# RESIDUAL VIBRATION IN MODAL BALANCING

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Details are given of a practical technique that has been developed for the balancing of large flexible rotors. The special conditions that arise when such a rotor is borne in bearings from which vibration readings are taken are described. A modal balancing technique may be used for all modes through whose critical speeds the shaft runs, and then an averaging technique can account for the remaining modes.

## INTRODUCTION

THOSE FORCED VIBRATIONS of shafts which are caused by small defects of mass unbalance and initial bend have been discussed in a number of papers. A resumé is given in reference (**1**)‡. In addition a theoretical balancing technique has been proposed (**2**). In this technique, a shaft is balanced 'for the first mode' by observing the vibration of the shaft at some speed near its first critical speed. It can then be run smoothly up to the vicinity of its second critical speed, at which stage vibration measurements permit the second modal component of unbalance to be removed without upsetting the previously acquired balance in the first mode. Then the shaft may be run smoothly through the first and second critical speeds and up to the neighbourhood of the third, and so on.

In the balancing technique, the shaft is run near a critical speed in order to magnify the vibration in the corresponding principal mode. When its effects have been magnified in this way, the component of defect which corresponds to that mode can be nullified by a systematic process of adding balancing weights. (It is essential to recognize that one must distinguish between modal components of unbalance, or of 'defect', and of vibration.) If the shaft can be run through $r$ critical speeds, then $r$ components of unbalance may be removed in this way.

At this stage the vibration of the shaft may be sufficiently slight for the shaft to be regarded as balanced. It is possible, however, that the vibration in one or more of the higher modes (the $(r+1)$th, $(r+2)$th, . . . .) may exceed the permitted level in the range of operating speed. If this 'residual' vibration is confined effectively to the $(r+1)$th mode, then the corresponding component of unbalance can be removed in the usual way; but the helpful process of magnifying the vibration in this mode is not now available, unless the $(r+1)$th critical speed can be approached closely.

Difficulty arises, however, when the residual vibration is caused by components of unbalance in two or more of the modes corresponding to higher critical speeds than the $r$th. This situation has been found to occur in practice. One technique for dealing with the problem of mixed modes has been proposed previously (**3**). This method, however, is only applicable when the shaft can be run through at least some of the critical speeds of the mixed modes and thus it cannot be used to solve the present problem. The nullification of the remaining vibration with components in two or more modes is the subject of the present paper and it will be shown that the residual vibration may be removed in an average way. The actual results that will be quoted in support of the text have been published elsewhere by Moore and Dodd (**4**), to whom we gladly make acknowledgement, and this paper gives our interpretation of these results. It is considered that the repetition of the results is easily justified by the technical importance of the problem concerned.

It is common to measure the vibration of large rotors by means of transducers carried on the bearing pedestals. It will be shown that the points at which the readings are taken determine the nature of the average way in which the vibration is removed. It will be shown, too, how the removal of vibration is accurate only for a single running speed, although the accuracy may not depend sensitively on the speed.

A special case is of particular interest and it arises when a flexible rotor does not reach its first critical speed under normal running conditions. (If the shaft approaches this critical closely enough for vibration in the first mode to become perceptible, then the shaft is correctly referred to as a 'flexible' one.) In these circumstances the averaging

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‡ *References are given in Appendix 3.*

process that will be discussed covers all modes of the shaft. In a sense, conventional low-speed balancing in a balancing machine is of this type.

It is worth mentioning that, in theory, a residual vibration due to mixed components of unbalance can be handled by a straightforward development of standard modal balancing. The components may be separated out and balanced individually. The difficulty is a practical one, in that this approach requires some knowledge of modal shapes corresponding to higher speeds than the maximum obtainable, and it is difficult to calculate these with sufficient accuracy.

