Incorporating Satisfaction into Customer Value Analysis: Optimal Investment in Lifetime Value

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Abstract

We extend Schmittlein et al.'s model (1987) of customer lifetime value to include satisfaction. Customer purchases are modeled as Poisson events and their rates of occurrence depend on the satisfaction of the most recent purchase encounter. Customers purchase at a higher rate when they are satisfied than when they are dissatisfied. A closed-form formula is derived for predicting total expected dollar spending from a customer base over a time period $(0, T]$. This formula reveals that approximating the mixture arrival processes by a single aggregate Poisson process can systematically under-estimate the total number of purchases and revenue.

Interestingly, the total revenue is increasing and convex in satisfaction. If the cost is sufficiently convex, our model reveals that the aggregate model leads to an over-investment in customer satisfaction. The model is further extended to include three other benefits of customer satisfaction: (1) satisfied customers are likely to spend more per trip on average than dissatisfied customers; (2) satisfied customers are less likely to leave the customer base than dissatisfied customers; and (3) previously satisfied customers can be more (or less) likely to be satisfied in the current visit than previously dissatisfied customers. We show that all the main results carry through to these general settings.

Keywords: Customer Satisfaction, Customer Value Analysis, Hidden Markov Model, Nonstationarity, Stochastic Processes

1 Introduction

Customers are assets and their values grow and decline (Shugan 2005). Segmenting customers based on their lifetime value is a powerful way to target them because marketing mix activities can then aim at enhancing customer value. In fact, predicting and managing customer lifetime value has become central to marketing because the health of a firm is intimately linked to the health of its customer base. This paper develops an analytical framework for forecasting customer lifetime value based on her satisfaction with the firm.

The relevance of this research is evident in the burgeoning practitioner literature on customer relationship management. Industry observers have emphasized the importance of incorporating satisfaction metrics into customer valuation and exhorted firms to balance customer satisfaction and cost control (e.g., Forrester Research 2003, Jupiter Research 2000). This paper provides a formal methodology to assess investment in customer satisfaction by linking it to likely future shopping and purchase patterns and hence revenue flow.

Our analytical modeling framework rests on two premises. First, we posit that customer satisfaction is a key controllable determinant of customer lifetime value. That is, *ceteris paribus*, a satisfied customer has a higher lifetime value than a dissatisfied customer. This premise is both intuitively appealing and empirically sound. Several studies have shown that customer satisfaction is a good predictor for likelihood of repeat purchases and revenue growth (e.g., Anderson and Sullivan 1993, Jones and Sasser 1995). In addition, prior research has shown that customers react negatively to poor service (e.g., stockouts) by switching to another firm on subsequent shopping trips (Fitzsimons 2000).

Second, customer satisfaction can be increased by investing in costly technology or productive processes. For instance, a call center that increases its number of customer representatives will reduce queueing time. Similarly, a catalog firm can improve its logistics processes to shorten delivery time and reduce the incidence of wrong shipments. The investment in these costly productive processes, however, requires a formal quantification of their revenue implication. A goal of this research is to derive a precise relationship between revenue and customer satisfaction by developing a micro-level stochastic purchase model.

We build on the seminal work of Schmittlein et al. (hereafter, abbreviated as SMC) (1987) and Schmittlein and Peterson (1994). Their model assumes that customer purchase arrivals are

Poisson events. Customers are allowed to die (i.e., switch to another firm or leave the product category entirely) in a Poisson manner so that the number of active customers can decline over time. Customers are heterogeneous in their purchase intensity and death propensity. The amount spent on each purchase is normally distributed and is assumed to be independent of the arrival and death processes. They derive an elegant formula to predict the total expected dollar spending from a customer base over a time period.

There are three behavioral mechanisms by which customer satisfaction can affect this classical stochastic purchase model. First, a satisfied customer is likely to have a higher purchase arrival rate and make more trips to the firm. In other words, the firm can increase its market share of the product category by making the customer happy. Second, a satisfied customer is less likely to switch to another seller or leave the product category entirely. That is, a satisfied customer has a lower death rate. Third, a satisfied customer may increase her average spending in the product category on each purchase visit.

The basic model in Section 2 extends SMC to capture situations where arrival rate depends on satisfaction. We derive a closed-form formula for determining the total expected dollar spending and characterize the optimal level of customer satisfaction. Section 3 extends the basic model to capture the effects of satisfaction on death rate and average expenditure. Furthermore, we extend the basic model to allow satisfaction in the current visit to depend on satisfaction in the previous visit. We show analytically that most qualitative insights remain unchanged with these extensions. Section 4 conducts a comprehensive numerical analysis to illustrate the main theoretical results for the general case. Section 5 concludes and suggests future research directions.

This paper makes three contributions. First, we extend the SMC framework to include satisfaction in predicting customer lifetime value. We derive a closed-form formula to predict the total expected dollar spending from a customer base. This formula allows the firm to predict lifetime value based on customer satisfaction, a key indicator of customer health. Second, we show that the total number of purchases is convex and increasing in customer satisfaction. In addition, we find that one will systematically under-estimate the total expected dollar spending if one ignores the non-stationarity in customer arrivals and departures due to the variation in customer satisfaction. Third, the analytical framework allows the firm to actively manage its productive processes to increase customer lifetime value. We prove that if the cost is sufficiently convex, a firm will over-invest in productive processes when it fails to account for the variation in customer satisfaction.

2 The Basic Model

We consider a firm that offers a homogeneous product or service to a group of N customers. The customers are ordered such that customer i is a heavier user than customer j if $1 \leq i < j \leq N$. The production process is inherently stochastic so that a customer is satisfied with probability p, and dissatisfied with probability $(1 - p)$ at each purchase encounter.[1] We assume that the production process does not discriminate customers and that it is independent of previous purchase encounters so that customer satisfaction can be modeled as independently and identically distributed Bernoulli trials.

The inter-purchase time of a customer is assumed to have an exponential distribution. The exponential rate varies with the outcome of each purchase encounter and differs across customers.[2] For customer i, $i \in \{1, \ldots, N\}$, her next purchase comes with arrival rate λ_{iD} if she is dissatisfied, and λ_{iS} if she is satisfied.[3] Customers visit more often when they are satisfied such that $\lambda_{iS} > \lambda_{iD}$ for all i. Our model can accommodate any parameter values.[4]

We assume a Markovian property so that the arrival rate depends only on the most recent purchase encounter. This assumption is reasonable if customers exhibit a kind of "recency effect" and react strongly to the most recent purchase encounter. In Section 3.3, we will extend the basic model to allow a customer's current satisfaction to correlate with her satisfaction in the previous purchase encounter.[5] When a customer defects, she is "dead"; otherwise, the customer is "alive".

[1] Our model may be extended to investigate more than two levels of satisfaction (e.g., three levels such as below expectation, met expectation, above expectation). For ease of exposition, we restrict ourselves to dichotomous levels such as happy versus unhappy, above expectation versus below expectation, satisfied versus dissatisfied, and so on.

[2] Like Schmittlein et al. (1987), we assume exponential inter-purchase times for its analytical tractability. The exponential assumption (i.e., purchase events are Poisson arrivals) has been applied extensively in the marketing literature because of its parsimony and empirical performance (see for example Fader et al. 2003, 2004, Morrison and Schmittlein 1981, Park and Fader 2004). The exponential assumption appears to work well for some product categories (e.g., frequently purchased consumer goods) (Schmittlein et al. 1987). It should be noted that we do not have a stationary Poisson process. We allow purchase and death rates to be dependent on customer satisfaction. Besides customer satisfaction, other marketing mix variables can also influence the rate of purchase arrivals. For instance, Ho et al. (1998) show that rational customers tend to shop more often with more frequent price promotion. Similarly, Bawa and Shoemaker (2004) show that free sample can significantly increase a customer's future purchase arrival rate. Thus, the purchase and departure processes are non-stationary. In fact, our model shows customer satisfaction can be an important source of non-stationarity in customer value analysis.

[3] Anderson et al. (2004) empirically show that the difference in arrival rates can be significant in catalog purchases.

[4] We explicitly capture heterogeneity by allowing each customer to have a personal set of arrival and departure rates. Allowing heterogeneity is important because Rust and Verhoef (2005) show that response to marketing interventions in intermediate-term customer relationship management is highly heterogeneous.

[5] We do not consider the related questions of sizing of customer segment (i.e., which customers to serve) and optimal

Clearly, a customer's satisfaction may affect her propensity to defect. For ease of exposition, we assume that each customer i has a defection rate μ_i that is independent of satisfaction. In Section 3.2, we relax this assumption to allow the defection rate to vary by satisfaction level.

On each purchase encounter, a customer spends a random dollar amount that is independent of purchase outcome (i.e., whether she is satisfied or dissatisfied). The dollar spending follows a general random distribution with expectation $\bar{Q}$. This assumption is reasonable for necessity product markets (e.g., hospital visits) where spending is mainly driven by needs. In Section 3.1, we relax this assumption to allow it to be contingent on the purchase outcome in order to capture some discretionary product markets (e.g., restaurants) where customers may modify spending based on service outcome.

We are interested in addressing the following three managerial questions: (1) What is the expected number of customers remain "alive" at time T?; (2) What is the expected total dollar spending from the customer base during $(0, T]$?; (3) Given a cost of providing customer satisfaction, what is the optimal customer satisfaction probability p^* that maximizes total profit from the customer base?

We will address these questions one by one. (Proofs of all the results can be found in the Appendix.) We first derive the probability of a customer being alive at time T. Because the death rate for customer i, μ_i, is independent of the satisfaction level, the "departure time" for the customer has an exponential distribution with rate μ_i. Therefore,

$$\Pr[\text{Customer } i \text{ is alive at time } T] = e^{-\mu_i T}. \tag{1}$$

Since customers' departure processes are independent of each other, the expected number of customers who remain alive at time T is given by:

$$\text{E}[\text{Number of customers at time } T] = \sum_{i=1}^{N} e^{-\mu_i T}. \tag{2}$$

2.1 Total Expected Dollar Spending During $(0, T]$

Because customers purchase more frequently when they are satisfied, their total expected dollar spending during $(0, T]$ depends on whether they are satisfied or dissatisfied at time $t = 0$.

contact frequency (e.g., number of catalogs to send). See Elsner et al. (2004) for a nice model and application. We also do not study product choice on each visit. See Ho and Chong (2003) for a model of stock-keeping unit choice.

Define $\gamma_i = p\lambda_{iD} + (1-p)\lambda_{iS}$. If all the customers are dissatisfied at time 0, the total expected dollar spending from the customer base during $(0, T]$ is:

$$R_D = \bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) - \frac{p\lambda_{iD}(\lambda_{iS} - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i)T}\right) \right]. \quad (3)$$

If all the customers are satisfied at time 0, the total expected dollar spending from the customer base during $(0, T]$ is:

$$R_S = \bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{(1-p)\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i)T}\right) \right]. \quad (4)$$

Clearly, $R_S > R_D$ so that customers spend more when they are initially happy. Let $\Delta R = R_S - R_D$. It can be easily shown that ΔR increases with the difference in arrival rates $(\lambda_{iS} - \lambda_{iD})$. Thus, ΔR is higher if purchase rate is more responsive to satisfaction. Since ΔR decreases with the death rate μ_i, we infer that the impact of satisfaction is more pronounced in markets where customers have a longer expected life.

