1997

Residues and Resultants

E Cattani
University of Massachusetts - Amherst, cattani@math.umass.edu

Alicia Dickenstein

Bernd Sturmfels

Residues and Resultants

Eduardo Cattani[1]
Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003, U.S.A.
cattani@math.umass.edu

Alicia Dickenstein[2]
Departamento de Matemática, F.C.E. y N.
Universidad de Buenos Aires
Ciudad Universitaria - Pabellón I
(1428) Buenos Aires, Argentina
alidick@dm.uba.ar

Bernd Sturmfels[3]
Department of Mathematics
University of California
Berkeley, CA 94720, U.S.A.
bernd@math.berkeley.edu

Abstract

Resultants, Jacobians and residues are basic invariants of multivariate polynomial systems. We examine their interrelations in the context of toric geometry. The global residue in the torus, studied by Khovanskii, is the sum over local Grothendieck residues at the zeros of n Laurent polynomials in n variables. Cox introduced the related notion of the toric residue relative to $n+1$ divisors on an n-dimensional toric variety. We establish denominator formulas in terms of sparse resultants for both the toric residue and the global residue in the torus. A byproduct is a determinantal formula for resultants based on Jacobians.

[1] Supported in part by the N.S.F.
[2] Supported in part by UBACYT and CONICET, Argentina.
[3] Supported in part by the N.S.F. and the David and Lucile Packard Foundation.

§0. Introduction

Resultants, Jacobians and residues are fundamental invariants associated with systems of multivariate polynomial equations. We shall investigate relationships among these three invariants in the context of toric geometry. The study of *global residues in the torus* has its origin in the work of Khovanskii [K2]. The global residue is the sum over local Grothendieck residues at the common roots of n Laurent polynomials in n variables; see (3.8) and (3.10). The related notion of the *toric residue* was introduced by Cox [C2] and subsequently studied in [CCD]. The toric residue is associated with $n+1$ divisors on an n-dimensional projective toric variety. For our purposes here it suffices to consider divisors that are multiples of a fixed ample divisor β. An algorithmic link between these two notions of residue ("toric" versus "in the torus") was established in [CD].

The main results of this paper are denominator formulas for toric residues (Theorem 1.4) and for residues in the torus (Theorem 3.2). In each case the denominator is given in terms of *sparse resultants*. These resultants are naturally associated with sparse systems of Laurent polynomials, or with line bundles on toric varieties. They were introduced by Gel'fand, Kapranov and Zelevinsky [GKZ] and further studied in [KSZ],[PSt],[S1],[S2]. In §4 we present new determinantal formulas for sparse resultants based on Jacobians.

One general objective of our work is to develop computational techniques, which may ultimately enter into the design of algorithms for solving polynomial equations. Classical results on residues, Jacobians and resultants are limited to dense equations, in which case the underlying toric variety is complex projective n-space $\mathbf{P}^n$. In that classical case our denominator formula appeared already in the work of Angéniol [A] and Jouanolou [J1],[J3]. Our results also extend the work of Gel'fond-Khovanskii [GK] and Zhang [Z], who studied residues in the torus for the special case when all facet resultants are monomials.

We illustrate our results for two generic quadratic equations in two complex variables:

$$\begin{aligned} f_1 &= a_0x^2 + a_1xy + a_2y^2 + a_3x + a_4y + a_5\,, \\ f_2 &= b_0x^2 + b_1xy + b_2y^2 + b_3x + b_4y + b_5\,. \end{aligned} \tag{0.1}$$

They have four common zeros $(x_i, y_i), i = 1, \ldots, 4$, in the algebraic torus $(\mathbf{C}^*)^2$, and the (affine toric) Jacobian $J^T(x,y) := xy\big(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\big)$ is non-zero at these four points. Consider any Laurent monomial x^iy^j. The *global residue* is the expression

$$\mathrm{Res}_f^T(x^iy^j) \quad := \quad \frac{x_1^iy_1^j}{J^T(x_1,y_1)} + \frac{x_2^iy_2^j}{J^T(x_2,y_2)} + \frac{x_3^iy_3^j}{J^T(x_3,y_3)} + \frac{x_4^iy_4^j}{J^T(x_4,y_4)}\,. \tag{0.2}$$

This is a rational function in the twelve indeterminates $a_0, \ldots, a_5, b_0, \ldots, b_5$. Theorem 3.2 implies that there exists a polynomial $P_{ij}(a_0, \ldots, a_5, b_0, \ldots, b_5)$ such that (0.2) equals

$$\frac{P_{ij}}{\mathcal{R}_\infty^{\max\{0,i+j-3\}} \cdot \mathcal{R}_x^{\max\{0,1-i\}} \cdot \mathcal{R}_y^{\max\{0,1-j\}}}\,, \tag{0.2$'$}$$

where the prime divisors in the denominator are the *facet resultants*

$$\begin{aligned} \mathcal{R}_\infty &= a_0^2b_2^2 - a_0a_1b_1b_2 - 2a_0a_2b_0b_2 + a_0a_2b_1^2 + a_1^2b_0b_2 - a_1a_2b_0b_1 + a_2^2b_0^2\,, \\ \mathcal{R}_x &= a_0^2b_5^2 - a_0a_3b_3b_5 - 2a_0a_5b_0b_5 + a_0a_5b_3^2 + a_3^2b_0b_5 - a_3a_5b_0b_3 + a_5^2b_0^2\,, \\ \mathcal{R}_y &= a_2^2b_5^2 - a_2a_4b_4b_5 - 2a_2a_5b_2b_5 + a_2a_5b_4^2 + a_4^2b_2b_5 - a_4a_5b_2b_4 + a_5^2b_2^2\,. \end{aligned}$$

For instance, for $i = 3$ and $j = 2$ we find $\mathrm{Res}_f^T(x^3y^2) = P_{32}/\mathcal{R}_\infty^2$, where

$$\begin{aligned} P_{32} \;=\;\; & a_0^2a_1b_2^2b_4 \,-\, 2a_0^2a_2b_1b_2b_4 \,+\, a_0^2a_2b_2^2b_3 \,-\, a_0^2a_3b_2^3 \,+\, a_0^2a_4b_1b_2^2 \,-\, a_0a_1^2b_2^2b_3 \\ & + 2a_0a_1a_2b_1b_2b_3 \,-\, 2a_0a_1a_4b_0b_2^2 \,+\, 2a_0a_2^2b_0b_1b_4 \,-\, 2a_0a_2^2b_0b_2b_3 \\ & - a_0a_2^2b_1^2b_3 \,+\, 2a_0a_2a_3b_0b_2^2 \,+\, a_1^2a_3b_0b_2^2 \,-\, a_1a_2^2b_0^2b_4 \,-\, 2a_1a_2a_3b_0b_1b_2 \\ & + 2a_1a_2a_4b_0^2b_2 \,+\, a_2^3b_0^2b_3 \,-\, a_2^2a_3b_0^2b_2 \,+\, a_2^2a_3b_0b_1^2 \,-\, a_2^2a_4b_0^2b_1\,. \end{aligned} \tag{0.3}$$

It is convenient to review the toric algorithm of [CD] for computing global residues by means of this example. First introduce the homogeneous polynomials $F_s(x,y,z) := z^2 \cdot f_s(x/z, y/z)$ for $s = 1, 2$. Next consider the following meromorphic 2-form on $\mathbf{P}^2$:

$$\frac{x^{i-1}y^{j-1}}{z^{i+j-3}F_1\,F_2} \cdot \Omega\,, \tag{0.4}$$

where $\Omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ denotes the Euler form on $\mathbf{P}^2$. The residue (0.2) in the torus $(\mathbf{C}^*)^2$ coincides with the toric residue of (0.4) in $\mathbf{P}^2$.

Suppose, for simplicity, that $i \geq 1$, $j \geq 1$ and $i + j > 3$. Consider the homogeneous ideal $I = \langle z^{i+j-3}, F_1, F_2 \rangle$ in the polynomial ring $K[x,y,z]$ over the field $K = \mathbf{Q}(a_0, a_1, \ldots, b_5)$. The quotient modulo this ideal is a one-dimensional K-vector space in the socle degree $i+j-2$. The homogenized Jacobian $J(x,y,z) := (z^{i+j}/xy)\, J^T(x/z, y/z)$ has degree $i + j - 2$ and is non-zero modulo I. Thus, the monomial $x^{i-1}y^{j-1}$ may be written as $\lambda\, J(x,y,z)$ modulo I, where $\lambda \in K$. The desired residue $\mathrm{Res}^T(x^iy^j)$ is then given by 4λ. The coefficient λ may be computed, for example, as the ratio of the normal form of $x^{i-1}y^{j-1}$ and the normal form of J relative to a Gröbner basis of I.

To prove a denominator formula like (0.2') we use the following technique. We replace the form z^{i+j-3} by a generic homogeneous polynomial $F_0(x,y,z)$ of degree $i+j-3$. Note that F_0 has $\binom{i+j-1}{2}$ indeterminate coefficients, say, $c_0, c_1, c_2, \ldots$. Consider the 2-form

$$\frac{x^{i-1}y^{j-1}}{F_0F_1F_2} \cdot \Omega\,. \tag{0.5}$$

Now all three forms in the denominator of (0.5) are generic relative to their degrees. In § 1 we study this situation for an arbitrary projective toric variety in the role of $\mathbf{P}^2$. Theorem 1.4 implies that the denominator of the toric residue of (0.5) equals the resultant $\mathcal{R} = \mathcal{R}(F_0, F_1, F_2)$. We now apply the specialization $F_0 \mapsto z^{i+j-3}$, which sets all but one of the variables $c_0, c_1, \ldots$ to zero. It takes (0.5) to (0.4), and by Lemma 3.4, it takes $\mathcal{R}$ to R_∞^{i+j-3}, as desired. Such a specialization from a generic polynomial F_0 to a monomial will connect residues in the torus (§3) to toric residues (§1). This technique will reduce Theorem 3.2 to Theorem 1.4.

In §2 we express the sparse resultant as the determinant of a Koszul-type complex which involves the Jacobian. In some special cases (Corollary 2.4) we obtain Sylvester-type formulas which generalize the approach in [GKZ, §III.4.D] (see also [Ch]).

Acknowledgements: We are grateful to David Cox, Fernando Cukierman, Irena Peeva, and Richard Stanley for their very helpful suggestions. Part of the work on this paper was done while Eduardo Cattani was visiting the University of Grenoble and the University of Buenos Aires; he is thankful for their support and hospitality.

