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Residence time in a semi-enclosed domain from the solution of an adjoint problem

Éric J.M. Delhez[a,*], Arnold W. Heemink[b], Éric Deleersnijder[c]

[a] Université de Liège, Modélisation et Méthodes Mathématiques, Sart Tilman, B37 B-4000 Liège, Belgium
[b] Delft University of Technology, Applied Mathematical Analysis, PO Box: 5031, NL-2600 GA, Delft, The Netherlands
[c] Université Catholique de Louvain, Louvain-la-Neuve, G. Lemaitre Institute of Astronomy and Geophysics (ASTR) & Centre for Systems Engineering and Applied Mechanics (CESAME), 4 Avenue G. Lemaître, B-1348 Louvain-la-Neuve, Belgium

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Abstract

The residence time measures the time spent by a water parcel or a pollutant in a given water body and is therefore a widely used concept in environmental studies. While many previous studies rely on severe hypotheses (assuming stationarity of the flow and/or neglecting diffusion) to evaluate the residence time, the paper introduces a general method for computing the residence time and/or the mean residence time without such simplifying hypotheses. The method is based on the resolution of an adjoint advection–diffusion problem and is therefore primarily meant to be used with numerical models.

The method and its implications are first introduced using a simplified one-dimensional analytical model. The approach is then applied to the diagnostic of the three-dimensional circulation on the Northwest European Continental Shelf.

Keywords: residence time; advection–diffusion; diagnostic; adjoint modeling; North Sea

1. Introduction

The residence time is one of the most widely used concepts to quantify the renewal of water in semi-enclosed water bodies. This is usually defined as 'the time it takes for a water parcel to leave the domain of interest' (e.g., Bolin and Rhode, 1973; Takeoka, 1984; Zimmerman, 1988). Together with the close concepts of flushing time, transit time and age, the residence time is often regarded as a very useful measure of the influence of the hydrodynamic processes on the aquatic systems. As such, it is included in many environmental studies (e.g., Nixon et al., 1996; Jay et al., 2000). The comparison of the residence time with the biochemical activity rates helps to understand the dynamics of a system (Braunschweig et al., 2003). In estuaries, in particular, the residence time is seen as a measure of the time of exposure of the living biomass, nutrients and contaminants to the prevailing particular biochemical conditions. It is therefore used to characterize and classify the estuaries into different groups (e.g., Dyer, 1998). A similar approach is used to understand eutrophication problems in lakes (e.g., Vollenweider, 1976).

From the definition, measuring the residence time in a given domain only requires to mark the water with some dye and follow it until it leaves the domain. While a priori simple, this definition of the residence time has, however, a number of rather complex implications. A first complication comes from the fact that the residence

* Corresponding author.
E-mail address: e.delhez@ulg.ac.be (É.J.M. Delhez).

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time is a function of space and time; tracers released at different locations and times inside the domain of interest will leave this domain at different times. Then, one must take into account the fact that every water parcel has its own residence time (in a well-defined domain). If some tracer is discharged at a given time and location, the different tracer parcels will follow different paths, have different histories … and leave the domain of interest at different times. Therefore, the situation should ideally be described by a distribution of residence times.

As a result of these complications, the effect of the hydrodynamics cannot be easily summarized into one single figure, as often sought by those who have recourse to the concept of residence time. Simplification hypotheses are therefore often introduced which ignore spatial and temporal variations and reduce the residence distribution to a single value. The residence time in a bounded system is therefore often approximated by the flushing time:

$$T_{\mathrm{f}} = \frac{V}{Q} \tag{1}$$

where V is the volume of the water body and Q is the flow rate through the system (e.g., Soetaert and Herman, 1996; Steen et al., 2002). Alternatively, the residence time is approximated by fitting a decreasing exponential curve:

$$m(t) = m(0)\mathrm{e}^{-t/T_{\mathrm{f}}} \tag{2}$$

to the time series of the measured mass $m(t)$ of a tracer that is progressively flushed out of the system (e.g., Monsen et al., 2003). The corresponding e-folding time, which is sometimes called 'turnover time' (e.g., Arneborg, 2004), represents the time taken for the initial mass of the tracer to be decreased by a factor 1/e. The flushing time T_{f} defined in this way represents the mean residence time of a tracer whose mass varies according to Eq. (2).

The previous approaches ignore the temporal variability of the flow, assume a perfect and immediate mixing in the studied domain and/or disregard spatial variations of the residence time (Monsen et al., 2003). In this paper, we present a generic method for evaluating the residence time and/or the mean residence time without simplifying hypotheses. The method relies on the concepts of adjoint state and adjoint problem, i.e. on the analysis of the time-reversed flow. It is therefore primarily meant to be used with numerical models, but some aspects can also be useful to understand experimental results.

The use of the adjoint state to compute transit times and tracer ages is not new. In their diagnostic study of atmospheric flows, Holzer and Hall (2000) introduce a boundary propagator to model the influence of open boundary conditions backwards in time. The interpretation of the boundary propagator as a flux in the time-reversed flow allows the computation of transit time distributions. Hill et al. (in press) use the adjoint approach to carry out sensitivity studies and estimate the efficiency of carbon sequestration in the ocean.

The adjoint approach is used here to derive evolution equations which describe the distribution of residence times and/or the mean residence time in arbitrary semi-enclosed water bodies.

2. Evaluation of the residence time in a one-dimensional problem

To introduce a clear definition of the residence time and understand its implications and way of calculations, let us consider a one-dimensional flow with constant and uniform velocity $u(>0)$ and diffusion coefficient $\kappa(>0)$. Let x be the spatial coordinate. This highly simplified problem could be seen as a model of a river or a marine channel. We are concerned with the residence time of water or dissolved constituents in a segment $x \in [-L,L]$ of this one-dimensional system. In the sequel, this region of interest $[-L,L]$ will be referred to as the control region.