## PEDESTAL-MOUNTED TRANSDUCERS

In order to isolate flexure of the rotor from that of the driving motor, it has become standard practice to couple the two rotating bodies by means of a double Hooke's joint. If the rotor is run near its first critical speed, it therefore deflects in the form of its first principal mode. This is indicated in Fig. 1*a*, where A and B are the centres of the two bearing pedestals. The principal modes shown in Fig. 1 are illustrations and do not relate to an actual rotor. It will be seen that the deflections at A and B are not zero, since the pedestals themselves deflect. (Indeed, it is from the pedestal deflection that vibration is detected.) Moreover, the deflections at A and B are not equal.

Fig. 1*b* shows the second principal mode of a shaft and Fig. 1*c* shows the third. In all cases the deflection of the shaft at a pedestal is not zero and there is no reason why it should be equal at the two ends of the shaft.

It has been found that the deflections of the pedestals at A and B may be taken as proportional to the deflections of the shaft, independent of speed. Of course, strictly, the modal theory is only valid for flexible pedestals if they are massless. This condition is approximately realized if the maximum rotor speed corresponds to a frequency well below the lowest natural frequencies of the pedestals.

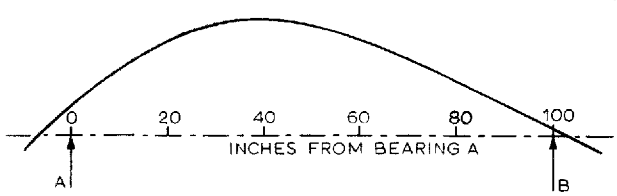
Suppose that horizontal vibration is measured at each of the two bearing pedestals*. Now the vibration of the bearing pedestals A and B is the only information available on the motion of the shaft. If there is no vibration of A and B then, to all intents and purposes, the shaft is running perfectly straight.

Since readings are only taken at the pedestals, it is natural to refer all vibration measurements to A and B. Consider, therefore, the vibration at these points. If the distortion of the shaft occurs in the first principal mode—i.e. if the shaft bends as shown in Fig. 1*a* and rotates bent—then, since the bend occurs in a single plane, the vibrations at A and B will be in phase as indicated by $\overrightarrow{OA}$, $\overrightarrow{OB}$ in Fig. 2*a*. If the distortion was in the second mode, then the vectors would be of the general form shown in Fig. 2*b*. And so one can go on for the higher principal modes.

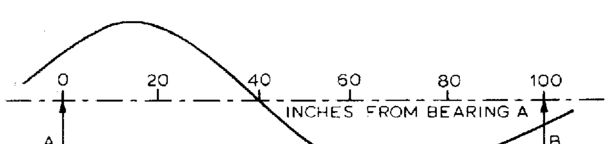
In the vector figures of Fig. 2, the relative lengths of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are determined by the ordinates at A and B of the curves in Fig. 1. The absolute lengths of the vectors are determined by the intensity of the corresponding modal components of unbalance while their orientation relative to the rotor is fixed by the orientation of the corresponding modal components of unbalance. The magnitudes and directions of the vectors also depend upon the shaft speed. This is discussed in reference (1).

In practice, measurements are made of speed, the displacement of the pedestal in primary (i.e. once-per-revolution) vibration and the phase difference between the vibration and the position of the rotor. In effect, this means that the *deflection* of the shaft is measured at A and B

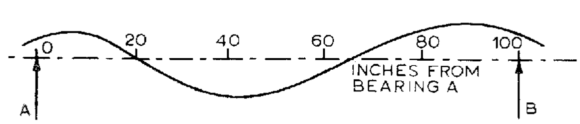
* *It is tacitly assumed in all work in this field that the vibration in the vertical direction is of equal magnitude since the system is assumed to possess axial symmetry. The fact that this is not strictly borne out in practice has not been found to detract from the usefulness of the technique under discussion.*




*a* First principal mode.

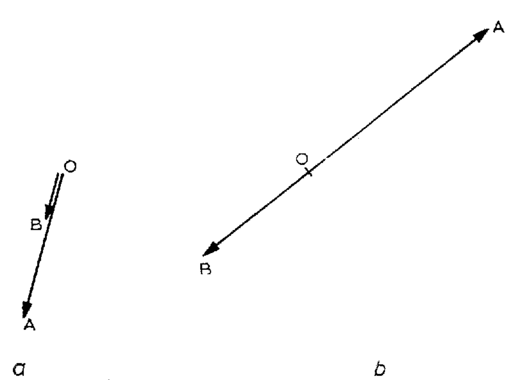



*b* Second principal mode.