§1. Residues, Jacobians and resultants in toric varieties

We begin with a review of basic concepts from toric geometry including the *toric residue*. For details and proofs see [F],[O],[C1],[C2], and [CCD]. Let $X = X_P$ denote the projective toric variety defined by an integral, n-dimensional polytope

$$P \;:=\; \big\{\, m \in \mathbf{R}^n \;:\; \langle m, \eta_i \rangle \geq -b_i \;\text{ for }\; i = 1, \ldots, s \,\big\}, \tag{1.1}$$

where the η_i are the first integral vectors in the inner normals to the facets of P. Thus, X is the toric variety associated with the lattice $M = \mathbf{Z}^n$ and the inner normal fan $\Sigma(P)$ as in [F, §1.5]. We introduce the polynomial ring $S := \mathbf{C}[x_1, \ldots, x_s]$, where the variable x_i is associated to the generator η_i and hence to a torus-invariant irreducible divisor D_i of X. The Chow group $A_{n-1}(X)$ of invariant Weil divisors is presented by the exact sequence

$$0 \to M \to \mathbf{Z}^s \to A_{n-1}(X) \to 0 \tag{1.2}$$

where the left morphism sends $m \in M$ to the s-tuple $\langle m, \eta \rangle \;:=\; (\langle m, \eta_1 \rangle, \ldots, \langle m, \eta_s \rangle)$.

Let Z denote the algebraic subset of $\mathbf{C}^s$ defined by the radical monomial ideal

$$\langle \prod_{\eta_i \notin \sigma} x_i \;,\; \sigma \text{ a cone of } \Sigma(P) \;\rangle \quad \subset \quad S\,.$$

The algebraic group $G := \mathrm{Hom}_{\mathbf{Z}}(A_{n-1}(X), \mathbf{C}^*) \hookrightarrow (\mathbf{C}^*)^s$ acts naturally on $\mathbf{C}^s$ leaving Z invariant. The toric variety X may be realized as the categorical quotient of $\mathbf{C}^s \backslash Z$ by G (see [C1]). When X is simplicial (i.e. P is simple), then the G-orbits are closed and X is the geometric quotient of $\mathbf{C}^s \backslash Z$ by G. The torus $(\mathbf{C}^*)^s$ lies in $\mathbf{C}^s \backslash Z$ and maps onto the dense torus in X under the quotient map.

Given $a \in \mathbf{N}^s$ we write x^a for the monomial $\Pi_{i=1}^s x_i^{a_i}$. As in [C1] the right morphism in (1.2) defines an $A_{n-1}(X)$-valued grading of the polynomial ring S:

$$\deg(x^a) \;:=\; [\sum_{i=1}^s a_i\, D_i] \;\in\; A_{n-1}(X)\,. \tag{1.3}$$

Let S_α denote the graded component of S of degree α. We abbreviate $\beta_0 := [\sum_i D_i]$ and $\beta := [\sum_i b_i D_i] \in A_{n-1}(X)$. The divisor β is ample and $S_\beta \cong H^0(X, \mathcal{L})$, where $\mathcal{L} = \mathcal{O}_X(\beta)$ is the line bundle associated to β (see [F, §3.4]). Thus, a homogeneous polynomial F of degree $k\beta$ represents a global section of $\mathcal{L}^k$, and we may consider its zero set in X.

A monomial x^a has degree $k\beta$, $k \in \mathbf{N}$, if and only if there exists $m(a) \in \mathbf{Z}^n$ such that

$$\langle m(a), \eta_i \rangle + k b_i \;=\; a_i \quad \text{for} \quad i = 1, \ldots, s\,.$$

The point $m(a)$ is unique and, since $a_i \geq 0$, it lies in $kP \cap \mathbf{Z}^n$. Therefore, the map

$$kP \cap \mathbf{Z}^n \;\to\; S_{k\beta}\,, \quad m \;\mapsto\; \prod_{i=1}^s x_i^{\langle m, \eta_i \rangle + k b_i} \tag{1.4}$$

defines a bijection between integral points in kP and monomials of degree $k\beta$ or, equivalently, between Laurent polynomials supported in kP and homogeneous polynomials of degree $k\beta$ in S. If $f(t_1,\ldots,t_n)$ is supported in kP then its image is the *kP-homogenization*

$$F(x_1,\ldots,x_s) \;=\; (\prod_{i=1}^{s} x_i^{kb_i}) \cdot f(t_1(x),\ldots,t_n(x)) \;\in\; S_{k\beta}\,, \tag{1.5}$$

$$\text{where} \qquad t_j(x) \;=\; \prod_{i=1}^{s} x_i^{\langle e_j,\eta_i\rangle} \qquad (j=1,\ldots,n)$$

and $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbf{Z}^n$. By restricting (1.4) we also get a bijection between monomials x^a of degree $k\beta-\beta_0$ and integral points in $(kP)^\circ$, the interior of kP.

Proposition 1.1. *The ring $S_{*\beta} = \bigoplus_{k=0}^{\infty} S_{k\beta}$ is Cohen-Macaulay of dimension $n+1$, with canonical module $\omega_{S_{*\beta}} = \bigoplus_{k=0}^{\infty} S_{k\beta-\beta_0}$. Fix positive integers $k_0,\ldots,k_n$ and let $\kappa = k_0+\cdots+k_n$, $\rho = \kappa\beta-\beta_0$. Given $F_i \in S_{k_i\beta}$ for $i=0,\ldots,n$ such that $F_0,\ldots,F_n$ have no common zeroes in X, then:*
(i) $F_0,\ldots,F_n$ are a regular sequence in $S_{*\beta}$ and, hence, in $\omega_{S_{*\beta}}$.
(ii) The degree ρ component R_ρ of the quotient $R = S_{*\beta}/\langle F_0,\ldots,F_n\rangle$ has $\mathbf{C}$-dimension 1.

Proof: See [B, Theorem 2.10 and Proposition 9.4] and [C2, Proposition 3.2]. $\diamond$

We next recall the construction of the Euler form Ω and the toric Jacobian $J(F)$ (see [BC,§9], [C2,§4]). For any subset $I=\{i_1,\ldots,i_n\}$ of $\{1,\ldots,s\}$ we abbreviate

$$\det(\eta_I) \;:=\; \det(\langle e_\ell,\eta_{i_j}\rangle_{1\le\ell,j\le n})\,, \quad dx_I = dx_{i_1}\wedge\cdots\wedge dx_{i_n}\,, \quad \hat{x}_I = \Pi_{j\notin I} x_j\,.$$

Note that the product $\det(\eta_I)dx_I$ is independent of the ordering of $i_1,\ldots,i_n$. The *Euler form* on X is the following sum over all n-element subsets $I\subset\{1,\ldots,s\}$:

$$\Omega \quad := \quad \sum_{|I|=n} \det(\eta_I)\,\hat{x}_I\,dx_I\,.$$

The Euler form Ω may be characterized by the property that $\Omega/(x_1\cdots x_s)$ is the rational extension to X of the T-invariant holomorphic form $\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}$ on the torus T.

As in Proposition 1.1, consider homogeneous polynomials $F_0,F_1,\ldots,F_n$ where $\deg(F_i) = k_i\beta$ and $\kappa = k_0+\cdots+k_n$. Then there exists a polynomial $J(F)\in S_{\kappa\beta-\beta_0}$ such that

$$\sum_{i=0}^{n}(-1)^i F_i\cdot dF_0\wedge\cdots\wedge dF_{i-1}\wedge dF_{i+1}\wedge\cdots\wedge dF_n \quad = \quad J(F)\cdot\Omega\,. \tag{1.6}$$

Furthermore, if $I=\{i_1,\ldots,i_n\}$ is such that $\eta_{i_1},\ldots,\eta_{i_n}$ are linearly independent, then

$$J(F) \quad = \quad \frac{1}{\det(\eta_I)\,\hat{x}_I}\ \det\begin{pmatrix} k_0F_0 & k_1F_1 & \ldots & k_nF_n \\ \partial F_0/\partial x_{i_1} & \partial F_1/\partial x_{i_1} & \ldots & \partial F_n/\partial x_{i_1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial F_0/\partial x_{i_n} & \partial F_1/\partial x_{i_n} & \ldots & \partial F_n/\partial x_{i_n} \end{pmatrix}. \tag{1.7}$$

The polynomial $J(F)$ is called the *toric Jacobian* of $F = (F_0, F_1, \ldots, F_n)$.

In the special case $k_0 = k_1 = \cdots = k_n = 1$ the toric Jacobian can also be computed as follows. Let $f_0, \ldots, f_n$ be Laurent polynomials supported in P and let $F_0, \ldots, F_n$ denote their P-homogenizations as in (1.5). Let $P \cap \mathbf{Z}^n = \{m_1, \ldots, m_\mu\}$ and

$$j(t) \quad := \quad \det \begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ t_1 \frac{\partial f_0}{\partial t_1} & t_1 \frac{\partial f_1}{\partial t_1} & \cdots & t_1 \frac{\partial f_n}{\partial t_1} \\ \vdots & \vdots & \ddots & \vdots \\ t_n \frac{\partial f_0}{\partial t_n} & t_n \frac{\partial f_1}{\partial t_n} & \cdots & t_n \frac{\partial f_n}{\partial t_n} \end{pmatrix} . \tag{1.8}$$

Proposition 1.2. *Let $f_j = \sum_{i=1}^{\mu} u_{ji} t^{m_i}$ and set $\tilde{m}_i = (1, m_i) \in \mathbf{Z}^{n+1}$. Then,*

$$j(t) \quad = \sum_{1 \le i_0 < i_1 < \ldots < i_n \le \mu} [i_0 i_1 \ldots i_n] \cdot \det(\tilde{m}_{i_0}, \tilde{m}_{i_1}, \ldots, \tilde{m}_{i_n}) \cdot t^{m_{i_0} + m_{i_1} + \cdots + m_{i_n}} ,$$

where the brackets denote the maximal minors of the coefficient matrix:

$$[i_0 i_1 \ldots i_n] \quad := \quad \det \begin{pmatrix} u_{0 i_0} & u_{0 i_1} & \cdots & u_{0 i_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n i_0} & u_{n i_1} & \cdots & u_{n i_n} \end{pmatrix} .$$

Moreover, $j(t)$ is supported in $((n+1)P)^\circ$ and its $(n+1)P$-homogenization is $x_1 \cdots x_s J(F)$.