2.1. Forward procedure

If some dye or tracer is injected at a particular point $x_0 \in [-L,L]$ at the initial time t_0, this will be progressively removed from the control region by the combined actions of advection and diffusion. Fig. 1 (solid curves) shows the progressive movement and spreading of the initial patch as time goes by. The inventory of the tracer present in the segment $x \in [-L,L]$

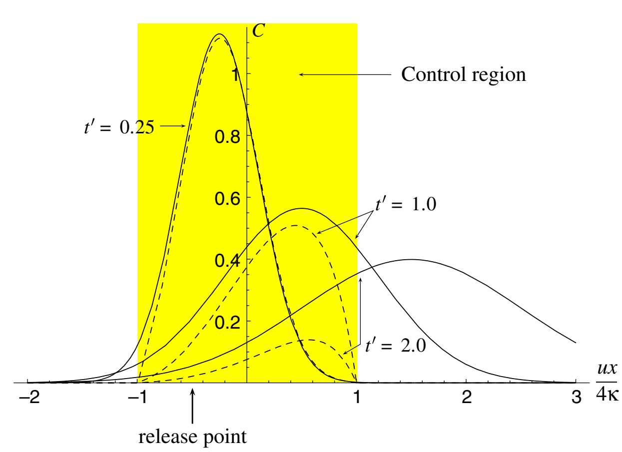

Fig. 1. Spatial distribution of the concentration of a passive tracer for different times $t' = u^2t/(4\kappa)$ after a point release (solid curves: infinite domain; dashed curves: with concentration clamped to zero at the boundary of the control region).

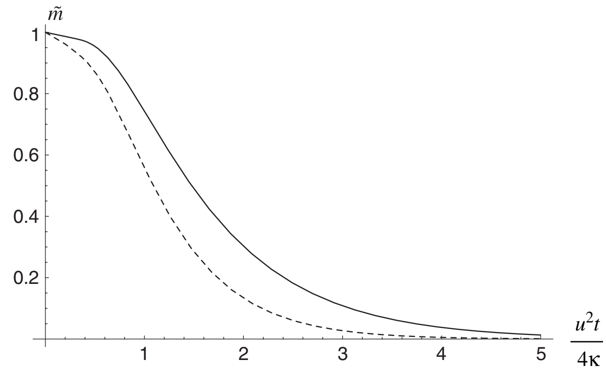

Fig. 2. Evolution of the mass of the tracer in the control region (solid curves: infinite domain; dashed curves: with concentration set to zero at the boundary of the control region).

allows to plot the mass $m(t)$ of the tracer remaining in the control region as a function of time (Fig. 2).

The residence time is defined as the time taken by the tracer to leave the control region. Obviously, the different parcels of tracer released initially take different times to leave the domain. The mass $m(t)$ plotted in Fig. 2 can be seen as the cumulative distribution function of the residence times; for any fixed time $t = t_0 + \tau \geq t_0$, $m(t_0 + \tau)/m(t_0)$ measures the fraction of the initial release with a residence time larger than τ. It is, however, common to focus on the mean residence time $\bar{\theta}$ of the tracer parcels. Following Bolin and Rhode (1973) and Takeoka (1984), we write:

$$\bar{\theta} = -\frac{1}{m(t_0)} \int_{m(t_0)}^{0} t \,\mathrm{d}m = -\frac{1}{m(t_0)} \int_{t_0}^{\infty} t \frac{\mathrm{d}m}{\mathrm{d}t}(t) \,\mathrm{d}t \quad (3)$$

Observing that the mass $m(t)$ decreases exponentially to zero for large times t, $\bar{\theta}$ can be further expressed as:

$$\bar{\theta} = \frac{1}{m(t_0)} \int_{t_0}^{\infty} m(t)\mathrm{d}t = \int_{t_0}^{\infty} \tilde{m}(t)\mathrm{d}t \quad (4)$$

where

$$\tilde{m}(t) = \frac{m(t)}{m(t_0)} \quad (5)$$

can be interpreted as the mass of tracer remaining at time t after a unit point release or, alternatively, as the normalized cumulative distribution of residence times.

With the constant hydrodynamics of our simplified system, the residence time does not depend on the time of release t_0. It does, however, vary with the location of the initial release. Fig. 3 shows the kind of distribution of the mean residence times that would result from such a procedure (in the particular case of $L = 4\kappa/u$).

The average of the mean residence time over the control region can be evaluated using a single – in situ or numerical – experiment. To show this, let us first clarify the notations and denote by $\tilde{m}(t;\, t_0, x_0)$ the mass of a tracer remaining inside the domain at time t after

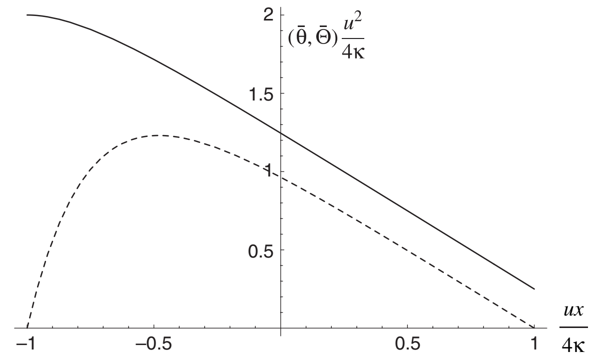

Fig. 3. Spatial distribution of the mean residence time $\bar{\theta}$ (solid curve) and the mean strict residence time $\bar{\Theta}$ (dashed curve).

a unit release at time t_0 and location x_0. Using Eq. (4), the average residence time can be expressed as:

$$\begin{aligned}\langle\bar{\theta}\rangle &= \frac{1}{2L} \int_{-L}^{L} \left(\int_{t_0}^{\infty} \tilde{m}(t;\, t_0, x_0) \,\mathrm{d}t \right) \mathrm{d}x_0 \\ &= \int_{t_0}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} \tilde{m}(t;\, t_0, x_0) \mathrm{d}x_0 \right) \mathrm{d}t \\ &= \int_{t_0}^{\infty} \tilde{M}(t;\, t_0) \,\mathrm{d}t \end{aligned} \quad (6)$$

where

$$\tilde{M}(t, t_0) = \frac{1}{2L} \int_{-L}^{L} \tilde{m}(t;\, t_0, x_0) \mathrm{d}x_0 \quad (7)$$

denotes the mass of the tracer remaining inside the control domain at time t after a unit discharge uniformly distributed in the control region at time t_0. The average mean residence time $\langle\bar{\theta}\rangle$ can therefore be obtained from the inventory following a uniform discharge.

2.2. Backward procedure by means of the adjoint model

The procedure described above for the evaluation of the mean residence time can be based on experimental data or model simulations. In both cases, the determination of the mean residence time at different points of the control region requires repeated and separate experiments with different releases at these points. The method is therefore not appropriate to get the complete spatial distribution of $\bar{\theta}$ over the control region. A different and much cheaper approach is applicable to numerical studies.