The total expected dollar spending from the customer base is $R = pR_S + (1-p)R_D$. Proposition 1 provides a closed-form expression for predicting the total expected dollar spending:

Proposition 1 *The total expected dollar spending during (0,T] from the customer base is:*

$$R = \bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i)T}\right) \right]. \quad (5)$$

Since the purchase arrival rate depends on the satisfaction outcome of one's previous visit, the inter-purchase time is a hyper-exponential random variable. It is exponential with mean $1/\lambda_{iD}$ with probability $(1-p)$ and with mean $1/\lambda_{iS}$ with probability p. In the rest of the paper, we will use the term SMC-p to refer to the model that ignores the non-stationarity in purchase arrivals and that estimates the dollar spending by using the SMC formula with an aggregate arrival rate. The aggregate arrival rate of customer i in the SMC-p model, λ_i^e, should give an average inter-purchase time that is identical to that of the hyper-exponential random variable. Consequently, we have:

$$\frac{1}{\lambda_i^e} = \frac{1-p}{\lambda_{iD}} + \frac{p}{\lambda_{iS}} \qquad \Rightarrow \qquad \lambda_i^e = \frac{\lambda_{iD}\lambda_{iS}}{p\lambda_{iD} + (1-p)\lambda_{iS}} = \frac{\lambda_{iD}\lambda_{iS}}{\gamma_i}. \quad (6)$$

Note that the SMC-p model indirectly captures customer satisfaction through the aggregate purchase arrival rate, λ_i^e. The SMC formula predicts that the expected total dollar spending is:

$$R^e = \bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_i^e}{\mu_i} \left(1 - e^{-\mu_i T}\right) \right]. \quad (7)$$

The importance of the three revenue formulas, equations (3)–(5), can be assessed by a comparison with that of the SMC-p model, equation (7). Equations (3)–(5) and (7) imply the following simple revenue equalities: $R_D = R^e - \eta_D$, $R_S = R^e + \eta_S$, and $R = R^e + \eta$, where η_D, η_S and η are strictly positive, or equivalently, $R_D < R^e$, $R_S > R^e$ and $R > R^e$.[6] The inequalities involving R_D and R_S are intuitive. The first suggests that when the customer base is initially dissatisfied, their dollar spending is less than that given by the SMC-p model. The second implies that when the customer base is initially satisfied, their dollar spending is more than that given by the SMC-p model. Both results simply capture the fact that customers buy more when they are satisfied.

The revenue inequality involving R is surprising. It shows that the total expected dollar spending from the customer base is higher than that given by the SMC-p model. We state this important result formally below.

Proposition 2 *The SMC-p model under-estimates the total dollar spending by an amount η, where*

$$\eta = \bar{Q}\sum_{i=1}^{N}\left[\frac{p(1-p)(\lambda_{iS}-\lambda_{iD})^2}{\gamma_i(\gamma_i+\mu_i)}\left(1-e^{-(\gamma_i+\mu_i)T}\right)\right]. \qquad (8)$$

Several points are worth noting. First, when $p = 0$ (i.e., the customer is always dissatisfied), $p = 1$, (i.e., the customer is always satisfied) or $\lambda_{iS} = \lambda_{iD}$ (i.e., customer arrival rates are not affected by satisfaction), the bias vanishes. That is, our model and the SMC-p model give the same prediction. To the best of our knowledge, this is the first generalization of SMC model to incorporate customer satisfaction. The formula allows one to quantify the marginal value of customer satisfaction so that a firm can weigh this value against the incremental cost of providing a better service. In addition, the quantification fills an important gap in operations management literature where the marginal value of customer satisfaction is often assumed exogenously. Second, the bias increases in the total number of customers (N) and increases linearly in the average dollar spent ($\bar{Q}$) per trip. Third, the bias increases in a quadratic manner in the incremental purchase rate from satisfaction, i.e., $\lambda_{iS} - \lambda_{iD}$. This result implies that the bias is more dramatic in markets

[6]The expression for η is given in Proposition 2, and η_D and η_S are given by:

$$\eta_D = \bar{Q}\sum_{i=1}^{N}[\frac{p\lambda_{iD}(\lambda_{iS}-\lambda_{iD})}{\gamma_i(\gamma_i+\mu_i)}(1-e^{-(\gamma_i+\mu_i)T})],$$

$$\eta_S = \bar{Q}\sum_{i=1}^{N}[\frac{(1-p)\lambda_{iS}(\lambda_{iS}-\lambda_{iD})}{\gamma_i(\gamma_i+\mu_i)}(1-e^{-(\gamma_i+\mu_i)T})].$$

where customers are more sensitive to service quality. Fourth, the bias is larger when customers have a longer expected life (i.e., a low death rate, μ_i).

From Proposition 1, one can show that the total expected dollar spending as a function of the probability of adequate service p is increasing. The proposition below establishes a stronger result that it has an increasing return to scale in p.

Proposition 3 *The total expected dollar spending during (0,T] from the customer base is convex in the probability of adequate service. That is,* $\frac{d^2R}{dp^2} > 0$.

This is an important and surprising result. One would have expected customer satisfaction to have a diminishing return. The result provides a formal justification for why many firms invest relentlessly in customer satisfaction. Our result suggests that this is optimal as long as the costs are either linear or not "too convex" in p. We shall show below (see Proposition 4) that it can be problematic if the costs are sufficiently convex (which can occur in practice).

2.2 Optimal Investment in Customer Satisfaction

Proposition 3 suggests that the total expected dollar spending as a function of the probability of adequate service p is convex. To determine the optimal service, one must know the shape of the cost function. In general, it is reasonable to expect the total cost to be increasing in the probability of adequate service (p) and the total expected number of customer visits (Λ). An informal survey of several call center outsourcing firms suggests that the following two-part tariff pricing structure is often employed:

$$TC(p) \quad = \quad F(p) + c(p) \cdot \Lambda.$$

For analytical tractability, we assume a constant marginal cost per purchase encounter and set $c(p) = c$. From Proposition 1, we have $R = \bar{Q}\Lambda$. Thus, the profit function can be written as follows:

$$\pi = R - TC = (\bar{Q} - c) \cdot \Lambda - F(p) \stackrel{\text{def}}{=} R_m - F(p). \tag{9}$$

It is clear that the modified revenue function, R_m, is convex in p (just as R is). We consider two separate cases of $F(p)$. First, $F(p)$ is concave (possibly linear) in p. As the company invests more in customer satisfaction, it receives an increasing marginal revenue but incurs a constant or decreasing marginal cost. Therefore, the better the service, the higher the profit. It is optimal for companies to seek to achieve a perfect customer satisfaction of 100%. Second, $F(p)$ is strictly convex in p. Here, it costs more to improve each additional incremental level of customer satisfaction. More often than not, this is the case we face in reality. It is not immediately clear what the shape of the profit function looks like as a function of p. Intuitively, when the cost function is "less convex" than the revenue function (i.e., both marginal revenue and marginal cost are increasing in p, but the former outpaces the latter), it again makes sense to pursue a perfect customer satisfaction. If the cost function is "more convex" than the revenue function, however, the profit will eventually decrease as p becomes higher and higher. This means that an interior optimal point exists for p. That is, it is optimal to invest in customer satisfaction up to a level less than 100%.

We now analyze the latter case in detail. Examples of service delivery systems that have a convex cost function are common. They include:

- $M/M/m$ queueing systems: If the cost is directly proportional to either the service rate of the individual servers or the number of servers, m, the cost function is convex as long as customer satisfaction is measured by the probability of not having to wait in the queue at all or by the average waiting time in the system (for details, see Kleinrock 1975).

- $M/M/m/K$ finite-waiting-space queueing systems: If the cost is directly proportional to either the service rate of the individual servers or the number of servers, m, the cost function is convex as long as customer satisfaction is measured by the proportion of lost customers due to finite waiting space.

- The classical single-period newsvendor inventory setup: Under this setup, customer satisfaction is defined by the probability that the demand is being fully met. Thus, if the uncertain demand is distributed with a cumulative distribution $G(\cdot)$ and the newsvendor carries x units of inventory, customer satisfaction, p, is given by $G(x)$. To achieve this level of customer satisfaction, the newsvendor must incur an inventory holding cost of $h \cdot x$ where h is the unit inventory holding cost per unit time. Consequently, the inventory holding cost necessary to achieve a customer satisfaction of p is $h \cdot G^{-1}(p)$. As long as G^{-1} is convex in p (or equivalently, G is concave in x), the cost function is convex. Any distribution that has a monotonically non-increasing probability density function satisfies such a condition. For example, exponential distribution, some Weibull and Gamma distributions, and uniform distribution all have a monotonically non-increasing probability density function.

To obtain managerial insights, we use a simple convex cost function that is quadratic: $F(p) = a + bp^2$. Here, parameter a represents the cost necessary to achieve the lowest customer satisfaction, and b represents how fast the cost increases in p. As we stated before, when b is small such that the cost function is less convex than the revenue function, the profit is convex, and the optimal service level is achieved at either $p = 0$ or $p = 1$. The interesting case is when the cost function is more convex than the revenue function. Again, we compare our optimal investment in customer satisfaction with that of SMC-p model. Similar to equation (9), we define the SMC-p profit to be $\pi^e = R^e - TC(p) = R_m^e - F(p)$, where

$$R_m^e = (\bar{Q} - c) \sum_{i=1}^{N} \left[\frac{\lambda_i^e}{\mu_i} \left(1 - e^{-\mu_i T}\right) \right]. \tag{10}$$

From Proposition 2, we have $\pi = \pi^e + \eta_m$ where

$$\eta_m \quad = \quad (\bar{Q} - c) \sum_{i=1}^{N} [\frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i + \mu_i)}(1 - e^{-(\gamma_i + \mu_i)T})].$$

One can show that there exists a threshold value of customer satisfaction, $\bar{p} \in (0, 1)$, such that η_m is decreasing in p for $p > \bar{p}$ (i.e., $\eta_m' < 0$ for $p > \bar{p}$).

The following proposition states that the firm may over-invest in customer satisfaction when they do not explicitly account for the non-stationarity in customer arrivals due to variation in customer satisfaction.

Proposition 4 *Let p^* be the maximizer of π, and p^{e^*} be the maximizer of π^e. If*

$$b > (\bar{Q} - c) \max \left\{ \sum_{i=1}^{N} \left(\frac{\lambda_{iS}(\lambda_{iS} - \lambda_{iD})^2}{\lambda_{iD}^2} \right) \left(\frac{1 - e^{-\mu_i T}}{\mu_i} \right), \sum_{i=1}^{N} \left(\frac{\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{2\lambda_{iD}} \right) \left(\frac{1 - e^{-\mu_i T}}{\mu_i} \right) \right\},$$

and $p^{e^*} > \bar{p}$ then $p^{e^*} \geq p^*$.[7]

Proposition 4 states that when the firm pursues a high customer satisfaction strategy, it tends to over-invest if it uses the SMC-p model to determine the optimal customer satisfaction (i.e., if it ignores non-stationarity in purchase arrivals due to variation in customer satisfaction).

This result appears counter-intuitive. Since the SMC-p model under-estimates the total profit by η_m, one would expect it to prescribe a lower optimal customer satisfaction level. However, it

[7]Similarly we can show that there exists a $\underline{p}$ such that if $p^{e^*} \leq \underline{p}$ then $p^{e^*} \leq p^*$. This case is less important because customer satisfaction is often high in practice. We choose to leave this out.

is the first derivative of η_m, η_m', not η_m itself, that matters in determining the optimal customer satisfaction. To see this, we note that marginal cost equals marginal revenue at the optimal customer satisfaction. Both models have the same identical marginal cost; they differ only in their marginal revenue. Since $\eta_m = \pi - \pi^e$, the derivative of η_m plays an important role. As indicated above, there exists a threshold $\bar{p}$ such that $\eta_m' < 0$ for $p > \bar{p}$. Therefore, when $p^{e^*} > \bar{p}$, the additional negative marginal revenue η_m' makes the total marginal revenue of our model smaller than that of SMC-p. Consequently, our model prescribes a lower optimal customer satisfaction.

We can show that as the departure rate μ_i gets smaller, the threshold $\bar{p}$ also gets smaller (see the Proof of Proposition 4 in the Appendix). This yields an interesting implication: If the firm has a more loyal customer base, it is more likely to over-invest in customer satisfaction if it uses the SMC-p model. Therefore, the importance of capturing non-stationarity in purchase arrivals is even more critical when the firm enjoys a high customer loyalty.

3 Model Extensions

Section 2 presents a model that explicitly captures the impact of customer satisfaction on the rate of purchase arrival. In this section, we extend the model in three important ways. First, we allow customer satisfaction to influence the average expenditure on each visit so that customers spend more on the current visit if they were satisfied with their previous visit. Second, we let customers' departure processes to be contingent on satisfaction. This extension captures the intuition that unhappy customers are more likely to switch to another firm or leave the product category entirely. Third, we allow customer satisfaction to correlate over time so that customers' past satisfaction may influence their current satisfaction.