Proof: We consider the $(n+1) \times \mu$ matrix $\tilde{A} = (\tilde{m}_1, \ldots, \tilde{m}_\mu)$, the $\mu \times \mu$ diagonal matrix $D = \mathrm{diag}(t^{m_1}, \ldots, t^{m_\mu})$ and the $\mu \times (n+1)$ matrix U, obtained by transposing the matrix of coefficients (u_{ji}). Their product $\tilde{A} \cdot D \cdot U$ equals the $(n+1) \times (n+1)$ matrix in (1.8). The first assertion amounts to the *Cauchy-Binet formula* for $j(t) = (\wedge_{n+1} \tilde{A}) \cdot (\wedge_{n+1} D) \cdot (\wedge_{n+1} U)$. If the sum $m_{i_0} + m_{i_1} + \cdots + m_{i_n}$ lies in the boundary of $(n+1)P$, then all m_{k_j} lie in a facet of P and the determinant $\det(\tilde{m}_{i_0}, \tilde{m}_{i_1}, \ldots, \tilde{m}_{i_n})$ must vanish. Consequently, $j(t)$ is supported in the interior of $(n+1)P$. The final statement follows from (1.6) together with

$$j(t) \, \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \quad = \quad \sum_{j=0}^{n} (-1)^j f_j \, df_0 \wedge \cdots \wedge df_{j-1} \wedge df_{j+1} \wedge \cdots \wedge df_n . \quad \diamond$$

We now return to general $k_0, \ldots, k_n$. Suppose that $F_0, \ldots, F_n$ have no common zeroes in X. Then $R_\rho \cong \mathbf{C}$ by (ii) in Proposition 1.1. In [C2] Cox constructs an explicit isomorphism $\mathrm{Res}_F^X : R_\rho \to \mathbf{C}$ whose value on the toric Jacobian is the positive integer

$$\mathrm{Res}_F^X(J(F)) \quad = \quad \left(\prod_{j=0}^{n} k_j \right) \cdot n! \cdot \mathrm{vol}(P) , \tag{1.9}$$

where $\mathrm{vol}(\,\cdot\,)$ denotes the standard volume in $\mathbf{R}^n$. The isomorphism $\mathrm{Res}_F^X(\,\cdot\,)$ is called the *toric residue.* From (1.9) we conclude that

$$J(F) \text{ defines a non-zero element in } R_\rho \,. \tag{1.10}$$

We next present an affine interpretation of the toric residue. Let f_j be a generic Laurent polynomial with Newton polytope k_jP. Let $F_j \in S_{k_j\beta}$ be the k_jP-homogenization of f_j. Given a homogeneous polynomial H of *critical degree* $\rho = \kappa\beta - \beta_0$, the expression

$$\frac{H\ \Omega}{F_0 \cdots F_n}$$

defines a meromorphic n-form on X. Its restriction to T may be written as

$$\frac{h}{f_0 \cdots f_n}\ \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\,,$$

where h is a Laurent polynomial supported in $(\kappa P)^\circ$. Our generic choice of $f_0, \ldots, f_n$ guarantees (cf. [K1,§2]) the following properties for each $i = 0, \ldots, n$: The finite set $V_i := \{x \in X \,:\, F_j(x) = 0\,;\, j \neq i\}$ lies in the torus T, hence $V_i = \{t \in T \,:\, f_j(t) = 0\,;\, j \neq i\}$, and the function h/f_i is regular at the points of V_i.

The following result is a consequence of Theorem 0.4 in [CCD]:

Proposition 1.3. *For any fixed $i \in \{0, \ldots, n\}$, the toric residue equals*

$$\mathrm{Res}_F^X(H) \quad = \quad (-1)^i \sum_{\xi \in V_i} \mathrm{Res}_\xi\Big(\frac{h/f_i}{f_0 \cdots f_{i-1} f_{i+1} \cdots f_n}\ \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\Big)\,. \tag{1.11}$$

Here the right-hand side is a sum of Grothendieck residues ([GH], [T]*; see also* §3) *relative to the divisors $\{f_j(t) = 0\} \subset T$, $j \neq i$.*

Remarks.

i) Even though Theorem 0.4 in [CCD] is only stated for simplicial toric varieties, it is valid for arbitrary complete toric varieties provided V_i lies in T, by passing to a desingularization.

ii) Note that while the right side of (1.11) makes sense for every Laurent polynomial h, Proposition 1.3 asserts that, if h is supported in $(\kappa P)^\circ$, then that expression is independent of i.

We next consider $n+1$ polynomials having indeterminate coefficients:

$$F_i(u;x) \quad := \quad \sum_{a \in \mathcal{A}_{k_i\beta}} u_{ia}\, x^a \qquad \text{for} \quad i = 0, \ldots, n\,, \tag{1.12}$$

where $\mathcal{A}_{k_i\beta} := \{\, a \in \mathbf{N}^s \,:\, \deg(x^a) = k_i\beta \,\}$. We shall work in the polynomial ring

$$C \,:=\, A[x_1, \ldots, x_s] \qquad \text{over} \qquad A \,:=\, \mathbf{Q}\,[\,u_{ia}\,;\ i = 0, \ldots, n\,;\ a \in \mathcal{A}_{k_i\beta}]\,.$$

We endow the polynomial ring C with the $A_{n-1}(X)$-grading given by (1.3). For any $H \in C_\rho$, the expression (1.11) depends rationally on the coefficients of $F_0, \ldots, F_n$ and hence defines an element in the field of fractions of A, which we also denote $\mathrm{Res}_F^X(H)$.

As in [GKZ, 3.3; 8.1] we define the *resultant* associated with the bundles $\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}$. It is an irreducible polynomial $\mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u) \in A$ with integral coefficients, uniquely defined up to sign, which vanishes for some specialization of the coefficients if and only if the corresponding sections $F_0, \ldots, F_n$ have a common zero in X. Via the correspondence (1.4) between homogeneous polynomials of degree $k\beta$ and Laurent polynomials supported in kP, the resultant $\mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u)$ agrees with the *mixed sparse resultant* (see [PSt],[S2]) associated with the support sets $k_0 P \cap \mathbf{Z}^n, \ldots, k_n P \cap \mathbf{Z}^n$.

The degree of the resultant is computed as follows. Suppose $k_0 \geq \ldots \geq k_n$. Consider the lattice affinely generated by the integral points in $k_0 P$. It has finite index in $\mathbf{Z}^n$:

$$\ell \quad := \quad [\, \mathbf{Z}^n \,:\, \mathrm{aff}_{\mathbf{Z}}(k_0 P \cap \mathbf{Z}^n)\,] \,. \tag{1.13}$$

Note that $\ell = 1$ if $\mathcal{L}^{k_0}$ is very ample. The degree of $\mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u)$ in the coefficients of the i-th form F_i equals, by [PSt, Corollary 1.4],

$$k_0 \cdots k_{i-1} k_{i+1} \cdots k_n \cdot n\,! \cdot \frac{1}{\ell} \cdot \mathrm{vol}(P) \,. \tag{1.14}$$

We now state and prove the main result of this section:

Theorem 1.4. *For any $H \in C_\rho$, the product $\mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u) \cdot \mathrm{Res}_F^X(H)$ lies in A.*

Proof: As noted above, for values of u in a Zariski open set, $F_0, \ldots, F_n$ have no common zeroes in X and, for every $i = 0, \ldots, n$, the set $V_i = \{x \in X \,:\, F_j(x) = 0,\ j \neq i\}$ is finite and contained in T. Thus, setting for simplicity $i = 0$, we have, as in (1.11):

$$\mathrm{Res}_F^X(H) \quad = \quad \sum_{\xi \in V_0} \mathrm{Res}_\xi \Big(\frac{h/f_0}{f_1 \cdots f_n} \, \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \Big) \,. \tag{1.15}$$

We may further assume that the zeroes of $f_1, \ldots, f_n$ are simple and, therefore, each term in the right hand side of (1.15) may be written as (see [GH, page 650]):

$$\mathrm{Res}_\xi \Big(\frac{h/f_0}{f_1 \cdots f_n} \, \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \Big) \;=\; \frac{h(\xi)}{f_0(\xi) \,\cdot\, J^T_{f_1, \ldots, f_n}(\xi)} \;=\; \frac{a_\xi(u_1, \ldots, u_n)}{f_0(\xi) \,\cdot\, b_\xi(u_1, \ldots, u_n)} \,, \tag{1.16}$$

where $J^T_{f_1, \ldots, f_n} = \det(t_j \dfrac{\partial f_i}{\partial t_j})$, the symbol u_i stands for the vector $(u_{ia} \,:\, a \in \mathcal{A}_{k_i \beta})$ of coefficients of f_i, and a_ξ, b_ξ are algebraic functions in these coefficients.

We now sum (1.16) over all points ξ in V_0. To get the best possible denominator even if $\ell > 1$, we must organize the sum (1.15) as follows. First, we may assume that P contains the origin. Then the affine lattice agrees with the linear lattice,

$$\mathrm{aff}_{\mathbf{Z}}(k_0 P \cap \mathbf{Z}^n) \quad = \quad \mathrm{lin}_{\mathbf{Z}}(k_0 P \cap \mathbf{Z}^n) \,, \tag{1.17}$$

and the inclusion of (1.17) in $\mathbf{Z}^n$ defines a morphism of tori $\pi : T \to (\mathbf{C}^*)^n$. The map π is a finite cover of degree ℓ, and the Laurent polynomial f_0 is constant along the fibers of π. Hence, if $\eta = \pi(\xi)$ for $\xi \in V_0$, then we can define $f_0(\eta) := f_0(\xi)$. Therefore,

$$\mathrm{Res}_F^X(H) \quad = \quad \sum_{\eta \in \pi(V_0)} \frac{1}{f_0(\eta)} \sum_{\xi \in \pi^{-1}(\eta)} \frac{a_\xi(u_1, \ldots, u_n)}{b_\xi(u_1, \ldots, u_n)} .$$

This expression depends rationally on $u_0, u_1, \ldots, u_n$. This implies

$$\mathrm{Res}_F^X(H) \quad = \quad \frac{A(u_0, u_1, \ldots, u_n)}{(\prod_{\eta \in \pi(V_0)} f_0(\eta)) \cdot B(u_1, \ldots, u_n)} ,$$

where A and B are polynomials. It follows from [PSt, Theorem 1.1] that

$$\prod_{\eta \in \pi(V_0)} f_0(\eta) \quad = \quad \mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u_0, u_1, \ldots, u_n) \cdot C(u_1, \ldots, u_n)$$

for some rational function C. Therefore, there exist polynomials A_0, B_0 such that

$$\mathrm{Res}_F^X(H) \quad = \quad \frac{A_0(u_0, u_1, \ldots, u_n)}{\mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u_0, u_1, \ldots, u_n) \cdot B_0(u_1, \ldots, u_n)} .$$

