Annals of Mathematics, **108** (1978), 461-505

# Residues and zero-cycles on algebraic varieties

By PHILLIP GRIFFITHS* and JOSEPH HARRIS**

**Table of Contents**



## 0. Introduction

In a general sense this paper is concerned with exceptional configurations of points on a smooth projective algebraic variety $M$. If $|D|$ is a linear system of divisors and $Z$ a set of distinct points, we denote by $|\mathcal{I}_Z(D)|$ the sub-linear system of divisors in $|D|$ which pass through $Z$. Then we say that $Z$ is *superabundant* in case the superabundance

$$\omega(Z, |D|) = \dim|\mathcal{I}_Z(D)| - (\dim|D| - \deg Z)$$

is positive; this means that the points of $Z$ fail to impose independent conditions on $|D|$. In case the linear system $|D|$ induces a projective embedding

$$\iota: M \longrightarrow \mathbf{P}^r ,$$

the superabundance exactly measures the failure of the points $\boldsymbol{p} = \iota(p)$ $(p \in Z)$

0003-486X/78/0108-0003 $02.25



* Research supported by NSF Grant MCS 72-05154 and the Forschungsinstitut für Mathematik ETH Zürich.

** NSF Predoctoral Fellow.

to be in general position, so that superabundant sets may be thought of as generalized multisecant planes. In case $M$ is an algebraic curve and $|D|$ is the canonical linear series we are discussing special divisors, a beautiful subject with a venerable history. For surfaces the simplest interesting example of a superabundant configuration are the nine points of intersection of two cubics in the projective plane.

There is one central idea in this paper, which we will explain following the introduction of some notation. We assume given a holomorphic line bundle $L \to M$ whose associated complete linear system induces a projective embedding

$$\iota_L: M \to \mathbf{P}^r .$$

We set $\boldsymbol{p} = \iota_L(p)$, $\mathbf{Z} = \iota_L(Z)$, $\mathbf{M} = \iota_L(M)$ and denote by $\{\mathbf{Z}\}$ the linear span in $\mathbf{P}^r$ of the points $\boldsymbol{p}$ where $p \in Z$. More generally, we denote by $\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^\mu\mathbf{Z}\}$ the linear span of the $\mu^{\text{th}}$ osculating spaces to $\mathbf{M}$ at the points $\boldsymbol{p}$. We call $\{\mathbf{Z}\}, \cdots, \{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^\mu\mathbf{Z}\}$ the *osculating sequence associated to the* 0*-cycle* $Z$. We shall also sometimes denote by $\boldsymbol{p}$ a point in $\mathbf{C}^{r+1}$ lying over $\iota_L(p) \in \mathbf{P}^r$.

Now suppose that $M$ has dimension $n$ and that $E \to M$ is a rank $n$ holomorphic vector bundle with a section $s \in H^0(\mathcal{O}(E))$ having a set $Z = (s)$ of distinct isolated zeroes. If we set

$$\begin{cases} Z = p_1(s) + \cdots + p_d(s) & \text{and} \\ L = K \otimes \det E \end{cases}$$

then the residue theorem (Section I) gives a linear relation

$$\sum_{i=1}^d \lambda_i(s)\mathbf{p}_i(s) = 0 , \qquad \lambda_i \neq 0. \tag{0.1}$$

In particular the superabundance $\omega(Z, |K \otimes \det E|) > 0$, a relation which has a converse at least in case $n = 1, 2$. Differentiation of (0.1) $\mu$ times with respect to $s \in H^0(\mathcal{O}(E))$ gives similar linear relations among the $\mu^{\text{th}}$ osculating spaces to $\mathbf{M}$ at points $\boldsymbol{p} \in Z$, relations in which the values of the polynomials in $\operatorname{Sym}^\mu(H^0(\mathcal{O}(E)))$ appear as coefficients. *Our basic observation is that* 0*-cycles* $Z = (s)$ *for* $s \in H^0(\mathcal{O}(E))$ *have osculating sequences whose growth is much slower than for a generic* 0*-cycle, and that the quantitative measure of this is reflected in the graded ideal in* $\bigoplus_{\mu \geq 0} \operatorname{Sym}^\mu H^0(\mathcal{O}(E))$ *defined by* $M$. The precise statement is given in Section II b).

This then is the idea behind the paper. Following a recollection of the local properties of residues, the global residue theorem and a converse are proved in Section I b). It is this converse which motivates our feeling that, in general, the conditions imposed by the residue theorem are sufficient for a configuration of points to have certain global properties. In Section I c) we interpret geometrically the constraints imposed by the residue theorem in terms of "multisecant planes," somewhat by analogy with the picture one has of special divisors on curves as multisecant planes on the canonical curve. Then in Sections I d) and e) we specialize the residue theorem to line bundles on curves and rank-two vector bundles on surfaces respectively. In the curve case the connection with Abel's theorem and special divisors is established. In the surface case we continue a project initiated by Schwarzenberger [S] of associating rank-two bundles to zero-cycles on surfaces. Here we are able to give a reasonably complete existence and uniqueness result, one which is especially aimed at superabundant 0-cycles.

Turning to Section II we first define precisely the osculating sequence associated to a 0-cycle on a variety in projective space. Then in Part II b) we give the fundamental relation bounding the growth of this osculating sequence in terms of the graded ideal mentioned above.

Applying this bound necessitates having information on this ideal. Actually, what is needed here are just inequalities, this because the mechanism works in the somewhat surprising way:

$$\left\{\begin{array}{c}\text{Fewer relations in the graded ring}\\ \Downarrow\\ \text{Slower growth of osculating sequence.}\end{array}\right\} \tag{0.2}$$

It is the tension created by the inequalities in (0.2) going in the non-obvious direction which makes our method work. The simplest case in which the homogeneous ideal can be estimated is when

$$M = \underbrace{L \oplus \cdots \oplus L}\qquad (n\text{-times})$$

where $L \to M$ is a line bundle whose complete linear system $|L|$ induces a projective embedding

$$\iota_L \colon M \longrightarrow \mathbf{P}^N .$$

If we let $M_L = \iota_L(M)$, the zero-cycles $Z = (s)$ for $s \in H^0(\mathcal{O}(E))$ are intersections of $M_L$ with a linear space $\mathbf{P}^{N-n}$ of complementary dimension. We are led by the residue theorem to consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{\iota_L} & \mathbf{P}^N \\ \big\downarrow{\scriptstyle \iota_{K+mL}} & & \\ \mathbf{P}^r & & \end{array} , \qquad 0 \leqq m \leqq n ,$$

and in particular to the growth of the osculating sequence of $\mathbf{Z} = \iota_{K+mL}(Z)$ in $\mathbf{P}^r$. Since the osculating spaces eventually exhaust $\mathbf{P}^r$, a corollary is the estimate

$$\dim|K + mL| \leqq \kappa(n, N, d, m) \tag{0.3}$$

given in Part II c) below. In particular, taking $m = 0$ we obtain the bound on $h^{n,0}(M)$ given in [H] and also in [C-G, 2] where it is derived for general webs as a consequence of Abel's theorem.

In Part II d) we examine in closer detail the growth of the osculating sequence in the case of curves. The bound (0.3) turns out to be essentially equivalent to Castelnuovo's inequality on the genus of a non-degenerate curve in projective space.

In Part II e) we turn again to surfaces. By the Kodaira vanishing theorem the L.H.S. of (0.3) is given by a topological number according to the Hirzebruch-Riemann-Roch theorem. We examine the resulting inequalities for low degrees, characterizing some of the extreme cases such as the $K3$ surfaces $S \subset \mathbf{P}^n$ which are uniquely specified by

$$\begin{cases} \deg S = 2n - 2 \\ p_g(S) \neq 0 \end{cases}$$

in almost exact analogy with elliptic curves.

As indicated by its title, Section III is concerned with proving the sufficiency of the conditions imposed by the residue theorem. The simplest case here contains the conditions imposed on a complete intersection

$$Z = C \cdot C' , \qquad C, C' \in |L| , \tag{0.4}$$

where $L \to S$ is a line bundle whose complete linear system $|L|$ induces a projective embedding in $\mathbf{P}^n$. Except for the case when $S$ has minimal degree $n - 1$ in $\mathbf{P}^n$, the residue theorem turns out to impose non-trivial conditions on a complete intersection (0.4), and in Part III a) we prove that these are "in general" sufficient. Actually, we give two completely different proofs of this "converse to the Bezout theorem," the second of which may lead to the best precise meaning of "in general" but we are not able to establish this.

In Parts III b) and c) we set about characterizing extremal varieties; i.e., those for which equality holds in (0.3) for some $m$, under the assumptions on the codimension $k$ and degree $d$,

$$\begin{cases} k = N - n \geqq 2 \\ d \geqq 2k + 3 \, . \end{cases} \tag{0.5}$$

It is first shown that the extremal property is independent of $m$, and is characterized by the linear section

$$Z = \mathbf{P}^k \cdot M_L$$

having the maximum superabundance

$$\omega(Z, |\mathcal{O}_{\mathbf{P}^k}(2)|) = d - (2k+1) \tag{0.6}$$

on the linear system of quadrics in $\mathbf{P}^k$. Under the conditions (0.5) and (0.6) on a configuration $Z$ of points in general position in $\mathbf{P}^k$, it follows that the $\infty^{k(k-1)/2}$ quadrics containing $Z$ intersect in a rational normal curve $C$. This fact is usually proved by a synthetic argument, but here we give an analytic proof in keeping with the general spirit of the paper. Once this is established we then show that as $\mathbf{P}^k$ varies the curves $C$ trace out an $(n+1)$-dimensional variety $V_L \subset \mathbf{P}^N$ of minimal degree $k = \operatorname{codim} V_L + 1$ on which $M_L$ is a divisor.

Once we know where to look it is reasonable to hope to explicitly construct extremal varieties, thereby proving that the estimates (0.3) are sharp. The details of this can be found in [H].

Finally, in the appendix we give some informal observations on the general problem of superabundance and on how our results fit in. Then we discuss three open problems which arise from this work.

As mentioned above, this paper is based on the one idea of differentiating the residue theorem (0.1) and then seeing what comes out. The results we find overlap somewhat with the thesis [H] by one of us. There the methods are quite different and may be roughly described as algebro-geometric using the Riemann-Roch for curves and linear series techniques similar to the proof of Castelnuovo's bound given in Chapter II of [G-H]. In fact, our techniques are closer to those used in the study of webs in [C-G, 1] and [C-G, 2], in that there is a common theme of applying local differential geometry to obtain global conditions on algebraic varieties. It is in the theory of abelian equations associated to webs that the osculating sequence first appeared (cf. the references cited in [C-G, 1]).

The use of residues provides a direct and self-contained method for arriving at the essential mechanism in understanding the superabundance in the present context. In principle the technique works equally for vector bundles, but this depends on finding a good concept of "non-degenerate" for subvarieties of a Grassmannian. For this reason the method has potentially wider applicability than standard algebro-geometric techniques, which rely on the curve sections and therefore have to do with line bundles. On the other hand, the web methods are restricted to abelian equations coming from $H^0(\mathcal{O}(K)) \otimes H^0(\mathcal{O}(\det E))$ whereas the residues give abelian equations associated to $H^0(\mathcal{O}(K \otimes \det E))$, and so are more general in scope.

Finally we should like to call special attention to the very interesting paper [A-S], which gives information on the osculating sequence in the general, as opposed to the extreme, case.

## I. The residue theorem and interpretations

a) *Local properties of residues.*

We shall summarize, in a form convenient for use in this paper, some local properties of residues from Chapter V of [G-H]. We denote by $\mathcal{O}$ the local ring at the origin in $\mathbf{C}^n$ and by $\boldsymbol{m} = \{z_1, \cdots, z_n\}$ the maximal ideal. Let $f_1(z), \cdots, f_n(z) \in \boldsymbol{m}$ be functions with the origin as isolated common zero generating an ideal $\mathcal{I} \subset \mathcal{O}$. Equivalently, the $f_i(z) \in \boldsymbol{m}$ constitute a regular sequence. Geometrically, we may think of the $f_i$ as defining a 0-dimensional scheme which is set-theoretically the origin but whose additional infinitesimal structure is given by the finite-dimensional C-algebra $\mathcal{O}/\mathcal{I}$.

Given $g(z) \in \mathcal{O}$ we set

$$\omega = \frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)}$$

and define the point residue by

$$\operatorname{Res}_{\{0\}} \omega = \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \omega \tag{1.1}$$

where $\Gamma$ is the real $n$-cycle given by $\{z: |f_i(z)| = \varepsilon\}$ for sufficiently small $\varepsilon$, and with orientation $d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geqq 0$. The following are the properties of (1.1) which we shall require.

i) The first one, which is obvious, is the *Cauchy integral formula*

$$\operatorname{Res}_{\{0\}}\left(\frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{z_1^{k_1+1} \cdots z_n^{k_n+1}}\right) = \frac{1}{k_1! \cdots k_n!}\left(\frac{\partial^{k_1+\cdots+k_n} g}{\partial z_1^{k} \cdots \partial z_n^{k_n}}\right)(0) . \tag{1.2}$$

ii) Next, if the $f_i(z)$ are defined in a neighborhood of $U = \{z: \|z\| \leqq \varepsilon\}$, and if $f_i(z, t)$ is analytic in $z \in U$ and $t$ for $|t| < \delta$ with $f_i(z, 0) = f_i(z)$, then for $\delta$ and $\varepsilon$ sufficiently small the functions $f_i(z, t)$ will have a finite set of isolated common zeroes $p_\nu(t)$ in the interior of $U$. With

$$\omega(t) = \frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z, t) \cdots f_n(z, t)} ,$$

the sum

$$\textstyle\sum_\nu \operatorname{Res}_{p_\nu(t)}(\omega(t))$$

is a holomorphic function of $t$ and

$$\frac{\partial}{\partial t}\left(\textstyle\sum_\nu \operatorname{Res}_{p_\nu(t)}(\omega(t))\right) = \textstyle\sum_\nu \operatorname{Res}_{p_\nu(t)}\left(\dfrac{\partial \omega(t)}{\partial t}\right) . \tag{1.3}$$

iii) If $\omega$ is as above, if

$$f'_i = \sum_j a_{ij} f_j$$

is also a regular sequence, and if

$$\omega' = \frac{\det \| a_{ij}(z) \| g(z) \cdot dz_1 \wedge \cdots \wedge dz_n}{f'_1(z) \cdots f'_n(z)} ,$$

then the *transformation formula*

$$\operatorname{Res}_{\{0\}} \omega = \operatorname{Res}_{\{0\}} \omega' \tag{1.4}$$

is valid.

iv) The pairing

$$\operatorname{Res}_f \mathcal{O}/\mathcal{I} \otimes_{\mathbf{C}} \mathcal{O}/\mathcal{I} \longrightarrow \mathbf{C}$$

defined by

$$\operatorname{Res}_f (g, h) = \operatorname{Res}_{\{0\}} \left( \frac{g(z)h(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} \right) \tag{1.5}$$

is non-degenerate (*local duality theorem*).

v) The pairing (1.5) depends on the coordinate system in $\mathbf{C}^n$ and choice of generators $f_i$ for the regular ideal $\mathcal{J}$. To make it intrinsic, we recall that for any regular sequence $g_1, \cdots, g_k \in m$ generating an ideal $\mathfrak{Y}$,

$$\begin{cases} \operatorname{Ext}^i_{\mathcal{O}}(\mathcal{O}/\mathfrak{Y}, \mathcal{O}) = 0 & i \neq k \\ \operatorname{Ext}^k_{\mathcal{O}}(\mathcal{O}/\mathfrak{Y}, \mathcal{O}) \cong \mathcal{O}/\mathfrak{Y} , \end{cases} \tag{1.6}$$

where the isomorphism in (1.6) transforms by $\det(b_{ij})$ when we change to new generators $g'_i = \sum_j b_{ij} g_j$ for $\mathfrak{Y}$. If we let $\Omega^n$ denote the stalk at the origin of the sheaf of holomorphic $n$-forms on $\mathbf{C}^n$, it follows that the pairing

$$\operatorname{Res}\colon \mathcal{O}/\mathcal{I} \otimes_{\mathbf{C}} \operatorname{Ext}^n_{\mathcal{O}}(\mathcal{O}/\mathcal{I}, \Omega^n) \longrightarrow \mathbf{C} \tag{1.7}$$

is intrinsic and non-degenerate.

b) *The residue theorem and a converse.*

Now we shall give a global residue theorem for vector bundles. Let $M$ be a compact, complex manifold of dimension $n$, $E \to M$ a homomorphic vector bundle of rank $n$, and $s \in H^0(\mathcal{O}_M(E))$ a section with zero set $Z$ a discrete set of points. More precisely, the image of

$$\mathcal{O}(E^*) \xrightarrow{s} \mathcal{O}$$

defines a sheaf of ideals $\mathcal{I}_Z$ and $Z$ is the 0-dimensional scheme with structure sheaf $\mathcal{O}_Z = \mathcal{O}/\mathcal{I}_Z$.