The numerical experiment aiming at the evaluation of the residence time at location x_0 according to the forward procedure described above requires the solution of the linear advection–dispersion problem:

$$\begin{cases} \dfrac{\partial C}{\partial t} + u \dfrac{\partial C}{\partial x} = \kappa \dfrac{\partial^2 C}{\partial x^2} \\ C(t_0, x) = \delta(x - x_0) \end{cases} \quad (8)$$

(where δ is the Dirac generalized function) with appropriate boundary conditions for the concentration field $C(t,x)$. The solution to this differential problem in the infinite domain $x \in]-\infty, \infty[$ is given by:

$$C(t,x) = \mathcal{G}_u(t,x,t_0,x_0) \tag{9}$$

where

$$\mathcal{G}_u(t,x,t_0,x_0) = \frac{1}{\sqrt{4\pi\kappa(t-t_0)}} \exp\left[-\frac{(x-x_0-u(t-t_0))^2}{4\kappa(t-t_0)}\right] \tag{10}$$

is the so-called Green's function of the problem.

The resolution of Eq. (8) amounts to advance the initial concentration field in time using the forward operator $\mathcal{A}_{t,t_0}$ such that:

$$C(t,x) = \mathcal{A}_{t,t_0} C(t_0,x) \tag{11}$$

For the simplified one-dimensional model considered here, the forward operator can be written as:

$$\mathcal{A}_{t,t_0} f(x) = \int_{-\infty}^{\infty} \mathcal{G}_u(t,x,t_0,x') f(x') \mathrm{d}x' \quad \text{for } t > t_0 \tag{12}$$

With the definition (Eq. (11)) of the forward operator and the notation:

$$\langle f,g \rangle = \int_{-\infty}^{\infty} f(x) g(x) \mathrm{d}x \quad \forall f,g \in \mathbb{L}_2 \tag{13}$$

of the scalar product on the space of square-Lebesgue-integrable real functions $\mathbb{L}_2$, $\tilde{m}$ can be expressed as:

$$\tilde{m}(t;\ t_0,x_0) = \int_{-\infty}^{\infty} C(x,t) d(x) \mathrm{d}x = \langle C(t,x), d(x) \rangle = \langle \mathcal{A}_{t,t_0} \delta(x-x_0), d(x) \rangle \tag{14}$$

where

$$d_{[-L,L]}(x) = \begin{cases} 1 & \forall x \in [-L,L] \\ 0 & \text{elsewhere} \end{cases} \tag{15}$$

is the characteristic function of the control region $[-L,L]$.

Using the above results, one gets:

$$\tilde{m}(t;\ t_0,x_0) = \frac{1}{2}\left(\mathrm{Erf}\left[\frac{x_0+L+u(t-t_0)}{\sqrt{4\kappa(t-t_0)}}\right] - \mathrm{Erf}\left[\frac{x_0-L+u(t-t_0)}{\sqrt{4\kappa(t-t_0)}}\right]\right) \tag{16}$$

and

$$\bar{\theta}(t_0,x_0) = \int_{t_0}^{\infty} \tilde{m}(t;t_0,x_0) \mathrm{d}t = \frac{L-x_0}{u} + \frac{\kappa}{u^2}\left(1 - \exp\left[-\frac{u(L+x_0)}{\kappa}\right]\right) \tag{17}$$

These expressions were used to plot Figs. 2 and 3.

Notice that the solution (Eq. (16)) differs from the decreasing exponential law (Eq. (2)) used in simple assessment procedures. The main difference is the delay that appears in Fig. 2 before the mass starts to decrease significantly. The assumptions of perfect and immediate mixing used to justify an exponential decrease are indeed not met here and Eq. (2) is not a valid approximation of $m(t;\ t_0,x_0)$.

Now, to derive the backward procedure for the computation of the residence time, we introduce the adjoint operator $\mathcal{A}^*_{t,t_0}$ of $\mathcal{A}_{t,t_0}$ such that:

$$\langle \mathcal{A}_{t,t_0} f, g \rangle = \langle f, \mathcal{A}^*_{t,t_0} g \rangle \quad \forall f,g \in \mathbb{L}_2 \tag{18}$$

With this definition, Eq. (14) takes the final form:

$$\tilde{m}(t;\ t_0,x_0) = \langle \delta(x-x_0), \mathcal{A}^*_{t,t_0} d_{[-L,L]}(x) \rangle = \int_{-\infty}^{\infty} \delta(x-x_0) \mathcal{A}^*_{t,t_0} d_{[-L,L]}(x) \mathrm{d}x = \left[\mathcal{A}^*_{t,t_0} d_{[-L,L]}(x)\right]_{x=x_0} \tag{19}$$

Using Eq. (19), the values taken by $\tilde{m}(t;\ t_0,x_0)$ for different release points $x_0 \in [-L,L]$ can all be obtained by a single application of the adjoint operator $\mathcal{A}^*_{t,t_0}$ to the characteristic function $d(x)$ of the control region.

For the simplified problem at stake, the adjoint of Eq. (12) can be written explicitly as:

$$\mathcal{A}^*_{t,t_0} f(x) = \int_{-\infty}^{\infty} \mathcal{G}_{-u}(t,x,t_0,x') f(x') \mathrm{d}x' \tag{20}$$

Using Eq. (19), one has therefore:

$$\begin{aligned} \tilde{m}(t;\ t_0,x_0) &= \mathcal{A}^*_{t,t_0} d_{[-L,L]}(x) \\ &= \int_{-\infty}^{\infty} \mathcal{G}_{-u}(t,x,t_0,x_0) d_{[-L,L]}(x_0) \mathrm{d}x_0 \\ &= \int_{-L}^{L} \frac{1}{\sqrt{4\pi\kappa(t-t_0)}} \times \exp\left[-\frac{(x-x_0+u(t-t_0))^2}{4\kappa(t-t_0)}\right] \mathrm{d}x_0 \quad \text{for } t > t_0 \end{aligned} \tag{21}$$

which is identical to Eq. (16). This shows the equivalence of the forward and backward approaches.