Replacing the role played by the index 0 by any other index $i = 1, \ldots, n$, we deduce that

$$\mathrm{Res}_F^X(H) \quad = \quad \frac{P(u_0, u_1, \ldots, u_n)}{\mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}}(u_0, u_1, \ldots, u_n)}$$

for some polynomial $P \in A$. $\diamond$

Remark 1.5. Suppose P is the standard simplex in $\mathbf{R}^n$. Then $X \cong \mathbf{P}^n$, β is the hyperplane class, $s = n+1$, and $F_j(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree k_j. The assumption that $F_0, \ldots, F_n$ have no common zeroes in $\mathbf{P}^n$ means that their only common zero in $\mathbf{C}^{n+1}$ is 0. For any homogeneous polynomial H of degree $\rho = \kappa - (n+1)$, the toric residue $\mathrm{Res}_F^{\mathbf{P}^n}(H)$ associated with the n-rational form $\frac{H}{F_0 \cdots F_n}\Omega$ on $\mathbf{P}^n$, coincides ([PS], [CCD,§5]) with the Grothendieck residue at the origin of $\mathbf{C}^{n+1}$ of the $(n+1)$-form

$$\frac{H}{F_0 \cdots F_n} \; dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n .$$

In this situation, it has been observed by Angéniol [A] that Theorem 1.4 follows from the work of Jouanolou (see, for example, [J1, 3.5]).

§2. Jacobian formulas for the sparse resultant

Let $F_0, \ldots, F_n$ be generic forms as in (1.12), let A be the polynomial ring on their coefficients, and let $C = A[x_1, \ldots, x_s]$ be graded by the Chow group $A_{n-1}(X)$ via (1.3). The given forms together with their toric Jacobian $J(F)$ define a map of free A-modules

$$\begin{aligned} \Phi \; : \; C_{\rho - k_0\beta} \times \cdots \times C_{\rho - k_n\beta} \times A \quad &\rightarrow \quad C_\rho \,, \\ (\, \Lambda_0 \,, \, \ldots \,, \, \Lambda_n, \, \Theta \,) \quad &\mapsto \quad \sum_{i=0}^{n} \Lambda_i \, F_i + \Theta \, J(F) \,. \end{aligned} \tag{2.1}$$

For any particular choice of complex coefficients $u = c$ we abbreviate $F_i^c(x) := F_i(\, c\,; \, x\,)$. The resultant $\mathcal{R} = \mathcal{R}_{\mathcal{L}^{k_0}, \ldots, \mathcal{L}^{k_n}} \in A$ considered in Theorem 1.4 satisfies $\mathcal{R}(c) = 0$ if and only if the forms $F_0^c, \ldots, F_n^c$ have a common zero in the toric variety X. Let

$$\Phi_c \;\; : \;\; S_{\rho - k_0\beta} \times \cdots \times S_{\rho - k_n\beta} \times \mathbf{C} \, \rightarrow \, S_\rho \tag{2.2}$$

denote the $\mathbf{C}$-linear map derived from (2.1) by substituting c for u.

Proposition 2.1. *The map Φ_c is surjective if and only if $\mathcal{R}(c) \neq 0$.*

Proof: For the if direction suppose $\mathcal{R}(c) \neq 0$. Then $F_0^c, \ldots, F_n^c$ have no common zeroes in X. Proposition 1.1 (ii) together with (1.10) implies the surjectivity of Φ_c.

For the converse, let $\mathcal{V}$ denote the affine variety in the space of coefficients consisting of all c such that the polynomials $F_0^c, \ldots, F_n^c$ have a common zero in the torus $(\mathbf{C}^*)^s$. Fix $c \in \mathcal{V}$ and let $p \in (\mathbf{C}^*)^s$ be such a common zero. It follows from (1.7) that $x_1 x_2 \cdots x_s \cdot J(F)$ lies in the ideal generated by $F_0, \ldots, F_n$ in S and hence $J(F)$ vanishes at p. If a monomial x^a of degree ρ were in the image of Φ_c then $x^a(p) = 0$ which is impossible. Thus, for $c \in \mathcal{V}$, Φ_c is not surjective. We conclude that $\mathcal{V}$ is contained in the algebraic variety defined by the vanishing of all maximal minors of Φ_c. Since the closure of $\mathcal{V}$ is the locus where the resultant $\mathcal{R}$ vanishes, the only if-direction follows. $\diamond$

For any subset $J \subseteq \{0, \ldots, n\}$ we set $k_J := \sum_{i \in J} k_i$. For $0 \leq j \leq n+1$ denote

$$W_j \quad := \quad \bigoplus_{|J| = j} C_{k_J\beta - \beta_0} \,. \tag{2.3}$$

From the Koszul complex on $F_0, \ldots, F_n$ we derive the following complex of free A-modules:

$$0 \longrightarrow W_0 \xrightarrow{\varphi_0} W_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{n-1}} W_n \xrightarrow{\varphi_n} W_{n+1} \longrightarrow 0 \,. \tag{2.4}$$

This construction is an instance of [GKZ, §3.4.A]. Note that $W_0 = 0$, $W_{n+1} = C_\rho$, and $W_n = C_{\rho - k_0\beta} \times \cdots \times C_{\rho - k_n\beta}$. Define $(\varphi_{n-1}, 0) : W_{n-1} \longrightarrow W_n \oplus A$ by adding 0 in the coordinate corresponding to A, and consider the modified complex

$$0 \longrightarrow W_1 \xrightarrow{\varphi_1} W_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-2}} W_{n-1} \xrightarrow{(\varphi_{n-1}, 0)} W_n \oplus A \xrightarrow{\Phi} W_{n+1} \longrightarrow 0 \,. \tag{2.5}$$

For any particular choice of coefficients $u = c$ in (2.5) we get a complex of $\mathbf{C}$-vector spaces:

$$0 \longrightarrow \bigoplus_i S_{k_i\beta - \beta_0} \xrightarrow{\varphi_1^c} \cdots \xrightarrow{(\varphi_{n-1}^c, 0)} \bigoplus_{|J| = n} S_{k_J\beta - \beta_0} \times \mathbf{C} \xrightarrow{\Phi_c} S_\rho \longrightarrow 0 \,. \tag{2.6}$$

Let D denote the determinant (see [GKZ, Appendix A]) of the complex of A-modules (2.5) with respect a fixed choice of monomial bases for the A-modules $W_1, \ldots, W_{n+1}$. This is an element in the field of fractions of A. We shall prove that it is a polynomial in A. Suppose $k_0 \geq \ldots \geq k_n$ and let ℓ be the lattice index defined in (1.13).

Theorem 2.2.

(i) *The complex of* $\mathbf{C}$*-vector spaces (2.6) is exact if and only if* $\mathcal{R}(c) \neq 0$*.*

(ii) *The determinant* D *of the complex (2.5) equals the greatest common divisor of all (not identically zero) maximal minors of a matrix representing the* A*-module map* Φ*.*

(iii) *The determinant* D *equals* $\mathcal{R}^\ell$*.*

(iv) *If* $\mathcal{L}^{k_0}$ *is very ample then the resultant* $\mathcal{R}$ *may be computed as the greatest common divisor of all maximal minors of any matrix representing* Φ*.*

Proof: We first prove the if-direction in part (i). Let β be an ample divisor and $F_0^c, \ldots, F_n^c$ homogeneous polynomials of respective degrees $k_i\beta$ without common zeroes in X, i.e. such that $\mathcal{R}(c) \neq 0$. By Proposition 1.1 (i), $F_0^c, \ldots, F_n^c$ is a regular sequence in $S_{*\beta}$ and in $\omega_{S_{*\beta}}$; consequently, the corresponding Koszul complex is acyclic [BH, page 49]. Setting $I = \langle F_0^c, \ldots, F_n^c \rangle$ this implies that

$$0 \longrightarrow \bigoplus_i S_{k_i\beta-\beta_0} \xrightarrow{\varphi_1^c} \cdots \xrightarrow{\varphi_{n-1}^c} \bigoplus_{|J|=n} S_{k_J\beta-\beta_0} \xrightarrow{\varphi_n^c} S_\rho \longrightarrow S_\rho/I_\rho \longrightarrow 0 \tag{2.7}$$

is an exact sequence of $\mathbf{C}$-vector spaces. Proposition 2.1 implies that Φ_c is surjective. Also, by (1.10), $\Phi_c(\lambda_1, \ldots, \lambda_n, \theta) = \sum_i \lambda_i F_i + \theta J(F) = 0$ implies $\theta = 0$. These two facts imply that (2.6) is exact. For the converse of (i) suppose $\mathcal{R}(c) = 0$. Then the map Φ_c is not surjective by Proposition 2.1, and hence (2.6) is not exact.

We next prove part (ii). We claim that $F_0, \ldots, F_n$ is a homogeneous regular sequence in the graded Cohen-Macaulay ring $C_{*\beta} := \bigoplus_{k=0}^\infty C_{k\beta}$. We extend scalars and consider $C_{*\beta} \otimes_{\mathbf{Q}} \mathbf{C}$ instead. Let N be the total number of terms in $F_0, \ldots, F_n$. The spectrum of $C_{*\beta} \otimes_{\mathbf{Q}} \mathbf{C}$ equals affine space $\mathbf{C}^N$ times the $(n+1)$-dimensional affine toric variety $\mathcal{X}_\beta := \mathrm{Spec}(S_{*\beta})$. Let $\mathcal{V}$ denote the algebraic set defined by $F_0, \ldots, F_n$ in $\mathbf{C}^N \times \mathcal{X}_\beta$.