*c* Third principal mode.

Fig. 1. *Possible principal modes of a flexible rotor supported in flexible bearings*




*a* Vibration in first principal mode.
*b* Vibration in second principal mode.

Fig. 2. *Deflection of bearing pedestals*

together with the angles between these deflections and some datum radius which rotates with the shaft.

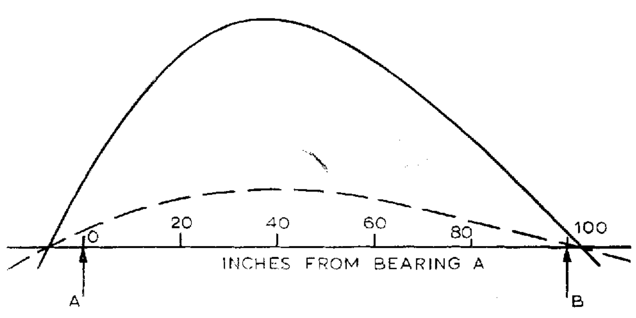
So much for the vibration vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ for the shaft's initial unbalance. Consider now a simple experiment. Suppose that a mass $m$ is attached to the shaft at a radius $R$. Let this mass be attached at a section $x_m$ along the shaft. In general the mass will augment all the modal components of unbalance. The supplementation of the $r$th component of unbalance will occur in the plane of attachment of $m$ and will be proportional to $mR\phi_r(x_m)$, where $\phi_r(x)$ is the $r$th characteristic function, i.e. the $r$th curve of the type shown in Fig. 1.

As mentioned already, each component of unbalance causes the shaft to bend in the form of the corresponding principal mode. The shaft rotates with this bent form, thereby setting up the vibrations at A and B. For any given running speed, the augmentation of the length of the vibration vector $\overrightarrow{OA}$—i.e. that component of the vibration of A which is due to the $r$th mode—is proportional to $mR\phi_r(x_m) \cdot \phi_r(x_A)$.

It follows that influence curves of vibration amplitude at A due to the attachment of the mass may be drawn for each mode. For a given $mR$, the influence curves are proportional to the appropriate characteristic functions, and the vertical scales of each are proportional to the relative magnitudes of deflection at A and B in the appropriate modes.

To illustrate this point, Fig. 3 shows what might be obtained from a shaft whose characteristic functions are shown in Fig. 1. The full-line curve represents the vibration at A in the first mode caused by a given $mR$ and the dotted curve represents the same thing for B. It will be seen from Fig. 1$a$ that the relative deflection at A is greater than that at B. Accordingly the full-line curve in Fig. 3 has larger ordinates than the dotted one.

So far, we have dealt merely with the matter of interpretation. These various features all follow immediately from the original theoretical balancing technique.




———— Deflection of bearing pedestal A.
— — — Deflection of bearing pedestal B.

Both curves are geometrically similar to the curve of Fig. 1$a$, such that the ratio of the ordinates of the full-line curve to the broken-line curve equals $\phi_1(x_A)/\phi_1(x_B)$.

Fig. 3. *Variation in the response of the bearing pedestals with the location of an unbalance mass along the shaft*

As already explained, the use of this technique with alternators has involved the taking of certain measurements. These are—to recapitulate—speed, the amplitudes of the vibrations at the pedestals and the phase of the vibrations relative to the position of some fixed radial line in the rotating body. It is perhaps more helpful to think of these measurements as giving essentially two things:

(1) speed, and

(2) radial amplitude and angular orientation of the distortion of the shaft relative to axes rotating at the driving speed.

In other words, if some arbitrary radial line is taken as a datum for angular measurement, the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ can be drawn.