Now, in the same way as the propagation of the initial conditions of Eq. (12) by the forward operator (Eq. (11)) was considered equivalent to the resolution of problem (12), the propagation in time (Eq. (19)) of $d(x)$ by the adjoint operator $\mathcal{A}^*_{t,t_0}$ corresponds to the resolution of the adjoint problem:

$$\begin{cases} \frac{\partial C_t^*}{\partial t_0} + u\frac{\partial C_t^*}{\partial x} + \kappa\frac{\partial^2 C_t^*}{\partial x^2} = 0 \\ C_t^*(t,x) = d_{[-L,L]}(x) \end{cases} \tag{22}$$

where the solution $C_t^*(t_0,x)$ defines the adjoint state (see Appendix A or Morse and Feshbach, 1953). This adjoint problem is the so-called 'reverse flow' problem. It must be integrated backwards in time from 'initial conditions' given at t until time t_0.

The spatial distribution of the residence time can then be computed from Eq. (4) where $\tilde{m}(t)$ is given by:

$$\tilde{m}(t) = \tilde{m}(t;\ t_0, x_0) = C_t^*(t_0, x_0)$$

i.e. from the solution of the single adjoint differential problem (22).

2.3. Boundary conditions

The resolution of the forward and adjoint problems presented above requires appropriate boundary conditions. Different boundary conditions can lead to different interpretation of the residence time.

In Fig. 1, we considered that the boundary of the control region had no influence on the dynamics of the tracer. In particular, the tracer continues to be advected and diffused after leaving the control domain $[-L,L]$. Although the velocity is continuously directed to the right, some tracer parcels that have been flushed out can re-enter the domain because of diffusion. The mass $\tilde{m}(t)$ shown in Fig. 1 (solid curve) is the total mass of the tracer parcels present in the control region, irrespective of the particular paths followed by these parcels; some of the tracer parcels contributing to $\tilde{m}(t)$ have left and re-entered the control domain (once or more) between t_0 and t. Therefore, the time scale computed from this inventory and whose spatial distribution is plotted in Fig. 3 (solid curve) does not correspond with the usual definition of the residence time as the time to exit the domain *for the first time* but with the time to *definitively* exit the domain.

From here, we will call *strict residence time* Θ the time taken by a parcel to exit the domain for the first time. This definition requires to completely ignore tracer parcels as soon as they leave the domain. This cannot usually be done experimentally but the mathematical and numerical approaches described above can be adapted to cope with this requirement. To do so, the forward problem (Eq. (8)) must be solved in $[-L,L]$ only with the additional boundary conditions:

$$C(t,-L) = C(t,L) = 0, \quad \forall t > t_0 \tag{23}$$

that ensures that the water parcels touching the boundary are immediately removed from the computation. As shown in the appendix, the corresponding boundary conditions for the adjoint problem are

$$C_t^*(t_0,-L) = C_t^*(t_0,L) = 0, \quad \forall t_0 < t \tag{24}$$

By comparison, the only boundary conditions used to derive Eqs. (16) and (17) are to request that the solutions do not grow exponentially at infinity. Except for these modifications, the computation procedure and the advantage of the backward approach are unchanged.

The solutions with this new approach are plotted with dashed lines in Figs. 1–3 where they are compared with the results of the first approach.

As expected from the definitions, the cumulative distribution function of the strict residence time is always smaller than the corresponding distribution function of the residence time (Fig. 2). Also, the mean strict residence time is everywhere smaller than the residence time (Fig. 3). The difference is particularly significant close to the upstream boundary where the mean residence time is maximum while the mean strict residence time is close to zero. The mean true residence time vanishes at the boundaries of the control region because the parcels located close to the boundaries are rapidly flushed out by advection or by diffusion. The mean strict residence time shows a maximum in the upstream half of the domain.

The decision about which of the two approaches is the most appropriate has no simple answer. Both concepts are useful in different contexts.

If the parcels undergo a dramatic change of their properties when leaving the domain, the strict residence time is certainly appropriate to characterize the dynamics. This could be the case, for instance, if one follows some organic material in an anaerobic region; leaving the control region would mean entering an aerobic regime which would permit rapid bacterial activity. The particles re-entering the anaerobic domain would be so different that it would not be appropriate to count them any longer. Also, if the problem is to evaluate the time taken before the parcels touch the surface where rapid gas exchanges take place, the strict residence time approach is appropriate.

If the boundaries of the control region are somehow artificial in that they do not correspond with strong dynamical changes, the residence time approach can be preferable. If the aim is to quantify the time during which a region will be exposed to a pollutant discharged into the control region, it is desirable to take into account the fraction of the pollutant that returns to the control domain and use the residence time concept. The strict residence time approach would indeed underestimate the duration of the pollution event. Also, the strict residence time is inappropriate when the flow meanders around the boundary of the control region. The residence time appears on the contrary as a way to remove part of the arbitrariness in the definition of the boundaries of the control region. The importance of the return flow is also well-known in tidal systems in which the tracers are advected back and forth though the boundary; when the tidal prism method (e.g., Dyer, 1998) is used to evaluate the flushing time of tidal estuaries, for instance, a return flow factor is introduced to estimate the fraction of effluent water returning to the domain each flood tide.

In spite of the significant differences between the two residence times, the modeler has sometimes no other choice but to use the strict residence time in his numerical studies. In particular, this happens when the control region coincides with the domain of integration. In this case, appropriate information is often lacking to describe the return path of tracer parcels that have left the control domain. The parcels are therefore ignored as soon as they exit the domain for the first time and the strict residence time is computed.

Note also that, strictly speaking, the concept of mean residence time as the mean time taken by water parcels to leave definitely the control region is only valid in an infinite domain. The mean residence time can be computed using Eq. (4) only if $\tilde{m}(t)$ goes to zero when t tends to infinity. This is only possible in a case of pure advection (no diffusion) or if the initial released is mixed into an infinite volume of water (or if some external non-linear process removes the material from the system). In other situations, the mass $\tilde{m}(t)$ will tend toward a very small value (1 over the volume of the system) which is not zero. In theory, the distribution of the residence time will still make sense but the mean will be unbounded. The strict residence time does not suffer from such problems and its mean value can always be computed using Eq. (4), at least if water parcels are allowed to leave the control region, i.e. if this is not disconnected from the world ocean. In practice, the concentration of the dye/tracer will fall below the detection limit or the computation accuracy or will leave the computation domain so that the problem associated with the definition of the residence time used here will be avoided.

3. Generalization

The forward and backward procedures described in the previous section for the computation of the residence time are easily extended to general three-dimensional time-dependent flows.

In this section, the discussion will be kept at a practical level. Detailed mathematical developments are deferred to Appendix A.