We shall prove that $\mathcal{V}$ has codimension $n+1$, by describing the two irreducible components of $\mathcal{V}$. Let O be the origin in $\mathcal{X}_\beta$ and $\mathcal{M}$ its maximal ideal. Hence $\mathcal{M}$ is spanned by all non-constant monomials in $S_{*\beta}$. For any $i \in \{0, \ldots, n\}$, the x-monomials appearing in F_i all lie in $\mathcal{M}^{k_i}$, and their radical equals $\mathcal{M}$. In other words, $F_i(p) \neq 0$ for all $p \in \mathcal{X}_\beta \setminus \{O\}$. Consider the projection from $\mathbf{C}^N \times \mathcal{X}_\beta$ onto its second factor and let π denote its restriction to $\mathcal{V}$. For $p \in \mathcal{X}_\beta \setminus \{O\}$, the fiber $\pi^{-1}(p)$ is a linear subspace of codimension $n+1$ in $\mathbf{C}^N \times \{p\}$. The fiber $\pi^{-1}(O)$ equals $\mathbf{C}^N \times O$, which has codimension $n+1$ in $\mathbf{C}^N \times \mathcal{X}_\beta$. We have shown that $\mathrm{codim}(\mathcal{V}) = n+1$, as desired.

Since $C_{*\beta} \otimes_{\mathbf{Q}} \mathbf{C}$ is graded and Cohen-Macaulay, we may conclude that $F_0, \ldots, F_n$ is a regular sequence. The Koszul complex on $F_0, \ldots, F_n$ is exact, and therefore (2.4) and (2.5) are exact sequences of A-modules except at W_{n+1}. By Theorem 34 in [GKZ, Appendix A], the determinant D equals the greatest common divisor of all maximal minors of Φ.

Part (iv) of Theorem 2.2 follows directly from (ii) and (iii) and the observation that $\ell = 1$ if $\mathcal{L}^{k_0}$ is very ample. It remains to prove part (iii). Part (i) implies that $D(c) = 0$ if and only if $\mathcal{R}(c) = 0$. We also deduce from the irreducibility of the resultant that D is a power of $\mathcal{R}$. In order to prove $D = \mathcal{R}^\ell$, we must show that the total degree of D equals

$$\ell \cdot \deg(\mathcal{R}) \quad = \quad \Big(\sum_{i=0}^n k_0 \cdots k_{i-1} k_{i+1} \cdots k_n\Big) \cdot n\,! \cdot \mathrm{vol}(P)\,. \tag{2.8}$$

Let us consider the *Erhart polynomial* for the interior of P:

$$p(j) \quad := \quad |(jP)^\circ \cap \mathbf{Z}^n| \quad = \quad \mathrm{vol}(P)\cdot j^n \;+\; \sum_{i=0}^{n-1} a_i j^i\,.$$

The rank of the free A-module W_j equals $\sum_{|J|=j} p(k_J)$. Taking into account the fact that any non-zero maximal minor of Φ has to involve the last column and $\deg(J(F)) = n+1$ in the coefficients of $F_0,\ldots,F_n$, we deduce from Theorem 14 in Appendix A in [GKZ] that

$$\begin{aligned} \deg(D) \quad &= \quad \sum_{j=0}^{n+1}(-1)^{n+1-j}\cdot j\cdot\left(\sum_{|J|=j} p(k_J)\right) \quad = \\ \mathrm{vol}(P)\cdot &\underbrace{\left(\sum_{j=0}^{n+1}(-1)^{n+1-j}\cdot j\cdot\sum_{|J|=j}k_J^n\right)}_{\gamma_n} \;+\; \sum_{i=0}^{n-1} a_i\cdot\underbrace{\left(\sum_{j=0}^{n+1}(-1)^{n+1-j}\cdot j\cdot\sum_{|J|=j}k_J^i\right)}_{\gamma_i}\,. \end{aligned} \tag{2.9}$$

To prove the equality of (2.8) and (2.9), it suffices to show the combinatorial identities:

$$\gamma_n \;=\; n\,!\cdot(\sum_{j=0}^{n+1}\prod_{\nu\neq j}k_\nu) \quad \text{and} \quad \gamma_i \;=\; 0 \quad \text{for } 0\le i\le n-1\,. \tag{2.10}$$

Following a suggestion made to us by Richard Stanley, we prove a more general identity:

Lemma 2.3. *Let $u_{i,j}$ be indeterminates indexed by $i=0,\ldots,n$ and $j=0,\ldots,r$. Then*

$$\sum_{I\subseteq\{0,1,\ldots,n\}}(-1)^{|I|}\prod_{j=0}^{r}(\sum_{i\in I}u_{i,j}) \qquad = \qquad (-1)^{n+1}\sum_{\substack{\phi:\{0,\ldots,r\}\to\{0,\ldots,n\}\\ \text{surjective}}}\;\prod_{j=0}^{r}u_{\phi(j),j}\,.$$

Proof: The terms in the expansion of the left side correspond to maps from $\{0,\ldots,r\}$ to subsets I of $\{0,\ldots,n\}$. Any term which appears at least twice gets cancelled. What remains are the terms corresponding to surjective maps from $\{0,\ldots,r\}$ to the full set $I=\{0,\ldots,n\}$. $\diamond$

We are interested in the special case $u_{i,0}=1$ for $0\le i\le n$ and $u_{i,j}=k_i$ for $0\le i\le n$ and $1\le j\le r$. Under this specialization, Lemma 2.3 implies (2.10) and hence part (iii). This completes the proof of Theorem 2.2. $\diamond$

Theorem 2.2 expresses the ℓ-th power of the resultant as an alternating product of determinants. Of particular interest are those cases when one determinant is involved. Such formulas are called *Sylvester-type*. They have been studied systematically by Weyman and Zelevinsky [WZ] in the case when X is a product of projective spaces.

Corollary 2.4. *Suppose that $(n-1)P$ has no interior lattice points and either*
(a) $k_0 = \cdots = k_n = 1$, *or*
(b) nP *has no interior lattice points and* $k_0 + \cdots + k_n = n+2$, *or*
(c) $n = 2$ *and* P *is a primitive triangle and* $k_0, k_1, k_2 \leq 2$.
Then, the matrix of Φ *is square and* $\mathcal{R}^\ell = \det(\Phi)$.

Let us discuss the formulas in Corollary 2.4 for the case of toric surfaces ($n = 2$). Suppose $k_0 = k_1 = k_2 = 1$ and the polygon P has no interior lattice points. Then the matrix of Φ is square and $\mathcal{R} = \det(\Phi)$. A lattice polygon P has no interior lattice points if and only if (X, β) is either the Veronese surface in $\mathbf{P}^5$ or any rational normal scroll (Hirzebruch surface). In the former case we recover Sylvester's formula for the resultant of three ternary quadrics [GKZ, §3.4.D]. In the latter case we get a new formula of Sylvester type for the Chow form of any rational normal scroll. Here is an explicit example.

Example 2.5. *(The Chow form of a Hirzebruch surface)* Consider the quadrangle

$$P \quad = \quad \left\{ (m_1, m_2) \in \mathbf{R}^2 \, : \, \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The corresponding toric surface is the rational normal scroll $S_{1,3}$; cf. [Ha, Example 8.17]. Let β be the divisor on $S_{1,3}$ defined by P. Consider three generic elements of $K[x_1, \ldots, x_4]_\beta$:

$$\begin{aligned} F_0 &= a_1 x_1 x_2^3 + a_2 x_1 x_2^2 x_4 + a_3 x_1 x_2 x_4^2 + a_4 x_1 x_4^3 + a_5 x_2 x_3 + a_6 x_3 x_4 \, , \\ F_1 &= b_1 x_1 x_2^3 + b_2 x_1 x_2^2 x_4 + b_3 x_1 x_2 x_4^2 + b_4 x_1 x_4^3 + b_5 x_2 x_3 + b_6 x_3 x_4 \, , \\ F_2 &= c_1 x_1 x_2^3 + c_2 x_1 x_2^2 x_4 + c_3 x_1 x_2 x_4^2 + c_4 x_1 x_4^3 + c_5 x_2 x_3 + c_6 x_3 x_4 \, . \end{aligned} \tag{2.11}$$

The quadrangle $3P$ has 10 interior lattice points, corresponding to the 10 monomials of critical degree. The map Φ in (2.1) is given by the following 10×10-matrix:

$$\begin{array}{c} \\ x_1x_2^5 \\ x_1x_2^4x_4 \\ x_1x_2^3x_4^2 \\ x_1x_2^2x_4^3 \\ x_1x_2x_4^4 \\ x_1x_4^5 \\ x_2^3x_3 \\ x_2^2x_3x_4 \\ x_2x_3x_4^2 \\ x_3x_4^3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} x_2^2 & x_2x_4 & x_4^2 & x_2^2 & x_2x_4 & x_4^2 & x_2^2 & x_2x_4 & x_4^2 & 1 \end{array} \\ \begin{pmatrix} a_1 & 0 & 0 & b_1 & 0 & 0 & c_1 & 0 & 0 & [125] \\ a_2 & a_1 & 0 & b_2 & b_1 & 0 & c_2 & c_1 & 0 & [126]+2[135] \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 & [235]+2[136]+3[145] \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 & [236]+2[245]+3[146] \\ 0 & a_4 & a_3 & 0 & b_4 & b_3 & 0 & c_4 & c_3 & [345]+2[246] \\ 0 & 0 & a_4 & 0 & 0 & b_4 & 0 & 0 & c_4 & [346] \\ a_5 & 0 & 0 & b_5 & 0 & 0 & c_5 & 0 & 0 & -[156] \\ a_6 & a_5 & 0 & b_6 & b_5 & 0 & c_6 & c_5 & 0 & -[256] \\ 0 & a_6 & a_5 & 0 & b_6 & b_5 & 0 & c_6 & c_5 & -[356] \\ 0 & 0 & a_6 & 0 & 0 & b_6 & 0 & 0 & c_6 & -[456] \end{pmatrix} \end{array}$$

The border column lists the monomials of critical degree. The border row gives the multipliers of F_0, F_1, F_2 and $J(F)$. For the coefficients of the Jacobian $J(F)$ we use the abbreviation

$$[i\,j\,k] \quad := \quad \det \begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix} \qquad \text{for } 1 \le i < j < k \le 6\,.$$

The determinant of the above 10×10-matrix equals the sparse unmixed resultant of (2.11), i.e., the Chow form of $S_{1,3}$ relative to the given embedding into P^5, by Corollary 2.4. $\diamond$

We close this section with an alternative proof of Theorem 1.4, based on Theorem 2.2.