Given $\psi \in H^0(\mathcal{O}_M(K \otimes \det E))$ we shall define for each $p \in Z$ a point residue

$$\operatorname{Res}_p\left(\frac{\psi}{s}\right).$$

To do this choose a local holomorphic frame $e_1, \cdots, e_n$ for $E$ and local holomorphic coordinates $z = (z_1, \cdots, z_n)$ centered at $p$. Then

$$\begin{cases} s(z) = f_1(z)e_1 + \cdots + f_n(z)e_n \\ \psi(z) = g(z)dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n , \end{cases}$$

and we set

$$\operatorname{Res}_p\left(\frac{\psi}{s}\right) = \operatorname{Res}_{\{0\}}\left(\frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)}\right). \tag{1.8}$$

By the transformation formula (1.4), this is well-defined. We also note from the property (1.5) of the point residue that (1.8) may be defined if we are given only

$$\psi_p \in \mathcal{O}_{Z,p}(K \otimes \det E) .$$

THEOREM. *With the above notations,*

$$\sum_{p \in Z} \operatorname{Res}_p\left(\frac{\psi}{s}\right) = 0 . \tag{1.9}$$

*Conversely, if we assume the vanishing theorem*

$$H^q\left(\Omega_M^n(\wedge^p E)\right) = 0 \qquad \text{for} \quad q > 0, 1 \leqq p \leqq n , \tag{1.10}$$

*then for given $\psi_p \in \mathcal{O}_{Z,p}(K \otimes \det E)$ the relation*

$$\sum_{p \in Z} \operatorname{Res}_p\left(\frac{\psi_p}{s}\right) = 0$$

*is necessary and sufficient for there to exist $\psi \in H^0\left(\mathcal{O}_M(K \otimes \det E)\right)$ inducing each $\psi_p$.*

We note that (1.10) is satisfied in case

$$E = \underbrace{L_1 \oplus \cdots \oplus L_n}$$

where the $L_\alpha \to M$ are positive line bundles, by the Kodaira vanishing theorem.

*Proof.* The result will be a formal consequence of duality. That is we let

$$\omega_Z = \operatorname{Ext}^n_{\mathcal{O}_M}(\mathcal{O}_Z, \Omega^n_M)$$

be the dualizing sheaf and recall the canonical identification (cf. (1.7))

$$\omega_{Z,p} \cong \operatorname{Hom}_{\mathbf{C}}(\mathcal{O}_{Z,p}, \mathbf{C}) .$$

Then we define

$$\operatorname{Res}\colon \mathcal{O}_M(K \otimes \det E) \longrightarrow \omega_Z$$

by

$$\langle \operatorname{Res} \psi_p, \varphi_p \rangle = \operatorname{Res}\left(\frac{\psi_p \varphi_p}{s}\right)$$

where $\psi_p \in \mathcal{O}_{M,p}(K \otimes \det E)$ and $\varphi_p \in \mathcal{O}_{Z,p}$. According to the local duality theorem and the standard Koszul complex, the sheaf sequence

$$\begin{gathered} 0 \longrightarrow \Omega^n_M \xrightarrow{s} \Omega^n_M(E) \xrightarrow{\wedge s} \Omega^n_M(\wedge^2 E) \longrightarrow \\ \cdots \xrightarrow{\wedge s} \Omega^n_M(\det E) \xrightarrow{\operatorname{Res}} \omega_Z \longrightarrow 0 \end{gathered} \tag{1.11}$$

is exact. It is then a formal result that the exact complex of sheaves (1.11) induces both maps in

$$H^0(\mathcal{O}_M(K \otimes \det E)) \xrightarrow{\operatorname{Res}} H^0(\omega_Z) \xrightarrow{\delta} H^n(\Omega^n_M) , \tag{1.12}$$

and that the composite is zero. In the present circumstances we may canonically make the identifications

$$\begin{cases} H^0(\omega_Z) = \operatorname{Ext}^n(M; \mathcal{O}_Z, \Omega^n_M) \\ H^n(\Omega^n_M) = \operatorname{Ext}^n(M; \mathcal{O}_M, \Omega^n_M) \end{cases}$$

and then (1.12) becomes

$$H^0(\mathcal{O}_M(K \otimes \det E)) \xrightarrow{\operatorname{Res}} \operatorname{Ext}^n(M; \mathcal{O}_Z, \Omega^n_M) \xrightarrow{\delta} \operatorname{Ext}^n(M; \mathcal{O}_M, \Omega^n_M) .$$

Because duality is functorial this sequence is the dual of

$$H^n(\mathcal{O}_M(\det E^*)) \xleftarrow{\operatorname{Res}^*} H^0(\mathcal{O}_Z) \xleftarrow{\delta^*} H^0(\mathcal{O}_M) .$$

Finally, since according to (1.7) the pairing between

$$\operatorname{Ext}^n(M; \mathcal{O}_Z, \Omega^n_M) \quad \text{and} \quad H^0(\mathcal{O}_Z)$$

is given by residues,

$$\begin{aligned} \sum\nolimits_{p \in Z} \operatorname{Res}_p\left(\frac{\psi}{s}\right) &= \langle \operatorname{Res} \psi, \delta^* 1 \rangle \\ &= 0 , \end{aligned}$$

thus proving the residue theorem.

Turning now to the converse, we see that the right hand end of (1.11) is

$$0 \longrightarrow \mathcal{I}_Z(K \otimes \det E) \longrightarrow \mathcal{O}_M(K \otimes \det E) \longrightarrow \omega_Z \longrightarrow 0 . \tag{1.13}$$

Under the vanishing assumption (1.10)

$$H^1(\mathcal{I}_Z(K \otimes \det E)) \cong H^n(\Omega^n_M) \cong \mathbf{C} . \tag{1.14}$$

Combining (1.14) with the exact cohomology sequence of (1.13) gives

$$H^0(\mathcal{O}_M(K \otimes \det E)) \xrightarrow{\operatorname{Res}} H^0(\omega_Z) \longrightarrow \mathbf{C} \longrightarrow 0 ,$$

which implies the converse. Q.E.D.

c) *Cayley-Bacharach property and multisecant varieties.*

We want to interpret the residue theorem in geometric language. To begin with, we assume given on our variety $M$ a zero-dimensional subvariety

$$Z = p_1 + \cdots + p_d$$

consisting for the moment of distinct points. Let $L \to M$ be a holomorphic line bundle with complete linear system $|L|$ of effective divisors $D$ with $[D] \cong L$.

*Definition. We shall say that $Z$ has the Cayley-Bacharach property relative to $|L|$ if any divisor $D \in |L|$ passing through all but one point of $Z$ necessarily contains $Z$. Additionally, $Z$ satisfies the strong Cayley-Bacharach property if any $Z' = Z - p_\nu$ fails to satisfy the Cayley-Bacharach property.*

The Cayley-Bacharach property implies that the points of $Z$ fail to impose independent conditions on $|L|$, i.e.,

$$\dim |L| - \deg Z < \dim |\mathcal{I}_Z(L)| \tag{1.15}$$

where $|\mathcal{I}_Z(L)|$ is the complete linear system associated to the subsheaf $\mathcal{I}_Z(L)$ of $\mathcal{O}(L)$. The condition (1.15) makes sense for any 0-dimensional scheme $Z$, where by definition

$$\deg Z = \sum_{p \in Z} \dim_{\mathbf{C}}(\mathcal{O}_{Z,p}) ,$$

and we may use it to define the Cayley-Bacharach property in this case. The strong Cayley-Bacharach property is equivalent to (1.15) together with

$$\dim |L| - \deg Z' = \dim |\mathcal{I}_{Z'}(L)|$$

for any $\mathcal{I}_{Z'} \supset \mathcal{I}_Z$ with $\mathcal{I}_{Z'} \neq \mathcal{I}_Z$.

From the exact cohomology sequence of

$$0 \longrightarrow \mathcal{I}_Z(L) \longrightarrow \mathcal{O}_M(L) \longrightarrow \mathcal{O}_Z(L) \longrightarrow 0$$

and $\dim H^0(\mathcal{O}_Z(L)) = \deg Z$, we note that the Cayley-Bacharach property implies

$$H^1(\mathcal{I}_Z(L)) \neq 0 . \tag{1.16}$$

When also $H^1(\mathcal{O}_M(L)) = 0$, (1.15) is equivalent to (1.16), and in this case the strong Cayley-Bacharach property is equivalent to (1.16) together with

$$H^1(\mathcal{I}_{Z'}(L)) = 0 \tag{1.17}$$

for ideal sheaves $\mathcal{I}_{Z'}$ properly contained in $\mathcal{I}_Z$.

Somewhat more geometrically, we consider the rational mapping

$$\iota_L : M \longrightarrow \mathbf{P}^r$$

given by the complete linear system $|L|$. Specifically, if $s_0, \cdots, s_r \in H_0(\mathcal{O}_M(L))$ constitute a basis then

$$\iota_L(p) = [s_0(p), \cdots, s_r(p)]\,, \qquad p \in M\,.$$

For a set of points $Z = p_1 + \cdots + p_d$ we denote by $\{Z\}_L$ the linear span of their images $\iota_L(p_\nu)$. Here we assume that none of the $p_\nu$ is a base point. For a 0-dimensional scheme $Z$ we define $\{Z\}_L$ to be the intersection of the hyperplane sections $D \in |L|$ which contain $Z$ in the ideal-theoretic sense; i.e., $D \in |\mathcal{I}_Z(L)|$. In all cases, if we define the *superabundance*

$$\omega(Z, L) = \dim|\mathcal{I}_Z(L)| - (\dim|L| - \deg Z) \tag{1.18}$$

to be the numerical measure of the failure of $Z$ to impose independent conditions on $|L|$, then by elementary linear algebra

$$\dim\{Z\}_L = \deg Z - 1 - \omega(Z, L)\,. \tag{1.19}$$

Summarizing, assuming no $p_\nu \in Z$ is a base point of $|L|$, we have the following:

*The Cayley-Bacharach condition implies*

$$\dim\{Z\}_L \leqq \deg Z - 2\,. \tag{1.20}$$

*The strong Cayley-Bacharach property is equivalent to* (1.20) *together with*

$$\dim\{Z'\}_L = \deg Z' - 1 \tag{1.21}$$

*for all $\mathcal{I}_{Z'}$ properly contained in $\mathcal{I}_Z$.*

Linear subspaces $\{Z\}_L$ satisfying (1.20) may be thought of as *multisecant planes* for the map $\iota_L: M \to \mathbf{P}^r$. For example, the simplest of these are trichords. If set-theoretically $Z$ consists of a single point $p$ and if (1.20) is satisfied, then $p$ is some sort of *inflection point* on the image variety. For spaces $\{Z\}_L$ satisfying (1.20) and (1.21) we have

$$\begin{cases} \dim\{Z\}_L = \deg Z - 2\,, & \text{and} \\ \dim\{Z'\}_L = \deg Z' - 1 & \end{cases}$$

for any proper subvariety $Z' \subset Z$, and we may think of $\{Z\}_L$ as a *simple multisecant plane.*

The residue Theorem (1.9) implies the

(1.22) PROPOSITION. *The zero locus $Z$ of a holomorphic section $s \in H^0(\mathcal{O}_M(E))$ satisfies the Cayley-Bacharach property relative to the complete linear system $|K \otimes \det E|$. If the vanishing theorem* (1.10) *holds, then the strong Cayley-Bacharach property is satisfied.*

d) *Points and line bundles on curves.*

We want to examine the residue theorem for curves and surfaces. For this a preliminary general lemma will be useful. Let $M$ be a compact, complex manifold and $L_i \to M$ $(i = 1, 2)$ a pair of holomorphic line bundles with $L = L_1 \otimes L_2$. Let $Z$ be a 0-dimensional scheme on $M$, assumed disjoint from the base loci of $|L_1|$ and $|L|$, and assume that

$$(1.23) \qquad \begin{cases} \text{The Cayley-Bacharach property holds for } Z \text{ relative to } |L| \,; \\ \dim [\text{image of } H^0(\mathcal{O}_M(L_2)) \longrightarrow H^0(\mathcal{O}_Z(L_2))] \geqq \sigma \,. \end{cases}$$

LEMMA. *With the preceding notations and assumption* (1.23),

$$(1.24) \qquad \dim \{Z\}_{L_1} \leqq \deg Z - 1 - \sigma \,.$$

*Proof.* By trivializing the line bundle $L$ around each $p_\nu \in Z$, the points $\iota_L(p) \in \mathbf{P}^r$ for $p$ near to $p_\nu$ may be considered as vectors $f(p) \in \mathbf{C}^{r+1}$. We initially assume the $p_\nu$ are distinct and, by suitable trivializations of $L$, may put the first assumption in (1.23) in the form

$$(1.25) \qquad \textstyle\sum_{p_\nu \in Z} f(p_\nu) = 0 \,.$$

Similarly, by trivializations of the $L_i$ the mapping $\iota_{L_1}$ is given for $p$ close to $p_\nu$ by $g(p) \in \mathbf{C}^{r_1+1}$ and the sections of $L_2$ by functions $s(p)$. By (1.25), and with suitable trivializations,

$$(1.26) \qquad \textstyle\sum_{p_\nu \in Z} s(p_\nu) g(p_\nu) = 0 \,, \qquad s \in H^0(\mathcal{O}_M(L_2)) \,.$$

By the second assumption in (1.23) there are $\geqq \sigma$ independent relations (1.26), and this implies the lemma in case the $p_\nu$ are distinct.

The general case is proved by the same linear algebra argument, but dualized so that the Cayley-Bacharach conditions are expressed in terms of the number of sections of a line bundle which contain $Z$ in the ideal-theoretic sense rather than the simple linear dependence condition (1.25).

Q.E.D.