In order to make clear how this information may be used to balance a shaft in its first mode, Appendix 1 contains a worked example taken from an actual rotor.

## RESIDUAL VIBRATION

For the sake of explanation, suppose—as is sometimes the case with alternators—that a rotor can be balanced in its first mode by nullification of the first component of the rotor's defects. Suppose, too, that the rotor's second critical speed lies just above the maximum running speed of the rotor. The vibration at A and B is quite perceptible and cannot be permitted to remain at the running speed of 3000 rev/min. A technique will now be described by which this residual vibration at bearings A and B may be removed in an average sense.

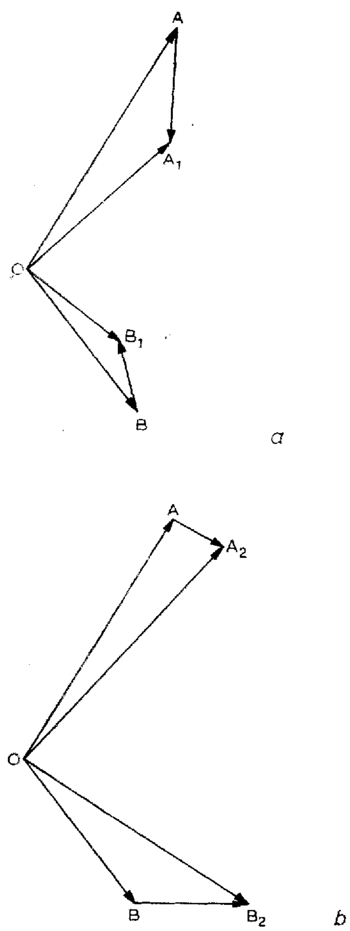
By following the procedure described in Appendix 1, the component of the rotor's defects corresponding to the first principal mode may be balanced out. What remains is an aggregate of contributions from all the remaining modes, each lying in some diametral plane.

If a set of masses is now attached to the shaft such that all members of the set lie in a single diametral plane, then a fresh radial contribution of defect will be added to the original defect (which is now minus its first modal component). Let this fresh component of defect be referred to as the $p$-component. A second similar distribution of masses, all lying in a single diametral plane, can be made up to provide a $q$-component.

Since the rotor has passed through its first critical speed, sufficient information has been obtained to permit a curve like Fig. 3 to be drawn. It follows therefore that $p$ and $q$ can both be made orthogonal to the first principal mode. In other words the masses can be chosen so that neither the $p$ nor the $q$ distribution will reintroduce any defect in the first principal mode. This is explained in numerical terms in Appendix 2.

Consider first the $p$ distribution. It is obtained by the attachment of masses to the surface of the shaft. It must now be agreed that the ratio between the $p$-masses will always be held constant and that the cross-sections at which they are applied will always be kept the same. These two properties identify the $p$-distribution. For reasons that will become apparent, the $p$-component can most conveniently be chosen in such a way as to produce a bend in the shaft that resembles the second principal mode of the rotor or, more accurately, to give displacements at A and B which are substantially in antiphase.

The $p$-component will give a distortion of the rotating shaft which depends upon the rotation speed $\Omega$. The distortion caused by the $p$ distribution is a mixture of distortions in all principal modes except the first and it contains a dominant contribution from the second. The deflection is not necessarily in the diametral plane of the $p$-masses since the shaft speed may be approaching the critical speed of one of the principal modal components of this $p$ distribution (the second). The distortion of the shaft in the second principal mode is then out of phase with the corresponding component of the $p$ distribution. For the same reason, the variation of the $p$-component deflection as a function of distance along the shaft will also be speed-dependent since the relative weighting of the principal modes varies with speed.




*a* $p$-response at A and B.
*b* $q$-response at A and B.

Fig. 4. *Calculation of the response of the bearing pedestals to an additional mass distribution*

If, however, some running speed $\Omega^*$ is selected, it can be said that the $p$ distribution of masses will produce a distortion and that if all the masses in the distribution were multiplied by some factor, then the distortion would also be multiplied by that same factor without any other change.