3.1. Extension to three-dimensional problems

The generalization to three-dimensional flows is straightforward. Let ω denote the control region. The forward problem amounts to solve the advection–dispersion problem:

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C = \nabla\cdot[\mathbf{K}\cdot\nabla C] \\ C(t_0,x) = \delta(x-x_0) \end{cases} \tag{25}$$

in the spatial domain Ω with appropriate conditions on the boundary $\delta\Omega$. If the strict residence time is to be computed, Eq. (26) needs only to be solved in the control region ω (a subset of Ω) with $C = 0$ at the inflow and outflow boundaries of Ω.

The problem for the adjoint variable $C^*_T(t,x)$ is then:

$$\begin{cases} \dfrac{\partial C^*_T}{\partial t} + \mathbf{v}\cdot\nabla C^*_T + \nabla\cdot[\mathbf{K}\cdot\nabla C^*_T] = 0 \\ C^*_T(T,x) = \delta_\omega(x) \end{cases} \tag{26}$$

where $\delta_\omega(x)$ is the characteristic function of the control region ω and where the model has to be integrated backward in time from the 'initial conditions' prescribed at some time T. Here again, the problem must be restricted to the control region ω if the strict residence time is to be computed. The boundary conditions for the adjoint problem are derived from those of the forward model according to Table 1.

The solution of the adjoint problem can still be interpreted according to:

$$C^*_T(T-\tau,\mathbf{x}_0) = \tilde{m}(T;\ T-\tau,x_0) \quad \forall\tau>0 \tag{27}$$

and can be used to compute the mean residence time at any point x_0 in the control region.

3.2. Variable hydrodynamics

In Section 2.2, we claimed that $\tilde{m}(t)$ could be obtained from one single simulation carried out with the adjoint model (Eq. (22)). A careful reader would have noticed that running the adjoint model backward from 'initial conditions' at time T delivers the total mass $\tilde{m}(T;t_0,x_0) = C^*_T(t_0,x_0)$ in one run for the different release points x_0 but for a single time T. The evaluation of $\tilde{m}(t;t_0,x_0)$ for different times t requires successive simulations with the adjoint model by varying the 'initial time' $T = t$. The problem of having to apply the forward method to different release points is now moved into the similar problem of having to deal with different 'initial times' for the adjoint problem.

Table 1
Correspondence between the boundary conditions for the forward and adjoint problems

Forward problem	Adjoint problem
$C = 0$	$C^* = 0$
$\nabla C\cdot\mathbf{n} = 0$	$[\mathbf{v}C^* + \mathbf{K}\cdot\nabla C^*]\cdot\mathbf{n} = 0$
$[\mathbf{v}C - \mathbf{K}\cdot\nabla C]\cdot\mathbf{n} = 0$	$\mathbf{K}\cdot\nabla C^*\cdot\mathbf{n} = 0$

In the case of constant hydrodynamic fields, the problem can be avoided because the solution does not depend explicitly on t_0 and t, but only on the difference $t-t_0$ (see Eqs. (10), (16) and (21)). In this case, running the adjoint model only once from an arbitrary 'initial time' T provides:

$$C^*_T(T-\tau,x_0)=\tilde{m}(T;\ T-\tau,x_0)=\tilde{m}(t_0+\tau;\ t_0,x_0)\quad \forall\tau>0 \tag{28}$$

where the last equality is valid because the translation of the observation time t and the release time t_0 by a common delay does not modify the solution. In other words, the solution $C^*_T(T-\tau,x_0)$ of the adjoint problem at successive times $T-\tau$ but from a fixed (arbitrary) origin T provides the fraction of the mass that is present in the control domain after a delay τ following a unit release at x_0. The mean residence time can then be computed using Eq. (4).

With variable hydrodynamic conditions, the problem given above cannot be avoided and a set of adjoint problems (Eq. (26)) must be solved by varying the 'initial time' T: the determination of the full distribution of residence times requires the solution of a one-parameter family problem. As suggested by the dependency of $\tilde{m}$ on the observation time t, the release time t_0 and the three spatial coordinates of the release point $x_0=(x,y,z)$, the problem can be restated as a differential problem in a five-dimensional space which is similar to the space introduced by Delhez et al. (1999a) and Deleersnijder et al. (2001) in their theory of age.

To render this five-dimensional space more explicit, we define the cumulative distribution function $D(t_0,\tau,x_0)$ as the fraction of the mass of the tracer released at time t_0 and location x_0 whose residence time is larger or equal to τ, i.e. the mass of tracer in the control region at time $t_0+\tau$. From this definition, it comes:

$$D(t_0,\tau,x_0)=\tilde{m}(t_0+\tau;\ t_0,x_0)=C^*_{t_0+\tau}(t_0,x_0) \tag{29}$$

Introducing this definition into Eq. (26), the differential equation for $D(t,\tau,x_0)$ can be expressed as:

$$\begin{cases}\dfrac{\partial D}{\partial t}-\dfrac{\partial D}{\partial \tau}+\mathbf{v}\cdot\nabla D+\nabla\cdot[\mathbf{K}\cdot\nabla D]=0\\ D(t,0,x)=\delta_\omega(x)\end{cases} \tag{30}$$

Eq. (30) with boundary conditions taken from Table 1 must be solved in a five-dimensional space. It plays the same role as the equation for the age distribution function discussed by Delhez et al. (1999a) and Delhez and Deleersnijder (2002) in the theory of age.

To simplify the problem, one can content oneself with the determination of the mean residence time. In terms of the cumulative distribution function, this can be expressed as:

$$\overline{\theta}(t,x_0)=\int_0^\infty D(t,\tau,x_0)\mathrm{d}\tau \tag{31}$$

Assuming that $D(t,\tau,x_0)$ decreases to zero when τ tends to infinity, i.e. the whole material is eventually flushed out of the control region, Eq. (30) can be integrated with respect to τ to simplify the problem into the more classical differential problem:

$$\frac{\partial\overline{\theta}}{\partial t}+\delta_\omega+\mathbf{v}\cdot\nabla\overline{\theta}+\nabla\cdot\left[\mathbf{K}\cdot\nabla\overline{\theta}\right]=0 \tag{32}$$

for the mean (strict) residence time $\overline{\theta}(t,x)$.