Suppose now that $L \to C$ is a line bundle over an algebraic curve of genus $g$ and that we are given a section $s \in H^0(\mathcal{O}_C(L))$ whose divisor

$$(s) = Z = p_1 + \cdots + p_d \qquad (d \geqq 1) \,,$$

where some of the $p_\nu$ may be repeated. Since $h^1(\mathcal{O}(K + L)) = 0$, by the Riemann-Roch theorem for curves

$$(1.27) \qquad \dim |K + L| = g + d - 2 \,,$$

while by the residue theorem in the form (1.22),

$$(1.28) \qquad \dim \{Z\}_{K+L} = d - 2 \,.$$

Comparing (1.27) and (1.28) we find:

i) In case $g = 0$ there is no implication. This is consistent with the following observation:

*The rational normal curves are the only algebraic curves having no multisecant planes.*

ii) In case $g = 1$ the canonical bundle is trivial, so that $K + L = L$ and (1.28) is a tautology.

iii) In case $g \geqq 2$ there is a non-trivial conclusion. This follows from (1.24) by taking $Z$ to be the divisor of $s \in H^0(\mathcal{O}(L))$, $L_1 = K$, and $L_2 = L$. Then we easily find $\sigma \geqq r = \dim |L|$, so that we obtain

$$\dim \{Z\}_K \leqq d - 1 - r \;. \tag{1.29}$$

The image of $\iota_K \colon C \to \mathbf{P}^{g-1}$ is called the *canonical curve*, denoted by $\mathbf{C}$ and with $\iota_K(p) = \boldsymbol{p}$. In case $C$ is non-hyperelliptic the mapping $\iota_K$ is birational; in the hyperelliptic case $\mathbf{C}$ is a rational normal curve and $\iota_K$ is two-to-one. The statement (1.29) has the geometric interpretation:

(1.30) *If the divisor $p_1 + \cdots + p_d$ varies in a linear system of dimension $r$, then the canonical images $\boldsymbol{p}_\nu \in \mathbf{C}$ span at most a $\mathbf{P}^{d-1-r}$ in $\mathbf{P}^{g-1}$.*

Now this result is usually derived from

ABEL'S THEOREM: *If $p_\nu = p_\nu(\lambda)$ varies with $r$ linear degrees of freedom with parameters $\lambda = [1, \lambda_1, \cdots, \lambda_r] \in \mathbf{P}^r$, then for any $\omega \in H^0(\Omega^1_C)$ the abelian sum*

$$\sum\nolimits_\nu \int_{p_\nu(\lambda_0)}^{p_\nu(\lambda)} \omega = \text{constant} \tag{1.31}$$

*modulo periods.*

The partial derivatives $\partial/\partial\lambda_\alpha$ of (1.31) give

$$\sum\nolimits_\nu s_\alpha(p_\nu)\omega(p_\nu) = 0 \qquad (\alpha = 1, \cdots, r) \;,$$

which implies (1.29). In fact, for curves our residue theorem is essentially equivalent to Abel's theorem (cf. [G] for further discussion).

Before leaving curves, at least for the time being, it is perhaps instructive to show how to prove equality in (1.29), at least in case

$$r \geqq d - g + 1 \tag{1.32}$$

so that (1.29) is non-vacuous. Suppose that $\mathbf{Z} = \boldsymbol{p}_1 + \cdots + \boldsymbol{p}_d$ spans a $\mathbf{P}^{d-1-\rho}$ $(\rho \geqq r)$ in $\mathbf{P}^{g-1}$. We will show that $Z$ then varies with $\infty^\rho$ linear degrees of freedom; i.e., $h^0(\mathcal{O}(L)) = \rho + 1$. Now $\mathbf{Z}$ lies on

$$\begin{aligned}(g-1) - (d - 1 - \rho) &= g - d + \rho \\ &= \rho' + 1\end{aligned}$$

independent hyperplanes where $\rho' \geqq 0$ by (1.32). For any such hyperplane $H$ containing the fixed linear space $\{\mathbf{Z}\} = \{Z\}_K$,

$$H \cdot \mathbf{C} = \mathbf{Z} + \mathbf{Z}'$$

where the residual divisor

$$\mathbf{Z}' = \boldsymbol{p}_1' + \cdots + \boldsymbol{p}_{d'}' , \qquad d + d' = 2g - 2 ,$$

varies with $\infty^{\rho'}$ linear degrees of freedom. By (1.29) applied to $Z'$,

$$\begin{aligned} \dim \{\mathbf{Z}'\} &\leqq d' - 1 - \rho' \\ &= g - 2 - \rho . \end{aligned}$$

So there are

$$(g - 1) - (g - 2 - \rho) = \rho + 1$$

independent hyperplanes containing $\mathbf{Z}'$, and by letting these vary we see that $Z$ moves with $\infty^{\rho}$ linear degrees of freedom as desired.

What we have done here is to prove the converse of Abel's theorem (1.31) for special divisors, i.e., those whose canonical images fail to span $\mathbf{P}^{g-1}$. By proof analysis one sees that this argument does *not* use the Riemann-Roch theorem, and in fact it gives a proof of this theorem using (1.29) and elementary linear algebra.

e) *Points and rank-two vector bundles on surfaces.*

It is well-known that on a complex manifold $M$ an effective divisor $D$ defines a holomorphic line bundle $[D] \to M$ and, up to constants, a holomorphic section $s \in H^0(\mathcal{O}_M([D]))$ with divisor $(s) = D$. We shall prove a partial analogue in higher codimension by giving a converse to the residue theorem in the case of points on a surface. This discussion is a refinement of Section iv), Chapter V of [G-H] where references to the original papers are given. A similar result was known to Barth and van de Ven some time ago and has been used by W. Barth [B] in connection with his work on stable bundles on $\mathbf{P}^2$.

Let $S$ be a smooth algebraic surface with structure sheaf $\mathcal{O} = \mathcal{O}_S$, $L \to S$ a holomorphic line bundle, and $Z$ a 0-dimensional scheme defined by a sheaf of regular ideals $\mathcal{I}_Z \subset \mathcal{O}$. We ask for a pair $(E, s)$ consisting of a rank-2 holomorphic vector bundle $E \to S$ and section $s \in H^0(\mathcal{O}(E))$ whose scheme-theoretic divisor is $Z$ and where $\det E = L$.

(1.33) PROPOSITION. *The pair $(E, s)$ with $(s) = Z$ and* $\det E = L$ *exists* $\Leftrightarrow Z$ *satisfies the Cayley-Bacharach property relative to $|K + L|$. The pair is (essentially) unique if $Z$ satisfies the strong Cayley-Bacharach property.*

*Proof.* We shall give the argument in case $Z$ consists of distinct points, and shall explain the meaning of essential uniqueness during the course of the proof.

If $(E, s)$ exists then with $L = \det E = \wedge^2 E$, the Koszul resolution

$$0 \longrightarrow \mathcal{O}(L^*) \xrightarrow{s} \mathcal{O}(E^*) \xrightarrow{s} \mathcal{I}_Z \longrightarrow 0 \tag{1.34}$$

gives a class

$$e \in \operatorname{Ext}^1(S; \mathcal{I}_Z, \mathcal{O}(L^*)) . \tag{1.35}$$

For each $p \in Z$ we denote by $\mathcal{O}_p$ the local ring $\mathcal{O}_{S,p}$ and by $\mathcal{I}_p$ the image of $\mathcal{I}_Z$ in $\mathcal{O}_p$; then $\mathcal{I}_p$ is the maximal ideal and by use of the local Koszul resolution,

$$\operatorname{Ext}^1_{\mathcal{O}_p}(\mathcal{I}_p, \mathcal{O}_p(L^*)) \cong L_p^* \otimes K_p^* . \tag{1.36}$$

Under the identification (1.36) the localizations

$$e_p \in L_p^* \otimes K_p^*$$

of $e$ are all non-zero. We recall here that in the local-to-global spectral sequence

$$E_2^{k,l} = H^k(S, \operatorname{Ext}^l_{\mathcal{O}}(\mathcal{I}_Z, \mathcal{O}(L^*))) \Longrightarrow \operatorname{Ext}^{k+l}(S; \mathcal{I}_Z, \mathcal{O}(L^*)) \tag{1.37}$$

the class $e$ maps naturally to

$$\bigoplus_{p \in Z} e_p \in H^0(S, \operatorname{Ext}^1_{\mathcal{O}}(\mathcal{I}_Z, \mathcal{O}(L^*))) \cong \bigoplus_{p \in Z} \operatorname{Ext}^1_{\mathcal{O}_p}(\mathcal{I}_p, \mathcal{O}_p(L^*)) .$$

Conversely, a class $e$ as in (1.35) defines a short exact sequence (1.34) with a coherent sheaf $\mathcal{F}$ in the middle. The condition $e_p \neq 0$ on the localizations is equivalent to $\mathcal{F}$ being locally free, in which case $\mathcal{F} \cong \mathcal{O}(E^*)$ for a rank-two bundle $E \to S$. The map $\mathcal{F} \to \mathcal{I}_Z \to 0$ gives $s \in H^0(\mathcal{O}(E))$ with $(s) = Z$, and since codim $Z = 2$ implies

$$\operatorname{Pic}(S - Z) \cong \operatorname{Pic}(S)$$

we see that $L = \det E$. Summarizing:

*Finding $(E, s)$ is equivalent to having $e \in \operatorname{Ext}^1(S; \mathcal{I}_Z, \mathcal{O}(L^*))$ with non-zero localizations $e_p \in L_p^* \otimes K_p^*$.*

Setting $\mathcal{L}^* = \mathcal{O}(L^*)$ and applying the exact sequence of $\operatorname{Ext}^*(S; \cdot, \mathcal{L}^*)$ to

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

give

(1.39)

$$\begin{array}{c} 0 \to \operatorname{Ext}^1(S; \mathcal{O}, \mathcal{L}) \to \operatorname{Ext}^1(S; \mathcal{I}_Z, \mathcal{L}^*) \xrightarrow{\rho} \operatorname{Ext}^2(S; \mathcal{O}_Z, \mathcal{L}^*) \to \operatorname{Ext}^2(S; \mathcal{O}, \mathcal{L}^*) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\!\wr \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \bigoplus_{p \in Z} L_p^* \otimes K_p^* \end{array}$$

where the local-to-global spectral sequence together with

$$\operatorname{Ext}^k_{\mathcal{O}_p}(\mathcal{O}_{Z,p}, \mathcal{L}_p^*) = \begin{cases} 0 & k \neq 2 \\ \cong L_p^* \otimes K_p^* & k = 2 \end{cases}$$

has been used. The dual of (1.39) is

$$(1.40)\qquad 0 \longleftarrow H^1(\mathcal{O}(L\otimes K)) \longleftarrow \mathrm{Ext}^1(S; \mathcal{I}_Z, \mathcal{L}^*)^* \\ \longleftarrow \bigoplus_{p\in Z} L_p \otimes K_p \xleftarrow{\rho^*} H^0(\mathcal{O}(K\otimes L)) .$$

If we are given $(E, s)$ defining (1.34) and (1.35), then (1.39) and (1.40) imply, for all $\psi \in H^0(\mathcal{O}(K\otimes L))$,

$$\begin{aligned} 0 &= \langle \rho(e), \psi\rangle \\ &= \langle e, \rho^*\psi\rangle \\ &= \sum\nolimits_{p\in Z} \mathrm{Res}_p\left(\frac{\psi}{s}\right) , \end{aligned}$$

giving another proof of the global residue theorem in this case. Conversely, if we are given

$$(1.41)\qquad 0 \neq e_p \in L_p^* \otimes K_p^* \quad \text{satisfying} \quad \sum\nolimits_{p\in Z} \langle e_p, \rho^*\psi\rangle = 0$$

for all $\psi \in H^0(\mathcal{O}(K\otimes L))$, then we obtain $e \in \mathrm{Ext}^1(S; \mathcal{O}, \mathcal{L}^*)$ with localizations $e_p$. But (1.41) is equivalent to the Cayley-Bacharach property, since a set of points $x_p \in \mathbf{P}^r$ has this property relative to the hyperplanes $\Leftrightarrow$ there are $X_p \in \mathbf{C}^{r+1}$ projecting onto $x_p$ and satisfying

$$(1.42)\qquad \sum\nolimits_p X_p = 0$$

in $\mathbf{C}^{r+1}$. This gives the existence half of our proposition.

The strong Cayley-Bacharach property is equivalent to a unique, up to scalar multiples, relation (1.42). In this case there is, again up to scalars, a unique $e \in \mathrm{Ext}^1(S; \mathcal{I}_Z, \mathcal{L}^*)$ with the desired properties. It follows that for any two pairs $(E_i, s_i)$ satisfying the conditions of the proposition there is a bundle isomorphism $E_1 \cong E_2$ taking $s_1$ to $s_2$. Q.E.D.

## II. Residues and the osculating sequence

a) *The osculating sequence.*

Let $M$ be a compact, complex manifold and

$$\iota_L: M \longrightarrow \mathbf{P}^r$$

the meromorphic mapping given by the complete linear system $|L|$ associated to a holomorphic line bundle $L \to M$. If $s_0, \cdots, s_r \in H^0(\mathcal{O}(L))$ are a basis for the sections, then locally we may choose a non-vanishing section $e$ of $L \to M$ and write

$$s_\lambda(z) = f_\lambda(z)\cdot e .$$

By slight abuse of notation we set

$$(2.1)\qquad f(z) = (f_0(z), \cdots, f_r(z)) \in \mathbf{C}^{r+1}$$

with the understanding that, if $p \in M$ has coordinate $z$, then $\boldsymbol{p} = \iota_L(p)$ is the image of $f(z)$ under the projection $\mathbf{C}^{r+1} \to \mathbf{P}^r$. Such local liftings of $\iota_L$ exist away from the base locus $B$ of the linear system $|L|$, and unless otherwise mentioned we shall restrict our attention to $M - B$.

*Definition. The span in* $\mathbf{P}^r$ *of the vectors*

$$\frac{\partial^\lambda f(z)}{\partial z_1^{\lambda_1} \cdots \partial z_n^{\lambda_n}}, \qquad \lambda = \lambda_1 + \cdots + \lambda_n \leqq \mu,$$

*is the* $\mu^{\text{th}}$ *osculating space* $\delta^\mu \boldsymbol{p}$ *to the image* $\mathbf{M} = \iota_L(M)$ *at the point* $\boldsymbol{p} = \iota_L(p)$.

Of course $\delta^\mu \boldsymbol{p}$ is well-defined independently of the local lifting $f$. The osculating spaces give an intrinsically defined increasing sequence of linear spaces attached to each point $\boldsymbol{p} \in \mathbf{M}$. We note the

(2.2) LEMMA. *For* $\mu$ *sufficiently large*

$$\delta^\mu \boldsymbol{p} = \mathbf{P}^r .$$

*Proof.* If not there is a linear function $\xi$ on $\mathbf{C}^{r+1}$ such that all osculating spaces lie in the hyperplane defined by $\xi$. Equivalently, the analytic function $\langle \xi, f(z) \rangle$ vanishes to infinite order at $z = 0$, and is hence identically zero. This contradicts the non-degeneracy of $\mathbf{M} \subset \mathbf{P}^r$. Q.E.D.

We now come to our basic construction. Let $E \to M$ be a rank-$n$ holomorphic vector bundle and $s \in H^0(\mathcal{O}(E))$ a holomorphic section whose divisor $(s) = Z$ consists of distinct points. We write

$$Z = p_1 + \cdots + p_d \qquad \big(d = c_n(E)\big) ;$$

and remark that what is essential here is not that the $p_i$ should be distinct but that the divisor of $s$ should be zero-dimensional. We consider the rational map

$$\iota_{K \otimes \det E} \colon M \longrightarrow \mathbf{P}^r$$

associated to the complete linear system $|K \otimes \det E|$ and assume that none of the $p_i$ is a base point.

We denote by

$$\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^\mu \mathbf{Z}\} = \{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_d; \delta\boldsymbol{p}_1, \cdots, \delta\boldsymbol{p}_d; \cdots; \delta^\mu\boldsymbol{p}_1, \cdots, \delta^\mu\boldsymbol{p}_d\}_{K \otimes \det E}$$

the linear span of the $\mu^{\text{th}}$ osculating spaces at the corresponding points $\boldsymbol{p}_i \in \mathbf{P}^r$.

*Definition. The sequence of linear spaces*

$$\{\mathbf{Z}\}, \{\mathbf{Z}, \delta\mathbf{Z}\}, \cdots, \{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^\mu\mathbf{Z}\}$$

*is called the* osculating sequence *associated to the section* $s \in H^0(\mathcal{O}(E))$.

The residue theorem

$$\sum_{i=1}^{d} \operatorname{Res}_{p_i}\left(\frac{\psi}{s}\right) = 0\,, \qquad \psi \in H^0(\mathcal{O}(K \otimes \det E))\,, \tag{2.3}$$

implies that

$$\dim\{\mathbf{Z}\} \leqq d - 2\,.$$

Our basic idea is to obtain bounds on the entire osculating sequence by successively differentiating the residue theorem, a program we shall begin in the next section and refine in the succeeding ones.