Exactly the same reasoning may be applied to the $q$-component of distortion which is caused by the $q$ distribution of masses. But now the latter should be chosen to produce a distortion that is such that the ends of the shaft at A and B are both deflected in substantially the same radial direction. For the sake of explanation, the system concerned in this discussion would be given a $q$-distortion largely in the third principal mode of the shaft.

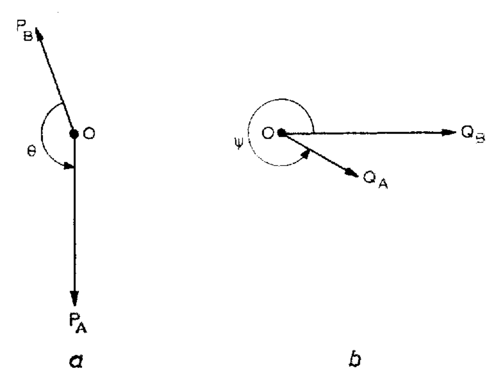
The $p$-response at A and B at a speed $\Omega^*$ can be found by vector subtraction (Fig. 4*a*). Thus let the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the initial vibration at A and B before any additional mass distribution is attached to the shaft. Further let the vectors $\overrightarrow{OA_1}$ and $\overrightarrow{OB_1}$ represent the deflections at A and B, when the $p$ distribution is attached. Then the $p$-response at A and B may be written in the form

$$\overrightarrow{OP_A} = \overrightarrow{OA_1} - \overrightarrow{OA} = \overrightarrow{AA_1}$$
$$\overrightarrow{OP_B} = \overrightarrow{OB_1} - \overrightarrow{OB} = \overrightarrow{BB_1}$$

The form of the $p$-response is shown in Fig. 5*a*. It will be noted that the two vectors are roughly in antiphase, as has been agreed should be the case.

If the $p$ distribution is removed and the $q$ distribution is attached, then the $q$-response can be obtained at the speed $\Omega^*$ (Fig. 4*b*). If the vibration at A and B, when the $q$ distribution is attached, is described by the vectors $\overrightarrow{OA_2}$ and $\overrightarrow{OB_2}$, then the $q$-response at A and B has the form

$$\overrightarrow{OQ_A} = \overrightarrow{OA_2} - \overrightarrow{OA} = \overrightarrow{AA_2}$$
$$\overrightarrow{OQ_B} = \overrightarrow{OB_2} - \overrightarrow{OB} = \overrightarrow{BB_2}$$




*a* Response to the $p$ distribution.
*b* Response to the $q$ distribution.

Fig. 5. *Response of the bearing pedestals to a distribution of additional masses*

The form of this response is indicated in Fig. 5*b*, where it can be seen that the two vectors are approximately in phase. It will be shown that the total residual vibration, described by the vectors $\vec{OA}$ and $\vec{OB}$ can be removed by suitable contributions of the types shown in Fig. 5*a* and *b*.

We must first contemplate the following problem:

Can the vectors $\vec{OA}$, $\vec{OB}$ be broken down into pairs of components which are such that

(1) one pair of components have the relative lengths and relative orientations of the $p$-vectors shown in Fig. 5*a*, and

(2) the other pair have the relative orientations and lengths of the $q$-vectors shown in Fig. 5*b*?

This resolution of vectors is in fact easy to make. A graphical technique is described in Fig. 8 of reference (4), while an analytical one will be presented here.

Now the relative lengths and orientations of the $p$-vectors can be simply expressed by the complex number $\bar{p}$ such that

$$\bar{p} = p\,e^{i\theta} = \frac{\vec{OP_A}}{\vec{OP_B}}$$

where the angle $\theta$ is defined in Fig. 5*a* and is also the obtuse angle between the vectors $\vec{AA_1}$ and $\vec{BB_1}$ in Fig. 4*a*.