Alternative Proof of Theorem 1.4: We assume for simplicity that $\ell = 1$. The case $\ell > 1$ can be dealt with by showing that the matrix of Φ has a block decomposition. We must show that $\mathcal{R} \cdot \mathrm{Res}_F^X(H)$ lies in A for any $H \in C_\rho$. Let $\mathcal{U}'$ be the intersection of $\mathcal{U}$ with the Zariski open set where all (non identically zero) maximal minors of Φ do not vanish. For $u \in \mathcal{U}'$, the $\mathbf{C}$-linear map Φ_u is surjective and we can write

$$H(x) \quad = \quad \sum_{i=0}^{n} \lambda_i(u;x)\, F_i(u;x) \;+\; \theta(u)\, J(F^u)\,,$$

where θ depends rationally on u. By (1.9) we have

$$\mathrm{Res}_{F^u}^X(H) \quad = \quad \gamma \cdot \theta(u)\,,$$

where γ is a rational constant independent of H and $F_0, \ldots, F_n$. This implies that every maximal minor of Φ which is not identically zero must involve the last column and that $\theta(u)$ is unique. Thus, it follows from Cramer's rule that $\mathrm{Res}_F^X(H)$ may be written as a rational function with denominator M for all non-identically zero maximal minors M. Consequently it may also be written as a rational function with denominator $\mathcal{R}$. $\diamond$

§3. Residues and resultants in the torus

In this section we apply the results of §1 to study the global residue associated with n Laurent polynomials in n variables. Let $\Delta_1, \ldots, \Delta_n$ be integral polytopes in $\mathbf{R}^n$. We form the Minkowski sum $\Delta := \Delta_1 + \cdots + \Delta_n$ and we consider its irredundant presentation

$$\Delta \quad = \quad \{\, m \in \mathbf{R}^n : \langle m, \eta_i \rangle + a_i \ge 0\,;\; i = 1, \ldots, s \,\}\,, \tag{3.1}$$

where, as in (1.1), the η_i are the first integral vectors in the inner normals to the facets of Δ. Writing $a_i^j = -\min_{m \in \Delta_j} \langle m, \eta_i \rangle$, we get a (generally redundant) inequality presentation

$$\Delta_j \quad = \quad \{\, m \in \mathbf{R}^n : \langle m, \eta_i \rangle + a_i^j \ge 0\,;\; i = 1, \ldots, s \,\} \quad \text{for all } j = 1, \ldots, n\,.$$

The facet normal η_i of Δ supports a (generally lower-dimensional) face of Δ_j:

$$\Delta_j^{\eta_i} \quad := \quad \{\, m \in \Delta_j \;:\; \langle m, \eta_i \rangle = -a_i^j \,\}\,. \tag{3.2}$$

Consider Laurent polynomials with indetermined coefficients and Newton polytopes Δ_j,

$$f_j \quad = \sum_{m \in \Delta_j \cap \mathbf{Z}^n} u_{jm} \cdot t^m \,, \tag{3.3}$$

and introduce the polynomial ring on their coefficients:

$$A' \quad := \quad \mathbf{Q}[\, u_{jm} \; ; \; j = 1 \ldots, n \; ; \; m \in \Delta_j \cap \mathbf{Z}^n] \,.$$

The leading form of f_j in the direction η_i equals

$$f_j^{\eta_i} \quad := \quad \sum_{m \in \Delta_j^{\eta_i}} u_{jm} \cdot t^m \,. \tag{3.4}$$

Since $\Delta^{\eta_i} = \Delta_1^{\eta_i} + \ldots + \Delta_n^{\eta_i}$ is a facet of Δ, we may regard $f_1^{\eta_i}, \ldots, f_n^{\eta_i}$ as a system of n polynomial functions on an $(n-1)$-dimensional torus. We define $\mathcal{R}^{\eta_i}$ to be their resultant *relative to the ambient lattice* $\mathbf{Z}^n$. More precisely, consider the sparse resultant $\mathcal{R}_{\Delta_1^{\eta_i}, \ldots, \Delta_n^{\eta_i}}$ for the support sets $\Delta_1^{\eta_i} \cap \mathbf{Z}^n, \ldots, \Delta_n^{\eta_i} \cap \mathbf{Z}^n$. This is the unique irreducible polynomial in A' which vanishes whenever $f_1^{\eta_i}, \ldots, f_n^{\eta_i}$ have a common zero in $(\mathbf{C}^*)^n$. Let $L_j^{\eta_i} := \mathrm{aff}_{\mathbf{Z}}(\Delta_j^{\eta_i} \cap \mathbf{Z}^n)$ be the affine lattice spanned by the integral points in $\Delta_j^{\eta_i}$, and let $L^{\eta_i} = \mathrm{aff}_{\mathbf{R}}(\Delta^{\eta_i}) \cap \mathbf{Z}^n$ be the restriction of $\mathbf{Z}^n$ to the i-th facet hyperplane of Δ. The index $\ell_i := [\, L^{\eta_i} : L_1^{\eta_i} + \ldots + L_n^{\eta_i} \,]$ is finite. We define the i-th *facet resultant* to be

$$\mathcal{R}^{\eta_i} \quad := \quad \left(\mathcal{R}_{\Delta_1^{\eta_i}, \ldots, \Delta_n^{\eta_i}} \right)^{\ell_i} \quad \text{for} \quad i = 1, \ldots, s \,. \tag{3.5}$$

We now specialize the coefficients u_{jm} in (3.4) to complex numbers such that

$$\mathcal{R}^{\eta_i}(u) \quad \neq \quad 0 \quad \text{for} \;\; i = 1, \ldots, s \,. \tag{3.6}$$

By Bernstein's Theorem [GKZ, §6.2.D, Thm. 2.8], the hypothesis (3.6) is equivalent to

$$\dim_{\mathbf{C}} \left(\mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] / \langle f_1, \ldots, f_n \rangle \right) \quad = \quad \mathrm{MV}(\Delta_1, \ldots, \Delta_n) \,, \tag{3.6$'$}$$

where $\mathrm{MV}(\cdots)$ denotes the *mixed volume*. Let V be the (finite) set of common zeros of $f_1, \ldots, f_n$ in the torus $T = (\mathbf{C}^*)^n$. Given any Laurent polynomial $q \in \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the *global residue* of the differential form

$$\phi_q \quad = \quad \frac{q}{f_1 \cdots f_n} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \,, \tag{3.7}$$

is defined as the sum of the local Grothendieck residues of ϕ_q, at each of the points in V:

$$\mathrm{Res}_f^T(q) \quad = \quad \sum_{p \in V} \mathrm{Res}_{p,f}(\phi_q) \,. \tag{3.8}$$

We refer to [GH], [AY], and [T] for the classical analytic definition of residues and to [H], [Ku] or [SS] for the algebraic definition of the Grothendieck residue.

Note that $\mathrm{Res}_f^T(J_f^T) = \mathrm{MV}(\Delta_1, \ldots, \Delta_n)$, where J_f^T denotes the *affine toric Jacobian*

$$J^T(f) \quad := \quad \det\big(t_k \, \frac{\partial f_j}{\partial t_k}\big)_{1 \le j,k \le n} . \tag{3.9}$$

If all the roots of $f_1, \ldots, f_n$ are simple, i.e. if V has cardinality $\mathrm{MV}(\Delta_1, \ldots, \Delta_n)$, then

$$\mathrm{Res}_f^T(q) \quad = \quad \sum_{\xi \in V} \frac{q(\xi)}{J^T(f)(\xi)} . \tag{3.10}$$

We conclude from (3.8) or (3.10) that, for fixed $q \in \mathbf{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the global residue $\mathrm{Res}_f^T(q)$ depends rationally on the coefficients u. In particular, for any $m \in \mathbf{Z}^n$, $\mathrm{Res}_f^T(t^m)$ is a rational function in u with $\mathbf{Q}$-coefficients.

Gel'fond and Khovanskii [GK] give a formula for evaluating that rational function, provided the Newton polytopes $\Delta_1, \ldots, \Delta_n$ satisfy the following genericity hypothesis:

$$\forall\, i \in \{1, \ldots, s\} \quad \exists\, j \in \{1, \ldots, n\} \, : \, \dim(\Delta_j^{\eta_i}) = 0 . \tag{3.11}$$

The Gel'fond-Khovanskii formula implies the following result, which appears also in [Z]:

Proposition 3.1. *Suppose the Newton polytopes $\Delta_1, \ldots, \Delta_n$ satisfy (3.11). Then, for any $m \in \mathbf{Z}^n$, the residue $\mathrm{Res}_f^T(t^m)$ is a Laurent polynomial in the coefficients of $f_1, \ldots, f_n$.*

If (3.11) is violated then $\mathrm{Res}_f^T(t^m)$ is generally not a Laurent polynomial. In particular, it is never a non-zero Laurent polynomial in the unmixed case $\Delta_1 = \ldots = \Delta_n$, $n \ge 2$.

Our aim is to characterize the denominator of $\mathrm{Res}_f^T(t^m)$. For each $m \in \mathbf{Z}^n$ we define

$$\mu_i^-(m) \quad := \quad -\min\{0, \langle m, \eta_i \rangle + a_i - 1\} \quad ; \quad i = 1, \ldots, s . \tag{3.12}$$

Geometrically, $\mu_i^-(m) > 0$ if m lies beyond the facet Δ^{η_i}. We state the main result of this section:

Theorem 3.2. *Let $f_1, \ldots, f_n$ be generic polynomials with Newton polytopes $\Delta_1, \ldots, \Delta_n$. For any $m \in \mathbf{Z}^n$, the following expression is a polynomial in A':*

$$\mathrm{Res}_f^T(t^m) \cdot \prod_{i=1}^{s} \mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})^{\mu_i^-(m)} .$$

It is easy to derive Proposition 3.1 from Theorem 3.2: If $\Delta_j^{\eta_i} = \{m\}$ in (3.11) then $\mathcal{R}^{\eta_i} = u_{jm}$ or $\mathcal{R}^{\eta_i} = 1$. In fact, (3.11) holds if and only if $\mathcal{R}^{\eta_1}\mathcal{R}^{\eta_2} \ldots \mathcal{R}^{\eta_s}$ is a monomial. We present an example where some facet resultants $\mathcal{R}^{\eta_i}$ are monomials and others are not.