We also note that the converse to the residue theorem implies the strong Cayley-Bacharach property

$$\begin{cases} \dim\{Z\} = d - 2 & \\ \dim\{Z'\} = d - 2 & \text{if} \quad Z' = Z\text{-point} \end{cases}$$

in case the vanishing theorem (1.10) is satisfied.

b) *The fundamental relation.*

We retain the notation and assumptions from the previous section. For a $\mu^{\text{th}}$ order differential operator

$$\wedge = \sum_{\mu_1+\cdots+\mu_n=\mu} \wedge_{\mu_1\cdots\mu_n} \frac{\partial^\mu}{\partial z_1^{\mu_1}\cdots\partial z_n^{\mu_n}}$$

considered as a vector in $\operatorname{Sym}^\mu(T_p(M))$ we use the convention (2.1) with $L = K \otimes \det E$ and set

$$\delta_\wedge \boldsymbol{p} = \wedge \cdot f(z)$$

projected into $\mathbf{P}^r$. This is well-defined modulo $\delta^{\mu-1}\boldsymbol{p}$, and the $\mu^{\text{th}}$ osculating space is clearly given by

$$\delta^\mu \boldsymbol{p} = \{\delta_\wedge \boldsymbol{p} \colon \lambda \leqq \mu \quad \text{and} \quad \wedge \in \operatorname{Sym}^\lambda(T_p(M))\}\,.$$

At each $p_i \in Z$ the differential $ds \in E_{p_i} \otimes T^*_{p_i}(M)$ induces an isomorphism

$$E_{p_i} \cong T_{p_i}(M) \tag{2.4}$$

since the zeroes of $s$ are assumed non-degenerate. Using (2.4) each $\wedge \in \operatorname{Sym}^\mu(H^0(\mathcal{O}(E)))$ induces

$$\wedge_i \in \operatorname{Sym}^\mu(T_{p_i}(M))$$

and we set

$$\delta_\wedge \boldsymbol{p}_i = \delta_{\wedge_i} \boldsymbol{p}_i\,.$$

THEOREM. *With the preceding notations*

$$\sum_{i=1}^{d} \delta_\wedge \boldsymbol{p}_i \equiv 0 \quad \text{modulo} \quad \{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^{\mu-1}\mathbf{Z}\} \tag{2.5}$$

*for all* $\wedge \in \operatorname{Sym}^\mu(H^0(\mathcal{O}(E)))$.

*Proof.* The residue theorem (2.3) gives

$$\sum_{i=1}^{d} \boldsymbol{p}_i = 0$$

for a suitable choice of local lifting (2.1), and this is the $\mu = 0$ case of the theorem. Let now

$$s = s_0, s_1, \cdots, s_N$$

be a basis for $H^0(\mathcal{O}(E))$ and set

$$s(t) = s_0 + \sum_{\rho=1}^{N} t_\rho s_\rho \,.$$

We shall apply $\partial/\partial t_\rho$ at $t = 0$ to

$$\sum \operatorname{Res}\left(\frac{\psi}{s(t)}\right) = 0 \,.$$

By (1.3) this is

$$\sum_{i=1}^{d} \operatorname{Res}_{p_i}\left(\frac{\partial}{\partial t_\rho}\left(\frac{\psi}{s(t)}\right)_{t=0}\right) = 0 \,, \tag{2.6}$$

and it remains to examine the individual terms in this sum.

Let $z_1, \cdots, z_n$ be local holomorphic coordinates centered at $p_i \in M$ and $e_1, \cdots, e_n$ a local holomorphic frame for $E \to M$ such that

$$s_0(z) = \sum_{\alpha=1}^{n} z_\alpha e_\alpha \,.$$

For $1 \leqq \rho \leqq N$ set

$$s_\rho(z) = \sum_{\alpha=1}^{n} A_{\rho\alpha}(z) e_\alpha \,.$$

The $i^{\text{th}}$ term in (2.5) is

$$\operatorname{Res}_{\{0\}}\left(\frac{\partial}{\partial t_\rho}\left(\frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{s_1(t) \cdots s_n(t)}\right)_{t=0}\right) \tag{2.7}$$

where

$$\begin{cases} s_\alpha(t) = z_\alpha + \sum_{\rho=1}^{N} t_\rho A_{\rho\alpha}(z) \\ \psi(z) = g(z) dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n \,. \end{cases}$$

Using Cauchy's formula (1.2),

$$\begin{aligned} (2.7) &= \sum_{\alpha=1}^{n} \operatorname{Res}_{\{0\}}\left(\frac{A_{\rho\alpha}(z) g(z) dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_\alpha^2 \cdots z_n}\right) \\ &= \sum_{\alpha=1}^{n} A_{\rho\alpha}(0) \frac{\partial g}{\partial z_\alpha}(0) + (\cdots) g(0) \,. \end{aligned}$$

Using our identifications this equation is

$$\sum_{i=1}^{d} \partial_{s_\rho} \boldsymbol{p}_i \equiv 0 \quad \text{modulo} \quad \{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_d\} \,,$$

which is the $\mu = 1$ case of the theorem.

The general argument proceeds in the same manner. Q.E.D.

We would like to explain the geometric meaning of (2.5) when $E \to M$ is generated by its global sections. If $\mathbf{G}(N-n, N)$ denotes the Grassmannian of $\mathbf{P}^{N-n}$'s in $\mathbf{P}^N$, then the assignment $p \to \{s \in H^0(\mathcal{O}(E)) \colon s(p) = 0\}$ gives a mapping $\iota_E: M \to \mathbf{G}(N-n, N)$ such that $E \to M$ is induced from the universal quotient bundle. The zero divisors of sections $s \in H^0(\mathcal{O}(E))$ are the inverse images under $\iota_E$ of Schubert cycles on the Grassmannian. In the diagram

$$\begin{array}{ccc} M & \xrightarrow{\iota_E} & \mathbf{G}(N-n, N) \\ \big\downarrow{\scriptstyle \iota_{K\otimes \det E}} & & \\ \mathbf{P}^r & & \end{array},$$

for such a generic Schubert cycle $\sigma$,

$$\iota_E^{-1}(\sigma) = p_1 + \cdots + p_d$$

is a zero cycle $Z$ on $M$, and we may consider the linear span $\{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_d\} = \{\mathbf{Z}\}$ in $\mathbf{P}^r$ of the points $\boldsymbol{p}_i = \iota_{K\otimes\det E}(p_i)$. The residue theorem (2.3) says that $\{\mathbf{Z}\}$ is a multisecant plane. By varying $\sigma$, we successively obtain the relations (2.5) bounding the growth of the sequence of osculating spaces to $\mathbf{M} = \iota_{K\otimes\det E}(M)$ at corresponding points $p_i$. The number of independent equations (2.5) is

$$\dim\big(\mathrm{Sym}^\mu H^0(\mathcal{O}(E))\big) - r(\mu, \sigma)$$

where $r(\mu, \sigma)$ is the dimension of the space of $\wedge \in \mathrm{Sym}^\mu\big(H^0(\mathcal{O}(E))\big)$ satisfying $\wedge(\boldsymbol{p}_i) = 0$. In other words, and this is the fundamental geometric point to our paper, *the growth of the osculating sequence at corresponding points* $\boldsymbol{p}_i$ *is governed by the relations in the graded ring* $\bigoplus_\mu \mathrm{Sym}^\mu\big(H^0(\mathcal{O}(E))\big)$. Applying this principle necessitates estimating $r(\mu, \sigma)$, and in the next section we shall do this in the simplest case.

c) *The fundamental bound for complete intersections.*

We want to pursue the consequences of the fundamental relation (2.5) in the simplest case when $E = L \underbrace{\oplus \cdots \oplus}_{} L$ ($n$-times) is a sum of line bundles. To focus on the essential aspects we assume that the complete linear system $|L|$ has a base locus in codimension $\geqq 2$ and induces a birational mapping

$$\iota_L: M \longrightarrow \mathbf{P}^N$$

onto a non-degenerate algebraic variety $M_L \subset \mathbf{P}^N$ of codimension $k = N - n$ and degree $d$. For a generic $\mathbf{P}^k$ in $\mathbf{P}^N$ the intersection

$$\begin{aligned} \mathbf{P}^k \cdot M_L &= p_1 + \cdots + p_d \\ &= Z \end{aligned} \tag{2.8}$$

is the divisor of a generic section $s \in H^0(\mathcal{O}(E)) \cong \bigoplus H^0(\mathcal{O}(L))$. We recall that the $p_i$ are $d \geqq k + 1$ points in general position in $\mathbf{P}^k$ (cf. Lemma (2.13) below).

Now choose a basis

$$\{s_1, \cdots, s_n; s_{n+1}, \cdots, s_{N+1}\} = \{s_\alpha; s_\rho\} \qquad (1 \leqq \alpha \leqq n,\ n+1 \leqq \rho \leqq N+1)$$

for $H^0(\mathcal{O}(L))$ and consider the zero-cycle (2.8) defined by $s_1 = \cdots = s_n = 0$. For $0 \leqq m \leqq n$ the residue theorem (1.9) implies the

PROPOSITION. *For* $\psi \in H^0(\mathcal{O}(K + mL))$

$$\sum_{i=1}^{d} \mathrm{Res}_{p_i}\left(\frac{s_{\rho_1} \cdots s_{\rho_l}\psi}{s_1^{\mu_1+1} \cdots s_n^{\mu_n+1}}\right) = 0 , \qquad l = \mu + n - m . \tag{2.9}$$

*Proof.* In (1.9) we take $E = L^{\mu_1+1} \oplus \cdots \oplus L^{\mu_n+1}$, $s = s_1^{\mu_1+1} \oplus \cdots \oplus s_n^{\mu_n+1}$ in the denominator, and $s_{\rho_1} \cdots s_{\rho_l}\psi$ $(n + 1 \leqq \rho_\nu \leqq N + 1)$ in the numerator. Q.E.D.

We may apply (2.9) to give another proof of (2.5) in the present situation. To explain the connection with the previous argument, remark that (2.9) may also be obtained by setting

$$s_\alpha(t_\alpha) = s_\alpha + \sum_\rho t_{\alpha\rho} s_\rho$$

and successively differentiating

$$\sum \mathrm{Res}\left(\frac{s_{\rho_1} \cdots s_{\rho_{n-m}}\psi}{s_1(t_1) \cdots s_n(t_n)}\right) = 0 .$$

Now consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{\iota_L} & \mathbf{P}^N \\ \Big\downarrow{\scriptstyle \iota_{K+mL}} & & \\ \mathbf{P}^r & & \end{array} \tag{2.10}$$

and assume that none of the $p_i$ is a base point of the linear system $|K+mL|$. We set

$$\begin{cases} \iota_{K+mL}(M) = \mathbf{M} \\ \iota_{K+mL}(p) = \boldsymbol{p} \\ \delta^\mu \boldsymbol{p} = \mu^{\text{th}} \text{ osculating space to } \mathbf{M} \text{ at } \boldsymbol{p} . \end{cases}$$

Suppose that $z_1, \cdots, z_n$ are local holomorphic coordinates centered at $p_i \in M$ and $e$ is a local holomorphic frame for $L \to M$ such that near $p_i$

$$s_\alpha = z_\alpha \cdot e , \qquad 1 \leqq \alpha \leqq n .$$

If

$$\psi = g(z) dz_1 \wedge \cdots \wedge dz_n \otimes e^m ,$$

then by (1.2)

$$(\mu_1)! \cdots (\mu_n)! \mathrm{Res}_{p_i}\left(\frac{s_{\rho_1} \cdots s_{\rho_l}\psi}{s_1^{\mu_1+1} \cdots s_n^{\mu_n+1}}\right) = s_{\rho_1}(0) \cdots s_{\rho_l}(0) \frac{\partial^\mu g(0)}{\partial z_1^{\mu_1} \cdots \partial z_n^{\mu_n}} + (\cdots) \tag{2.11}$$

where $(\cdots)$ are linear combinations of lower derivates of $g(z)$ evaluated at $z = 0$. We may use (2.11) to evaluate the individual terms in (2.9), the result being that if we set

$$\delta^{\mu}\boldsymbol{p}_i = \frac{\partial^{\mu} g(0)}{\partial z_1^{\mu_1} \cdots \partial z^{\mu_n}}\ , \qquad \mu = (\mu_1, \cdots, \mu_n),\ \mu = \mu_1 + \cdots + \mu_n$$

we obtain the relation

$$\sum_{i=1}^{d} s_{\rho_1}(p_i) \cdots s_{\rho_l}(p_i)\delta^{\mu}\boldsymbol{p}_i \equiv 0\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^{\mu-1}\mathbf{Z}\}\ . \tag{2.12}$$

This equation is, of course, a special case of (2.5).

To apply (2.12) we need an estimate on the number of independent coefficients which appear. This is provided by the following lemma which may be found in Section iii), Chapter II of [G-H] or in Section I of [C-G, 1]. For completeness we shall give the proof here.

(2.13) GENERAL POSITION LEMMA. *The generic section*

$$\mathbf{P}^k \cdot M_L = p_1 + \cdots + p_d$$

*consists of* $d \geqq k + 1$ *points in general position in* $\mathbf{P}^k$*. Any subset of* $\min(d, kl+1)$ *imposes independent conditions on the complete linear system* $|\mathcal{O}_{\mathbf{P}^k}(l)|$ *of hypersurfaces of degree* $l$ *in* $\mathbf{P}^k$*.*

To rephrase the lemma in cohomological terms, we set $Z = p_1 + \cdots + p_d$ and consider the exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{I}_Z(l)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^k}(l)) \longrightarrow H^0(\mathcal{O}_Z(l)) \longrightarrow H^1(\mathcal{I}_Z(l)) \longrightarrow 0 \tag{2.14}$$

of the exact sheaf sequence

$$0 \longrightarrow \mathcal{I}_Z(l) \longrightarrow \mathcal{O}_{\mathbf{P}^k}(l) \longrightarrow \mathcal{O}_Z(l) \longrightarrow 0\ .$$

The lemma implies that

$$\dim[\text{image of } (H^0(\mathcal{O}_{\mathbf{P}^k}(l)) \longrightarrow H^0(\mathcal{O}_Z(l)))] \geqq \min(d, kl + 1)\ , \tag{2.15}$$

which by (2.14) is the same as

$$h^1(\mathcal{I}_Z(l)) \leqq d - \min(d, kl + 1). \tag{2.16}$$

*Proof.* By non-degeneracy we may choose $(N+1)$ points $p_1, \cdots, p_{N+1} \in M$ whose images $p'_\nu = \iota_L(p_\nu)$ span $\mathbf{P}^N$. Then $p'_1, \cdots, p'_{k+1}$ $(k = N - n)$ span a $\mathbf{P}^k$ such that the intersection

$$\mathbf{P}^k \cdot M_L = p'_1 + \cdots + p'_{k+1} + \cdots + p'_d\ .$$

This implies that a generic $\mathbf{P}^k$ meets $M_L$ in $d \geqq k + 1$ points of which *some* subset of $(k + 1)$ is linearly independent. We want to show that *any* subset of $(k + 1)$ is independent.

If not, then we may fix a $\mathbf{P}_0^k$ and assume that for some neighborhood $U$ of $\mathbf{P}_0^k$ in the Grassmannian with coordinate $\xi$, the intersections

$$\mathbf{P}^k(\xi) \cdot M_L = p'_1(\xi) + \cdots + p'_{k+1}(\xi) + \cdots + p'_d(\xi)$$

where $p_\nu(\xi)$ varies in an open neighborhood $V_\nu$ on $M$, the $V_\nu$ being pairwise disjoint, and where some $(k+1)$ of the $p'_\nu(\xi)$ are linearly dependent. By perhaps shrinking neighborhoods we may assume that $p'_1(\xi), \cdots, p'_{k+1}(\xi)$ are linearly dependent.