For the multiplication implicit in this and the following equations not only the quantities $\bar{p}$ and $\bar{q}$, but also the displacement vectors $\vec{OA}$, $\vec{OA_1}$ etc., must be treated as complex numbers. Thus, this equation means, in words, that the vector $\vec{OP_A}$ giving the effect at A of the $p$ distribution's attachment may be obtained from that at B (i.e. from $\vec{OP_B}$) by multiplying its length by $p$ and rotating it counterclockwise through an angle $\theta$ (Fig. 5*a*).

In the same way we have

$$\bar{q} = q\,e^{i\psi} = \frac{\vec{OQ_A}}{\vec{OQ_B}}$$

That is, the vector $\vec{OQ_A}$ in Fig. 5*b* is obtained by multiplying $\vec{OQ_B}$ by $q$ and rotating it counterclockwise though an angle $\psi$.

We now wish to separate the vectors $\vec{OA}$ and $\vec{OB}$ into components (Fig. 6) such that

$$\vec{OA} = \vec{OA'} + \vec{A'A}$$

$$\vec{OB} = \vec{OB'} + \vec{B'B}$$

where

$$\vec{OA'} = \bar{p}\,\vec{OB'}$$

$$\vec{A'A} = \bar{q}\,\vec{B'B}$$

The required components $\vec{OB'}$ and $\vec{B'B}$ can now be calculated from the relations

$$\vec{OB'} = \frac{\vec{OA} - \bar{q}\vec{OB}}{\bar{p} - \bar{q}}$$

$$\vec{B'B} = \frac{\vec{OA} - \bar{p}\vec{OB}}{\bar{q} - \bar{p}}$$

Strictly, only one of these components need be calculated through the above expressions. For if one component is known, the other is directly obtainable by subtraction of the known vector from $\vec{OB}$.

Once $\vec{OB}$ has been resolved in this way, the components $\vec{OA'}$, $\vec{A'A}$ of $\vec{OA}$ may readily be found by use of the complex numbers $\bar{p}$, $\bar{q}$.

Let the components be renamed $\vec{A_p}$, $\vec{A_q}$, $\vec{B_p}$ and $\vec{B_q}$, as in Fig. 6. It is now possible to nullify the contributions $\vec{A_p}$ and $\vec{B_p}$ by increasing the $p$-masses at all points proportionally, and re-orientating them in a fresh diametral plane. Equally the second pair of vectors $\vec{A_q}$, $\vec{B_q}$ may be nullified by increasing or decreasing the $q$-masses and re-orienting them.

To illustrate the annulment of a residual vibration consider the results given by the full lines in Fig. 7. (They are taken from reference (4) where they form the basis of a graphical solution of a balancing problem.) Figs 4, 5 and 6 have in fact been constructed on the basis of this example. By measurement of the given vectors, it is found that

$$\bar{p} = 1{\cdot}5\,e^{i160°}$$

$$\bar{q} = 0{\cdot}5\,e^{i330°}$$

whence $\quad \bar{p} - \bar{q} = -1{\cdot}9 + 0{\cdot}7i$

If a real axis is taken as indicated by the line OR, it is found that

$$\vec{OA} = 7{\cdot}4\,e^{i58°} \qquad \vec{OB} = 4{\cdot}7\,e^{-i51°}$$

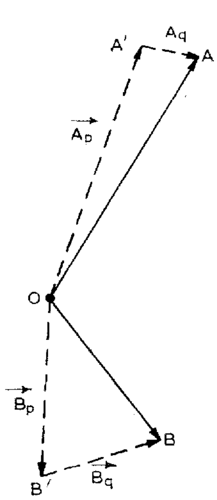



Fig. 6. *Separation of the vibration of the bearing pedestals into pairs of components*