Eq. (32) forms the main result of this paper and defines the procedure to compute the mean residence time. The computation of the mean residence time at any time and location only requires the integration of Eq. (32) with appropriate boundary conditions as discussed above. This equation is nothing but an advection–diffusion equation with a unit (negative) source term in the control region ω. Usual numerical techniques, which are already available in the hydrodynamic model, can therefore be applied to solve it. The only subtlety is the fact that the equation must be integrated backward in time and with the reversed flow, i.e. with $\mathbf{v}$ changed to $-\mathbf{v}$. This requires minor additional implementation efforts. In a first step, the hydrodynamic model used to generate the velocity and turbulence fields is integrated forward, as usual, and the intermediate results are stored. Then, in a second step, Eq. (32) is integrated backward in time using the hydrodynamic fields computed in the first step.

Some words of caution are required here. Eq. (32) must be integrated backward from some 'initial time' T. The real conditions at time T will never be known precisely because they require the knowledge of what happens between T and $+\infty$. Theoretically, Eq. (32) should be integrated backward from $T=+\infty$. In practice, however, T has to be finite and will be chosen as large as possible. As a result, the solution of Eq. (32) will never produce the exact mean residence time. If the simulation is carried out for a sufficient long time, however, the influence of the initial conditions will disappear, the solution of Eq. (32) will be representative of a larger and larger proportion of the tracer and it will approach the strict mean residence time.

4. Application to the English Channel and southern North Sea

In this section, the backward approach described above is applied to the evaluation of the mean residence time in the eastern English Channel and the southern North Sea.

The results are obtained by means of the three-dimensional, hydrodynamic model of the North Western European Continental Shelf developed at the GHER

(GeoHydrodynamics and Environment Research laboratory, University of Liège). This model is baroclinic and includes a robust and versatile turbulence closure scheme (Delhez et al., 1999b). It covers the whole shelf to the east of the 200 m isobath, from 48°N to 61°N, including the Skagerrak and Kattegat, with an horizontal resolution of 10′ in longitude and latitude, i.e. about 10 km × 16 km, and 10 vertical σ-levels (Delhez and Martin, 1992). The numerical implementation is based on a finite volume approach and uses a TVD scheme with superbee limiter for the advection of scalar quantities (James, 1996). The model is forced at its open ocean boundaries, which are located far away from the regions of interest, by nine tidal constituents and the inverse barometer effect. The meteorological forcing data (6-h air temperature, surface pressure, relative humidity, wind speed, cloud cover; horizontal resolution 1.5° × 1.5°) are extracted from the NCEP/NCAR reanalysis of surface data from NOAA/CDC (http://www.cdc.noaa.gov/cdc/reanalysis/).

The simulations span the period of about two years between January 1983 and September 1984. The hydrodynamic model is first integrated forward in time from adjusted hydrodynamic fields. Then Eq. (32) for the mean residence time is integrated backward in time.

Two different control regions are considered: the Eastern part of the English Channel and the southern North Sea. The precise boundaries of these control regions are plotted in Fig. 4 together with a schematic view of the residual circulation. In addition to the general circulation shown in this figure, the region is also characterized by strong tidal currents – with a tidal excursion of more than 10 km – and a variable wind forcing dominated at a time scale of three to five days by the passage of Atlantic depressions over the shelf (e.g., Lee, 1980; Pingree, 1980).

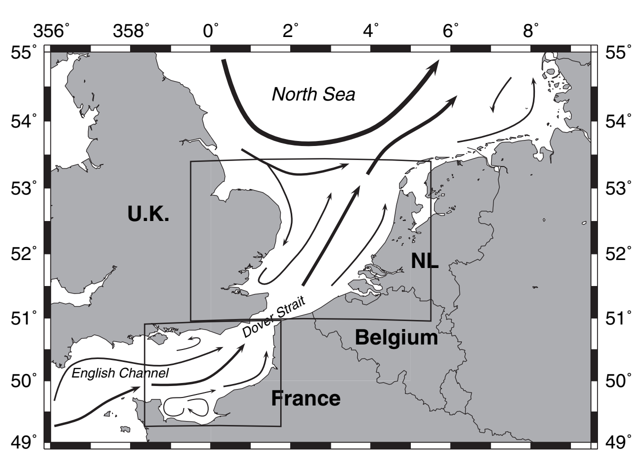

Fig. 4. Boundaries of the control regions and schematic description of the residual circulation (after Lee, 1980; Pingree, 1980; Salomon et al., 1988).

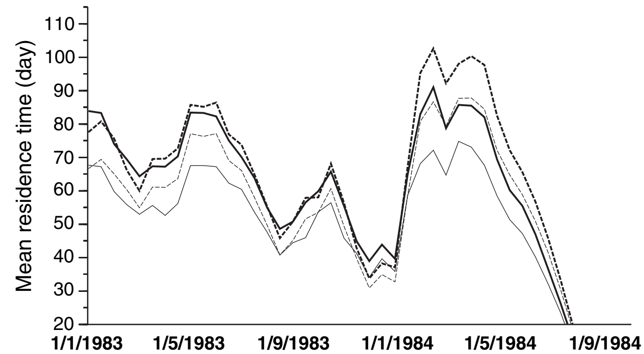

Fig. 5. Time series of the average mean residence time (thick curves) and average mean strict residence time (light curves) at the surface in the two control regions (solid line = Eastern English Channel, dashed line = southern North Sea).

The average values of the residence times in the two control regions are shown in Fig. 5 as a function of time. The backward simulations started on 1 September 1984. During a first phase, the mean residence time increased from the prescribed zero initial value. It takes about 6 months to remove the effects of this initialisation. During the rest of the simulation period, the average mean residence time lies in both control regions between 40 and 85 days, with extreme values of 30 days in November/December 1983 and 105 days in March 1984.

Clearly, the results obtained between May and September 1984 are not significant. This is not unexpected: it is indeed impossible to get the mean residence time during this period from a simulation that ignores what happens after September 1984. Considering the physical meaning of the residence time it appears reasonable to trust the results obtained after an initialisation period equal to twice the computed mean residence time.

As expected from their respective definitions, the mean strict residence time is always smaller than the mean residence time.

The temporal variations of the average residence times in the two control regions are roughly in phase with each other and reflect the temporal variability of the flow. The enhanced winter circulation induces minimum residence times while the slowing down of the flow in spring and summer is responsible for the larger residence times of particles released during this period of the year. Because the residence time reflects the intensity of the horizontal exchange during some period of time following the observation time, relative minima (resp. maxima) occurs at the beginning of each period of strong (resp. weak) circulation.

The snapshots of the mean residence time taken in mid August 1983 and shown in Figs. 6 and 7 reflect the general circulation in this part of the shelf.

