beams which are mapped onto themselves by the symmetry operation. For the identity operation, E, all four electron beams are mapped onto themselves. For each of the reflection operations, σ_x and σ_y, two electron beams are mapped onto themselves. For none of the other symmetry operations are any electron beams mapped onto themselves. Table 1 summarizes these results.

TABLE 1. Number of Electron Beams Mapped onto Themselves by a Symmetry Operation

Symmetry Operation	E	C_4	C_2	$\bar{C}_4$	σ_x	σ_y	σ_p	σ_q
Number	4	0	0	0	2	2	0	0

Application of the orthogonality theorem for characters (reference 4, page 122) indicates that three of the five irreducible representations of C_{4v} will appear in the desired representation, and that it can be written as the direct sum of these,

$$D'(S) = D_1(S) \oplus D_3(S) \oplus D_5(S). \tag{9}$$

Here the $D_i(S)$ are the irreducible representations for symmetry operation S of C_{4v} (listed in the Appendix), and $D'(S)$ is the reduced form of the desired representation. This indicates that there will be two nondegenerate modes and a degenerate pair of modes for the coupled electron beam system corresponding to each space charge wave of an isolated electron beam.

The vector fields of the coupled electron beam system must provide a basis for any one of the irreducible representations that are contained in the set $\{D'(S)\}$. These bases give the eigenvectors of H. The bases may be constructed using the basis function generating technique (reference 4, page 128).

$$\sum_S D_1(S)\ S\ E_{z1} = 2[E_{z1} + E_{z2} + E_{z3} + E_{z4}] \tag{10a}$$

$$\sum_S D_3(S)\ S\ E_{z1} = 2[E_{z1} - E_{z2} + E_{z3} - E_{z4}] \tag{10b}$$

$$\sum_S D_5^{11}(S)\ S\ E_{z1} = 2[E_{z1} - E_{z3}] \tag{10c}$$

$$\sum_S D_5^{12}(S)\ S\ E_{z1} = 2[-E_{z2} + E_{z4}] \tag{10d}$$

$$\sum_S D_5^{21}(S)\ S\ E_{z1} = 0 \tag{10e}$$

$$\sum_S D_5^{22}(S)\ S\ E_{z1} = 0 \quad . \tag{10f}$$

These field configurations transform according to the irreducible representations of C_{4v} contained in $\{D'(S)\}$.

The slow space charge wave will be discussed in detail; a similar procedure applies to the fast space charge wave. For the slow space charge wave, H must have the form

$$H = \begin{pmatrix} \beta_o & h_1 & h_2 & h_1 \\ h_1 & \beta_o & h_1 & h_2 \\ h_2 & h_1 & \beta_o & h_1 \\ h_1 & h_2 & h_1 & \beta_o \end{pmatrix} \tag{11}$$

Here, $\beta_o = \beta_e + \beta_q$ is the phase constant of the slow space charge wave of an isolated electron beam. The coupling between nearest neighbor electron beams is denoted by h_1, and between opposite electron beams by h_2; these coupling factors are evaluated in the next section. From the results in Equations (10), the eigenvectors of H are

$$A_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad A_5^{11} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad A_5^{12} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{12}$$

These eigenvectors have the respective eigenvalues

$$\beta_1 = \beta_o + 2h_1 + h_2,$$

$$\beta_3 = \beta_o - 2h_1 + h_2,$$

$$\beta_5 = \beta_o - h_2 \text{ (a degenerate pair).} \tag{13}$$

Figure 2 shows how the phase constants for the slow space charge waves of the coupled electron beam system will vary as the coupling is increased from zero to a finite value.

The coupled mode corresponding to the eivenvalue β_1 has axial symmetry; that is, the fields on all four of the electron beams are in phase. For this coupled mode, $\beta_1 = \beta_e + R_1\beta_p$ is greater than $\beta_o = \beta_e + R\beta_p$ of an isolated electron beam. Thus the plasma frequency reduction factor R_1 for this mode of the coupled system is larger than the value R for an isolated electron beam. The other plasma frequency reduction factors for the coupled system, R_2 and R_3, are both smaller than R.