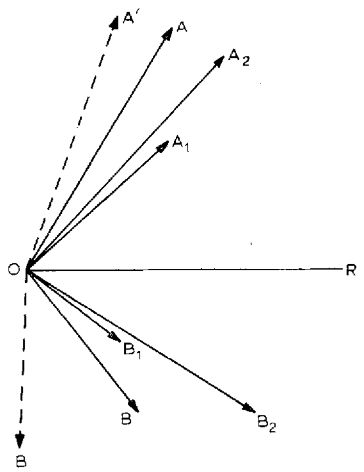



Fig. 7. *Example on the balancing of residual vibration*

whence $\overrightarrow{OB'} = 4{\cdot}6\,e^{-i92^\circ}$

so that $\overrightarrow{OA'} = 1{\cdot}5\,e^{i160^\circ} \times 4{\cdot}6\,e^{-i92^\circ} = 6{\cdot}9\,e^{i68^\circ}$

Having now located A′ and B′, we have only to interpret the results. Instead of producing the vectors $\overrightarrow{AA_1}$, $\overrightarrow{BB_1}$ the $p$ distribution is required to produce the vectors $-\overrightarrow{OA'}$ and $-\overrightarrow{OB'}$. This requires the masses of the $p$ distribution to be increased by a factor $OA'/AA_1$ and to be swung clockwise through the acute angle, $\alpha$ say, between $\overrightarrow{OA'}$ and $\overrightarrow{AA_1}$. In the same way, the masses of the $q$ distribution must be multiplied by $A'A/AA_2$ and be swung clockwise through the obtuse angle $A'AA_2$ or $\beta$ say.

This process is illustrated in Fig. 8 which shows a cross-section of the shaft perpendicular to the axis of the bearings. The shaft axis intersects this cross-section at the point O and OR represents the radial datum line taken as the real axis in Fig. 7. Suppose that the $p$ distribution was attached along the surface of the shaft in the diametral plane POP. This distribution may, of course, have been added on both sides of the shaft. Similarly suppose that the $q$ distribution was attached to the shaft in the QOQ plane. Then, in order to balance the residual vibration in the above manner, the plane of location of the $p$ distribution must be rotated clockwise through the angle $\alpha$ to the position P′OP′ (Fig. 8). Similarly the plane of attachment of the $q$ distribution must be rotated through the angle $\beta$ to Q′OQ′.

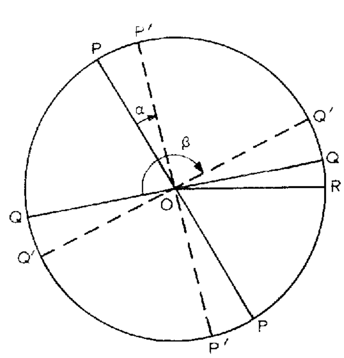



Fig. 8. *Location of the balancing planes relative to the initial planes of attachment of the p and q distributions*

## CONCLUSIONS

If in the above example $\Omega^*$ is close to the second critical speed, then it is likely that the balancing process described will be rather sensitively dependent on speed for its accuracy. If, on the other hand, $\Omega^*$ is remote from the next higher critical speed, it is unlikely that the nullification will be very speed-sensitive. In fact, in the latter circumstances, the $p$- and $q$-components will take on relatively simple forms. The $p$ and $q$ distortion shapes, amplitudes and orientations will not vary perceptibly with speed and the distortions will take place in the diametral planes containing the $p$- and $q$-mass distributions, that is in the planes POP and QOQ in Fig. 8.

If the rotor concerned is not to be operated at a constant speed, it may be necessary to improve upon a single-speed average balancing of this sort. In that event it will be necessary to balance for the next higher critical speed and only then to have recourse to the average technique described here. (The next higher mode can be identified by the fact that it is the only one with a perceptible phase change with speed.) As we have seen this is only likely to be the case when the next higher critical speed is fairly close to the operating speed range.

When the maximum running speed $\Omega^*$ of a shaft lies just below the lowest critical speed and average balancing of the present type is to be practised, the foregoing discussion still holds good in principle. But one would then, naturally, take a $q$-distortion to give a distortion largely in the first principal mode and a $p$ to give one largely in the second. The essential features of the two distributions is that they produce vectors of the general form shown in Fig. 4.