3.1 Contingent Spending Amount

We let the average expenditure on each visit to depend on whether the customer was previously satisfied with the firm. If a customer is satisfied, she spends a random amount with an average of Q_S; and when a customer is dissatisfied, she spends a random amount with an average of Q_D. Clearly, we have $Q_S \geq Q_D$.

Recall that the death rate of customer i remains at μ_i. Hence, the probability of customer i being alive at time T is still $e^{-\mu_i T}$. Similarly, the expected number of active customers at time T

is $\sum_{i=1}^{N} e^{-\mu_i T}$.

Since customer satisfaction at each visit is independently and identically distributed, p fraction of the visits will be satisfactory and $(1-p)$ fraction of the visits dissatisfactory. Therefore, the average amount a customer spends is simply Q_S times the number of satisfactory visits, plus Q_D times the number of dissatisfactory visits. The following proposition provides the revised formula for the total expected dollar spending during (0,T] from the customer base:

Proposition 5 *The total expected dollar spending during (0,T] from the customer base is:*

$$R = [pQ_S + (1-p)Q_D] \cdot \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i)T}\right) \right]. \quad (11)$$

Similarly, we show that the SMC-p model under-estimates the above expression by an amount given by:

$$\eta = [pQ_S + (1-p)Q_D] \cdot \sum_{i=1}^{N} \left[\frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i + \mu_i)} (1 - e^{-(\gamma_i + \mu_i)T}) \right]. \quad (12)$$

Again, the revised revenue function is convex in p. Also, the SMC-p model still leads to an over-investment in customer satisfaction. Hence, we conclude that all the main results presented in Section 2 generalize to this more realistic setting.

3.2 Contingent Death Rate

We allow the departure process to be contingent on whether customers are satisfied. Customer i defects with a rate of μ_{iD} if she is dissatisfied and with a rate of μ_{iS} if she is satisfied. As we shall see below, this extension has significantly increased the complexity of the analysis.

Let PA_i be the probability that customer i is alive at time T. Let β_{i1} and β_{i2} be the two roots of the quadratic equation:

$$\beta^2 + [p\lambda_{iD} + (1-p)\lambda_{iS} + \mu_{iD} + \mu_{iS}]\beta + [\mu_{iS}\mu_{iD} + (1-p)\lambda_{iS}\mu_{iD} + p\lambda_{iD}\mu_{iS}] \quad = \quad 0.$$

The probability that customer i is alive at time T is given in the following lemma:

Lemma 1

$$PA_i = A_i \, e^{\beta_{i1}T} + B_i \, e^{\beta_{i2}T}, \quad (13)$$

where

$$A_i \quad = \quad \frac{p\,(\beta_{i1} + \lambda_{iD} + \mu_{iD}) \left[(1-p)\lambda_{iS}(\mu_{iS} - \mu_{iD}) + \beta_{i2}\mu_{iS} + \mu_{iS}^2\right]}{p(1-p)\lambda_{iD}\lambda_{iS}\beta_{i1} - \beta_{i1}[\beta_{i1} + \mu_{iD} + p\lambda_{iD}][\beta_{i2} + \mu_{iS} + (1-p)\lambda_{iS}]},$$

$$B_i \quad = \quad \frac{(1-p)\left[\beta_{i1}\mu_{iD} + \mu_{iD}^2 + p\lambda_{iD}\,(\mu_{iD} - \mu_{iS})\right](\beta_{i2} + \lambda_{iS} + \mu_{iS})}{p(1-p)\lambda_{iD}\lambda_{iS}\beta_{i2} - \beta_{i2}[\beta_{i1} + \mu_{iD} + p\lambda_{iD}][\beta_{i2} + \mu_{iS} + (1-p)\lambda_{iS}]}.$$

It is clear that the expected number of customers remaining at the end of T is $\sum_{i=1}^{N} PA_i$.

Equation (13) is more complex than the one given in equation (1) where the death rate is independent of customer satisfaction. Note that the probability of being alive for each customer is a superposition of two exponential terms. It can be shown that when $\mu_S = \mu_D$, equation (13) reduces to equation (1).

Proposition 6 *The total expected dollar spending from the customer base during (0,T] is*

$$R = \bar{Q} \sum_{i=1}^{N} \left[C_i \left(1 - e^{\beta_{i1} T} \right) + D_i \left(1 - e^{\beta_{i2} T} \right) \right], \tag{14}$$

where

$$\begin{aligned} C_i &= \frac{p\lambda_{iS} \left(\beta_{i1} + \lambda_{iD} + \mu_{iD} \right) \left[\beta_{i2} + \mu_{iS} + (1-p)(\lambda_{iS} - \lambda_{iD}) \right]}{p(1-p)\lambda_{iD}\lambda_{iS}\beta_{i1} - \beta_{i1}[\beta_{i1} + \mu_{iD} + p\lambda_{iD}][\beta_{i2} + \mu_{iS} + (1-p)\lambda_{iS}]}, \\ D_i &= \frac{(1-p)\lambda_{iD} \left(\beta_{i2} + \lambda_{iS} + \mu_{iS} \right) \left[\beta_{i1} + \mu_{iD} + p(\lambda_{iD} - \lambda_{iS}) \right]}{p(1-p)\lambda_{iD}\lambda_{iS}\beta_{i2} - \beta_{i2}[\beta_{i1} + \mu_{iD} + p\lambda_{iD}][\beta_{i2} + \mu_{iS} + (1-p)\lambda_{iS}]}. \end{aligned}$$

Even though the revenue expression is much more complex, we can again show that it is convex in p. We cannot however analytically prove that the corresponding SMC-p model still leads to a systematic downward bias in revenue forecast and an over-investment in customer satisfaction. An extensive numerical analysis in Section 4 suggests that both results do carry through to this more realistic setting as well.

3.3 Serial Correlation in Customer Satisfaction

The basic model assumes that the satisfaction outcomes of successive purchase encounters are independent of each other. This is reasonable if customer satisfaction is primarily driven by factors determined by the service provider, such as the inventory level at a store and the number of servers at a counter. There exist scenarios in which customer satisfaction may correlate over time so that current satisfaction may depend on past satisfaction. This serial correlation could either be negative or positive. For example, if a customer was dissatisfied last time, her service expectation for the forthcoming visit may be lower as a consequence and hence she is more likely to feel satisfied. On the other hand, one can also argue that if the customer was satisfied last time, she is likely to be more positive in assessing the current visit and hence more likely to be satisfied.

We use a Hidden Markov Model (HMM) to model this serial correlation of customer satisfaction. Our extension is similar to a model developed by Netzer et al. (2005). Specifically, we use a two-state Markov chain to capture the transition of customer satisfaction over time. Let p_1 be the probability of satisfaction if the customer was dissatisfied last time and p_2 be the probability of satisfaction if the customer was satisfied last time. Then the transition probability of satisfaction is

$$\begin{pmatrix} 1-p_1 & p_1 \\ 1-p_2 & p_2 \end{pmatrix}.$$

Given the transition probability matrix, the steady-state (or long-run average) probabilities of a customer being dissatisfied and satisfied are $\frac{1-p_2}{1-p_2+p_1}$ and $\frac{p_1}{1-p_2+p_1}$ respectively. If one focuses on the "steady-state" behavior, one can use $p = \frac{p_1}{1-p_2+p_1}$ in our basic model to revise the total expected dollar spending formula.

Proposition 7 *Let $p = \frac{p_1}{1-p_2+p_1}$. Then the total expected dollar spending during (0,T] from the customer base is:*

$$R = (1-p_2+p_1)\cdot \bar{Q}\cdot \sum_{i=1}^{N}\left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i\mu_i}\left(1-e^{-\mu_i T}\right)+\frac{p(1-p)(\lambda_{iS}-\lambda_{iD})^2}{\gamma_i(\gamma_i+\mu_i)}\left(1-e^{-(\gamma_i+\mu_i)T}\right)\right]. \quad (15)$$

The revised revenue function differs from the original revenue function given in (5) only by a multiplicative factor, $(1-p_2+p_1)$. That is, ignoring the time-dependency of satisfaction will cause the revenue forecast to be off by a factor of $(1-p_2+p_1)$. If customer satisfaction is positively correlated over time, we have $p_1 < p_2$ and the basic model over-states the revenue. If satisfaction is negatively correlated over time, we have $p_1 > p_2$ and the basic model under-states the revenue. Clearly, both models give the same prediction when there is no time-dependence (i.e., $p_1 = p_2$). All the main results carry to this more general setting as we vary the steady state customer satisfaction, $p = \frac{p_1}{1-p_2+p_1}$, as long as we keep $1-p_2+p_1$ fixed.

4 Numerical Study

The general model allows two distinct arrival and death rates for a customer, one when the customer is satisfied and the other when she is not. As discussed in Sections 2 and 3, ignoring the non-stationarity in customer arrivals or departures due to variation in customer satisfaction (i.e., using the SMC-p model) can lead to a systematic downward bias in estimating the total expected dollar spending. In addition, we show that the SMC-p model leads to an over-investment in customer satisfaction and suboptimal profits.

To illustrate the above points, we present a systematic numerical analysis using a two-segment market (heavy versus light users). We use equation (6) to determine the "equivalent" arrival rates for the SMC-p model. We derive the "equivalent" death rates by equating (1) and (13). The numerical study also allows us to quantify the nature and magnitude of the potential biases of the SMC-p model.

We choose the model parameters such that satisfied customers are twice as quickly to return and half as quickly to defect. The light user segment has $\lambda_{LS} = 1.2$, $\lambda_{LD} = 0.6$, $\mu_{LS} = 0.3$, and $\mu_{LD} = 0.6$ and the heavy user segment has $\lambda_{HS} = 2.0$, $\lambda_{HD} = 1.0$, $\mu_{HS} = 0.5$, and $\mu_{HD} = 1.0$. These values are consistent with previous empirical estimates (see for example Morrison and Schmittlein 1981, Schmittlein et al. 1987). Without loss of generality, we normalize T to 1.0. The size of the customer base (N) is set to 1000 and the average expenditure $\bar{Q}$ is set to 1.0.

Based on the above parameters, we simulate hypothetical purchases for each of the 1000 customers. The probability of adequate service p and the size of the heavy user segment δ are varied systematically. We assess the predictive performance of our model and the SMC-p model in forecasting the total expected dollar spending. For each model, we measure prediction error by the mean absolute deviation. The relative performance of the two models is evaluated by the relative difference in their mean absolute deviations, i.e., $\frac{\text{MAD}^e - \text{MAD}}{\text{MAD}}$ where MAD^e and MAD are the prediction error of the SMC-p and our models respectively. If our model is better, we will see a positive value in the relative error measure (i.e., $\text{MAD}^e > \text{MAD}$). Table 1 shows the relative error measure across three levels of p (0.2, 0.5, 0.8) and δ (0.2, 0.5, 0.8) values. Clearly, our model dominates the SMC-p model. It also shows that the relative error measure can be as high as 31% and appears to be highest when $p = 0.5$.

Insert Table 1 about here

We investigate the relative magnitude of the total expected dollar spending between the SMC-p model and our model, i.e., $\frac{R - R^e}{R} = \frac{\eta}{R}$. If this difference is small, we have evidence that the SMC-p model is robust to misspecification involving non-stationarity in purchase arrivals and departures due to variation in customer satisfaction. Figure 1 shows the importance of accounting for non-stationarity in purchase arrivals due to variation in customer satisfaction. As shown, the relative difference in total expected dollar spending varies from 4% to 8% as we change the probability of adequate service. Similar to the relative error measure reported above, the difference appears to be highest when $p = 0.5$. This finding reinforces the analytical result that the SMC-p model leads to a downward bias in predicting total expected dollar spending.

Insert Figure 1 about here

We observe a smaller variation in the relative difference in revenue when we vary the size of the heavy segment (δ). For instance, when $p = 0.8$, the relative revenue difference $\frac{\eta}{R}$ is 5.54%, 5.09%, and 5.59% for $\delta = 0.2, 0.5, 0.8$ respectively. We conclude that the relative revenue difference is more sensitive to the probability of adequate service p than the segment size δ.

We now examine the degree of over-investment in customer satisfaction and its impact on profit for the SMC-p model. To this end, we set the constant marginal cost per purchase encounter (c) to 0.3 and the cost necessary to achieve the lowest customer satisfaction (a) to 100. We choose the cost parameter b to ensure that $F(p)$ is sufficiently convex to capture realistic scenarios. Consequently, we set b to 400, 425, 450.