The phase constants for the fast space charge waves are obtained by reversing the signs of β_p in β_o, and of h_1 and h_2 in Equation (13). The variation of the phase constants for the fast space charge waves of the coupled electron beam

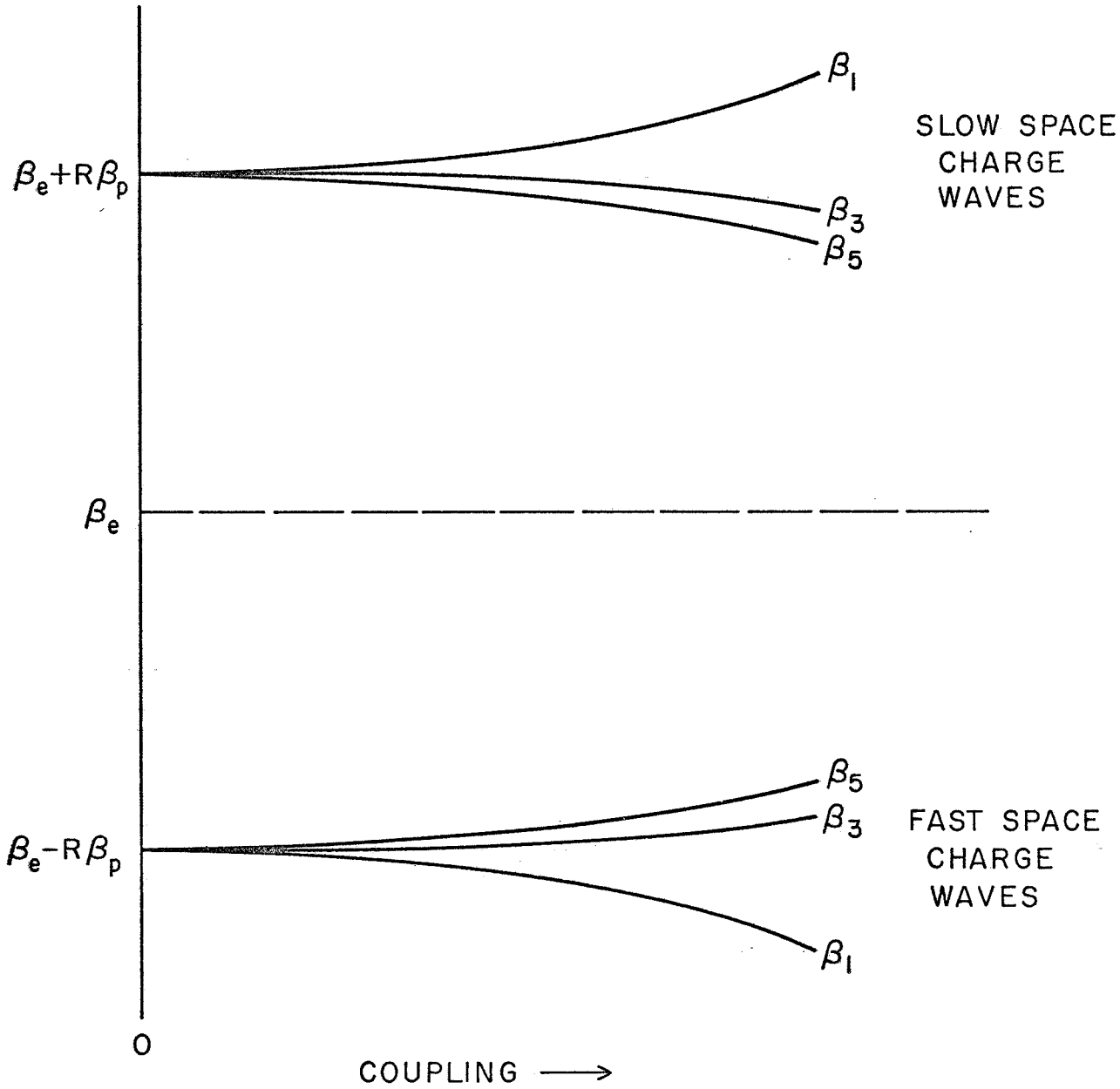

FIGURE 2. Qualitative Variation of Phase Constants for Space Charge Waves on a Four-Beam Coupled System.