Now, and this is the main observation, for any points $p_1 \in V_1, \cdots, p_{k+1} \in V_{k+1}$ the images perhaps will lie on several $\mathbf{P}^k$'s, at least one of which can be chosen to be a $\mathbf{P}^k(\xi)$ for some $\xi \in U$. Then $p'_1 = p'_1(\xi), \cdots, p'_{k+1} = p'_{k+1}(\xi)$ and consequently

$$p'_1 \wedge \cdots \wedge p'_{k+1} \equiv 0 .$$

By analytic continuation of this equation we conclude that any $(k+1)$ points of $M_L$ are linearly dependent, which contradicts our initial observation.

Now suppose that $d \geqq kl + 1$ and group the points as follows:

$$\underbrace{p_1, \cdots, p_k}_{G_1};\ \underbrace{p_{k+1}, \cdots, p_{2k}}_{G_2};\ \cdots;\ \underbrace{p_{k(l-1)+1}, \cdots, p_{kl}}_{G_l};\ p_{kl+1}, \cdots .$$

Each group $G_\alpha$ spans a hyperplane $H_\alpha$ in $\mathbf{P}_k$ not meeting any of the other points, and then

$$D = H_1 + \cdots + H_l$$

gives a hypersurface $D \in |\mathcal{O}_{\mathbf{P}^k}(l)|$ passing through $p_1, \cdots, p_{kl}$ but not $p_{kl+1}$. This proves (2.15) with any subset $Z'$ consisting of $kl+1$ points from $Z$ replacing $Z$ in the statement there.

The case $d < kl + 1$ is similar only easier. Q.E.D.

It follows from (2.12) and (2.15) that for each $\mu = (\mu_1, \cdots, \mu_n)$ the points $\delta^\mu_{\mathbf{P}_i}$ are subject to at least $\min(d, (N-n)(\mu+n-m)+1)$ independent relations modulo $\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^{\mu-1}\mathbf{Z}\}$ $(\mu = \mu_1 + \cdots + \mu_n)$. We define

(2.17)

$$\begin{aligned}\kappa(n, N, d, \mu, m) &= \binom{n+\mu-1}{n-1}[d - \min(d, (N-n)(\mu+n-m)+1)] \\ &= h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(\mu))[d - \min(d, (N-n)(\mu+n-m)+1)] .\end{aligned}$$

By (2.12)

(2.18) $\dim\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^\mu\mathbf{Z}\}$

$$\begin{aligned}&\leqq h^0(\mathcal{O}_{\mathbf{P}^{n-1}}(\mu)) \cdot h^1(\mathcal{I}_Z(\mu+n-m)) + \dim\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^{\mu-1}\mathbf{Z}\} \\ &\leqq \kappa(n, N, d, \mu, m) + \dim\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^{\mu-1}\mathbf{Z}\} .\end{aligned}$$

We note that

$$\kappa(n, N, d, \mu, m) = 0 \quad \text{for} \quad \mu \geqq \left[\frac{d-1}{N-n}\right] + (m-n) ,$$

so that for $\mu$ in this range the osculating sequence stabilizes. If we define

$$\kappa(n, N, d, m) = \textstyle\sum_{\mu \geq 0} \kappa(n, N, d, \mu, m) \tag{2.19}$$

then this is a finite sum and by Lemma (2.2) we obtain the

FUNDAMENTAL BOUND. *For any complete linear system* $|L|$ *for which* $\iota_L\colon M \to \mathbf{P}^N$ *is birational onto its image, and for* $0 \leqq m \leqq n$

$$\dim |K + mL| \leqq \kappa(n, N, d, m)\ . \tag{2.20}$$

*If* $Z = \mathbf{P}^{N-n} \cdot M_L$ *is a generic section of* $\iota_L(M)$, *then equality holds in* (2.20) *if, and only if,*

$$h^1\big(\mathcal{I}_Z(\mu + n - m)\big) = d - \min\big(d, (N - n)(\mu + n - m) + 1\big) \tag{2.21}$$

*for* $\mu \leqq [d - 1/N - n] + (m - n)$.

There are relations between these estimates for various $m$. For example, from (2.17) we see that

$$\kappa(n, N, d, \mu, m) = 0 \Longleftrightarrow \kappa(n, N, d, \mu + 1, m + 1) = 0\ . \tag{2.22}$$

Situations (2.10) for which equality holds in (2.20) will be said to be *extremal.* A priori it would appear that this depends on the particular $m$ ($0 \leqq m \leqq n$), but because of Lemma (1.24) and (2.22) it is at least plausible that the essential case is when $n = m$. This will be proved to be the case in Section III below. For the moment we note the

(2.23) COROLLARY. $H^0\big(\mathcal{O}(K + mL)\big) = 0$ *if* $d \leqq (n - m)(N - n) + 1$. *In particular the Hodge number*

$$h^{n,0}(M) = 0 \quad \textit{if} \quad d \leqq n(N - n) + 1\ .$$

d) *The osculating sequence for curves.*

We will examine the bound (2.20) for a curve $C$ of genus $g$. According to the discussion at the end of the preceding section we should consider the osculating sequence arising from the diagram

$$\begin{array}{c} C \xrightarrow{\iota_L} \mathbf{P}^n \\ \Big\downarrow {\scriptstyle \iota_{K+L}} \\ \mathbf{P}^r\ , \end{array} \tag{2.24}$$

which is (2.10) in the case $n = 1$, $N = n$, and $m = 1$. By the Riemann-Roch theorem (cf. the end of Part I a) for a proof of R-R)

$$\begin{cases} n = d - g + i \\ r = d + g - 2 \end{cases} \tag{2.25}$$

where $d$ is the degree of the image curve $\iota_L(C)$ and $i = h^0(K - L)$ is the index of speciality of the line bundle $L \to C$. Referring to (2.7), we have

$$\kappa(1, n, d, \mu, 1) = d - \min\big(d, (n - 1)\mu + 1\big)\ ,$$

and by (2.18) the bounds in the osculating sequence are

$$\begin{cases} \dim\{\mathbf{Z}\} \leqq d - 2\ , \\ \dim\{\mathbf{Z}, \delta\mathbf{Z}\} \leqq (d-1) + (d-n) - 1, \text{ since } d \geqq n\ , \\ \dim\{\mathbf{Z}, \delta\mathbf{Z}, \delta^2\mathbf{Z}\} \leqq (d-1) + (d-n) + (d-2n+1) - 1\ , \text{ if } d \geqq 2n-1. \end{cases}$$

In Part I d) we have already discussed the first estimate with the conclusion that it was interesting essentially in case the genus

$$g \geqq 2$$

which we now assume. The second estimate is non-trivial only if

$$2d - n - 2 < r\ ,$$

which by (2.25) is equivalent to $i > 0$. Consequently, *the higher order behavior in the osculating sequence is interesting only for special divisors.*

Suppose now that $i > 0$. We will use the third step in (2.26) to give a proof of *Clifford's theorem*:

$$2\dim|L| \leqq \deg L\ , \tag{2.27}$$

in case $|L|$ is birational. If (2.27) fails, i.e.,

$$d \leqq 2n - 1\ ,$$

then the osculating space sequence (2.26) stabilizes with $\{\mathbf{Z}, \delta\mathbf{Z}, \delta^2\mathbf{Z}\}$, and by (2.20)

$$r = \dim\{\mathbf{Z}, \delta\mathbf{Z}, \delta^2\mathbf{Z}\} = 2d - n - 1\ . \tag{2.28}$$

Comparing (2.28) and (2.25) gives $i \leqq 0$, which contradicts our assumption.

We remark that the general case of Clifford's theorem together with an analysis of the case when equality holds in (2.27) may be proved by a similar method.

Returning to the general osculating sequence, we have

$$\{\mathbf{Z}, \delta\mathbf{Z}, \cdots, \delta^{\mu}\mathbf{Z}\} = \mathbf{P}^r \quad \text{for} \quad \mu \geqq \left[\frac{d-1}{N-n}\right].$$

By (2.20)

$$\begin{aligned} r &\leqq \kappa(1, n, d, 1) - 1 \\ &= \kappa(1, n, d, 0) + (d-2) \end{aligned} \tag{2.29}$$

where

$$\kappa(1, n, d, 0) = g(d, n) = (d-n) + (d-2n+1) + (d-3n+2) + \cdots \tag{2.30}$$

will be called *Castelnuovo's number*. Either by applying (2.20) when $n = 1$, $N = n$, $m = 0$ or by comparing (2.25) and (2.29) we deduce

CASTELNUOVO'S BOUND. *For a non-degenerate curve of degree $d$ in $\mathbf{P}^n$ the genus satisfies*

$$g \leqq g(d, n) \tag{2.31}$$

*where $g(d, n)$ is Castelnuovo's number* (2.30).

The curves for which equality holds in (2.31) have been discussed in detail in [G-H] and [C-G, 1]. We shall examine them once again from our present viewpoint in Section III below.

e) *The osculating sequence for surfaces.*

We will to some extent parallel the discussion for curves in the preceding section, although the complete enumeration of cases is of course more complicated. Let $S$ be an algebraic surface and consider the diagram (2.10) in the case $n = 2$, $N = n$, $m = 2$,

$$\begin{array}{ccc} S & \xrightarrow{\iota_L} & \mathbf{P}^n \\ \big\downarrow{\scriptstyle \iota_{K+2L}} & & \\ \mathbf{P}^r . & & \end{array}$$

We use the standard notations from surface theory,

$$\begin{cases} p_g = h^0(K) = \text{geometric genus} \\ q = h^0(\Omega^1) = h^1(\mathcal{O}) = \text{irregularity} \\ p_a = p_g - q = \text{arithmetic genus}\ , \end{cases}$$

and for any line bundle $L \to S$

$$\begin{cases} s = h^1(L) = h^1(K - L) = \text{superabundance} \\ i = h^2(L) = h^0(K - L) = \text{index of speciality} \\ \pi(L) = \dfrac{1}{2}(L \cdot L + K \cdot L) + 1 \\ \qquad = g(C) \end{cases}$$

in case $|L|$ contains a smooth curve $C$. The Riemann-Roch theorem for surfaces is

$$\begin{aligned} \dim |L| &= \frac{L \cdot L}{2} - \frac{K \cdot L}{2} + p_a + s - i \\ &= d - \pi(L) + p_a + s - i + 1\ , \\ &\text{where } d = L \cdot L = \text{degree of } L \to S\ . \end{aligned} \tag{2.33}$$

Applying (2.33) to $L$, $K + 2L$, and $K + L$ using $h^i(K + mL) = 0$ for $i > 0$, $m > 0$ gives

$$(2.34)\qquad \begin{cases} n = \dim|L| = d - \pi(L) + p_a + s - i + 1 \\ r = \dim|K + 2L| = d + 2\pi(L) + p_a - 2 \\ t = \dim|K + L| = \pi(L) + p_a - 1\,. \end{cases}$$

By (2.17)

$$\begin{aligned} \kappa(2, n, d, \mu, 2) &= (\mu + 1)[d - \min(d, \mu(n-2) + 1)] \\ &= 0 \quad \text{for} \quad \mu \geqq \left[\frac{d-1}{n-2}\right], \end{aligned}$$

and by this together with (2.18) the bounds in the osculating sequence are

$$(2.35)\qquad \begin{cases} \dim\{\mathbf{Z}\} \leqq d - 2\,, \\ \dim\{\mathbf{Z}, \delta\mathbf{Z}\} \leqq (d-1) + 2(d - n + 1) - 1\,, \quad \text{since} \quad d \geqq n - 1\,, \\ \dim\{\mathbf{Z}, \delta\mathbf{Z}, \delta^2\mathbf{Z}\} \leqq (d-1) + 2(d-n+1) + 3(d - 2n + 3) - 1\,, \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad d \geqq 2n - 3\,. \end{cases}$$

In fact the first two are equalities as follows from the proof of the converse to the residue theorem and (2.13). We now examine cases.

i) The first inequality is non-vacuous unless

$$(2.36)\qquad r \leqq d - 2\,,$$

and we shall prove

(2.37) *The inequality* (2.36) *holds* $\Leftrightarrow$ *$S$ is a surface of minimal degree $n - 1$, in which case it is an equality.*

Before giving the argument we note the analogy with curves (cf. case i) below (1.28)):

*The surfaces of minimal degree, but not* e.g., *the image of $\mathbf{P}^2$ under the complete linear system $|\mathcal{O}_{\mathbf{P}^2}(k)|$ for $k \geqq 3$, are analogues of the rational normal curves.*

*Proof of* (2.37). If (2.36) holds, then by (2.34)

$$\begin{cases} 2\pi(L) + p_a \leqq 0 \\ -1 \leqq \pi(1) + p_a - 1\,, \qquad\qquad \text{since} \quad t \geqq -1\,, \end{cases}$$

which together imply $\pi(L) = 0$. By Noether's lemma (cf. Chapter IV of [G-H]), $S$ must be rational, from which it follows that $q = p_g = i = 0$. By (2.34) again

$$d + 1 \geqq n = d + s + 1$$

so that $s = 0$ and $d = n - 1$ as desired.

Conversely, the surfaces of minimal degree are known (loc. cit.) and for these $r = d - 2$. Q.E.D.

The geometric meaning of (2.37), which will be discussed in a general context in Part III a) is this:

*Given an ample line bundle $L \to S$ over a surface and of degree $d = L \cdot L$, a general set of $d$ points on $S$ is a complete intersection of two curves from $|L| \Leftrightarrow \dim |L| = d + 1$.*

ii) Suppose now that we are in the intermediate range

$$n \leqq d \leqq 2n - 3 \;, \tag{2.38}$$

so that the initial inequality in (2.35) is non-vacuous. By (2.2) and (2.35)

$$r \leqq d - 2n \;.$$

On the other hand, by (2.23)

$$p_g = i = 0 \;,$$

which together with (2.34) implies

$$\begin{aligned} d + 2\pi(L) - q - 2 &\leqq 3d - 2n \\ &= d + 2\pi(L) + 2q - 2s - 2 \\ \Longrightarrow \qquad s &= \frac{3q}{2} \;. \end{aligned}$$

This gives a version of Clifford's theorem for surfaces:

*If $|L|$ is birational and $q = 0$ but $h^1(L) \neq 0$, then*

$$2 \dim |L| \leqq d + 2 \;.$$

This may be proved directly by going to the usual Clifford's theorem on the curves in $|L|$.

So far as we can determine the surfaces in the degree range (2.38) do not have the the same simple description as for curves because of the various possibilities for the irregularity. So we shall examine these surfaces for the extreme values of $q$.

If $q = 0$ then $S$ is rational. In fact, by (2.34)

$$\begin{aligned} d &\leqq 2d - 2\pi(L) - 1 \\ &= d - K \cdot L - 3 \\ \Longrightarrow \qquad & (-K) \cdot L > 0 \end{aligned}$$

so that $|mK|$ is empty for $m > 0$. By the Castelnuovo-Enriques criterion (loc. cit.), $S$ must be rational.

At the other extreme, from (2.34) and $t \geqq -1$ we have for all degrees

$$-p_a \leqq \pi(L) \;.$$

On the other hand, $p_a = -q$ for $d \leqq 2n - 3$ so that we obtain

$$q \leqq \pi(L) ,$$

which also follows from the Lefschetz hyperplane theorem. By Castelnuovo's bound on the genus of a hyperplane section applied to $C = S \cdot \mathbf{P}^{n-1}$ we have

$$q \leqq \pi(L) \leqq d - n + 1 . \tag{2.39}$$

We will prove that:

*Equality holds in* (2.39) $\Leftrightarrow S$ *is a geometrically ruled surface over a curve of maximal genus in* $\mathbf{P}^{n-1}$.

What this means is that $S$ is a $\mathbf{P}^1$-bundle over a base curve $B$, and under the given projective embedding the fibres of $S \to B$ go into straight lines. A generic hyperplane section of $S$ then projects birationally onto $B$, and is hence a Castelnuovo curve.