Figure 2 shows the difference between the optimal levels of customer satisfaction between the SMC-p model and our model, i.e., $(p^{e^*} - p^*)$. As expected, the optimal customer satisfaction of the SMC-p model (p^{e^*}) is always greater than that of our model (p^*). In particular, when δ is 0.5, this difference is 22%, 30%, 37% for the three different levels of cost parameters used. Clearly, the cost parameter b plays an important role in deciding the degree of bias in the optimal investment in customer satisfaction.

Figure 3 translates the investment bias in customer satisfaction associated with the SMC-p model into the impact on profit. The figure reports the relative profit loss of the SMC-p model and shows that when δ is 0.5, the firm can increase its profit by 2.12%, 4.88%, 8.40% if it optimally provides a lower level of customer satisfaction under the three cost scenarios. When δ is close to 1.0, similar to Figure 2, the bias vanishes and the two models give the same prediction.

Insert Figures 2 and 3 about here

We also vary the relative magnitudes of arrival and death rates and study their impact on the three measures of interest: $\frac{\eta}{R}$, $(p^{e^*} - p^*)$, and $\frac{\pi(p^*) - \pi(p^{e^*})}{\pi(p^*)}$. In the simulation described below, δ and b are set to 0.5 and 400 respectively.

We vary the ratio in arrival rates between satisfied and dissatisfied customers, $\frac{\lambda_{yD}}{\lambda_{yS}}$, for $y = H, L$ from 0.4 to 0.6 (note that this ratio is set to 0.5 for the base case shown in Figures 1–3). Table 2 shows the sensitivity analysis results. As expected, we find that as the ratio increases (i.e., the arrival rates between satisfied and dissatisfied customers are closer in magnitude), the relative revenue difference $\frac{\eta}{R}$ becomes smaller. However, the relative difference between the optimal levels of customer satisfaction $(p^{e^*} - p^*)$ increases which results in a significant relative profit loss for the SMC-p model. For example, when $\frac{\lambda_{HD}}{\lambda_{HS}} = \frac{\lambda_{LD}}{\lambda_{LS}} = 0.6$, the relative profit loss, i.e., $\frac{\pi(p^*)-\pi(p^{e^*})}{\pi(p^*)}$ is 9.37%.

We also vary the ratio in arrival rates between heavy and light users, $\frac{\lambda_{Lx}}{\lambda_{Hx}}$, where $x = S, D$ from 0.5 to 0.7 (note that this ratio is set to 0.6 for the base case shown in Figures 1–3). Table 3 shows the sensitivity analysis results. The highest relative profit loss is as high as 11.00% and occurs when $\frac{\lambda_{LS}}{\lambda_{HS}} = 0.5$ and $\frac{\lambda_{LD}}{\lambda_{HD}} = 0.7$.

Insert Tables 2 and 3 about here

Similarly, we conduct a sensitive analysis involving the ratio in death rates between satisfied and dissatisfied customers, $\frac{\mu_{yS}}{\mu_{yD}}$, for $y = H, L$ from 0.4 to 0.6 in Table 4 and the ratio in death rates between heavy and light users, $\frac{\mu_{Lx}}{\mu_{Hx}}$ for $x = S, D$ from 0.4 to 0.6 in Table 5. Tables 4 and 5 show the simulation results. The impact on the relative profit loss is smaller when compared with the above sensitivity analysis involving ratios in arrival rates. For instance, the highest relative profit loss is 5.24% in Table 4 and is 3.26% in Table 5.

Insert Tables 4 and 5 about here

Taken together, the significant differences in revenue and profit between the SMC-p model and our model highlight the importance of accounting for non-stationarity in purchase arrivals due to variation in customer satisfaction. Our results show that firms must not use an aggregate approach in analyzing customer satisfaction. Also, they must weigh the benefits of customer satisfaction against its costs. It is not always optimal to pursue customer satisfaction relentlessly.

5 Discussion

In this paper, we present a model that incorporates satisfaction into customer value analysis. By doing so, we incorporate the behavioral customer satisfaction research into the quantitative customer value analysis literature. This is significant because customer satisfaction is an important, if not the most important, contributor of customer lifetime value. Also, customer lifetime value is inherently tied to repeat purchases and it seems odd to ignore customer satisfaction in estimating lifetime value.

We develop our model by building on the seminal work of Schmittlein et al. (1987) and Schmittlein and Peterson (1994). This generalized model allows the purchase rate to vary with satisfaction outcome so that a better service leads to a higher purchase rate. We also explicitly capture heterogeneity by allowing customers to have different purchase rates. Consequently, the purchase rate changes both across customer population and over time in our model.

We derive a formula for determining the total dollar spending from a customer base over a time period. This formula reveals a surprising result: Customer lifetime value has an increasing return to scale in the probability of receiving adequate service. This may explain why many firms pursue customer satisfaction relentlessly. Our formula also suggests a downward bias in revenue prediction if one approximates the mixture Poisson processes by a single aggregate Poisson process (i.e., using the SMC-p model). Finally, we examine how the firm should optimally invest in customer satisfaction when the latter can only be achieved via costly productive processes. We show that the SMC-p model leads to an over-investment in customer satisfaction.

To improve the applicability of our results, we extend our model to allow for satisfaction-dependent expenditure and death rate, and to allow customer satisfaction to be temporally correlated. While these extensions make the formula for the total expected dollar spending more complex, they do not change the qualitative predictions of the formula.

Our model has several managerial implications. First, our model implies that it is crucial to account for non-stationarity in purchase and departure processes due to variation in customer satisfaction into the prediction of total expected dollar spending from a customer base. This finding suggests a natural extension of the classical RFM (Recency, Frequency and Monetary value) model to the RFMS (Recency, Frequency, Monetary value, and Satisfaction) model of predicting total revenue. Second, our model yields a formula for quantifying the incremental benefits of increasing customer satisfaction. Firms can now use our formula to weigh the potential benefits against the costs of increasing customer satisfaction. Researchers in operations management now have a formal way to quantify the benefits of service quality. Third, we believe our model can serve as a useful backend engine for customer relationship management system since every purchase encounter outcome can be captured and used to modify the expected lifetime value of a customer. In this way, customer lifetime value can be updated dynamically and continuously to provide an accurate estimate of the value of a customer base.

Our model opens up several research opportunities. First, it will be useful to estimate our model on a field data set. Such estimation will allow us to study how purchase arrival rates differ across the satisfaction categories and provide a direct way to assess the usefulness of our model in field settings. The recent model by Fader et al. (2005) would serve as a benchmark in such applications. Second, the firm can offer distinct service classes (e.g., premium versus regular) based on lifetime value so that a premium customer receives a better service than a regular customer. It will be fruitful to examine how this kind of discriminating production processes will affect optimal investment in customer value. Third, it will be worthwhile to investigate how referrals by happy customers might affect the level of service to offer and the design of referral reward (e.g., Biyalogorsky et al. 2001). Finally, our model ignores active competition. It will be interesting to explore how optimal investment in lifetime value changes with active rivalry (e.g., Villas-Boas 2004).

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	p		
	0.2	0.5	0.8
δ 0.2	14.11%	30.78%	25.73%
0.5	9.15%	19.25%	14.46%
0.8	7.29%	12.33%	8.98%

Table 1: Relative Difference in the Mean Absolute Deviations

		$\frac{\eta}{R}$			$p^{e^*} - p^*$			$\frac{\pi(p^*)-\pi(p^{e^*})}{\pi(p^*)}$		
	$\frac{\lambda_{HD}}{\lambda_{HS}} \rightarrow$	0.4	0.5	0.6	0.4	0.5	0.6	0.4	0.5	0.6
$\frac{\lambda_{LD}}{\lambda_{LS}}$	0.4	13.72%	9.73%	6.90%	0%	11%	27%	0.00%	0.35%	3.44%
	0.5	11.48%	7.73%	5.07%	0%	22%	34%	0.00%	2.12%	6.31%
	0.6	9.91%	6.33%	3.81%	15%	31%	41%	0.65%	4.61%	9.37%

Table 2: The Impact of Changes in $\frac{\lambda_D}{\lambda_S}$ on Revenue Bias, Over-investment in Satisfaction, and Profit Loss

		$\frac{\eta}{R}$			$p^{e^*} - p^*$			$\frac{\pi(p^*)-\pi(p^{e^*})}{\pi(p^*)}$		
	$\frac{\lambda_{LS}}{\lambda_{HS}} \rightarrow$	0.5	0.6	0.7	0.5	0.6	0.7	0.5	0.6	0.7
$\frac{\lambda_{LD}}{\lambda_{HD}}$	0.5	7.31%	9.35%	11.31%	19%	13%	0%	5.45%	0.57%	0.00%
	0.6	6.01%	7.73%	9.47%	37%	22%	0%	8.18%	2.12%	0.00%
	0.7	5.12%	6.53%	8.04%	43%	30%	11%	11.00%	4.16%	0.32%

Table 3: The Impact of Changes in $\frac{\lambda_L}{\lambda_H}$ on Revenue Bias, Over-investment in Satisfaction, and Profit Loss

		$\frac{\eta}{R}$			$p^{e^*} - p^*$			$\frac{\pi(p^*)-\pi(p^{e^*})}{\pi(p^*)}$		
	$\frac{\mu_{HS}}{\mu_{HD}} \rightarrow$	0.4	0.5	0.6	0.4	0.5	0.6	0.4	0.5	0.6
$\frac{\mu_{LS}}{\mu_{LD}}$	0.4	8.21%	7.70%	7.19%	9%	19%	26%	0.25%	1.47%	3.34%
	0.5	8.24%	7.73%	7.22%	13%	22%	29%	0.58%	2.12%	4.26%
	0.6	8.27%	7.75%	7.24%	17%	25%	32%	1.02%	2.86%	5.24%

Table 4: The Impact of Changes in $\frac{\mu_S}{\mu_D}$ on Revenue Bias, Over-investment in Satisfaction, and Profit Loss

		$\frac{\eta}{R}$			$p^{e^*} - p^*$			$\frac{\pi(p^*)-\pi(p^{e^*})}{\pi(p^*)}$		
	$\frac{\mu_{HS}}{\mu_{HD}} \rightarrow$	0.5	0.6	0.7	0.5	0.6	0.7	0.5	0.6	0.7
$\frac{\mu_{LS}}{\mu_{LD}}$	0.5	7.20%	7.21%	7.22%	22%	24%	27%	2.00%	2.61%	3.26%
	0.6	7.70%	7.73%	7.75%	20%	22%	25%	1.57%	2.12%	2.73%
	0.7	8.18%	8.21%	8.24%	18%	20%	23%	1.20%	1.70%	2.26%

Table 5: The Impact of Changes in $\frac{\mu_L}{\mu_H}$ on Revenue Bias, Over-investment in Satisfaction, and Profit Loss

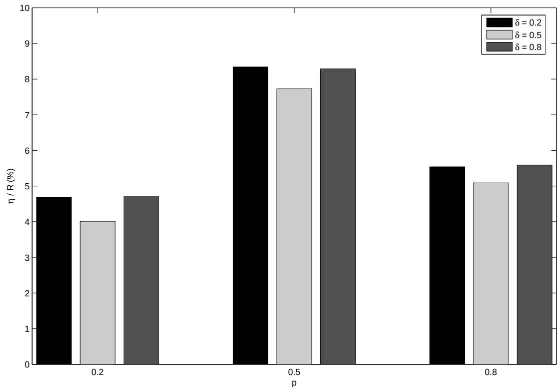

Figure 1: Relative Revenue Difference Between the General and the SMC-p Models

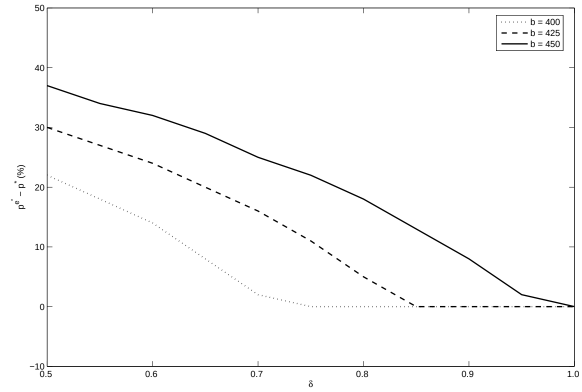

Figure 2: Difference Between Optimal Levels of Customer Satisfaction of the SMC-p and the General Models

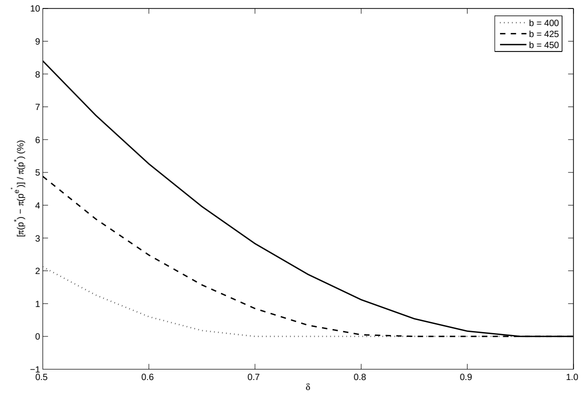

Figure 3: Relative Profit Loss of the SMC-p Model

Appendix

A Revenue Function

In this section, we derive the revenue function. Specifically, we prove Propositions 1, 2, and 3. To simplify analysis, we first derive the expected revenue from a generic customer and ignore her subscript i. We then aggregate the revenue function over all customers.