The main part of the general circulation in the Eastern English Channel is one-dimensional from the Western English Channel to the Southern Bight of the North Sea.

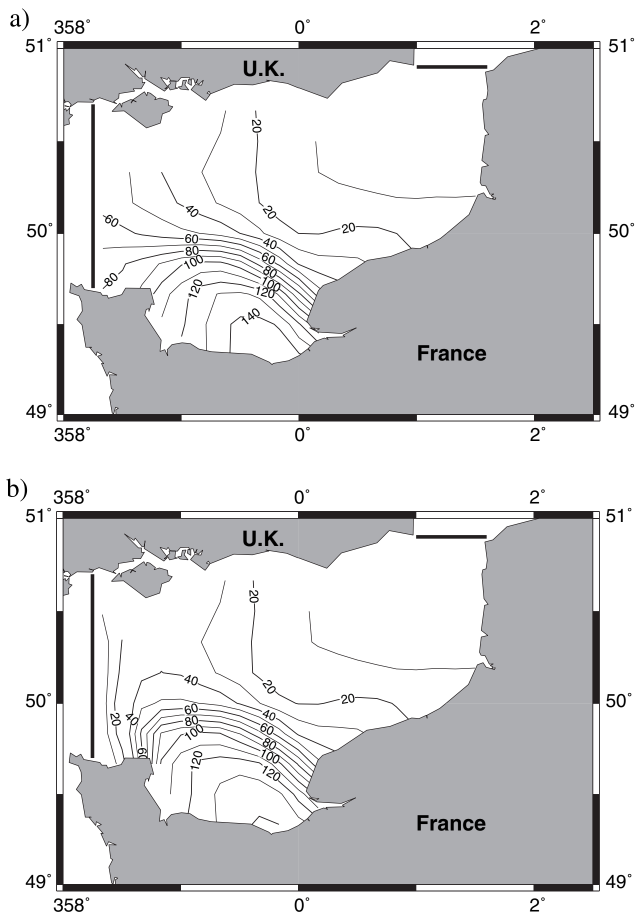

Fig. 6. Snapshot of the mean residence time (in days) (a – upper panel) and mean strict residence time (in days) (b – lower panel) at the surface in the Eastern English Channel on the 15/08/1983.

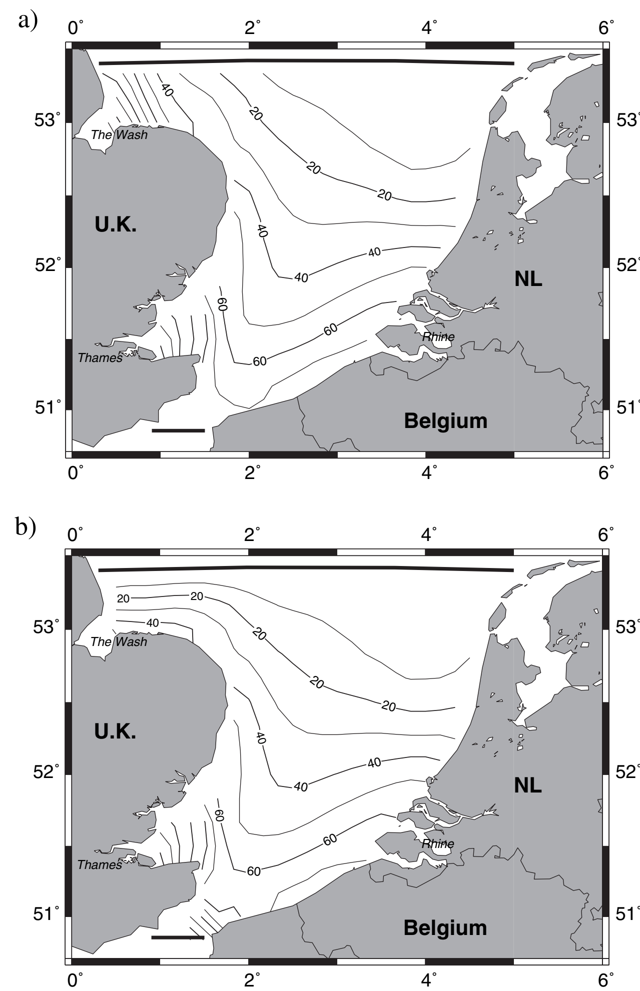

Fig. 7. Snapshot of the mean residence time (in days) (a – upper panel) and mean strict residence time (in days) (b – lower panel) at the surface in the southern North Sea on the 15/08/1983.

The residence time decreases therefore gradually from the Western boundary of the control region to the Strait of Dover. The mean residence time in the mid-channel waters is about 30 days.

Much larger residence times are found along the French coast (Fig. 6). These are related to the very slow net motion in this part of the Channel. The flow in this region is indeed dominated by persistent residual gyres induces by tidal non-linear interactions (e.g., Salomon et al., 1988; Delhez, 1996). As shown by Guéguéniat et al. (1995), pollutants can remain trapped in these residual gyres for a long time, hence the large mean residence time computed here.

The more distant iso-lines in the vicinity of the Strait of Dover reflect the acceleration of the residual flow in this part of the Channel.

The residence time (Fig. 6a) and the strict residence time (Fig. 6b) computed in the English Channel differ significantly in the vicinity of the western boundary of the control region, especially in the central and northern part of this section. As expected, the strict residence time is close to zero there because tracer parcels are rapidly flushed out towards the Western part of the Channel by diffusion and strong mesoscale currents. The same tracer parcels re-enter the control domain by the combined action of the mesoscale and residual currents. The return flow might therefore not be neglected and the residence time must be preferred to the strict residence time.

Similar results are obtained for the residence time in the southern North Sea (Fig. 7). Here, the main differences between the residence time and the strict residence time are concentrated in the Strait of Dover and at the western side of the northern boundary. These differences show that boundaries of the control region cannot be seen as natural boundaries of the flow. The residence time approach helps to circumvent this apparent level of arbitrariness.

At this time of the year, the main residence time reaches 70 days at the Strait of Dover. It decreases gradually along the main stream flowing across the Southern Bight of the North Sea. The mean residence time is about 40 days in the center of the control region.

Larger values are found along the Belgium and Dutch coasts, on one side, and along the English coast, on the other side. Maximum values are found in the shallow estuaries (Thames, Wash) but also at the mouth of the Rhine estuary. Note, however, that the spatial resolution of the hydrodynamic model is not appropriate to describe the complex baroclinic dynamics of the Rhine ROFI (e.g., Ruddick et al., 1995). Such processes are likely to accelerate the flow along the Dutch coast and reduce therefore the residence time in this region.