system as the coupling is increased from zero to a finite value are also shown in Figure 2. One way to vary the coupling is to vary the radius a of the coupled electron beam system. The coupling is zero for infinite radius and increases from zero as the radius decreases.

D. Calculation of the Coupling Factors

The calculation of the coupling factors, h_1 and h_2, for the coupled electron beam system is discussed now. It is only necessary to consider the coupling between two electron beams; from that result the coupling factors for an n-electron beam system follow directly. The two-beam system is shown in Figure 3; two identical, confined flow electron beams of radius b and separation d drift parallel to each other with a d-c velocity u_o.

It is assumed that the plasma frequency reduction factors for this two-beam coupled system are still proportional to the square root of the ratio between the axial electric field and the axial current density, as in Equation (7). Now, however, for given axial current densities in the two electron beams, the axial electric field in either beam has components produced by the axial currents in both beams. Thus, if R' is the plasma frequency reduction factor for the coupled system, then in the first electron beam,

$$R'^2 = [-\frac{j\omega\varepsilon_o E_z}{J_z}] = [-\frac{j\omega\varepsilon_o[E_{z1}(r_1) + E_{z2}(r_2)]}{j_{z1}(r_1)}] \quad . \qquad (14)$$

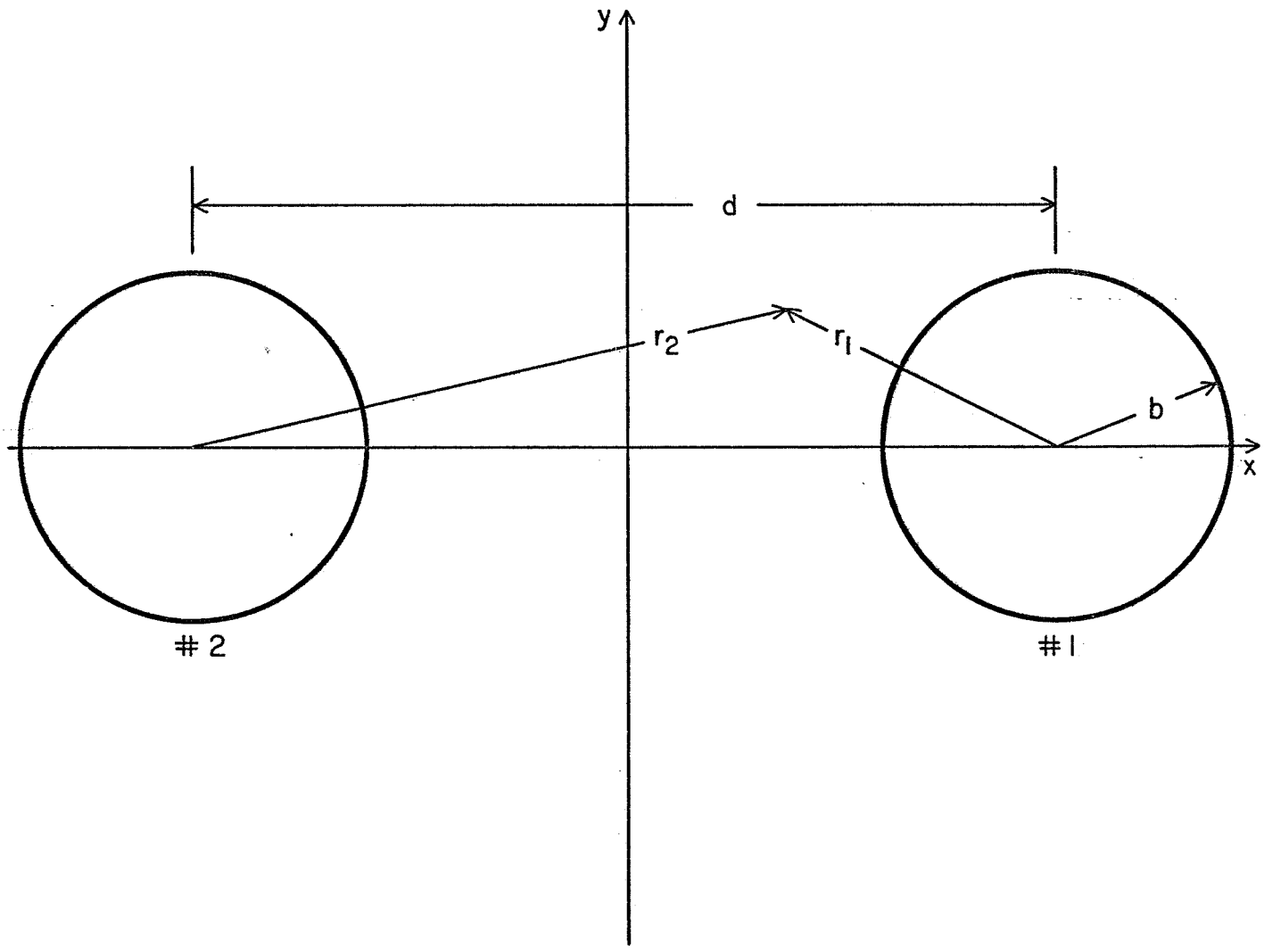

FIGURE 3. Two Coupled Electron Beams of Radius b and Separation d.

Here, r_1 is measured from the axis of the first beam and r_2 from the axis of the second beam.