The main technique we use to study the embedded Markov Chain of the Continuous Time Markov Chain (CTMC) is the so-called "uniformization" (see Ross 1996, pp. 282-284).

Because $\lambda_S > \lambda_D$, the uniformized rate is λ_S. Once uniformized, the CTMC spends an exponential(λ_S) amount of time in each state. Moreover, the transition probabilities in the uniformized chain are as follows (the rows and columns are ordered as: state D for dissatisfied and state S for satisfied):

$$\mathbf{P} = \begin{pmatrix} (1-p)\left(\frac{\lambda_D}{\lambda_S}\right) + \left(1 - \frac{\lambda_D}{\lambda_S}\right) = 1 - p\left(\frac{\lambda_D}{\lambda_S}\right) & p\left(\frac{\lambda_D}{\lambda_S}\right) \\ 1-p & p \end{pmatrix}. \tag{16}$$

Note that of all the transitions from state D into state D, only $\frac{(1-p)(\lambda_D/\lambda_S)}{1-p(\lambda_D/\lambda_S)}$ fraction are real transitions corresponding to customer purchases; the other $\frac{1-\lambda_D/\lambda_S}{1-p(\lambda_D/\lambda_S)}$ fraction are fictitious transitions due to uniformization.

A.1 Proof of Proposition 1

Suppose the customer starts in state i and has had n arrivals in the uniformized Markov chain. Because each real transition corresponds to a purchase by the customer, we would like to know how many of the n arrivals correspond to real transitions. Let $N_j^i(n)$ be the random number of real transitions into state j in the first n transitions of the uniformized embedded Markov chain, where $i, j \in \{D, S\}$. Moreover, let $\bar{N}_j^i(n)$ be its expected value.

As an example, let us examine $N_D^D(n)$. If the customer starts in state D, then with probability $(1-p)(\lambda_D/\lambda_S)$, she makes a real transition into state D, in which case $N_D^D(n) = 1 + N_D^D(n-1)$; with probability $(1-\lambda_D/\lambda_S)$, she makes a fictitious transition into state D, in which case $N_D^D(n) = 0 + N_D^D(n-1)$; and with probability $p\lambda_D/\lambda_S$, she makes a real transition into state S, in which case $N_D^D(n) = 0 + N_D^S(n-1)$. Overall, we have $\bar{N}_D^D(n) = (1-p)(\lambda_D/\lambda_S)(1 + \bar{N}_D^D(n-1)) + (1-\lambda_D/\lambda_S)\bar{N}_D^D(n-1) + p(\lambda_D/\lambda_S)\bar{N}_D^S(n-1) = (1-p)(\lambda_D/\lambda_S) + (1-p\lambda_D/\lambda_S)\bar{N}_D^D(n-1) +$

$p(\lambda_D/\lambda_S)\bar{N}_D^S(n-1)$. Repeating this analysis, we arrive at the following equation,

$$\begin{pmatrix} \bar{N}_D^D(n) & \bar{N}_S^D(n) \\ \bar{N}_D^S(n) & \bar{N}_S^S(n) \end{pmatrix} = \begin{pmatrix} (1-p)\frac{\lambda_D}{\lambda_S} & \frac{p\lambda_D}{\lambda_S} \\ 1-p & p \end{pmatrix} + \begin{pmatrix} 1-\frac{p\lambda_D}{\lambda_S} & \frac{p\lambda_D}{\lambda_S} \\ 1-p & p \end{pmatrix} \begin{pmatrix} \bar{N}_D^D(n-1) & \bar{N}_S^D(n-1) \\ \bar{N}_D^S(n-1) & \bar{N}_S^S(n-1) \end{pmatrix}.$$

If we use matrix notation $\bar{\mathbf{N}}(n)$ to denote $\begin{pmatrix} \bar{N}_D^D(n) & \bar{N}_S^D(n) \\ \bar{N}_D^S(n) & \bar{N}_S^S(n) \end{pmatrix}$, and let $\mathbf{W} = \begin{pmatrix} 0 & 0 \\ (1-p)/p & 1 \end{pmatrix}$, the above equation becomes $\bar{\mathbf{N}}(n) = \mathbf{PW} + \mathbf{P}\bar{\mathbf{N}}(n-1)$.

Noting that $\bar{\mathbf{N}}(0) = \mathbf{0}$, or $\bar{\mathbf{N}}(1) = \mathbf{PW}$, we conclude:

$$\bar{\mathbf{N}}(n) = \left(\sum_{k=1}^{n} \mathbf{P}^k\right)\mathbf{W}. \tag{17}$$

We need to diagonalize $\mathbf{P}$ in order to calculate $\mathbf{P}^k$. We first calculate its eigenvalues:

$$0 \;=\; |\alpha I - P| = (\alpha - 1)\left[\alpha - p\left(1 - \frac{\lambda_D}{\lambda_S}\right)\right].$$

The two eigenvalues are $\alpha_1 = 1$ and $\alpha_2 = p\left(1 - \frac{\lambda_D}{\lambda_S}\right)$. It is straightforward to calculate their corresponding eigenvectors: $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} \frac{p\lambda_D}{\lambda_S} \\ p-1 \end{pmatrix}$ respectively. Therefore, $(X_1, X_2) = \begin{pmatrix} 1 & \frac{p\lambda_D}{\lambda_S} \\ 1 & p-1 \end{pmatrix}$ and $(X_1, X_2)^{-1} = \begin{pmatrix} p-1, & -\frac{p\lambda_D}{\lambda_S} \\ -1, & 1 \end{pmatrix}/\Delta$, where $\Delta = p - 1 - \frac{p\lambda_D}{\lambda_S}$. It is clear that $1 + \Delta = \alpha_2$ and $1 - \alpha_2 = -\Delta$.

Now that $P = (X_1, X_2)\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}(X_1, X_2)^{-1}$, we have

$$\begin{aligned} \bar{\mathbf{N}}(n) &= (X_1, X_2)\begin{pmatrix} \sum_{k=1}^n \alpha_1^k & \\ & \sum_{k=1}^n \alpha_2^k \end{pmatrix}(X_1, X_2)^{-1}\mathbf{W} \\ &= \begin{pmatrix} \frac{-n(1-p)\lambda_D}{\lambda_S} + \frac{(1-p)\lambda_D}{\lambda_S}\left(\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\right), & \frac{-np\lambda_D}{\lambda_S} + \frac{p\lambda_D}{\lambda_S}\left(\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\right) \\ \frac{-n(1-p)\lambda_D}{\lambda_S} - \frac{(1-p)^2}{p}\left(\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\right), & \frac{-np\lambda_D}{\lambda_S} + (p-1)\left(\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\right) \end{pmatrix}/\Delta. \end{aligned} \tag{18}$$

A.1.1 Customer Starts "Dissatisfied"

We first condition on n, the number of transitions in the uniformized Markov chain. If a customer starts in state D, then the expected number of purchases (among the n transitions) is:

$$\begin{aligned} \bar{\mathbf{N}}_D^D(n) + \bar{\mathbf{N}}_S^D(n) &= \left[\frac{-n(1-p)\lambda_D}{\lambda_S} + \frac{(1-p)\lambda_D}{\lambda_S}\left(\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\right) - \frac{np\lambda_D}{\lambda_S} + \frac{p\lambda_D}{\lambda_S}\left(\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\right)\right]/\Delta \\ &= -K_1 n + K_2\alpha_2 - K_2\alpha_2^{n+1}, \end{aligned} \tag{19}$$

where $K_1 = \frac{\lambda_D}{\lambda_S \Delta} = \frac{-\lambda_D}{(1-p)\lambda_S + p\lambda_D} = -\frac{\lambda_D}{\gamma}$ and $K_2 = \frac{\lambda_D}{\lambda_S \Delta (1-\alpha_2)} = \frac{-\lambda_D \lambda_S}{[(1-p)\lambda_S + p\lambda_D]^2} = -\frac{\lambda_D \lambda_S}{\gamma^2}$. Recall that γ is defined as $\gamma = (1-p)\lambda_S + p\lambda_D$.

Next, we calculate the unconditioned expected number of purchases:

$$
\begin{aligned}
& \int_0^T \sum_{n=0}^{\infty} \left[\left(-K_1 n + K_2 \alpha_2 - K_2 \alpha_2^{n+1}\right) \frac{e^{-\lambda_S t} (\lambda_S t)^n}{n!} \right] \mu e^{-\mu t} dt \\
& + \int_T^{\infty} \sum_{n=0}^{\infty} \left[\left(-K_1 n + K_2 \alpha_2 - K_2 \alpha_2^{n+1}\right) \frac{e^{-\lambda_S T} (\lambda_S T)^n}{n!} \right] \mu e^{-\mu t} dt \\
= & \int_0^T \left[-K_1 \lambda_S t + K_2 \alpha_2 - K_2 \alpha_2 e^{-(1-\alpha_2)\lambda_S t} \right] \mu e^{-\mu t} dt + \int_T^{\infty} \left[-K_1 \lambda_S T + K_2 \alpha_2 - K_2 \alpha_2 e^{-(1-\alpha_2)\lambda_S T} \right] \mu e^{-\mu t} dt \\
= & \frac{\lambda_D \lambda_S}{\gamma \mu} \left(1 - e^{-\mu T}\right) - \frac{p \lambda_D (\lambda_S - \lambda_D)}{\gamma (\gamma + \mu)} \left(1 - e^{-(\gamma + \mu) T}\right). \qquad (20)
\end{aligned}
$$

Note that the purchase amount is independent of the number of purchases. Thus, the total expected purchase amount is a simple product of the two averages. Activating the subscript i, we see that the overall expected revenue from customer i becomes:

$$
r_{iD} = \bar{Q} \left[\frac{\lambda_{iD} \lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) - \frac{p \lambda_{iD} (\lambda_{iS} - \lambda_{iD})}{\gamma_i (\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i) T}\right) \right].
$$

A.1.2 Customer Starts "Satisfied"

Similarly, we first condition on n, the number of transitions in the uniformized Markov chain. If a customer starts in state S, the expected number of purchase visits (among the n transitions) is:

$$
\begin{aligned}
\bar{\mathbf{N}}_D^S(n) + \bar{\mathbf{N}}_S^S(n) &= \left[\frac{-n(1-p)\lambda_D}{\lambda_S} - \frac{(1-p)^2}{p} \left(\frac{\alpha_2 - \alpha_2^{n+1}}{1 - \alpha_2} \right) + \frac{-np\lambda_D}{\lambda_S} + (p-1) \left(\frac{\alpha_2 - \alpha_2^{n+1}}{1 - \alpha_2} \right) \right] / \Delta \\
&= -K_1 n + K_3 \alpha_2 - K_3 \alpha_2^{n+1}, \qquad (21)
\end{aligned}
$$

where $K_3 = \frac{p-1}{p\Delta(1-\alpha_2)} = \frac{(1-p)\lambda_S^2}{p\gamma^2}$.