5. Conclusion

The residence time is a very useful concept in many environmental studies. While previous approaches were often based on simplified hypotheses, a rigorous generic method is now available to study the residence time in a semi-enclosed domain by means of a numerical model. This method can be used to compute the distribution of residence times or only the mean value of this distribution. The procedure requires the solution of the adjoint problem to the advection–diffusion equation. It can be extended to compute the residence time of tracers with a radioactive decay or other tracers with a linear dynamics.

While the basin average residence time can be computed by other means, the backward procedure described here provides the spatial distribution of the residence time. This is a very valuable information as it allows to identify the regions where, for instance, pollution problems are likely to develop.

As for other similar concepts, a clear understanding of the implications of the definition is required for an appropriate interpretation of the results. The definitions of the residence time as the time before the parcel leaves the domain for the first time or for the last time can be used in different contexts. Both approaches are feasible with the procedure set up in this paper; only the boundary conditions must be adapted.

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Appendix A. Derivation of the adjoint problem

In this section, we present an alternative method to introduce the adjoint problem used to evaluate the residence time in a control domain ω. The method, relying on elementary calculus, provides directly the expression of the adjoint problem.

The forward problem consists in the general three-dimensional time-dependent advection–diffusion problem:

$$\begin{cases} \dfrac{\partial C}{\partial t}+\mathbf{v}\cdot\nabla C=\nabla\cdot[\mathbf{K}\cdot\nabla C] & \text{in } [t_0,t_\infty]\times\Omega \\ C(t_0,x)=\delta(x-x_0) \end{cases} \tag{33}$$

where t is time, x is the current point, $[t_0,t_\infty]$ denotes some appropriate integration time window (where t_∞ is chosen large enough for most of the initial released to be flushed out of the control domain), Ω is the spatial domain of integration, C is the concentration field produced by a unit point release at time t_0 and location x_0, $\mathbf{v}$ is the velocity field and $\mathbf{K}$ is the diffusion tensor which is assumed symmetric. Because the aim of the forward problem is to follow the fate of the material released into the control domain without additional inputs of material, the boundary Γ of the integration domain can be split into three parts Γ_1, Γ_2, Γ_3 ($\Gamma_1\cup\Gamma_2\cup\Gamma_3=\Gamma$) where homogeneous boundary conditions are prescribed according to the first column of Table 2.

For an arbitrary time horizon $T>t_0$, the mass of the tracer present in the control domain is given by:

$$m(T;\,t_0,x_0)=\iiint_\omega C(T,x)\mathrm{d}x \tag{34}$$

Introducing the so far arbitrary adjoint variable $C^*(t,x)$ (Lagrangian multipliers), one has also:

$$m(T;\,t_0,x_0)=\iiint_\omega C(T,x)\mathrm{d}x-\int_{t_0}^{T}\mathrm{d}t \times\iiint_\Omega C^*\left\{\frac{\partial C}{\partial t}+\mathbf{v}\cdot\nabla C-\nabla\cdot[\mathbf{K}\cdot\nabla C]\right\}\mathrm{d}x \tag{35}$$

By integration by parts, this expression is easily transformed into:

Table 2
Boundary conditions for the forward and adjoint problems on the different parts Γ_1, Γ_2 and Γ_3 of the boundary Γ of the integration domain

	Forward problem	Adjoint problem
Γ_1	$C=0$	$C^*=0$
Γ_2	$\nabla C\cdot\mathbf{n}=0$	$[\mathbf{v}C^*+\mathbf{K}\cdot\nabla C^*]\cdot\mathbf{n}=0$
Γ_3	$[\mathbf{v}C-\mathbf{K}\cdot\nabla C]\cdot\mathbf{n}=0$	$\mathbf{K}\cdot\nabla C^*\cdot\mathbf{n}=0$

$$m(T;\ t_0,x_0)=\iiint_\omega C(T,x)\mathrm{d}x-\int_{t_0}^{T}\mathrm{d}t\iiint_\Omega \times C\left\{-\frac{\partial C^*}{\partial t}-\mathbf{v}\cdot\nabla C^*-\nabla\cdot\left[\mathbf{K}\cdot\nabla C^*\right]\right\}\mathrm{d}x - \iiint_\Omega\left[CC^*\right]_{t_0}^{T}\mathrm{d}x-\int_{t_0}^{T}\mathrm{d}t \times\iint_\Sigma C^*[\mathbf{v}C-\mathbf{K}\cdot\nabla C]\cdot\mathbf{n}\,\mathrm{d}\Sigma - \int_{t_0}^{T}\mathrm{d}t\iint_\Sigma C\left(\mathbf{K}\cdot\nabla C^*\right)\cdot\mathbf{n}\,\mathrm{d}\Sigma \quad (36)$$

Now, assume that the adjoint variable C^* solves the problem:

$$\begin{cases}\frac{\partial C^*}{\partial t}+\mathbf{v}\cdot\nabla C^*+\nabla\cdot\left[\mathbf{K}\cdot\nabla C^*\right]=0 & \text{in } [t_0,T]\times\Omega \\ C^*(T,x)=d_\omega(x)\end{cases} \quad (37)$$

where

$$d_\omega(x)=\begin{cases}1 & \forall x\in\omega \\ 0 & \text{elsewhere}\end{cases} \quad (38)$$

is the characteristic function of the integration domain. Eq. (36) simplifies into:

$$m(T;\ t_0,x_0)=\iiint_\Omega C(t_0,x)C^*(t_0,x)\mathrm{d}x - \int_{t_0}^{T}\mathrm{d}t\iint_\Sigma C^*[\mathbf{v}C-\mathbf{K}\cdot\nabla C]\cdot\mathbf{n}\,\mathrm{d}\Sigma - \int_{t_0}^{T}\mathrm{d}t\iint_\Sigma C\left(\mathbf{K}\cdot\nabla C^*\right)\cdot\mathbf{n}\,\mathrm{d}\Sigma \quad (39)$$

Taking into account the initial condition of the forward problem (Eq. (33)) and using the boundary conditions listed in Table 2 for the adjoint variable C^*, Eq. (39) takes the final form:

$$m(T;\ t_0,x_0)=C^*(t_0,x_0) \quad (40)$$

which is the basic equation on which relies the backward method for the evaluation of the residence time.

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