Consistent with the weak coupling assumption inherent in coupled mode theory, it is assumed that $|E_{z2}(r_2)/E_{z1}(r_1)| << 1$, and that this ratio does not vary appreciably over the first electron beam cross section. This will be valid if both the following conditions are fulfilled. First, $d >> b$ so that $E_{z2}(r_2)$ is small compared to $E_{z1}(r_1)$ and varies slowly, and second, the electron beams are electrically thin ($\beta_e b/R \lesssim 1$) so that $E_{z1}(r_1)$ varies slowly over the cross section of the first electron beam. Then

$$R' \cong \left[-\frac{j\omega\varepsilon_o E_{z1}(r_1)}{J_{z1}(r_1)}\right]^{\frac{1}{2}} \left[1 + \frac{E_{z2}(r_2)}{E_{z1}(r_1)}\right]^{\frac{1}{2}}$$

$$R' \cong R\left[1 + \frac{J_o(\xi b)}{K_o(\gamma b)} K_o(\gamma d)\right]^{\frac{1}{2}} , \tag{15}$$

if the axial currents in the two beams are in phase (the axially symmetric case). However, the assumptions imply that $J_o(\xi b)K_o(\gamma d)/K_o(\gamma b) << 1$, so that to reasonable accuracy,

$$R' \cong R\left[1 + \frac{J_o(\xi b)}{2K_o(\gamma b)} K_o(\gamma d)\right]. \tag{16}$$

For the axially symmetric slow space charge wave of the two-beam system,

$$\beta_1 = \beta_o + h, \tag{17}$$

from symmetry analysis. But we also have

$$\beta_1 = \beta_e + R'\beta_p = \beta_e + R\beta_p + (R'-R)\beta_p$$

$$= \beta_o + (R'-R)\beta_p. \tag{18}$$

Thus, by comparing (17) and (18),

$$h = (R' - R)\beta_p = \frac{R\beta_p}{2} \frac{J_o(\xi b)}{K_o(\gamma b)} K_o(\gamma d). \tag{19}$$

In Figure 4, h/β_p versus γb is shown for two values of d/b.