Next, we calculate the unconditioned expected number of purchase during $(0, T]$:

$$
\begin{aligned}
& \int_t^T \sum_{n=0}^{\infty} \left[\left(-K_1 n + K_3 \alpha_2 - K_3 \alpha_2^{n+1}\right) \frac{e^{-\lambda_S t} (\lambda_S t)^n}{n!} \right] \mu e^{-\mu t} dt \\
& + \int_T^{\infty} \sum_{n=0}^{\infty} \left[\left(-K_1 n + K_3 \alpha_2 - K_3 \alpha_2^{n+1}\right) \frac{e^{-\lambda_S T} (\lambda_S T)^n}{n!} \right] \mu e^{-\mu t} dt \\
= & \int_t^T \left[-K_1 \lambda_S t + K_3 \alpha_2 - K_3 \alpha_2 e^{-(1-\alpha_2)\lambda_S t} \right] \mu e^{-\mu t} dt + \int_t^{\infty} \left[-K_1 \lambda_S T + K_3 \alpha_2 - K_3 \alpha_2 e^{-(1-\alpha_2)\lambda_S T} \right] \mu e^{-\mu t} dt \\
= & \frac{\lambda_D \lambda_S}{\gamma \mu} \left(1 - e^{-\mu T}\right) + \frac{(1-p) \lambda_S (\lambda_S - \lambda_D)}{\gamma (\gamma + \mu)} \left(1 - e^{-(\gamma + \mu) T}\right). \qquad (22)
\end{aligned}
$$

Similarly, the overall expected revenue from customer i becomes:

$$
r_{iS} = \bar{Q} \left[\frac{\lambda_{iD} \lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{(1-p) \lambda_{iS} (\lambda_{iS} - \lambda_{iD})}{\gamma_i (\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i) T}\right) \right].
$$

Finally, $R_D = \sum_i r_{iD}$ and $R_S = \sum_i r_{iS}$ give us (3) and (4) respectively. The total expected revenue from the whole customer base is calculated as:

$$R = \sum_{i=1}^{N} [p \cdot r_{iS} + (1-p) \cdot r_{iD}] = \bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i + \mu_i)T}\right) \right].$$

A.2 Proof of Proposition 2

Equation (7) gives the expected revenue in the SMC-p model. Taking difference between R^e and $R_D = \sum_{i=1}^{N} r_{iD}$, $R_S = \sum_{i=1}^{N} r_{iS}$, and R, we obtain:

$$\begin{aligned}
\eta_D &= \bar{Q} \sum_{i=1}^{N} \frac{p\lambda_{iD}(\lambda_{iS} - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)} (1 - e^{-(\gamma_i + \mu_i)T}) \\
\eta_S &= \bar{Q} \sum_{i=1}^{N} \frac{(1-p)\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)} (1 - e^{-(\gamma_i + \mu_i)T}) \\
\eta &= \bar{Q} \sum_{i=1}^{N} \frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i + \mu_i)} (1 - e^{-(\gamma_i + \mu_i)T}).
\end{aligned}$$

It is easy to see that all three are non-negative.

A.3 Proof of Proposition 3

Because $R = \sum_{i=1}^{N} [p \cdot r_{iS} + (1-p) \cdot r_{iD}]$, we have $R'' = \sum_{i=1}^{N} [r''_{iD} + 2\,(r_{iS} - r_{iD})' + p\,(r_{iS} - r_{iD})'']$. To show that $R'' \geq 0$, the following lemma suffices:

Lemma 2

(i) The first two derivatives of r_{iD} are non-negative.

(ii) The first two derivatives of $(r_{iS} - r_{iD})$ are non-negative.

Part (i) We will suppress the subscript i to simplify exposition. To show that r_D has non-negative first and second derivatives, it suffices to show that the $\bar{\mathbf{N}}_D^D(n) + \bar{\mathbf{N}}_S^D(n)$ has non-negative first and second derivatives. From the proof of Proposition 1, we have:

$$\bar{\mathbf{N}}_D^D(n) + \bar{\mathbf{N}}_S^D(n) = \frac{\lambda_D}{\lambda_S \Delta} \left(\sum_{k=1}^{n} \alpha_2^k - n \right) = \frac{\lambda_D}{\lambda_S} \sum_{k=1}^{n} \left(\frac{\alpha_2^k - 1}{\alpha_2 - 1} \right) = \frac{\lambda_D}{\lambda_S} \sum_{k=1}^{n} \left(\sum_{i=0}^{k-1} \alpha_2^i \right).$$

Because any derivative of $\alpha_2^i = p^i \left(1 - \frac{\lambda_D}{\lambda_S}\right)^i$ is non-negative with respect to p as long as $i \geq 0$, the sum clearly has non-negative first and second derivatives with respect to p.

Part *(ii)*

$$r_S - r_D \quad = \quad \bar{Q}\left(\frac{\lambda_S - \lambda_D}{\gamma + \mu}\right)\left(1 - e^{-(\gamma+\mu)T}\right).$$

Now define $f(x) = \frac{1-e^{-Tx}}{x}$.

- $f'(x) = \frac{Te^{-Tx}x - \left(1-e^{-Tx}\right)}{x^2}$. If we let $g(x) = Te^{-Tx}x - \left(1 - e^{-Tx}\right)$, then $g(0) = 0$ and $g'(x) = -T^2xe^{-Tx} \leq 0$. So $g(x) \leq 0$, and $f'(x) \leq 0$ for $x \geq 0$. Therefore, $f(\gamma+\mu)$ has a non-negative first derivative with respect to p because $\gamma'(p) = -(\lambda_S - \lambda_D) \leq 0$.
- $f''(x) = \frac{2-e^{-Tx}\left(T^2x^2+2Tx+2\right)}{x^3}$. If we let $g(x) = 2 - e^{-Tx}(T^2x^2 + 2Tx + 2)$, then $g(0) = 0$. Moreover, $g'(x) = T^3x^2e^{-Tx} \geq 0$. So $g(x) \geq 0$, and hence $f''(x) \geq 0$, for $x \geq 0$. It follows that $f''(\gamma + \mu)$ has a non-negative second derivative with respect to p.

B Profit Function

In this section, we prove Proposition 4. We need the following lemma first.

Lemma 3 *Let* $g(\gamma) = \frac{(\lambda_S-\gamma)(\gamma-\lambda_D)}{\gamma(\gamma+\mu)}\left(1 - e^{-(\gamma+\mu)T}\right)$*. Then* $g(\gamma)$ *is unimodal in* γ*.*

Proof: Let $f = \frac{(\lambda_S-\gamma)(\gamma-\lambda_D)}{\gamma(\gamma+\mu)}$, then (all derivatives are with respect to γ)

$$\frac{f'}{f} \quad = \quad (\ln(f))' = -\frac{1}{\lambda_S - \gamma} + \frac{1}{\gamma - \lambda_D} - \frac{1}{\gamma} - \frac{1}{\gamma + \mu}. \tag{23}$$

Let $u = \frac{1}{\lambda_S-\gamma} > 0$, $v = \frac{1}{\gamma-\lambda_D} > 0$, $w = \frac{1}{\gamma} > 0$, $y = \frac{1}{\gamma+\mu} > 0$, and $x = -u + v - w - y$. Then by equation (23), we have $\frac{f'}{f} = x$ (i.e., $f' = fx$). Note also that $v > w > y$.

Because $x' = -u^2 - v^2 + w^2 + y^2$,

$$x^2 - x' \quad = \quad 2(u^2 + v^2 - vu - vw - vy + uw + uy + wy). \tag{24}$$

There are two cases:

(1) $v \geq u + w + y$. In this case, equation (24) becomes:

$$x^2 - x' \quad = \quad 2(u^2 + v(v - u - w - y) + uw + uy + wy) \geq 0.$$

(2) $v < u + w + y$. In this case, equation (24) becomes:

$$x^2 - x' \quad = \quad 2(u(u + w + y - v) + (v - w)(v - y)) \geq 0.$$

So in either case, we have shown that $x^2 \geq x'$.

Next, $g' = \left(f\left(1-e^{-(\gamma+\mu)T}\right)\right)' = f' - (f' - fT)e^{-(\gamma+\mu)T}$. Whenever $g' = 0$, we must have $e^{-(\gamma+\mu)T} = \frac{f'}{f'-fT} = \frac{x}{x-T}$. Because $0 < e^{-(\gamma+\mu)T} < 1$, we must have $x < 0$. Moreover,

$$\begin{aligned} g'' &= f'' + [2f'T - fT^2 - f'']e^{-(\gamma+\mu)T} = f'' + \frac{[2f'T - fT^2 - f'']x}{x-T} \\ &= \frac{-f''T + 2f'Tx - fT^2x}{x-T} = \frac{-(fx)'T + 2f'Tx - fT^2x}{x-T} \\ &= \frac{f'xT - fx'T - f'T^2}{x-T} = f'T - \frac{fx'T}{x-T} = fxT - \frac{fx'T}{x-T} = fT\left[x - \frac{x'}{x-T}\right]. \end{aligned}$$

Because $x^2 \geq x'$ and $x \leq 0$, we know $x(x-T) \geq x'$ and $x \leq \frac{x'}{x-T}$. So by now we have shown that whenever $g' = 0$, $g'' \leq 0$. This means that g has local maxima. Moreover, there would be one (because g is a continuous function on a compact set) and only one (because there is a local minimum between any two local maxima) local maximum. This maximum is the unique solution to $e^{-(\gamma+\mu)T} = \frac{x}{x-T}$, which can be further simplified to

$$\frac{T}{e^{(\gamma+\mu)T} - 1} = \frac{1}{\lambda_S - \gamma} - \frac{1}{\gamma - \lambda_D} + \frac{1}{\gamma} + \frac{1}{\gamma+\mu}. \tag{25}$$

Because $\gamma = (1-p)\lambda_S + p\lambda_D$, g as a function of p has exactly one local maximum and no local minimum. Moreover, because $g(p)|_{p=0} = g(p)|_{p=1} = 0$ and $g(p)|_{0<p<1} > 0$, it must be true that the maximum is interior. To the left of local maximum, $g(p)$ is increasing in p and to the right of it, $g(p)$ is decreasing. $\square$.

Recall that because $\gamma_i = p\lambda_{iD} + (1-p)\lambda_{iS}$, η_m can be expressed as

$$\eta_m = (\bar{Q} - c)\sum_{i=1}^{N}\left[\frac{(\lambda_{iS} - \gamma_i)(\gamma_i - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)}(1 - e^{-(\gamma_i+\mu_i)T})\right]. \tag{26}$$

Lemma 3 shows that each summand in (26) is unimodal. Let the modes be $\bar{p}_i$ and let $\bar{p} = \max_i\{\bar{p}_i\}$. It is clear that when $p > \bar{p}$, $\eta' < 0$.

When the cost increases slowly, i.e., when b is small, the company would optimally invest to achieve the maximum possible service quality. This is intuitive because revenue is convex and increases quickly. So when b is small, the optimal p should be 1. To avoid such trivial conclusions, here we focus only on large b. Specifically, we assume that

$$b > (\bar{Q} - c)\max\left\{\sum_{i=1}^{N}\left(\frac{\lambda_{iS}(\lambda_{iS} - \lambda_{iD})^2}{\lambda_{iD}^2}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right), \sum_{i=1}^{N}\left(\frac{\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{2\lambda_{iD}}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right)\right\}.$$

Recall that

$$R_m^e = (\bar{Q} - c)\sum_{i=1}^{N} \lambda_i^e \left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right) = (\bar{Q} - c)\sum_{i=1}^{N} \left(\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right).$$

Because $\gamma_i'(p) = \lambda_{iD} - \lambda_{iS}$,

$$\begin{aligned}
[R_m^e(p)]' &= (\bar{Q} - c)\sum_{i=1}^{N} \left(\frac{\lambda_{iD}\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{\gamma_i^2}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right),\\
[R_m^e(p)]'' &= 2(\bar{Q} - c)\sum_{i=1}^{N} \left(\frac{\lambda_{iD}\lambda_{iS}(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i^3}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right),\\
[R_m^e(p)]''' &= 6(\bar{Q} - c)\sum_{i=1}^{N} \left(\frac{\lambda_{iD}\lambda_{iS}(\lambda_{iS} - \lambda_{iD})^3}{\gamma_i^4}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right).
\end{aligned}$$

Note that $[R_m^e(p)]''' \geq 0$, and when $p = 1$, $\gamma_i = \lambda_{iD}$. Therefore,

$$[R_m^e(p)]'' \leq [R_m^e(1)]'' = 2(\bar{Q} - c)\sum_{i=1}^{N} \left(\frac{\lambda_{iS}(\lambda_{iS} - \lambda_{iD})^2}{\lambda_{iD}^2}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right) < 2b = F''(p), \forall p.$$

Thus, we have shown that $\pi^e(p) = R_m^e(p) - F(p)$ is concave. We note that $[\pi^e(0)]' > 0$, so 0 is not a maximum. Also, because

$$[R_m^e(p)]'\,|_{p=1} = (\bar{Q} - c)\sum_{i=1}^{N} \left(\frac{\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{\lambda_{iD}}\right)\left(\frac{1 - e^{-\mu_i T}}{\mu_i}\right) < 2b = F'(1),$$

we have $[\pi^e(1)]' < 0$ so the maximizer p^{e^*} is interior. Moreover, it is the only p that satisfies $[\pi^e(p)]' = 0$.