*Proof.* If equality holds in (2.39), then when $C = \mathbf{P}^{n-1} \cdot S$ the line bundle $L_C \to C$ is non-special by Clifford's theorem for curves. Then by the Riemann-Roch theorem,

$$\begin{aligned} \dim |L_C| &= d - \pi(L) \\ &= n - 1 , \end{aligned}$$

so that $C \subset \mathbf{P}^{n-1}$ is normally embedded. The exact cohomology sequence of $0 \to \mathcal{O} \to \mathcal{O}(L) \to \mathcal{O}(L_C) \to 0$ gives, since $H^0(\mathcal{O}(L)) \to H^0(\mathcal{O}(L_C))$ is surjective,

$$q = h^1(L) . \tag{2.40}$$

Now consider the Albanese map

$$\alpha : S \longrightarrow A , \qquad A = \mathrm{Alb}(S) .$$

Since $p_g = 0$ the image is a curve $B$ of genus $q = \pi(L)$. By the Riemann-Hurwitz formula the restriction

$$\alpha : C \longrightarrow B$$

is birational, and hence the fibres of $\alpha$ are straight lines. This together with (2.40) proves our claim.

iii) In case $d \geqq 2n - 2$ we obtain from (2.21) the bound

$$\begin{aligned} p_g(S) &\leqq \textstyle\sum_{\mu \geqq 0} (\mu + 1)[d - \min(d, (n-2)(\mu + 2) + 1)] \\ &= (d - 2n + 3) + 2(d - 3n + 5) + \cdots \\ &= p_g(d, n) \end{aligned}$$

where the last equality is a definition. In particular

$$p_g(S) \leqq 1 \qquad \text{if} \quad d = 2n - 2 .$$

We shall show that if equality holds then $S$ is a $K3$ surface. Since $h^0(K) > 0$,

$t \geqq n$ in (2.34) and using $p_a = 1 - q$ we have

$$n \leqq \pi(L) - q$$
$$\leqq n - q ,$$

the second step being Castelnuovo's bound on the genus of a non-degenerate curve in $\mathbf{P}^{n-1}$. It follows that $q = 0$ and $\pi(L) = n$. Then

$$n = \frac{1}{2}(L \cdot L + K \cdot L) + 1$$
$$= n + \frac{K \cdot L}{2}$$

implies that $K \cdot L = 0$, and consequently $K$ is trivial. The properties

$$q = 0 , \qquad K = 0$$

serve to characterize $K3$ surfaces, and prove our claim.

## III. Inverting the residue theorem

a) *Complete intersections on surfaces.*

Perhaps the most näive question which one can ask about a configuration of points $Z$ on a smooth variety $M$ is the converse of the Bezout theorem: *When is $Z$ a complete intersection?* More precisely, we should be given line bundles $L_\alpha \to M$ ($\alpha = 1, \cdots, n = \dim M$) together with a 0-dimensional scheme $Z$ whose ideal sheaf is locally a complete intersection, and we ask for divisors $D_\alpha \in |L_\alpha|$ such that

$$Z = D_1 \cdots D_n$$

in the sense of schemes; i.e.,

$$\mathcal{I}_Z = \mathcal{I}_{D_1} + \cdots + \mathcal{I}_{D_n} .$$

In this section we shall for simplicity concentrate on the case when $Z$ consists of distinct points, but the results carry over to the general situation.

The topological constraint is given by the usual Bezout theorem

$$\deg Z = \#(D_1, \cdots, D_n) , \tag{3.1}$$

the right hand side being the intersection number of the divisors $D_\alpha$. The formula (3.1) holds under the usual proviso that the $D_\alpha$ intersect in a 0-dimensional variety. An additional constraint is furnished by the residue theorem in the form (1.22):

(3.2) *$Z$ satisfies the Cayley-Bacharach property relative to the complete linear system $|K + L_1 + \cdots + L_n|$.*

In this section we will show that:

(3.3) *In case $M$ is a surface $S$ and $L_1 = L_2 = L$ is a line bundle whose complete linear system $|L|$ gives a birational mapping of $S$, the conditions* (3.1) *and* (3.2) *are "in general" sufficient that $Z$ be a complete intersection.*

The intuitive meaning of "in general" is

$$\text{(3.4)} \qquad \begin{cases} \dim \{Z\}_{K+2L} = d - 2\ , & \textit{but} \\ Z \quad \textit{is otherwise generic}\ . \end{cases}$$

In particular, a proper subset $Z' \subset Z$ should impose independent conditions on $|K + 2L|$; i.e., the strong Cayley-Bacharach property should hold.

The precise meaning can be put in several forms, and for us it will be this:

(3.5) *Given $Z$ satisfying* (3.1) *and* (3.2), *either $Z$ lies on a pencil from $|L|$ or else $|\mathcal{I}_Z(K + 2L)|$ is a positive-dimensional linear system having a fixed curve. In the former case, either $Z$ is a complete intersection or else the pencil in $|L|$ has a fixed curve.*

What is going on here is that in the $d$-fold symmetric product $S^{(d)}$ the condition (3.2) defines a closed subvariety which may not be irreducible but which in any case has a Zariski open $U$ in common with the set of complete intersections. What is desirable is to characterize the points in the closure of $U$ which remain complete intersections; (3.5) gives one sufficient condition that this be so.

Before giving the proof of (3.3) we wish to record several remarks. The first is that, by (2.37) above, the condition (3.2) is vacuous $\Leftrightarrow$ the complete linear system $|L|$ embeds $S$ as a surface of minimal degree in $\mathbf{P}^n$.

The second is that (3.3) should not be thought of as giving an explicit answer to the problem of characterizing complete intersections. Rather, it should be interpreted as a *reciprocity* or *duality* statement which is "in general" equivalent to a cycle being a complete intersection. The flavor of (3.3) is perhaps best illustrated by the special case of the Pascal theorem: In order that 6 points in $\mathbf{P}^2$ lie on a conic, it is necessary and sufficient that, for the hexagon they define, the pairs of opposite sides should meet in 3 collinear points.

The final remark is that in case

$$|K + L| \subset |L|$$

we deduce from (3.4) that the conditions (3.1) and (3.2) imply that $Z$ lies on a pencil in $|L|$, and is therefore a complete intersection provided this pencil has no fixed curve. In particular, this applies to $\mathbf{P}^2$ and covers the converse to the Cayley-Bacharach theorem given in Section ii), Chapter V of [G-H].

We now give the proof of (3.3). Suppose

$$\iota_L : S \longrightarrow \mathbf{P}^n$$

is the birational embedding given by $|L|$. Proving that $Z$ lies on a pencil in $L$ is equivalent to showing that

$$\dim \{Z\}_L \leqq n - 2 \ . \tag{3.6}$$

Setting $d = \deg Z$, our assumption (3.2) implies

$$\dim \{Z\}_{K+2L} \leqq d - 2 \ , \tag{3.7}$$

so that if (3.6) were false then by Lemma (1.24)

$$\dim \{Z\}_{K+L} \leqq d - n - 1 \ . \tag{3.8}$$

Now for $W$ a generic complete intersection

$$\begin{cases} \dim \{W\}_{K+2L} = d - 2 \\ \dim \{W\}_{K+L} \ = d - n \ , \end{cases}$$

so that we have, so to speak, proved the converse of (3.1) and (3.2) for 0-cycles which satisfy (3.7) but are otherwise general.

To complete the proof of (3.5) we assume (3.7) but not (3.6), so that (3.8) must hold and therefore $|\mathcal{I}_Z(K + L)|$ is non-empty. Suppose there is an irreducible curve $C \in |\mathcal{I}_Z(K + L)|$; we will derive from this a contradiction.

If we let $\omega_C = \mathrm{Ext}^2_{\mathcal{O}_S}(\mathcal{O}_C, \Omega^2_S)$ be the dualizing sheaf, the exact cohomology sequence of

$$0 \longrightarrow \mathcal{O}_S(K + L) \longrightarrow \mathcal{O}_S(K + 2L) \longrightarrow \omega_C(L) \longrightarrow 0$$

together with $h^1(\mathcal{O}_S(K + L)) = 0$ gives a surjection

$$H^0(\mathcal{O}_S(K + 2L)) \longrightarrow H^0(\omega_C(L)) \longrightarrow 0 \ . \tag{3.9}$$

If $s \in H^0(\omega_C(L))$ vanishes at all but one point of $Z$, then by (3.9) we may find $\boldsymbol{s} \in H^0(\mathcal{O}_S(K + 2L))$ vanishing at all but one point of $Z$. By the assumption (3.2) $\boldsymbol{s}$, and hence $s$, must vanish at the remaining point of $Z$. This implies that $Z \in |L_C|$ (cf. (1.28); the argument is given there only in case $C$ is smooth, but the general case may be deduced in the same way as Abel's theorem for singular curves in [$G$]).

Using $\mathcal{O}_C([Z]) \cong \mathcal{O}_C(L)$, we consider the exact cohomology sequence of

$$0 \longrightarrow \mathcal{I}_Z(K + L) \longrightarrow \mathcal{I}_Z(K + 2L) \longrightarrow \omega_C \longrightarrow 0 \ ,$$

which because of $h^2(\mathcal{I}_Z(K + L)) = 0$ is

$$\begin{aligned} 0 \to H^0(\mathcal{I}_Z(K + L)) \to H^0(\mathcal{I}_Z(K + 2L)) \to H^0(\omega_C) \\ \to H^1(\mathcal{I}_Z(K + L)) \to H^1(\mathcal{I}_Z(K + 2L)) \to H^1(\omega_C) \to 0 \ . \end{aligned}$$

The dimensions $h_Z^i(\mathcal{O}(K+2L))$ and $h^i(\omega_C)$ are the same as if $Z$ were a complete intersection. Hence the same must be true of the $h^i(\mathcal{I}_Z(K+L))$, and this contradicts (3.8). Q.E.D.

It is perhaps of interest to give a variational form of (3.3) based on Proposition (1.33). Precisely what we shall prove is:

(3.10) *Let $S$ be a regular surface and $Z_t$ $(|t|<\varepsilon)$ an analytic family of 0-cycles satisfying*

$$\begin{cases} Z_0 \text{ is complete intersection, and} \\ \dim\{Z_t\}_{K+2L} \leqq d-2 \,. \end{cases}$$

*Then the $Z_t$ are also complete intersections.*

*Proof.* From deformation theory [K-S] we recall that the Zariski tangent space to the moduli of a vector bundle $E \to S$ is $H^1(S, \mathcal{H}om(E, E))$. In particular, if this group is zero then any local deformation is trivial.

In case $E = L \oplus L$, $\mathcal{H}om(E, E) \cong \oplus \mathcal{O}$ and by the regularity assumption on the surface $S$ the bundle $L \oplus L$ is rigid.

Now by Proposition (1.33) there are rank-two bundles $E_t \to S$ together with $s_t \in H^0(\mathcal{O}(E_t))$ such that $\det E_t = 2L$ and $(s_t) = Z_t$. Since $\dim\{Z_t\} = d-2$ for $t$ sufficiently small, we even have the uniqueness of $(E_t, s_t)$ so that we may assume holomorphic dependence on $t$. Because of $E_0 \cong L \oplus L$ and rigidity it follows that all $E_t \cong L \oplus L$, and this implies that $Z_t$ is a complete intersection. Q.E.D.

This argument suggests an alternate proof of (3.3). Suppose that $Z$ is a 0-cycle with

$$\dim\{Z\}_{K+2L} = d-2 \,.$$

Then by (1.33) there is a unique $(E, s)$ with $(s) = Z$ and $\det E = 2L$. We note that the discriminant $\Delta = c_1^2(E) - 4c_2(E) = 0$. Under the additional assumptions

$$\begin{cases} \dim\{Z'\}_{K+2L} = d-2 \qquad \text{for} \quad Z' = Z\text{-point} \\ h^1(\mathcal{O}_S) = 0 \,, \end{cases}$$

we think it quite likely that $E \cong L \oplus L$ by a semi-stable bundle argument. If so, this would provide a more satisfactory converse to the Bezout theorem.

To conclude this section we wish to make one final observation. Recall that the canonical curve $C \subset \mathbf{P}^{g-1}$ has the remarkable property that if $p_1, \cdots, p_d \in C$ is a set of points spanning a $\mathbf{P}^{d-1-n}$, then $Z = p_1 + \cdots + p_d$ varies in an $\infty^n$-dimensional family of such multisecant planes. The following corollaries of (3.3) are analogues for 0-cycles on surfaces:

*If $Z$ is a 0-cycle of degree $d$ with $\{Z\}_{L+2L} = \mathbf{P}^{d-2}$ but otherwise $Z$ is generic, then $Z$ varies in an $\infty^{2n-2}$-dimensional family of such multisecant planes where $\{Z\}_{K+L} = \mathbf{P}^{d-n}$; and if $Z$ is as above, then the osculating sequence $\{Z\}_{K+2L}, \{Z, \delta Z\}_{K+2L}, \cdots$ is subject to* (2.35).

b) *Structure of extremal varieties,* i)

*Definition. A mapping*

$$\iota_L: M \longrightarrow \mathbf{P}^N$$

*of a compact, complex manifold into $\mathbf{P}^N$ given by a complete linear system $L$ for some line bundle $L \to M$ is said to be* extremal *if $\iota_L$ is birational onto its image $M_L$ and if equality holds in the fundamental bound* (2.21),

$$\dim|K + mL| \leqq \kappa(n, N, d, m)$$

*for some $m$ with $0 \leqq m \leqq n$.*

Letting $k = N - n$ and $d \geqq k + 1$ be the codimension and degree of $M_L$, we have discussed extremal curves and surfaces satisfying

$$d \leqq 2k + 2\ ,$$

and probably this discussion generalizes to higher dimensions. In this and the following section we shall determine the structure of extremal varieties when $d \geqq 2k + 3$.

We begin with a lemma.

(3.11) LEMMA. *Let $Z = p_1 + \cdots + p_d$ be a set of $d > kl + 1$ points in general position in $\mathbf{P}^k$ where $k \geqq 2$, $l \geqq 2$. If $Z$ imposes the minimum number $kl + 1$ of conditions on $|\mathcal{O}_{\mathbf{P}^k}(l)|$, then this is already true for $l = 2$.*

*Proof.* Recall from Lemma (2.13) that *any* $kl + 1$ points from $Z$ impose independent conditions. Under the assumptions of Lemma (3.11) then, any hypersurface $D \in |\mathcal{O}_{\mathbf{P}^k}(l)|$ passing through $kl + 1$ points from $Z$ must contain $Z$ entirely.

Let $Q \in |\mathcal{O}_{\mathbf{P}^k}(2)|$ be any quadric passing through $2k + 1$ points of $Z$, say $p_1, \cdots, p_{2k+1}$. We must show that $Q$ contains $Z$, i.e., that $Q$ passes through any remaining point, say $p_{kl+2}$. Group the points in $Z$ as follows:

$$\underbrace{p_1, \cdots, p_{2k+1}}; \underbrace{p_{2k+2}, \cdots, p_{3k+1}}_{G_3}; \cdots; \underbrace{P_{(l-1)k+2}, \cdots, P_{lk+1}}_{G_l}; P_{lk+2}, \cdots .$$

By general position the points in $G_\alpha$ span a hyperplane $H_\alpha$ containing no other points from $Z$, and then

$$D = Q + H_3 + \cdots + H_l \in |\mathcal{O}_{\mathbf{P}^k}(l)|$$

passes through $p_1, \cdots, p_{kl+1}$ and hence through $p_{kl+2}$. So $P_{kl+2} \in Q$ as desired.

Q.E.D.

From this lemma and (2.21) we deduce the

(3.12) LEMMA. *Suppose that* $d \geqq 2k+2$ *and* $M_L \subset \mathbf{P}^N$ *is extremal. Then equality holds in* (2.21) *when either*

$$\begin{cases} m = n & \text{and} \quad \mu = 2\,, \quad \text{or} \\ m = n-1 & \text{and} \quad \mu = 1\,. \end{cases}$$

To see how equality can hold in (2.21) we shall prove the

(3.13) LEMMA. *A set* $Z$ *of* $d \geqq kl+1$ *points lying on a rational normal curve* $C$ *in* $\mathbf{P}^k$ *is always in general position and imposes exactly* $kl+1$ *independent conditions on* $|\mathcal{O}_{\mathbf{P}^k}(l)|$.