In the four-beam coupled system, the coupling factors are determined by replacing the distance d in (19) by the appropriate beam separation value; $d = \sqrt{2}\, a$ for h_1 and $d = 2a$ for h_2.

$$h_1 = \frac{R\beta_p}{2} \frac{J_o(\xi b)}{K_o(\gamma b)} K_o\,(\sqrt{2}\,\gamma a), \tag{20a}$$

$$h_2 = \frac{R\beta_p}{2} \frac{J_o(\xi b)}{K_o(\gamma b)} K_o(2\,\gamma a). \tag{20b}$$

Note that in evaluating h_1 and n_2, one can set

$$\xi b = \gamma b \sqrt{\frac{1}{R^2} - 1} \tag{21}$$

with γb the value for an isolated electron beam; this is consistent with the various approximations made.

E. Coupled Electron Beams with a Finite Magnetic Focusing Field

Although confined flow focusing of electron beams can be approximated in practice, the case of a finite magnetic focusing field is of considerable interest also. If a finite magnetic focusing field is used, then, in general, to obtain an uniform cross section as the beam drifts, it is necessary to balance the radial space charge force in the beam by a centripetal force produced by beam rotation. To obtain beam rotation, the magnetic flux linking the cathode is established at a lower value than the main focusing magnetic

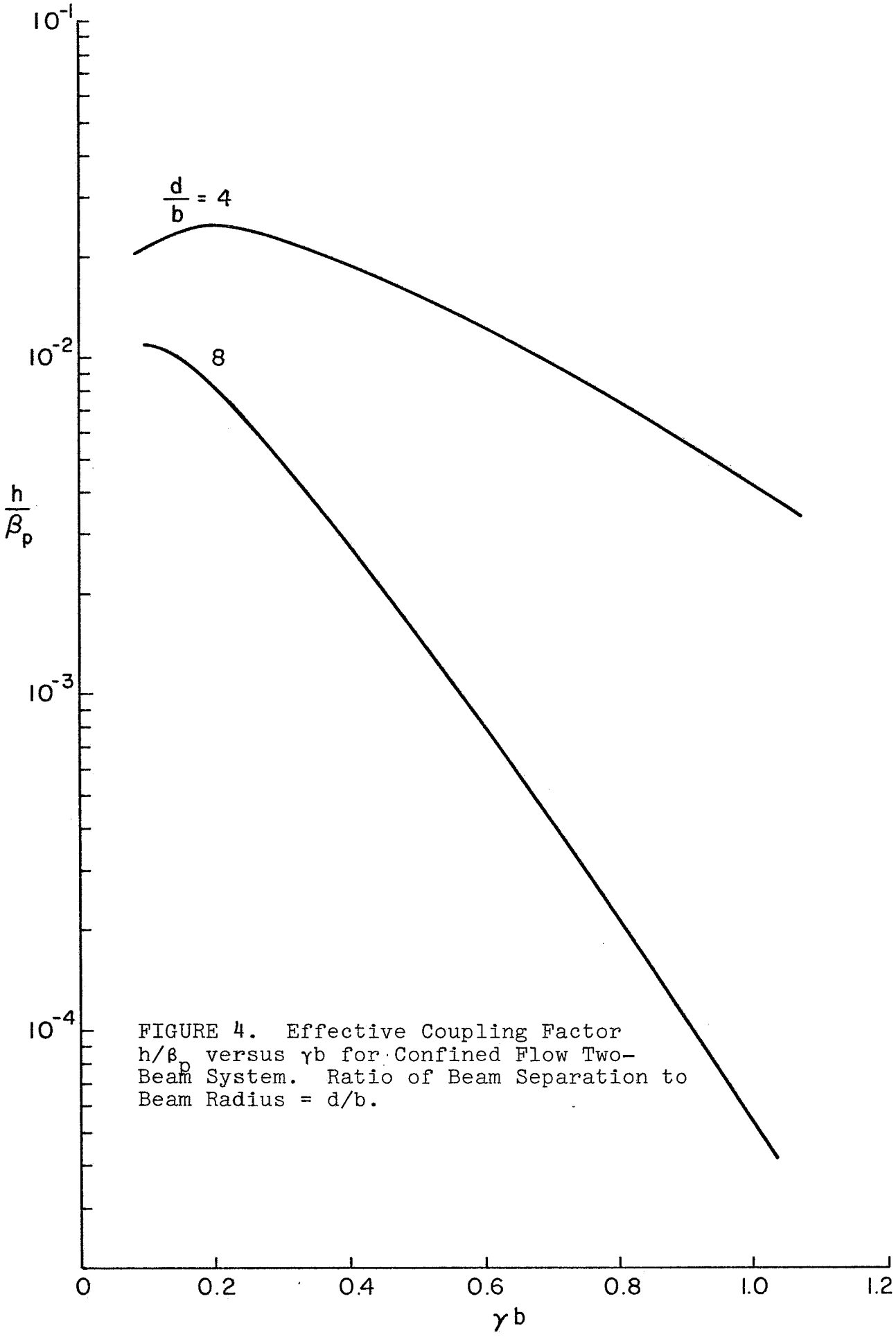

FIGURE 4. Effective Coupling Factor h/β_p versus γb for Confined Flow Two-Beam System. Ratio of Beam Separation to Beam Radius = d/b.

flux. The limiting case is Brillouin focusing[5] where the cathode flux is zero; this case also requires the least main magnetic flux to focus an electron beam of given radius, current density, and beam voltage.