Next, we study $\pi(p)$. It is clear that 0 won't be the maximizer of $\pi(p)$ because $R_m'(0) > 0, F'(0) = 0 \Rightarrow \pi'(0) = R_m'(0) - F'(0) > 0$. Moreover, because $[\pi^e(1)]' \leq 0$ and $\eta_m'(1) < 0$, we also have $\pi'(1) = [\pi^e(1)]' + \eta_m'(1) < 0$, so 1 won't be the maximizer of $\pi(p)$ either. So now we study the interior p^*, which must satisfy $\pi'(p^*) = 0$.

Finally, we show that $p^{e^*} \geq \bar{p} \Rightarrow p^{e^*} \geq p^*$. We use contradiction. Suppose $p^{e^*} \geq \bar{p}$ and $p^{e^*} < p^*$. Because $\pi^e(p)$ is concave in p and $[\pi^e(p_e^*)]' = 0$, we must also have $[\pi^e(p^*)]' < 0$. Moreover, $p^* > p^{e^*} \geq \bar{p} \Rightarrow \eta_m'(p^*) < 0$. Therefore, we have $\pi'(p^*) = [\pi^e(p^*)]' + \eta_m'(p^*) < 0$, which is a contradiction.

Equation (25) helps us to get more insights into the property of $\bar{p}$. For example, we can rearrange it to be:

$$\frac{T}{e^{(\gamma+\mu)T} - 1} - \frac{1}{\gamma + \mu} = \frac{1}{\lambda_S - \gamma} - \frac{\lambda_D}{\gamma(\gamma - \lambda_D)}.$$

We know $e^{xT}-1$ increases in x much faster than xT, so as μ decreases the left-hand side of this equation increases. We also know that the right-hand side is increasing in γ. So as μ gets smaller, γ should change in such a way to balance the left-hand side, or make the right-hand side increase as well. Either way, γ should increase. And the corresponding p should decrease. This implies that when the customers in general are more loyal, this threshold $\bar{p}$ is lower, and the companies are more likely to over-invest in service quality.

C Extension: Contingent Death Rate

Again, in this section we will suppress the customer subscript i whenever there is no possibility of confusion. Now that the death rates also vary by which state the customer is in, we need to modify the Markov chain we developed in Appendix A accordingly. First of all, we introduce a new state 0 to indicate that a customer has departed, thus increasing the state space by one dimension. Second, because $\mu_D > \mu_S$, the uniformization rate becomes $\omega = \mu_D + \lambda_S$. Then the original CTMC is equivalent to a Markov process that spends an *i.i.d.* exponential(ω) time in each state and has the following transition probabilities among states:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\mu_D}{\omega} & \frac{(1-p)\lambda_D+(\lambda_S-\lambda_D)}{\omega}=\frac{\lambda_S-p\lambda_D}{\omega} & \frac{p\lambda_D}{\omega} \\ \frac{\mu_S}{\omega} & \frac{(1-p)\lambda_S}{\omega} & \frac{p\lambda_S+(\mu_D-\mu_S)}{\omega} \end{pmatrix}. \tag{27}$$

C.1 Proof of Lemma 1

Denote by $q_{ij}(n)$ the probability that if a customer starts in state i she will end up in state j after n transitions. Note that the one-step transition probabilities, $q_{ij}(1)$, are the probabilities in (27). The quantities of interest to us are $1 - q_{D0}(n)$ and $1 - q_{S0}(n)$. They are the probabilities that if the customer is alive and dissatisfied (satisfied, respectively) currently, the probability that she will still be alive after n transitions.

It is straightforward to derive the following:

$$q_{D0}(n) = \frac{\mu_D}{\omega} + \frac{\lambda_S - p\lambda_D}{\omega} q_{D0}(n-1) + \frac{p\lambda_D}{\omega} q_{S0}(n-1) \ , \tag{28}$$

$$q_{S0}(n) = \frac{\mu_S}{\omega} + \frac{(1-p)\lambda_S}{\omega} q_{D0}(n-1) + \frac{p\lambda_S + (\mu_D - \mu_S)}{\omega} q_{S0}(n-1) \ . \tag{29}$$

If we let $Q(n) = \begin{pmatrix} q_{D0}(n) \\ q_{S0}(n) \end{pmatrix}$ and $Y = \begin{pmatrix} \frac{\mu_D}{\omega} \\ \frac{\mu_S}{\omega} \end{pmatrix}$, then equations (28) and (29) are equivalent to

$$Q(n) = Y + \hat{P}Q(n-1) = \left(\sum_{k=0}^{n-1} \hat{P}^k\right) Y, \tag{30}$$

where $\hat{P} = \begin{pmatrix} \frac{\lambda_S - p\lambda_D}{\omega} & \frac{p\lambda_D}{\omega} \\ \frac{(1-p)\lambda_S}{\omega} & \frac{p\lambda_S + (\mu_D - \mu_S)}{\omega} \end{pmatrix}$ is a sub-matrix of the transition matrix (27).

To calculate $\sum_{k=0}^{n-1} \hat{P}^k$, we need to diagonalize $\hat{P}$. We redefine $\alpha_1 \geq \alpha_2$ to be the two eigenvalues of $\hat{P}$. They satisfy

$$0 = \alpha^2 + \frac{p\lambda_D - (1+p)\lambda_S - (\mu_D - \mu_S)}{\omega}\alpha + \frac{p\lambda_S(\lambda_S - \lambda_D) + (\lambda_S - p\lambda_D)(\mu_D - \mu_S)}{\omega^2}. \tag{31}$$

Let X_1, X_2 be the eigenvectors, then we have $\hat{P} = (X_1, X_2) \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} (X_1, X_2)^{-1}$, and

$$Q(n) = (X_1, X_2) \begin{pmatrix} \frac{1-\alpha_1^n}{1-\alpha_1} & \\ & \frac{1-\alpha_2^n}{1-\alpha_2} \end{pmatrix} (X_1, X_2)^{-1} Y.$$

Note that $Q(n)$ is the probability of a customer being "dead" conditioning on n events in the uniformized Markov chain. We now uncondition it.

We know that the uniformized Markov chain has exponential(ω) time between two events so during (0,T] the number of events in the uniformized Markov chain is Poisson with rate ω. So

$$\begin{aligned} \begin{pmatrix} 1 - PA_D \\ 1 - PA_S \end{pmatrix} &= \sum_{n=0}^{\infty} Q(n) \frac{e^{-\omega T}(\omega T)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[(X_1, X_2) \begin{pmatrix} \frac{1-\alpha_1^n}{1-\alpha_1} & \\ & \frac{1-\alpha_2^n}{1-\alpha_2} \end{pmatrix} (X_1, X_2)^{-1} Y \right] \frac{e^{-\omega T}(\omega T)^n}{n!} \\ &= (X_1, X_2) \begin{pmatrix} \frac{1-e^{-(1-\alpha_1)\omega T}}{1-\alpha_1} & \\ & \frac{1-e^{-(1-\alpha_2)\omega T}}{1-\alpha_2} \end{pmatrix} (X_1, X_2)^{-1} Y. \end{aligned}$$

Then, the probability of a customer (who may be satisfied or dissatisfied at time 0) being alive at time T is:

$$(1-p, p) \begin{pmatrix} 1 - PA_D \\ 1 - PA_S \end{pmatrix} = (1-p, p)\,(X_1, X_2) \begin{pmatrix} \frac{1-e^{-(1-\alpha_1)\omega T}}{1-\alpha_1} & \\ & \frac{1-e^{-(1-\alpha_2)\omega T}}{1-\alpha_2} \end{pmatrix} (X_1, X_2)^{-1} Y. \tag{32}$$

To simplify the expression, we will make the following substitutions: $\beta_1 = -(1-\alpha_1)\omega$ and $\beta_2 = -(1-\alpha_2)\omega$. Then, after some simple manipulation of (31), β_1 and β_2 are the two roots of:

$$0 = \beta^2 + (p\lambda_D + (1-p)\lambda_S + \mu_D + \mu_S)\beta + [\mu_S\mu_D + (1-p)\lambda_S\mu_D + p\lambda_D\mu_S].$$

With this substitution and some more arithmetic manipulation, (32) can be simplified to $Ae^{\beta_1 T} + Be^{\beta_2 T}$, where

$$\begin{aligned} A &= \frac{p\,(\beta_1 + \lambda_D + \mu_D)\left[(1-p)\lambda_S(\mu_S - \mu_D) + \beta_2\mu_S + \mu_S^2\right]}{p(1-p)\lambda_D\lambda_S\beta_1 - \beta_1[\beta_1 + \mu_D + p\lambda_D][\beta_2 + \mu_S + (1-p)\lambda_S]}, \\ B &= \frac{(1-p)\left[\beta_1\mu_D + \mu_D^2 + p\lambda_D\,(\mu_D - \mu_S)\right](\beta_2 + \lambda_S + \mu_S)}{p(1-p)\lambda_D\lambda_S\beta_2 - \beta_2[\beta_1 + \mu_D + p\lambda_D][\beta_2 + \mu_S + (1-p)\lambda_S]}. \end{aligned}$$

C.2 Proof of Proposition 6

Let $\bar{N}_D(n)$ be the expected number of purchases in the next n uniformized transitions when customer is dissatisfied at time 0, and $\bar{N}_S(n)$ be the expected number of purchases in the next n uniformized transitions when customer is satisfied at time 0. We then have the following

$$\bar{N}_D(n) = \frac{\lambda_D}{\omega} + \frac{p\lambda_D}{\omega}\bar{N}_S(n-1) + \frac{\lambda_S - p\lambda_D}{\omega}\bar{N}_D(n-1) \quad (33)$$

$$\bar{N}_S(n) = \frac{\lambda_S}{\omega} + \frac{(1-p)\lambda_S}{\omega}\bar{N}_D(n-1) + \frac{p\lambda_S + \mu_D - \mu_S}{\omega}\bar{N}_S(n-1). \quad (34)$$

If we let $\bar{N}(n) = \begin{pmatrix} \bar{N}_D(n) \\ \bar{N}_S(n) \end{pmatrix}$ and $Z = \begin{pmatrix} \frac{\lambda_D}{\omega} \\ \frac{\lambda_S}{\omega} \end{pmatrix}$, equations (33) and (34) are equivalent to:

$$\bar{N}(n) = Z + \hat{P}\bar{N}(n-1) = \left(\sum_{k=0}^{n-1} \hat{P}^k\right) Z.$$

We have already diagonalized $\hat{P}$, so

$$\bar{N}(n) = (X_1, X_2)\begin{pmatrix} \frac{1-\alpha_1^n}{1-\alpha_1} & \\ & \frac{1-\alpha_2^n}{1-\alpha_2} \end{pmatrix}(X_1, X_2)^{-1} Z.$$