*Proof.* If $t$ is a linear coordinate on $\mathbf{P}^1$ we recall that, for any integer $m$, $H^0(\mathcal{O}_{\mathbf{P}^1}(m))$ is the vector space of polynomials in $t$ of degree $\leqq m$; it has dimension $h^0(\mathcal{O}_{\mathbf{P}^1}(m)) = m+1$. A rational normal curve $C$ in $\mathbf{P}^k$ is the image of $\mathbf{P}^1$ under the complete linear system $|\mathcal{O}_{\mathbf{P}^1}(k)|$. In a suitable homogeneous coordinate system it is given parametrically by

$$t \longrightarrow [1, t, \cdots, t^k]\,.$$

Using a van der Monde determinant we see that distinct points on $\mathbf{P}^1$ go to points in general position in $\mathbf{P}^k$.

Now $\mathcal{O}_{\mathbf{P}^k}(l)|_C \cong \mathcal{O}_{\mathbf{P}^1}(kl)$ and the restriction mapping

$$H^0(\mathcal{O}_{\mathbf{P}^k}(l)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^1}(kl)) \longrightarrow 0$$

is visibly surjective. Since $h^0(\mathcal{O}_{\mathbf{P}^1}(kl)) = kl+1$ it follows that, for $\mathcal{I}_C$ the ideal sheaf of $C$,

$$h^0(\mathcal{I}_C(l)) = h^0(\mathcal{O}_{\mathbf{P}^k}(l)) - (kl+1)\,.$$

For $Z$ a set of distinct points on $C$,

$$h^0(\mathcal{I}_Z(l)) \geqq h^0(\mathcal{I}_C(l)) \geqq h^0(\mathcal{O}_{\mathbf{P}^k}(l)) - (kl+1)$$

while the reverse inequality is (2.23). Q.E.D.

The usual route to the determination of extremal varieties, which is the one used originally by Castelnuovo and presented in [C-G, 1], is the following converse to Lemma (3.13):

(3.14) LEMMA. *Given* $d \geqq 2k+3$ *points* $Z$ *in general position in* $\mathbf{P}^k$ *imposing exactly* $2k+1$ *independent conditions on the linear system* $|\mathcal{O}_{\mathbf{P}^k}(2)|$ *of quadrics, it follows that* $Z$ *lies on a unique rational normal curve.*

Combining (3.13) and (3.14) we deduce the converse to Lemma (3.12); namely, if equality holds in (2.21) when either

$$(3.15)\qquad \begin{cases} m = n & \text{and} \quad \mu = 2\,, \quad \text{or} \\ m = n - 1 & \text{and} \quad \mu = 1\,, \end{cases}$$

then $M_L$ is extremal and moreover a generic section $\mathbf{P}^k \cdot M_L$ lies on a unique rational normal curve. Since $\dim \mathrm{G}(k, N) = (k + 1)n$ there are $\infty^{(k+1)n}$ such rational normal curves, but in previous work one could show by additional arguments that they only fill up a variety $V_L$ of dimension $n + 1$, which then must be a variety of minimal degree $k = \operatorname{codim} V_L + 1$ in $\mathbf{P}^N$. Still further analysis shows that $M_L$ lies in a special way on $V_L$. For example, when $d \geqq 7$ an extremal algebraic curve of degree $d$ in $\mathbf{P}^3$ lies on a quadric surface where it has type $(m, m)$ or $(m, m + 1)$ according to whether the degree $d = 2m$ or $2m + 1$ is even or odd. The general rule is that $M_L$ should be as close as its degree allows to being an intersection of $V_L$ with a hypersurface in $\mathbf{P}^N$.

Now, although quite elegant, Lemma (3.14) so far requires a rather elaborate synthetic proof. Here we shall proceed differently, and shall basically use infinitesimal methods to replace Lemma 3.14 by a lemma in *tri*-linear algebra. In fact this is consistent with the present paper, which in some sense has as a theme the application of infinitesimal methods (calculus) to problems in algebraic geometry. The manner in which rational normal curves will turn up linearly is via the observation that, in $\mathbf{P}^k$ with homogeneous coordinates $[x_0, \cdots, x_k]$, the rational normal curves are images of straight lines $y_\alpha = A_\alpha t + B_\alpha$ under a Cremona transformation

$$(3.16)\qquad x_\alpha = \frac{1}{y_\alpha}\,.$$

In fact, (3.16) gives the unique rational normal curve through the vertices $[0, \cdots, 1, \cdots, 0]$ of the coordinate simplex in $\mathbf{P}^k$ and passing through the two additional points $[\cdots, 1/A_\alpha, \cdots]$, $[\cdots, 1/B_\alpha, \cdots]$.

In the next section, then, we will complete the determination of extremal varieties by infinitesimal methods.

c) *Structure of extremal varieties*, ii)

In this section we will prove the

(3.17) THEOREM. *Suppose that $\iota_L\colon M \to \mathbf{P}^N$ is an extremal algebraic variety of codimension $k \geqq 2$ and degree $d \geqq 2k + 3$. Then the image $M_L$ is a hypersurface in an $(n + 1)$-dimensional variety $V_L$ of minimal degree* $\operatorname{codim} V_L + 1$.

Before giving the proof we make a couple of remarks. The first is that once we know where to look for extremal varieties, it is not too hard to show first $M_L$ sets in a special way in $V_L$ (basically, as close to a complete intersection as possible), and then to actually construct extremal varieties verifying that the basic estimates (2.21) are sharp. The second is that we need not assume (2.21) for all $\mu$, but only either of the equivalent conditions in (3.15). This follows from the argument below.

To start the proof, our hypothesis suggests that we consider the diagram (2.10) in one of the two cases in (3.15), and because of (2.18) it seems better to take $m = n - 1$ (although either would do). Thus in

$$\begin{array}{ccc} M & \xrightarrow{\iota_L} & \mathbf{P}^N \\ \Big\downarrow{\scriptstyle \iota_{K+(m-1)L}} & & \\ \mathbf{P}^r \, , & & \end{array} \tag{3.18}$$

we denote by $\xi \in \mathrm{G}(k, N)$ a variable point, by $\mathbf{P}^k(\xi) \subset \mathbf{P}^N$ the corresponding linear space, and then the generic intersection

$$\mathbf{P}^k(\xi) \cdot M_L = p_1(\xi) + \cdots + p_d(\xi) \; ; \tag{3.19}$$

while over in $\mathbf{P}^r$ using the customary notations

$$\begin{cases} \boldsymbol{p} = \iota_{K+(n-1)L}(p) \; , \\ \mathbf{M} = \iota_{K+(n-1)L}(M) \; , \end{cases}$$

we have by (3.15) and (2.18)

$$\begin{cases} \dim \{\boldsymbol{p}_i(\xi)\} = d - k - 2 \\ \dim \{\boldsymbol{p}_i(\xi), \, \delta\boldsymbol{p}_i(\xi)\} = (d - k - 1) + n(d - 2k - 1) - 1 \, . \end{cases} \tag{3.20}$$

If we put $\mathbf{P}^{d-k-2}(\xi) = \{\boldsymbol{p}_i(\xi)\}$, then

$$\xi \longrightarrow \mathbf{P}^{d-k-2}(\xi)$$

gives a rational map

$$\pi \colon \mathrm{G}(k, N) \longrightarrow \mathrm{G}(d - k - 2, r) \tag{3.21}$$

analogous to the Poincaré mapping used in [C-G, 1]. The second equation in (3.20) will give information on the differential of $\pi$ using a tri-linear algebra lemma which we shall explain shortly.

By way of motivation, consider in general for a moment the Grassmannian $\mathrm{G}(m - 1, N - 1)$ as the set of $\mathbf{C}^m$'s through the origin in $\mathbf{C}^N$, and for $A \in \mathrm{G}(m - 1, N - 1)$ recall the canonical identification of the tangent space

$$T_A(\mathrm{G}(m - 1, N - 1)) \cong \mathrm{Hom}\,(A, \mathbf{C}^N/A) \, .$$

Suppose that $W$ is an open set in some $\mathbf{C}^l$ and that

$$\psi \colon W \longrightarrow \mathrm{G}(m - 1, N - 1)$$

is a holomorphic mapping (obviously we have (3.21) in mind). Then the differential of $\psi$ at $A = \psi(w)$ is

$$\psi_*\colon \mathbf{C}^l \longrightarrow \operatorname{Hom}(A, \mathbf{C}^N/A) . \tag{3.22}$$

If, in addition, $\psi$ has the property that for one (and hence any) holomorphically varying basis $e_1(w), \cdots, e_m(w)$ the $m(l+1)$ vectors $\{e_s(w), \partial e_s(w)/\partial w_\alpha\}$ $(1 \leqq s \leqq m, 1 \leqq \alpha \leqq l)$ span a $\mathbf{C}^{2m-l+1}$, then the differential (3.22) factors through the inclusion $\mathbf{C}^{2m-l+1} \subset \mathbf{C}^N$ and becomes

$$\begin{cases} \psi_*\colon \mathbf{C}^l \longrightarrow \operatorname{Hom}(A, \mathbf{C}^{2m-l+1}/A) \\ \qquad\ \ \| \qquad\qquad\qquad \|\wr \\ \psi_*\colon \mathbf{C}^l \longrightarrow \operatorname{Hom}(\mathbf{C}^m, \mathbf{C}^{m-l+1}) . \end{cases}$$

The tri-linear algebra lemma will pertain to maps

$$\varphi\colon \mathbf{C}^l \longrightarrow \operatorname{Hom}(\mathbf{C}^m, \mathbf{C}^{m-l+1})$$

having properties which will be consequences of the second equation in (3.20) when applied to the differential of $\pi$ in (3.21).

We assume that $m \geqq l+1$ and use the index ranges $1 \leqq \alpha, \beta \leqq l$; $1 \leqq s \leqq m$; $1 \leqq \mu \leqq l-1$. For $\varphi$ as above and $\sigma = (\sigma_1, \cdots, \sigma_l) \in \mathbf{C}^l$ denote by

$$\varphi_\sigma \in \operatorname{Hom}(\mathbf{C}^m, \mathbf{C}^{m-l+1})$$

the image of $\sigma$. We make the assumption that there is a basis $\{e_s\}$ for $\mathbf{C}^m$ and generators $(f_s)$ for $\mathbf{C}^{m-l+1}$ such that

$$\begin{aligned} \varphi_\sigma(e_s) &= \Phi_s(\sigma) \cdot f_s \qquad (\textit{no}\ \text{summation}) \\ &= \left(\textstyle\sum_{\alpha=1}^{l} \Phi_{s\alpha}\sigma_\alpha\right) f_s . \end{aligned} \tag{3.24}$$

We also assume that all

$$\det \| \Phi_{s_\alpha \beta} \| \neq 0 . \tag{3.25}$$

Intuitively, (3.21) is a reasonable alternative to having the $\varphi_\sigma$ simultaneously diagonalized.

*Definition. A knotpoint is* $0 \neq e \in \mathbf{C}^m$ *such that*

$$e \in (\ker \varphi_{\sigma_1}) \cap \cdots \cap (\ker \varphi_{\sigma_{l-1}}) \tag{3.26}$$

*where* $\sigma_1, \cdots, \sigma_{l-1} \in \mathbf{C}^l$ *are linearly independent.*

Equivalently, by (3.24) and (3.25), for $0 \neq \sigma \in \mathbf{C}^l$,

$$\ker \varphi_\sigma = \mathbf{C}^{l-1}(\sigma) \subset \mathbf{C}^m ,$$

and a knotpoint is

$$0 \neq e \in \textstyle\bigcap_{\langle \omega, \sigma \rangle = 0} \mathbf{C}^{l-1}(\sigma) \tag{3.27}$$

for some $0 \neq \omega \in \mathbf{C}^{l*}$.

In general there may be no solutions to (3.27), but in our application $\mathbf{C}^m$ will be spanned by knotpoints, and so we assume this.

(3.28) LEMMA. *The knotpoint locus is a rational normal curve in* $\mathbf{P}^{m-1}$.

*Proof*. Suppose that

$$\sum_{s=1}^{m} A_{s\mu} f_s = 0 , \qquad \mu = 1, \cdots, l-1$$

is a basis for the relations among the $\{f_s\}$. The condition that

$$e = \sum_{s=1}^{m} \lambda_s e_s$$

belong to $\ker \varphi_\sigma$ is, by (3.24),

$$\sum_{\alpha=1}^{l} \lambda_s \Phi_{s\alpha} \sigma_\alpha = \sum_{\mu=1}^{l-1} A_{s\mu} c_\mu , \qquad s = 1, \cdots, m$$

for some $\{c_\mu\}$. If we set $\rho_s = -(1/\lambda_s)$, the knotpoint condition (3.26) is that the system

$$\sum_\alpha \Phi_{s\alpha}\sigma_\alpha + \sum_\mu \rho_s A_{s\mu} c_\mu = 0 , \qquad s = 1, \cdots, m \tag{3.29}$$

of $m$ equations in the $2l-1$ unknowns $(\sigma_1, \cdots, \sigma_l; c_1, \cdots, c_{l-1})$ should have $(l-1)$ independent solutions. Viewing the coefficient matrix in the system (3.29) as a linear mapping $\mathbf{C}^{2l-1} \to \mathbf{C}^m$, the equivalent condition that the image have dimension $\leqq l$ is that all $(l+1) \times (l+1)$ minors of this coefficient matrix

$$\| \Phi_{s\alpha}; \rho_s A_{s\mu} \|$$

should be zero. By (3.25) these are *linear* equations in the $\rho_s$. In other words, under a suitable Cremona transformation (3.16) of $\mathbf{P}^{m-1}$ to itself, the knotpoint locus maps to a linear space. By (3.25) this linear space has dimension $\leqq 1$, while by assumption it is non-empty. Hence the knotpoints are the Cremona transform of a line. Q.E.D.

Using (3.28) we will prove the

(3.30) LEMMA. *The points* $p_i(\xi)$ *lie on a rational normal curve* $C(\xi)$ *in* $\mathbf{P}^{d-k-2}(\xi)$.

*Proof*. We will apply (3.28) not to $\pi$ in (3.21) considered on all of $\mathbf{G}(k, N)$, but rather we fix a $\mathbf{P}_0^{k+1}$ in $\mathbf{P}^N$ and consider the embedding of the dual projective space

$$\mathbf{P}_0^{k+1} \subset \mathbf{G}(k, N);$$

i.e., we take all $\mathbf{P}^k(\xi)$ contained as hyperplanes in $P_0^{k+1}$. Geometrically,

$$\mathbf{P}_0^{k+1} \cdot M_L = E_L$$

is a curve section of $M_L$, and we are considering the diagram

$$\begin{tikzcd} E \arrow[r, "\iota_L"] \arrow[d, "\iota_{K+(n-1)L}"] & \mathbf{P}^{k+1} \\ \mathbf{P}^r & \end{tikzcd} \tag{3.31}$$

obtained by restricting (3.18) to this curve. For $\xi \in \mathbf{P}_0^{k+1}$ the points $\boldsymbol{p}_i(\xi)$ vary on a 1-dimensional arc in $\mathbf{P}^r$, and we denote by $\boldsymbol{p}_i'(\xi)$ the tangent line. Then (3.20) applied to the diagram (3.31) becomes

$$(3.32) \qquad \begin{cases} \dim\{\boldsymbol{p}_i(\xi)\} = d - k - 2 \\ \dim\{\boldsymbol{p}_i(\xi), \boldsymbol{p}_i'(\xi)\} = (d - k - 1) + (d - 2k - 1) - 1 \\ \qquad\qquad\qquad\qquad = 2d - 3k - 3 \, . \end{cases}$$

Indeed, the coefficient $n$ in the second equation in (3.20) corresponds to the $n$ independent directions $\partial/\partial z_\alpha$ to $M$ at a point, and in the second equation in (3.32) we are only considering the direction which is tangent to $E$. The mapping (3.21) corresponding to (3.31) is

$$\mathbf{P}_0^{k+1} \longrightarrow \mathbf{G}(d - k - 2, r) \, .$$

Taking

$$l = k + 1 \, , \quad m = d - k - 1 \, , \quad N = r + 1 \, ,$$

we have

$$2m - l + 1 = 2d - 3k - 2 \, .$$

The above mapping is

$$\xi \longrightarrow \{\boldsymbol{p}_1(\xi), \cdots, \boldsymbol{p}_{d-k-2}(\xi)\} \, ,$$

and since $\boldsymbol{p}_i(\xi)$ varies on an arc,

$$\frac{\partial}{\partial \xi_\alpha}\bigl(\boldsymbol{p}_i(\xi)\bigr) = \boldsymbol{p}_i'(\xi)(\cdots) \, .$$

Together with (3.22) this last equation says first of all that the differential condition

$$\dim \left\{\boldsymbol{p}_i(\xi), \frac{\partial \boldsymbol{p}_i}{\partial \xi_\alpha}(\xi)\right\} = 2m - l$$

given just below (3.22) is satisfied, and secondly that the differential has the "pseudo-diagonalized" form (3.24), where $e_s = \boldsymbol{p}_s$ and $f_s = \boldsymbol{p}_s'$ $(1 \leqq s \leqq d-k-1)$. Lemma (3.30) will follow from (3.28) provided we can show that $\mathbf{P}^{d-k-2}(\xi)$ is spanned by knotpoints.