Because of the rotation of each electron beam around its own axis, the symmetry of an n-beam coupled system will now be C_n rather than C_{nv}. That is, reflection in a plane containing the system axis will reverse the direction of rotation of each electron beam, and hence this cannot be a symmetry operation. In particular, the symmetry group for a four-beam system is C_4. This group contains four symmetry operations: identity (E), rotation by +90° (C_4), rotation by 180° (C^2), and rotation by -90° ($\bar{C}_4$). The three irreducible representations of C_4 are given in the Appendix. There are two one-dimensional and one two-dimensional representations.

To obtain the characters of the matrix representations of C_4 for this system, we again count the number of electron beams mapped onto themselves by a symmetry operation. For the identity operation E, all four electron beams are mapped onto themselves. For none of the other symmetry operations, C_4, C_2 and $\bar{C}_4$, are any electron beams mapped onto themselves. Application of the orthogonality theorem for characters indicates that all three irreducible representations of C_4 will appear in the desired representation, and

$$D'(S) = D_1(S) \oplus D_2(S) \oplus D_3(S). \tag{22}$$

Thus in this case, as in the confined flow electron beam case, there will be two nondegenerate modes and a degenerate pair of modes corresponding to each space charge wave of an isolated electron beam. Therefore, by analogy with Equation (13),

$$\beta_1 = \beta_o + 2h_1 + h_2$$

$$\beta_2 = \beta_o - 2h_1 + h_2,$$

$$\beta_3 = \beta_o - h_2 \quad \text{(a degenerate pair).} \tag{23}$$

The behavior of the slow and fast space charge wave phase constants as the coupling is varied for this case has the form shown in Figure 2.