Example 3.3. Let $n = 2$ and consider the mixed system

$$f_1(t_1, t_2) \quad = \quad a_0 t_1 + a_1 t_1 t_2 + a_2 t_2^2 \quad , \quad f_2(t_1, t_2) \quad = \quad b_0 t_2 + b_1 t_1 t_2 + b_2 t_1^2 .$$

The Minkowski sum of their Newton triangles is the pentagon

$$\Delta \quad = \quad \Delta_1 + \Delta_2 = \left\{ (m_1, m_2) \in \mathbf{R}^2 \; : \; \begin{pmatrix} -1 & 0 \\ -1 & -1 \\ 0 & -1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 3 \\ -3 \\ -3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The Δ-homogenizations of the input polynomials are

$$F_1 \quad = \quad \frac{x_1 x_2^2 x_3^2}{x_4^2 x_5} \cdot f_1 \left(\frac{x_4^2 x_5}{x_1 x_2}, \frac{x_4 x_5^2}{x_2 x_3} \right) \quad = \quad a_0 x_2 x_3^2 + a_1 x_3 x_4 x_5^2 + a_2 x_1 x_5^3 \, ,$$

$$F_2 \quad = \quad \frac{x_1^2 x_2^2 x_3}{x_4 x_5^2} \cdot f_2 \left(\frac{x_4^2 x_5}{x_1 x_2}, \frac{x_4 x_5^2}{x_2 x_3} \right) \quad = \quad b_0 x_1^2 x_2 + b_1 x_1 x_4^2 x_5 + b_2 x_3 x_4^3 \, .$$

Consider the lattice point $m = (3,3)$, which lies beyond three facets of Δ. The global residue of the corresponding monomial $t_1^3 t_2^3$ is equal to

$$\mathrm{Res}_f^T(t_1^3 t_2^3) \quad = \quad \frac{a_0 a_1 a_2 b_0 b_1 b_2 + a_0 a_2^2 b_0 b_2^2 - a_1^3 b_0^2 b_2 - a_0^2 a_2 b_1^3}{a_2 b_2 (a_1 b_1 - a_2 b_2)^3} \, .$$

The denominator can be derived from Theorem 3.2, since $\mu_1^-(m) = \mu_3^-(m) = 1, \mu_2^-(m) = 3, \mu_4^-(m) = \mu_5^-(m) = 0$ and the five facets resultants are

$$\mathcal{R}^{\eta_1} = b_2 \, , \qquad \mathcal{R}^{\eta_2} = a_1 b_1 - a_2 b_2 \, , \qquad \mathcal{R}^{\eta_3} = a_2 \, , \quad \mathcal{R}^{\eta_4} = b_0 \, , \quad \text{and} \quad \mathcal{R}^{\eta_5} = a_0 \, . \ \diamond$$

We shall develop the proof of Theorem 3.2 in several steps. We first consider the unmixed case $P := \Delta_1 = \cdots = \Delta_n$. Let P be presented as in (1.1) and $\mathcal{L}$ the associated line bundle on X. Fix an integer $k_0 > 0$ such that $\mathcal{L}^{k_0}$ is very ample. Consider the mixed sparse resultant $\mathcal{R}_{k_0} := \mathcal{R}_{k_0 P, P, \ldots, P}$ associated with the support sets $k_0 P \cap \mathbf{Z}^n, P \cap \mathbf{Z}^n, \ldots, P \cap \mathbf{Z}^n$. Thus, $\mathcal{R}_{k_0}$ coincides with the resultant associated to the line bundles $\mathcal{L}^{k_0}, \mathcal{L}, \ldots, \mathcal{L}$. In the following formula we evaluate $\mathcal{R}_{k_0}$ at a special monomial section t^m of $\mathcal{L}^{k_0}$ and generic sections of $\mathcal{L}, \ldots, \mathcal{L}$. Note that the facet resultants $\mathcal{R}^{\eta_i}$ are irreducible if $\mathcal{L}$ is very ample.

Lemma 3.4. *For any $m \in k_0 P \cap \mathbf{Z}^n$ we have the following identity in A':*

$$\mathcal{R}_{k_0}(t^m, f_1, \ldots, f_n) \quad = \quad \prod_{i=1}^{s} \mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})^{\langle m, \eta_i \rangle + k_0 b_i} \, .$$

Proof: Theorem 1.1 in [PSt] gives the following identity of rational functions:

$$\mathcal{R}_{k_0}(f_0, f_1, \ldots, f_n) \quad = \quad \left(\prod_{\xi \in V(f_1, \ldots, f_n)} f_0(\xi) \right) \cdot \prod_{i=1}^{s} \mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})^{k_0 b_i} \, , \qquad (3.13)$$

where $f_0, f_1, \ldots, f_n$ are generic polynomials supported in $k_0 P, P, \ldots, P$. On the other hand, the same result applied to the support sets $\{m\}, P \cap \mathbf{Z}^n, \ldots, P \cap \mathbf{Z}^n$ gives

$$\prod_{\xi \in V(f_1,\ldots,f_n)} \xi^m \quad = \quad \prod_{i=1}^{s} \mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})^{\langle m,\eta_i\rangle} \tag{3.14}$$

since $\mathcal{R}_{\{m\},P,\ldots,P}(t^m, f_1, \ldots, f_n) = 1$. Now combine (3.13) and (3.14) for $f_0 = t^m$. $\diamond$

For $m \in \mathbf{Z}^n$ and $1 \leq i \leq s$ we abbreviate

$$\mu_i^+(m) := \max\{0, \langle m, \eta_i\rangle + nb_i - 1\} \quad \text{and} \quad \mu_i^-(m) := -\min\{0, \langle m, \eta_i\rangle + nb_i - 1\}\,.$$

This notation distinguishes the facets of nP visible from m from those not visible from m. The following lemma is the unmixed case of Theorem 3.2.

Lemma 3.5. *Let $f_1, \ldots, f_n$ be generic polynomials with support in P. Given $m \in \mathbf{Z}^n$,*

$$\mathrm{Res}_f^T(t^m) \cdot \prod_{i=1}^{s} \mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})^{\mu_i^-(m)} \quad \in \quad A'\,.$$

Proof: We denote by $F_1, \ldots, F_n$ the generic polynomials in S_β obtained from $f_1, \ldots, f_n$ by homogenization as in (1.5). More precisely, if $f_i = \sum_{m \in P\cap\mathbf{Z}^n} u_{im} t^m$ then

$$F_i \;=\; F_i(u;x) \;=\; \sum_{m \in P\cap\mathbf{Z}^n} u_{im}\,(\prod_{i=1}^{s} x_i^{\langle m,\eta_i\rangle + b_i})\,. \tag{3.15}$$

It is shown in [CD] that the differential form

$$\frac{x^{\mu^+(m)}}{x^{\mu^-(m)}\,F_1\cdots F_n} \cdot \Omega$$

is the meromorphic extension to the toric variety X of the form ϕ_{t^m} on the torus T defined in (3.7). By Theorem 4 in [CD] (or Lemma 3.6 below), there exist monomials x^c such that $\deg(x^{\mu^-(m)+c}) = k_0\beta$ for some (arbitrarily large) positive integer k_0. Whenever the coefficients of $f_1, \ldots, f_n$ lie in the Zariski open set where none of the facet resultants $\mathcal{R}^{\eta_i}$ vanishes, then $F_1, \ldots, F_n$ have no common zeroes at infinity. In this case, $\{\, x \in X : F_1(x) = \cdots = F_n(x) = 0 \,\} \subset T$ and, as shown in [CCD], [CD], the global residue in the torus of ϕ_{t^m} may be computed as

$$\mathrm{Res}_f^T(t^m) \quad = \quad \mathrm{Res}_F^X(x^{\mu^+(m)+c})\,,$$

where F denotes the $(n+1)$-tuple: $F_0 = x^{\mu^-(m)+c}$, $F_1, \ldots, F_n$.

By Theorem 1.4, the global residue $\text{Res}_f^T(t^m)$ is a rational function with denominator $\mathcal{R}_{k_0}(x^{\mu^-(m)+c}, F_1, \ldots, F_n)$. Lemma 3.4 implies that

$$\mathcal{R}_{k_0}(x^{\mu^-(m)+c}, F_1, \ldots, F_n) \quad = \quad \prod_{i=1}^{s} \left(\mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})\right)^{\mu_i^-(m)+c_i}. \tag{3.16}$$

We conclude that the residue $\text{Res}_f^T(t^m)$ may be written as a rational function with denominator the greatest common divisor of all expressions of the form (3.16), where $c = (c_1, \ldots, c_s)$ runs over all non-negative integer vectors such that $\deg(x^{\mu^-(m)+c}) = k_0\beta$ for some integer $k_0 > 0$. Since unmixed resultants depend on the coefficients of all polynomials (e.g. by [KSZ, Theorem 5.3]), the facet resultants $\mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})$ are powers of distinct irreducible polynomials. The proof of Lemma 3.5 follows from Lemma 3.6 below. $\diamond$

Lemma 3.6. *For any non-negative vector $a \in \mathbf{N}^s$ and any $i \in \{1, \ldots, s\}$ there exists a non-negative vector $c \in \mathbf{N}^s$ such that $c_i = 0$ and $\deg(x^{a+c}) = k_0\beta$ for some $k_0 \in \mathbf{N}$.*

Proof: Let $u^{(1)}, \ldots, u^{(\tau)} \in \mathbf{Z}^n$ be all the vertices of the lattice polytope P which lie on the facet $P^{\eta_i} = \{ m \in P : \langle m, \eta_i \rangle + b_i = 0 \}$. Their sum $u := u^{(1)} + \cdots + u^{(\tau)}$ satisfies $\langle u, \eta_i \rangle + \tau \cdot b_i = 0$ and $\langle u, \eta_j \rangle + \tau \cdot b_j \geq 1$ for all $j \neq i$. Since η_i is primitive, we can find $m \in \mathbf{Z}^n$ such that $\langle m, \eta_i \rangle = a_i$. Let k_0 be an integer divisible by τ such that

$$c_j \quad := \quad \frac{k_0}{\tau} \cdot (\langle u, \eta_j \rangle + \tau \cdot b_j) + \langle m, \eta_j \rangle \, - \, a_j$$

is non-negative for $j = 1, 2 \ldots, s$. Then $c = (c_1, \ldots, c_s)$ has the desired properties. $\diamond$

We now prove Theorem 3.2 for mixed systems of generic Laurent polynomials.

Proof of Theorem 3.2: We shall assume $MV(\Delta_1, \ldots, \Delta_n) > 0$. Otherwise the residue $\text{Res}_f^T(t^m)$ is zero and Theorem 3.2 trivially holds.