In fact, we claim that each $\boldsymbol{p}_i(\xi)$ is a knotpoint. By projective duality, each point $p \in \mathbf{P}_0^{k+1}$ defines a hyperplane $p^\perp$ in $\mathbf{P}_0^{k+1}$. Fixing $\xi_0$, as $\xi \in \mathbf{P}_0^{k+1}$ varies in the hyperplane $p_i(\xi_0)^\perp$ the point $\boldsymbol{p}_i(\xi_0)$ remains fixed. Then, $k = l - 1$ independent tangent vectors to $p_i(\xi_0)^\perp$ give exactly what is required in (3.26). Q.E.D.

(3.33) LEMMA. *The points $p_i(\xi)$ lie on a rational normal curve $C(\xi) \subset \mathbf{P}^k(\xi)$.*

*Proof.* This will be a formal consequence of (3.30), and we will omit reference to $\xi$. Choose $\boldsymbol{p}_{k+2}, \cdots, \boldsymbol{p}_d$ as vertices of a coordinate simplex in

$\mathbf{P}^{d-k-2}$, and use the index range $1 \leqq \alpha, \beta \leqq k+1, k+2 \leqq s \leqq d$. Multiplying $\boldsymbol{p}_s$ by a constant if necessary, we may assume that C goes through the $\boldsymbol{p}_s$ and $[1, \cdots, 1]$, and hence C is given parametrically by

$$t \longrightarrow \left[\cdots, \frac{1}{t-b_s}, \cdots\right].$$

In particular

$$\boldsymbol{p}_\alpha = \left[\cdots, \frac{1}{t_\alpha - b_s}, \cdots\right]. \tag{3.34}$$

Now suppose that $[x_1, x_2, \cdots, x_d]$ is a homogeneous coordinate system in $\mathbf{P}^k$ having $p_1, \cdots, p_{k+1}$ as vertices; i.e.,

$$x_\alpha(p_\beta) = \delta^\alpha_\beta .$$

According to (2.12), the relations on the points $\boldsymbol{p}_i \in \mathbf{P}^r$ are

$$\textstyle\sum_{i=1}^d x_\alpha(p_i)\boldsymbol{p}_i = 0 , \qquad \alpha = 1, \cdots, k+1 . \tag{3.35}$$

Comparing (3.34) and (3.35) we find

$$x_\alpha(p_s) = -\frac{1}{t_\alpha - b_s} .$$

Thus, the $p_i$ lie on the rational normal curve $C \subset \mathbf{P}^k$ given parametrically by

$$b \longrightarrow \left[\cdots, \frac{1}{t_\alpha - b}, \cdots\right],$$

with $p_\alpha$ corresponding to $b = t_\alpha$ and $p_s$ to $b = b_s$. Q.E.D.

To complete the proof we assume for simplicity that $\iota_L$ is biregular and write $M$ in place of $M_L$. For a generic $\mathbf{P}^k$ the intersection $M \cdot \mathbf{P}^k = Z$ is a set of $d$ points in general position lying on a rational normal curve $C$, which is the intersection of $\infty^{k(k-1)/2}$ quadrics in $\mathbf{P}^k$. Suppose we can prove that the restriction mapping

$$H^0(\mathcal{I}_M(2)) \xrightarrow{\rho} H^0(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^k}(2)) \cdots > 0 \tag{3.36}$$

is surjective. Geometrically this means that the quadrics in $\mathbf{P}^k$ which pass through $Z$ are intersections of quadrics in $\mathbf{P}^N$ which contain $M$. Now as mentioned above the intersection of the quadrics in $H^0(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^k}(2))$ is just the rational normal curve $C$, and the intersection of the quadrics in $H^0(\mathcal{I}_M(2))$ will be a variety $V$ such that

$$V \cdot \mathbf{P}^k = C .$$

It follows that $\dim V = n+1$, $\deg V = k$, and we are done. So it remains to prove the

(3.37) LEMMA. *The mapping $\rho$ in* (3.36) *is surjective.*

*Proof.* Let us choose coordinates $[x_0, \cdots, x_k; y_1, \cdots, y_n] = [x, y]$ in $\mathbf{P}^N$ such that $\mathbf{P}^k$ is given by $y = 0$. Then $H^0(\mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^k}(2))$ are the quadrics $Q(x)$ which pass through $z = \mathbf{P}^k \cdot M$. We want to find a quadric of the form

$$Q' = \sum_{s=1}^{n} y_s P_s(x, y) , \qquad \deg P_s = 1 , \tag{3.37}$$

such that $Q - Q'$ vanishes on $M$. For this it will suffice to show that every quadric in $H^0(\mathcal{I}_Z \otimes \mathcal{O}_M(L^2))$ has the form (3.37). The Koszul resolution of $\mathcal{I}_Z$ is

$$0 \longrightarrow \mathcal{O}_M(L^{*k}) \longrightarrow \cdots \longrightarrow \bigoplus \mathcal{O}_M(L^{*2}) \longrightarrow \bigoplus \mathcal{O}_M(L^*) \xrightarrow{\sigma} \mathcal{I}_Z \longrightarrow 0 . \tag{3.38}$$

We tensor (3.38) with $\mathcal{O}(L^2)$, and then to prove the lemma it will suffice to show that

$$\bigoplus H^0(\mathcal{O}_M(L)) \xrightarrow{\sigma} H^0(\mathcal{I}_Z \otimes \mathcal{O}_M(L^2)) \cdots > 0$$

is surjective. This is tedious but straightforward to check by writing everything out using (3.38).

## Appendix: Some observations and open problems

Let $M$ be a smooth projective algebraic variety on which we are given a configuration $Z$ of distinct points and a linear system $|D|$. We may take as a general aim that of estimating the superabundance $\omega(Z, |D|)$, and of saying something about the extreme cases.

When $M$ is an algebraic curve the superabundance is bounded by $h^1(\mathcal{O}([D - Z]))$, and hence is zero unless $D - Z$ is a special divisor. Then one can always estimate the superabundance by Clifford's theorem, and in case $|D - Z|$ is birational the much stronger bound provided by Castelnuovo's estimate on the genus can be used. In both circumstances the extreme cases can be identified. Especially interesting is the situation when $|D| = |K|$ is the canonical series; then the superabundance is zero unless $Z$ itself is a special divisor, and the Riemann-Roch formula gives

$$\dim |Z| = d - g + \omega(Z, |K|)$$

where $d = \deg Z$.

When $M$ is an algebraic surface the first observation is that the general problem is not particularly well-posed since e.g., in $\mathbf{P}^2$ the superabundance relative to $|\mathcal{O}_{\mathbf{P}^2}(k)|$ is maximized when $Z$ lies on a line. In practice one will want to estimate $\omega(Z, |D|)$ when $Z$ is not on a fixed component of an auxiliary linear system. Now there is a choice of going directly to the zero-cycles, or of working with the curves on the surface as an intermediary with the hope of using the previous estimates. The latter, which is the traditional approach used e.g., in the classification theory of surfaces, sometimes necessitates detailed case arguments involving singular and/or reducible curves. More seriously, this technique does not seem to generalize easily to higher dimensions, e.g., threefolds ([R] notwithstanding). One may suspect that, on account of the Lefschetz hyperplane theorem, the essential geometry occurs in dimension $[n/2]$ on an $n$-dimensional variety. So, with the residue theorem providing the analytic tool, we have gone directly to the zero-cycles, and our main result may be informally summarized as follows:

*Given a holomorphic vector bundle $E \to M$ of rank $n$ over an $n$-dimensional algebraic manifold $M$ and a section $s \in H^0(\mathcal{O}(E))$ having divisor $(s) = Z$ a zero-dimensional scheme, there is an inverse relation between the growth of the two sequences*

$$\text{(A.1)} \qquad \begin{cases} \omega(Z, |\mu E|) & \textit{and} \\ \omega(\delta^\mu Z, |K \otimes \det E|) & \end{cases}$$

*where $|\mu E|$ is the sequence of linear systems associated to the graded ring $\bigoplus_\mu \mathrm{Sym}^\mu H^0(\mathcal{O}(E))$ and $\delta^\mu Z$ is the zero-dimensional scheme with ideal sheaf $(\mathcal{I}_Z)^\mu$.*

More precisely, the fundamental relation (2.5) pertains to a bundle $E \to M$ induced by a holomorphic mapping

$$\text{(A.2)} \qquad \varphi\colon M \longrightarrow \mathrm{G}(k, N)\ , \qquad k = N - n\ ,$$

where $|E|$ is essentially the linear system obtained by composing $\varphi$ with the Plücker embedding.

We were able to utilize (A.1) effectively only in case $E = \underbrace{L \oplus \cdots \oplus L}$ is a sum of line bundles and, with the benefit of hindsight, it may be said that at a very basic level our results do not go significantly beyond these of Castelnuovo. Indeed, many of our applications could have been proved by going to the curve sections of $M$. On the other hand, the formalism (A.1) is of a more general character, and in its application several interesting questions arise which we should like to mention briefly.

i) *On the problem of non-degenerate maps to Grassmannians.*

Suppose we consider (A.2) (or rather the dual) in the case of a surface, say

$$\varphi\colon S \longrightarrow \mathrm{G}(1, N)\ ,$$

and assume that $\varphi$ is biregular onto its image. For $p \in S$ we denote by

$l(p) \subset \mathbf{P}^N$ the corresponding line. If $\mathbf{P}^{N-3} \subset \mathbf{P}^N$ is generic, the Schubert condition

$$l(p) \quad \text{meets} \quad \mathbf{P}^{N-3}$$

defines the 2nd Chern class $p_1 + \cdots + p_d$ on $S$. The problem we have in mind is a) to find a good definition of non-degeneracy for the mapping $\varphi$, and b) to show that for non-degenerate maps, $l(p_1), \cdots, l(p_d)$ give $d \geqq d(N)$ $(\stackrel{?}{=}[N/2])$ lines in general position in $\mathbf{P}^N$. It is not immediately clear to us what the definition of general position for lines $l_1, \cdots, l_d$ in $\mathbf{P}^N$ should be; certainly any $k \leqq [(N+1)/2]$ should span a $\mathbf{P}^{2k-1}$, but this is probably not sufficient.

It may be that this problem only has reasonable solutions for stable (or semi-stable) bundles $E \to S$.

ii) *Extending Castelnuovo's bound to the non-linear case.*

The general question here is to bound the genus of a curve $C$ in a general variety $V$, perhaps in terms of the homology cycle $\gamma_C \in H_2(V)$ carried by the curve. Two special cases came to mind, the first being to bound the genus of a curve

$$C \subset \mathbf{G}(1, 3)$$

assuming that the ruled surface in $\mathbf{P}^3$ defined by the $\infty^1$ lines $l(p)$ $(p \in C)$ is non-developable (the developable case is covered by Castelnuovo's bound). Another occurs when $V$ is a threefold with $q(V) = p_g(V) = 0$ and $h^2(V) = 1$ (e.g., $V$ may be a complete intersection in $\mathbf{P}^n$). Every curve on $V$ has a unique homology invariant, which we may call its degree.

A motivation for this second problem is this: In the study of the intermediate Jacobian of $V$, the essential difficulty lies in understanding the curves on $V$. An optimistic possibility is that the curves of maximum genus, being somehow a distinguished family, might be of help. For example, guided by the case of curves in $\mathbf{P}^3$ it seems possible that the curves of large fixed degree $d$ might form an irreducible family, and for those $d$ such that the curve cannot be a complete intersection of two surfaces on $V$, it may be the case that the residual curves $D$ such that $C + D$ is a complete intersection will contribute to the intermediate Jacobian.

Another related question is to bound the $p_g$ of a non-degenerate surface in $\mathbf{G}(1, 3)$, as this situation bears some formal resemblance to plane curves but is non-linear.

iii) *Constructing vector bundles of higher rank.*

This is the problem of extending Proposition (1.33) to bundles of rank

$n \geqq 3$. Suppose e.g., that $E \to V$ is a rank-3 bundle over a smooth threefold. If $s_1 \in H^0(\mathcal{O}(E))$ is a section with a set $Z_1$ of simple zeroes, then $Z_1$ represents $c_3(E)$ and satisfies the Cayley-Bacharach property relative to $|K \otimes \det E|$. If $s_2$ is another section, then the condition

$$s_1 \wedge s_2 = 0 \tag{A.3}$$

defines (in general) a curve $C$ representing $c_2(E)$. In fact, the section $s_1 \wedge s_2 \in H^0(\mathcal{O}(\wedge^2 E))$ is not generic, so that (A.3) defines a codimension-two subvariety (it is a determinantal variety) rather than one of codimension three. Moreover, $C$ is the locus of zeroes of the sections $s_0 - ts_1$ ($t \in \mathbf{P}^1$), and hence carries a linear pencil containing $Z_1$ and $Z_2$.

So extending (1.33) to bundles of higher rank might begin with the following: Given $Z \subset C$ on $V$ and a line bundle $L$ such that $Z$ satisfies the Cayley-Bacharach condition relative to $|K \otimes L|$, and moreover such that this condition defines a linear pencil on $C$ containing $Z$, then does there exist $E \to V$ and sections $s_1, s_2 \in H^0(\mathcal{O}(E))$ defining $Z_1$, $Z_2$, and $C$ as above? No doubt this formulation needs refinement, perhaps to take into account the determinantal character of $C$, but the geometric problem may be worthwhile.

HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS

BIBLIOGRAPHY

[A-S] E. ARBARELLO and E SERNESI, to appear.

[B] W. BARTH, Moduli of vector bundles on the projective plane, to appear.

[C-G, 1] S.S. CHERN and P. GRIFFITHS, Abel's theorem and webs, to appear in Jahr. Deut. Math. Ver.

[C-G, 2] ———, An inequality for the rank of a web and webs of maximal rank, to appear in Ann. Scuola Norm. Pisa.

[G] P. GRIFFITHS, Variations on a theorem of Abel, Invent. Math. **35** (1976), 321-390.

[G-H] P. GRIFFITHS and JOSEPH HARRIS, *Algebraic Geometry*, John Wiley, New York (1978).

[H] JOSEPH HARRIS, Thesis at Harvard University (1977).

[K-S] K. KODAIRA and D.C. SPENCER, Deformations of complex structures, Ann. of Math. **67** (1958), 328-466.

[R] L. ROTH, Algebraic threefolds with special regard to problems of rationality, Ergebnisse der Mathematik und ihre Grenzgebiete, Springer-Verlag, Bd. **6** (1955).

[S] R.E.L. SCHWARZENBERGER, Vector bundles on algebraic surfaces, Proc. London Math. Soc. **11** (1961), 601-622.

(Received October 15, 1977)