Although the general behavior of the phase constants for an electron beam system with a finite magnetic focusing field is the same as for confined flow focusing, the coupling factors vary with the magnetic field. For simplicity, only the case of Brillouin focusing will be considered here. For the space charge waves on an isolated Brillouin focused electron beam one finds (Trevena[6])

$$\frac{-j\omega\varepsilon_o E_z}{J_z} = R\,\frac{\omega}{\omega_p} \quad . \tag{24}$$

Again considering a two beam system with a beam separation d,

$$\frac{-j\omega\varepsilon_o E_z}{J_z} = R'\,\frac{\omega}{\omega_p} = R\,\frac{\omega}{\omega_p}\left[1 + \frac{I_o(\gamma b)}{K_o(\gamma b)}\,K_o(\gamma d)\right], \tag{25}$$

and

$$R' = R\left[1 + \frac{I_o(\gamma b)}{K_o(\gamma b)}\,K_o(\gamma d)\right] \quad . \tag{26}$$

Note that $I_o(\gamma b)$ rather than $J_o(\xi b)$ appears in these equations; this is a consequence of the somewhat different variation of the fields inside a Brillouin beam compared to a confined flow beam.[6]

Proceeding in a manner similar to that used in the previous section, one finds that

$$h = (R' - R)\,\beta_p = R\,\beta_p \frac{I_o(\gamma b)}{K_o(\gamma b)} K_o(\gamma d)\ . \tag{27}$$

In Figure 5, h/β_p versus γb is shown for two values of d/b. In the four-beam coupled system,

$$h_1 = R\,\beta_p \frac{I_o(\gamma b)}{K_o(\gamma b)} K_o\,(\sqrt{2}\,\gamma a), \tag{28a}$$

$$h_2 = R\,\beta_p \frac{I_o(\gamma b)}{K_o(\gamma b)} K_o\,(2\gamma a). \tag{28b}$$

F. Conclusions

One of the applications of symmetry analysis to the study of the interaction between electron beams and electromagnetic fields has been explored. Symmetry analysis was used to determine the phase constants for the space charge waves on a symmetric system of coupled electron beams.

At a qualitative level, symmetry analysis is able to predict the number of modes and the occurrence of possible degeneracies of these modes. Further, it predicts the direction in which the phase constant for each mode will shift, relative to the isolated electron beam value, as the coupling between the electron beams increases. It was found that the number of modes of the coupled electron beam system, and the general