Let $X = X_\Delta$ be the projective toric variety associated with Δ. We consider the homogenization of the Laurent polynomial $f_j(t_1, \ldots, t_n)$:

$$F_j(x_1, \ldots, x_s) \quad := \quad \sum_{m \in \Delta_j \cap \mathbf{Z}^n} u_{jm} \left(\prod_{i=1}^{s} x_i^{\langle m, \eta_i \rangle + a_i^j}\right).$$

Note that $F_j(x)$ is generic of degree $\alpha_j := [\sum_{i=1}^s a_i^j D_i]$. Let $\alpha := \alpha_1 + \cdots + \alpha_n = [\sum_{i=1}^s a_i D_i]$. For each $j = 1, \ldots, n$, let Q_j be a generic polynomial of degree $\alpha - \alpha_j$ and set $G_j = F_j \, Q_j$. Given a positive integer k_0, let F_0 be a generic polynomial of degree $k_0 \, \alpha$. Thus $F_0, G_1, \ldots, G_n$ are homogeneous polynomials of degrees $k_0\alpha, \alpha, \ldots, \alpha$. For all choices of complex coefficients in a Zariski open set, they have no common roots in X. Given a polynomial H of critical degree $\rho(F) := (k_0+1)\,\alpha - \beta_0$ relative to the $(n+1)$-tuple $F = (F_0, F_1, \ldots, F_n)$, we can compute the toric residue $\text{Res}_F^X(H)$ and, according to the Global Transformation Law [CCD, Theorem 0.1]:

$$\text{Res}_F^X(H) \quad = \quad \text{Res}_G^X(H \cdot Q_1 \cdots Q_n) \; ; \quad G = (F_0, G_1, \ldots, G_n).$$

Let $\mathcal{R}$ be the $(k_0\Delta, \Delta, \ldots, \Delta)$-resultant. It follows from Theorem 1.4 that the specialization $\mathcal{R}(F_0, G_1, \ldots, G_n)$ is a denominator for the rational function $\mathrm{Res}_F^X(H)$.

Let f_0 denote the dehomogenization of F_0, let q_j be the dehomogenization of Q_j, and set $g_j := f_j \cdot q_j$ for any $j = 1, \ldots, n$. Then, $\mathcal{R}(F_0, G_1, \ldots, G_n)$ agrees with the sparse resultant $\mathcal{R}(f_0, g_1, \ldots, g_n)$ arising from the support sets $k_0\Delta \cap \mathbf{Z}^n, \Delta \cap \mathbf{Z}^n, \ldots, \Delta \cap \mathbf{Z}^n$. Given a subset $J \subseteq \{1, \ldots, n\}$, we denote $\tilde{f}_j := f_j$ if $j \in J$, and $\tilde{f}_k := q_k$ if $k \notin J$. We let $\tilde{\Delta}_j$ stand for the Newton polytope of $\tilde{f}_j$, i.e. $\tilde{\Delta}_j = \Delta_j$ if $j \in J$, and

$$\tilde{\Delta}_k = \Delta_1 + \cdots + \Delta_{k-1} + \Delta_{k+1} + \cdots + \Delta_n \quad \text{if} \quad k \notin J\,.$$

It follows from the Product Formula for sparse mixed resultants [PSt, Proposition 7.1] that

$$\mathcal{R}(f_0, g_1, \ldots, g_n) \quad = \prod_{J \subseteq \{1,\ldots,n\}} \mathcal{R}^J(f_0, \tilde{f}_1, \ldots, \tilde{f}_n)\,, \tag{3.17}$$

where $\mathcal{R}^J$ denotes the sparse mixed resultant associated with the support sets

$$k_0\Delta \cap \mathbf{Z}^n, \tilde{\Delta}_1 \cap \mathbf{Z}^n, \ldots, \tilde{\Delta}_n \cap \mathbf{Z}^n \tag{3.18}$$

relative to the ambient lattice $\mathbf{Z}^n$ as in (3.5).

We now show that the factor $\mathcal{R}(f_0, \ldots, f_n)$ corresponding, in (3.17), to $J = \{1, \ldots, n\}$ is already a denominator of the rational function $\mathrm{Res}_F^X(H)$. Since this is a function of the coefficients of $f_0, \ldots, f_n$ only, it suffices to show that every additional factor in (3.17) must involve the coefficients of some q_k, $k = 1, \ldots, n$, i.e. if $J \neq \{1, \ldots, n\}$, the polynomial $\mathcal{R}^J(f_0, \tilde{f}_1, \ldots, \tilde{f}_n)$ has positive degree in the coefficients of some q_k, $k \notin J$. But this is a consequence of our assumption $MV(\Delta_1, \ldots, \Delta_n) > 0$. Indeed, according to Lemma 1.2 and Corollary 1.1 of [S2], it is enough to show that the collection of supports $k_0\Delta \cap \mathbf{Z}^n$, $\tilde{\Delta}_j \cap \mathbf{Z}^n$, $j \in J$ contains no proper *essential* subset. A subset which contains $k_0\Delta \cap \mathbf{Z}^n$ cannot be essential since $\dim(\Delta) = n$ and the cardinality of the subset is at most n. On the other hand, no collection of supports $\tilde{\Delta}_j \cap \mathbf{Z}^n$ can be essential because $MV(\Delta_1, \ldots, \Delta_n) > 0$.

We now complete the proof of Theorem 3.2 similarly to the proof of Lemma 3.5. The algorithm in [CD] computes $\mathrm{Res}_f^T(t^m)$ as the toric residue $\mathrm{Res}_F^X(x^\mu)$ for appropriate monomials x^μ and $F_0(x) = x^\nu$ of degree $(k_0+1)\alpha - \beta_0$ and $k_0\alpha$, respectively, where k_0 is a positive integer. For any such choice of μ and ν, the specialization

$$\mathcal{R}(x^\nu, F_1, \ldots, F_n) \quad = \quad \prod_{i=1}^{s} \mathcal{R}^{\eta_i}(f_1^{\eta_i}, \ldots, f_n^{\eta_i})^{\nu_i}$$

is a denominator of the rational function $\mathrm{Res}_f^T(t^m)$. Taking the greatest common divisor over all possible choices and applying Lemma 3.6 yields the theorem. $\diamond$

References

[AY] I. A. Aĭzenberg and A. P. Yuzhakov, *Integral representations and residues in multidimensional complex analysis*, Translations of Mathematical Monographs **58**. American Mathematical Society, 1983.

[A] B. Angéniol, *Résidus et effectivité*, Unpublished manuscript, 1983.

[B] V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Mathematical Journal **69** (1993) 349–409.

[BC] V. Batyrev and D. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke J. Math. **75** (1994) 293–338.

[BH] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Univ. Press, 1993.

[CCD] E. Cattani, D. Cox, and A. Dickenstein, *Residues in toric varieties*, Compositio Mathematica, to appear, alg-geom 9506024.

[CD] E. Cattani and A. Dickenstein, *A global view of residues in the torus*, Journal of Pure and Applied Algebra, to appear.

[Ch] M. Chardin, *The resultant via a Koszul complex.* In "*Computational Algebraic Geometry*" (F. Eysette, A. Galligo, eds.), Proceedings MEGA 92, Progress in Math. **109**, Birkhäuser, Boston, 1993, 29–39.

[C1] D. Cox, *The homogeneous coordinate ring of a toric variety*, Journal of Algebraic Geometry **4** (1995) 17–50.

[C2] D. Cox, *Toric residues*, Arkiv för Matematik **34** (1996) 73–96.

[F] W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, Princeton, 1993.

[GKZ] I. Gel'fand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser,Boston, 1994.

[GK] O. A. Gel'fond and A. G. Khovanskii, *Newtonian polyhedrons and Grothendieck residues*, Doklady Mathematics **54** (1996) 700–702.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[H] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. **20**, Springer, 1966.

[Ha] J. Harris, *Algebraic Geometry*, Springer Graduate Texts, New York, 1992.

[J1] J. P. Jouanolou, *Singularités rationnelles du résultant*, In Lecture Notes in Mathematics, Vol. 732, Springer, 1978, pp. 183–213.

[J2] J. P. Jouanolou, *Le Formalisme du Résultant*, Advances in Mathematics **90** (1991) 117–263.

[J3] J. P. Jouanolou, *Résultant, intersections complètes et résidu de Grothendieck*, Notes incomplètes, U. Strasbourg, 1994.

[KSZ] M.M. Kapranov, B. Sturmfels and A.V. Zelevinsky, *Chow polytopes and general resultants*, Duke Mathematical Journal **67** (1992) 189–218.

[K1] A. G. Khovanskii, *Newton polyhedra and toroidal varieties*, Functional Analysis and its Applications **11** (1977) 289–296.

[K2] A. G. Khovanskii, *Newton polyhedra and the Euler-Jacobi formula*, Russian Math. Surveys **33** (1978) 237–238.

[Ku] E. Kunz, *Kähler Differentials*, Advanced Lectures in Mathematics, Vieweg, 1986.

[M] F.S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, 1916.

[O] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, 1988.

[PSt] P. Pedersen and B. Sturmfels, *Product formulas for resultants and Chow forms*, Mathematische Zeitschrift **214** (1993) 377–396.

[PS] C. Peters and J. Steenbrink, *Infinitesimal variation of Hodge structure and the generic Torelli theorem for projective hypersurfaces*. In *Classification of Algebraic and Analytic Manifolds* (K. Ueno, ed.), Progress in Math. **39**, Birkhäuser, Boston, 1983, 399–463.

[SS] G. Scheja and U. Storch, *Über Spurfunktionen bei vollständigen Durchschnitten*, J. Reine u. Angewandte Mathematik **278/9** (1975) 174–190.

[Sta] R. Stanley, *Enumerative Combinatorics*, Volume 1, Wadsworth & Brooks, Monterey, 1986.

[S1] B. Sturmfels, *Sparse Elimination Theory*. In *Computational Algebraic Geometry and Commutative Algebra* (D. Eisenbud, L. Robbiano, eds.), Proceedings, Cortona, June 1991. Cambridge University Press, 1993.

[S2] B. Sturmfels, *On the Newton polytope of the resultant*, Journal of Algebraic Combinatorics **3** (1994) 207–236.

[T] A. Tsikh, *Multidimensional Residues and Their Applications*, AMS, Providence, 1992.

[WZ] J. Weyman and A. Zelevinsky, *Determinantal formulas for multigraded resultants*, Journal of Algebraic Geometry **3** (1994) 569–597.

[Z] H. Zhang, *Sur certains calculs de résidus pour des systèmes de polynômes de Laurent interpretés dans le cadre des variétés toriques*, Thèse, Université de Bordeaux, 1996.