Again, we need to uncondition $N(n)$ on n:

$$\begin{aligned} &\sum_{n=0}^{\infty} \begin{pmatrix} \bar{N}_1(n) \\ \bar{N}_2(n) \end{pmatrix} \frac{e^{-\omega T}(\omega T)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[(X_1, X_2)\begin{pmatrix} \frac{1-\alpha_1^n}{1-\alpha_1} & \\ & \frac{1-\alpha_2^n}{1-\alpha_2} \end{pmatrix}(X_1, X_2)^{-1} Z\right]\frac{e^{-\omega T}(\omega T)^n}{n!} \\ &= (X_1, X_2)\begin{pmatrix} \frac{1-e^{-(1-\alpha_1)\omega T}}{1-\alpha_1} & \\ & \frac{1-e^{-(1-\alpha_2)\omega T}}{1-\alpha_2} \end{pmatrix}(X_1, X_2)^{-1} Z. \end{aligned}$$

We again make the substitution $\beta_1 = -(1-\alpha_1)\omega$ and $\beta_2 = -(1-\alpha_2)\omega$. For a random customer, she is satisfied at time 0 with probability p and dissatisfied with probability $(1-p)$, so the expected revenue from her is

$$r = (1-p, p)\,(X_1, X_2) \begin{pmatrix} \frac{1-e^{\beta_1 T}}{-\beta_1/\omega} & \\ & \frac{1-e^{\beta_2 \omega T}}{-\beta_2/\omega} \end{pmatrix} (X_1, X_2)^{-1}\, Z.$$

After some straightforward arithmetic manipulation,

$$\begin{aligned} r &= \bar{Q}\left\{ \frac{p\lambda_S\,(\beta_1+\lambda_D+\mu_D)\,(\beta_2+\mu_S+(1-p)(\lambda_S-\lambda_D))}{p(1-p)\lambda_D\lambda_S\beta_1 - \beta_1[\beta_1+\mu_D+p\lambda_D][\beta_2+\mu_S+(1-p)\lambda_S]}\left(1-e^{\beta_1 T}\right)\right. \\ &+ \left.\frac{(1-p)\lambda_D\,(\beta_2+\lambda_S+\mu_S)\,(\beta_1+\mu_D+p(\lambda_D-\lambda_S))}{p(1-p)\lambda_D\lambda_S\beta_2 - \beta_2[\beta_1+\mu_D+p\lambda_D][\beta_2+\mu_S+(1-p)\lambda_S]}\left(1-e^{\beta_2 T}\right)\right\}. \end{aligned}$$

D Extension: Hidden Markov Model of Customer Satisfaction

D.1 Proof of Proposition 7

The proof is a straightforward extension of the derivation in Appendix A. For ease of exposition, we will first suppress the customer subscript i. Due to the dependency of customer satisfaction over time, the transition probability matrix in the uniformized chain, (16), now becomes:

$$\mathbf{P} = \begin{pmatrix} (1-p_1)\left(\frac{\lambda_D}{\lambda_S}\right) + \left(1-\frac{\lambda_D}{\lambda_S}\right) = 1 - p_1\left(\frac{\lambda_D}{\lambda_S}\right) & p_1\left(\frac{\lambda_D}{\lambda_S}\right) \\ 1-p_2 & p_2 \end{pmatrix}.$$

As in Appendix A, we will let $N_j^i(n)$ be the random number of real transitions into state j in the first n transitions of the uniformized embedded Markov chain if the system starts in state i. Moreover, let $\bar{N}_j^i(n)$ be its expected value.

If we use matrix-vector notation $\bar{\mathbf{N}}(n)$ to denote $\begin{pmatrix} \bar{N}_D^D(n) & \bar{N}_S^D(n) \\ \bar{N}_D^S(n) & \bar{N}_S^S(n) \end{pmatrix}$, then the solution is:

$$\bar{\mathbf{N}}(n) = \left(\sum_{k=1}^{n} \mathbf{P}^k\right)\mathbf{V}, \tag{35}$$

where $\mathbf{V} = \begin{pmatrix} \frac{(p_2-p_1)\lambda_D}{p_2\lambda_S - p_1\lambda_D} & 0 \\ \frac{(1-p_2)(\lambda_S-\lambda_D)}{p_2\lambda_S-p_1\lambda_D} & 1 \end{pmatrix}$.

Again, as in Appendix A, we first diagonalize $\mathbf{P}$. The two eigenvalues are found to be $\alpha_1 = 1$ and $\alpha_2 = p_2 - p_1\frac{\lambda_D}{\lambda_S}$, and their corresponding eigenvectors are $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} \frac{p_1\lambda_D}{\lambda_S} \\ p_2 - 1 \end{pmatrix}$

respectively. Therefore, $(X_1, X_2) = \begin{pmatrix} 1 & \frac{p_1\lambda_D}{\lambda_S} \\ 1 & p_2 - 1 \end{pmatrix}$ and $(X_1, X_2)^{-1} = \begin{pmatrix} p_2 - 1, & -\frac{p_1\lambda_D}{\lambda_S} \\ -1, & 1 \end{pmatrix} /\Delta$, where $\Delta = p_2 - 1 - \frac{p_1\lambda_D}{\lambda_S}$. Therefore,

$$\begin{aligned}
\bar{\mathbf{N}}(n) &= \left(\sum_{k=1}^{n} \mathbf{P}^k\right) \mathbf{V} \\
&= \begin{pmatrix} 1 & \frac{p_1\lambda_D}{\lambda_S} \\ 1 & p_2 - 1 \end{pmatrix} \begin{pmatrix} n & \\ & \frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2} \end{pmatrix} \begin{pmatrix} p_2 - 1 & -\frac{p_1\lambda_D}{\lambda_S} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{(p_2-p_1)\lambda_D}{p_2\lambda_S - p_1\lambda_D} & 0 \\ \frac{(1-p_2)(\lambda_S-\lambda_D)}{p_2\lambda_S - p_1\lambda_D} & 1 \end{pmatrix} /\Delta \\
&= \begin{pmatrix} \frac{-n(1-p_2)\lambda_D}{\lambda_S} - \frac{p_1\lambda_D}{\lambda_S}\left(1 - \frac{\lambda_D - \lambda_S}{p_1\lambda_D - p_2\lambda_S}\right)\frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}, & -\frac{np_1\lambda_D}{\lambda_S} + \frac{p_1\lambda_D}{\lambda_S}\frac{\alpha_2 - {\alpha_2}^{n+1}}{1-\alpha_2} \\ (1-p_2)\left(-\frac{n\lambda_D}{\lambda_S} + \frac{\alpha_2 - \alpha_2^{n+1}}{1-\alpha_2}\left(1 - \frac{\lambda_S - \lambda_D}{p_2\lambda_S - p_1\lambda_D}\right)\right), & -\frac{np_1\lambda_D}{\lambda_S} + (p_2 - 1)\frac{\alpha_2 - {\alpha_2}^{n+1}}{1-\alpha_2} \end{pmatrix} /\Delta.
\end{aligned}$$

We will again proceed in two separate cases.

1. When a customer starts "dissatisfied". Then the expected number of purchases (among the first n transitions) is:

$$\bar{\mathbf{N}}_D^D(n) + \bar{\mathbf{N}}_S^D(n) = -K_1 n + K_2\alpha_2 - K_2\alpha_2^{n+1}, \tag{36}$$

where $K_1 = -\frac{\lambda_D(1-p_2+p_1)}{\gamma}$ and $K_2 = -\frac{p_1(\lambda_D-\lambda_S)\lambda_D\lambda_S}{(p_1\lambda_D - p_2\lambda_S)\gamma^2}$. Recall that γ is defined as $\gamma = (1-p_2)\lambda_S + p_1\lambda_D = (1-\alpha_2)\lambda_S = -\lambda_S\Delta$.

Next, we calculate the unconditioned expected number of purchases:

$$\begin{aligned}
&\int_0^T \sum_{n=0}^{\infty} \left[\left(\bar{\mathbf{N}}_D^D(n) + \bar{\mathbf{N}}_S^D(n)\right) \frac{e^{-\lambda_S t}(\lambda_S t)^n}{n!}\right] \mu e^{-\mu t} dt + \int_T^{\infty} \sum_{n=0}^{\infty} \left[\left(\bar{\mathbf{N}}_D^D(n) + \bar{\mathbf{N}}_S^D(n)\right) \frac{e^{-\lambda_S T}(\lambda_S T)^n}{n!}\right] \mu e^{-\mu t} dt \\
= &\ \frac{\lambda_D\lambda_S(1-p_2+p_1)}{\gamma\mu}\left(1 - e^{-\mu T}\right) - \frac{p_1\lambda_D(\lambda_S - \lambda_D)}{\gamma(\gamma+\mu)}\left(1 - e^{-(\gamma+\mu)T}\right). \qquad (37)
\end{aligned}$$

Because the purchase amount is independent of the number of purchases, the total expected purchase amount is a simple product of the two averages. We find the overall expected revenue from customer i to be:

$$r_{iD} = \bar{Q}\left[\frac{\lambda_{iD}\lambda_{iS}(1-p_2+p_1)}{\gamma_i\mu_i}\left(1 - e^{-\mu_i T}\right) - \frac{p_1\lambda_{iD}(\lambda_{iS} - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)}\left(1 - e^{-(\gamma_i+\mu_i)T}\right)\right].$$

2. When a customer starts "satisfied". Similarly, the expected number of purchases (among the first n transitions) is:

$$\bar{\mathbf{N}}_D^S(n) + \bar{\mathbf{N}}_S^S(n) = -K_1 n + K_3\alpha_2 - K_3\alpha_2^{n+1}, \tag{38}$$

where $K_3 = \frac{(1-p_2)\lambda_S^2(\lambda_S-\lambda_D)}{(p_2\lambda_s - p_1\lambda_D)\gamma^2}$.

Next, we calculate the unconditioned expected number of purchase during $(0,T]$:

$$\int_0^T \sum_{n=0}^{\infty} \left[\left(\bar{\mathbf{N}}_D^S(n) + \bar{\mathbf{N}}_S^S(n) \right) \frac{e^{-\lambda_S t}(\lambda_S t)^n}{n!} \right] \mu e^{-\mu t} dt + \int_T^{\infty} \sum_{n=0}^{\infty} \left[\left(\bar{\mathbf{N}}_D^S(n) + \bar{\mathbf{N}}_S^S(n) \right) \frac{e^{-\lambda_S T}(\lambda_S T)^n}{n!} \right] \mu e^{-\mu t} dt$$
$$= \frac{\lambda_D \lambda_S (1-p_2+p_1)}{\gamma\mu} \left(1 - e^{-\mu T}\right) + \frac{(1-p_2)\lambda_S(\lambda_S - \lambda_D)}{\gamma(\gamma+\mu)} \left(1 - e^{-(\gamma+\mu)T}\right). \tag{39}$$

Similarly, the overall expected revenue from customer i to be:

$$r_{iS} = \bar{Q} \left[\frac{\lambda_{iD}\lambda_{iS}(1-p_2+p_1)}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{(1-p_2)\lambda_{iS}(\lambda_{iS} - \lambda_{iD})}{\gamma_i(\gamma_i + \mu_i)} \left(1 - e^{-(\gamma_i+\mu_i)T}\right) \right].$$

Finally, the total expected revenue from the whole customer base becomes

$$R = \sum_{i=1}^{N} [p \cdot r_{iS} + (1-p) \cdot r_{iD}],$$

where p is the long-run fraction of time a customer is satisfied, $p = \frac{p_1}{1-p_2+p_1}$.

$$\begin{aligned}
R &= \sum_{i=1}^{N} [\frac{p_1}{1-p_2+p_1} \cdot r_{iS} + \frac{1-p_2}{1-p_2+p_1} \cdot r_{iD}] \\
&= \bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}(1-p_2+p_1)}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{p_1(1-p_2)(\lambda_{iS} - \lambda_{iD})^2}{(1-p_2+p_1)\gamma_i(\gamma_i+\mu_i)} \left(1 - e^{-(\gamma_i+\mu_i)T}\right) \right] \\
&= (1-p_2+p_1)\bar{Q} \sum_{i=1}^{N} \left[\frac{\lambda_{iD}\lambda_{iS}}{\gamma_i \mu_i} \left(1 - e^{-\mu_i T}\right) + \frac{p(1-p)(\lambda_{iS} - \lambda_{iD})^2}{\gamma_i(\gamma_i+\mu_i)} \left(1 - e^{-(\gamma_i+\mu_i)T}\right) \right].
\end{aligned}$$