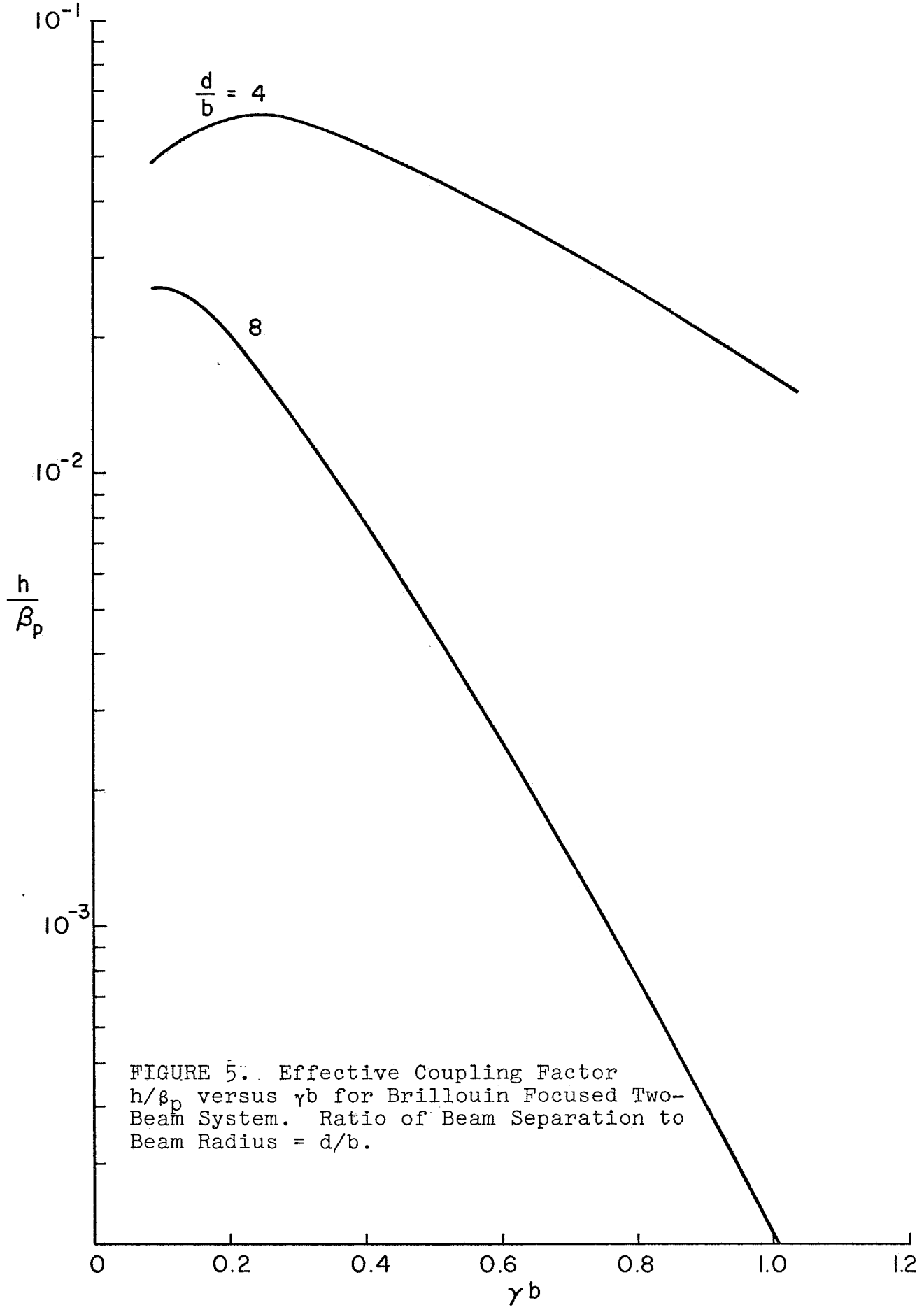

FIGURE 5. Effective Coupling Factor h/β_p versus γb for Brillouin Focused Two-Beam System. Ratio of Beam Separation to Beam Radius = d/b.

behavior of their phase constants with the magnitude of coupling, is independent of whether the magnetic focusing field is infinite or finite, even though the associated symmetry group is different for these two cases.

By combining coupled mode theory with symmetry analysis, quantitative estimates of the change in phase constant for the various modes of the coupled system can be obtained. These are expressed in terms of the isolated electron beam parameters and the separation of the electron beams in the coupled system. In particular, analytic expressions for the limiting cases of confined flow focusing and Brillouin focusing were developed. For the axially symmetric mode (E_z in phase in all the electron beams), the effective plasma frequency reduction factor always increases relative to the value for an isolated electron beam. This is true for any number of coupled electron beams. At least for the four-beam case, all the other modes have a reduced value of effective plasma frequency reduction factor.

APPENDIX

IRREDUCIBLE REPRESENTATIONS FOR SYMMETRY GROUPS C_4 and C_{4v}

1. C_4

S	$D_1(S)$	$D_2(S)$	$D_3(S)$
E	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_4	1	-1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
C_2	1	1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\bar{C}_4$	1	-1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Note: Four one-dimensional irreducible representations can be found for C_4, but two of them involve imaginary entries and are complex conjugates. For the purpose of this analysis, this complex conjugate pair are combined to produce a real two-dimensional representation.

2. C_{4v}

S	$D_1(S)$	$D_2(S)$	$D_3(S)$	$D_4(S)$	$D_5(S)$
E	1	1	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_4	1	1	-1	-1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
C_2	1	1	1	1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\bar{C}_4$	1	1	-1	-1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
σ_x	1	-1	1	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
σ_y	1	-1	1	-1	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
σ_p	1	-1	-1	1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
σ_q	1	-1	-1	1	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

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