## APPENDIX 1

### THE CHOICE OF A BALANCING MASS FOR THE FIRST MODAL COMPONENT

The following example relates to the measurement of vibration at one pedestal—say A—of a boiler feed pump rotor. The first critical speed of this rotor is in the region of 1180 rev/min, and the pedestal vibration, $\overrightarrow{OA}$, was measured at a shaft speed of 850 rev/min. This speed is appreciably lower than the first critical speed, but the unbalance in the first mode was too large to permit a closer approach to the critical value. In fact, after achieving a major reduction in the vibration by attaching the balancing mass determined below, a final minor adjustment was made by balancing the shaft at a higher speed of 1000 rev/min. A trial balancing mass of 12 oz was then attached to the surface of the rotor at a particular cross-section and the response, $\overrightarrow{OA'}$, of the pedestal again observed at a speed of 850 rev/min. The amplitude, in terms of arbitrary electrical units, and the phase, relative to an arbitrary radial line in the rotor, of each of these vectors are given in Table 1.

Table 1

| Vector | Amplitude | Phase |
|---|---|---|
| $\overrightarrow{OA}$ | 18·0 | 48° |
| $\overrightarrow{OA'}$ | 17·2 | 41·5° |

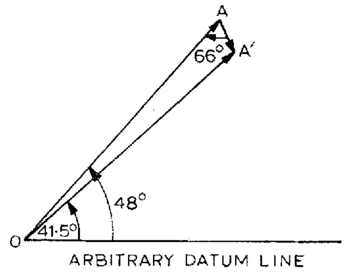



Fig. 9. *Vector calculation of the mass required to balance the first mode*

The response, $\overrightarrow{AA'}$, of the pedestal to the additional mass alone may be calculated by the simple vector subtraction of Fig. 9. From this it can be seen that the actual mass required to balance the first mode is of magnitude

$$12\times\frac{OA}{AA'} = 12\times 9 = 108 \text{ oz}$$

Further this mass, although still attached to the rotor at the same cross-section, must be rotated around the rotor through an angle of 66° clockwise relative to the datum line.

## APPENDIX 2

### ORTHOGONALITY OF THE $p$- AND $q$-MASS DISTRIBUTION TO THE FIRST COMPONENT OF UNBALANCE

The supplementation of the first component of unbalance by a mass $m$ attached at radius $R$ occurs in the plane of attachment and is proportional to $mR\phi_1(x_m)$, where $x_m$ is the position of the section at which the attachment is made. Fig. 1*a* shows the curve of $\phi_1(x)$ drawn to some arbitrary scale. Suppose that direct measurements from the curve give:

| $x_m$, inches from A | 10 | 25 | 44 | 70 | 90 |
|---|---|---|---|---|---|
| $\phi_1(x_m)$, units | 3·4 | 5·4 | 5·9 | 3·9 | 1·6 |

To form the $p$ distribution in the required manner, we may decide to attach masses to the shaft at points 25 and 70 inches from bearing A. The masses are both to be attached in the same diametral plane and it will be seen that they will not augment the first component of unbalance if they:

(1) are attached on opposite sides of the rotor, and

(2) satisfy the condition

$$\frac{(mR)_{25}}{(mR)_{70}} = \frac{3\cdot 9}{5\cdot 4} = 0\cdot 72$$

where the subscripts 25 and 70 have the obvious significance. These two requirements serve to define a suitable $p$ distribution.

The $q$ distribution may be formed in a similar way. It will be orthogonal to the first component of unbalance, for instance, if masses are added at distances 10, 44 and 90 in from bearing A such that

(1) $(mR)_{10} = (mR)_{90}$

(2) the masses at 10 and 90 in from A are on one side of the rotor while that placed 44 in from A is on the other, and

(3) $\dfrac{(mR)_{10}}{(mR)_{44}} = \dfrac{5\cdot 9}{3\cdot 4+1\cdot 6} = 1\cdot 18$

These three requirements represent one way of defining a suitable $q$ distribution. It will be noted that they are not unique, however, since the relation (1) is used only for convenience.

## APPENDIX 3

### REFERENCES

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