

Resistance distance in wheels and fans

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Abstract: *The wheel graph is the join of a single vertex and a cycle, while the fan graph is the join of a single vertex and a path. The resistance distance between any two vertices of a wheel and a fan are obtained. The resistances are related to Fibonacci numbers and generalized Fibonacci numbers. The derivation is based on evaluating determinants of submatrices of the Laplacian matrix. A combinatorial argument is also illustrated. A connection with the problem of squaring a rectangle is described.*

1 Introduction

We consider graphs which have no loops or parallel edges. Thus a *graph* $G = (V(G), E(G))$ consists of a finite set of *vertices*, $V(G)$, and a set of *edges*, $E(G)$, each of whose elements is a pair of distinct vertices. A *weighted graph* is a graph in which each edge is assigned a positive number, called its *weight*. We will assume familiarity with basic graph-theoretic notions, see, for example, Bondy and Murty [4]).

Given a graph, one associates a variety of matrices with the graph. Some of the important ones will be defined now. Let G be a graph with $V(G) = \{1, \dots, n\}$, $E(G) = \{e_1, \dots, e_m\}$. The *adjacency matrix* $A(G)$ of G is an $n \times n$

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matrix with its rows and columns indexed by $V(G)$, and with the (i, j) -entry equal to 1, if vertices i, j are adjacent (i.e., joined by an edge) and 0 otherwise. Thus $A(G)$ is a symmetric matrix with its i -th row (or column) sum equal to $d(i)$, which by definition is the degree of the vertex $i, i = 1, 2, \dots, n$. Let $D(G)$ denote the $n \times n$ diagonal matrix, whose i -th diagonal entry is $d(i), i = 1, 2, \dots, n$.

The *Laplacian matrix* of G , denoted by $L(G)$, is defined as $L(G) = D(G) - A(G)$. For a weighted graph G , the Laplacian is defined as follows. For $i \neq j$, the (i, j) -entry of the Laplacian matrix is set equal to 0, if i and j are not adjacent, and otherwise it is defined to be the negative inverse (reciprocal) of the weight of the edge $\{i, j\}$. The diagonal entries of the Laplacian matrix are defined so that they make the sum of all entries in their row to be zero.

The *resistance distance* r_{ij} between vertices i and j in a weighted graph G is defined as the effective electrical resistance between points i and j when we place a resistor along every edge (with the resistance offered by the resistor equal to the weight of the edge) and a potential difference is applied at i and j . The resistance matrix R is defined as the matrix with its (i, j) -entry equal to r_{ij} . There are several equivalent ways to define the resistance distance. We describe some such definitions which will be used. For proofs and more information about resistance distance we refer to [11,3, 1, 10, 14, 13].

We introduce some notation. If A is an $n \times n$ matrix, then for $i, j \in \{1, \dots, n\}$, $A(i|j)$ will denote the submatrix of A obtained by deleting row i and column j . For $i, j \in \{1, \dots, n\}, i \neq j$, $A(i, j|i, j)$ will denote the principal submatrix of A obtained by deleting rows i, j and columns i, j . If $S, T \subset \{1, \dots, n\}$, we denote by $A[S|T]$, the submatrix of A formed by the rows indexed by S and the columns indexed by T .

Lemma 1 *Let G be a weighted graph with $V(G) = \{1, \dots, n\}$, and let L be the Laplacian of G . Then for $i, j \in \{1, \dots, n\}, i \neq j$,*

$$r_{ij} = \frac{\det L(i, j|i, j)}{\det L(i|i)}.$$

It may be remarked that $\det L(i|i)$ equals the number of spanning trees of G , by the Matrix-Tree Theorem.

Let G be a connected graph. A spanning 2-tree of G is defined as a spanning forest of G with two components. A spanning 2-tree is said to separate vertices i and j if the vertices are in distinct components of the 2-tree. If G is a weighted graph, then the weight of a spanning tree, or a spanning 2-tree, is defined to be the product of the edge weights of the tree, or 2-tree.

Lemma 2 *Let G be a weighted graph with $V(G) = \{1, \dots, n\}$, and let $i, j \in \{1, \dots, n\}, i \neq j$. Then r_{ij} equals the sum of the weights of all spanning 2-trees of G separating i and j , divided by the sum of the weights of all spanning trees of G .*

If A is an $m \times n$ matrix, then an $n \times m$ matrix G is said to be a generalized inverse of A if $AGA = A$. For basic properties of generalized inverse see, for example, [2, 6].

Lemma 3 *Let G be a weighted graph with $V(G) = \{1, \dots, n\}$, and let L be the Laplacian of G . Let H be a generalized inverse of L . Then for $i, j \in \{1, \dots, n\}$,*

$$r_{ij} = h_{ii} + h_{jj} - h_{ij} - h_{ji}.$$

In this paper we consider two specific graphs, the wheel and the fan, and obtain explicit formulas for the resistance distance between any two vertices in these graphs.

2 Resistances in wheel and fan

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 , and edge sets X_1 and X_2 , is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2 .

A wheel graph W_n is by definition, $K_1 + C_n$, where C_n denotes the cycle with n vertices. In the wheel, the vertices corresponding to the cycle C_n are labeled from 1 to n in cyclic order, and the central vertex corresponding to K_1 is labeled as $n + 1$. We also consider a weighted wheel in which the edges

$\{n+1, i\}, i \in \{1, 2, \dots, n\}$, called the *spokes* of the wheel, have the same weight α .

The fan graph Fan_n is defined as $K_1 + P_n$, where P_n is the path on n vertices. All the vertices of the fan corresponding to the path P_n are labeled from 1 to n consecutively. The vertex in the fan corresponding to the point K_1 is labeled $n+1$.

We find the resistance distance between any two vertices in a wheel or a fan in terms of Fibonacci and generalized Fibonacci numbers. We have specifically considered the two classes of graphs together as they are related. The fan is just a wheel with an edge of the cycle removed.

We introduce some notation. Recall that α is the weight of a spoke in a weighted wheel. Let

$$\gamma = 2 + \frac{1}{\alpha}, \mu = \frac{\gamma + \sqrt{\gamma^2 - 4}}{2}, \nu = \frac{\gamma - \sqrt{\gamma^2 - 4}}{2}.$$

We define G_k as the generalized Fibonacci number,

$$G_k = \frac{\mu^k - \nu^k}{\mu - \nu}.$$

Note that when $\alpha = 1$, $G_k = F_{2k}$, where F_r is the r^{th} Fibonacci number. The next result is well-known. We include a proof for completeness.

Lemma 4 *Let*

$$M_n = \begin{bmatrix} \gamma & -1 & 0 & \cdots & 0 \\ -1 & \gamma & -1 & 0 & \cdots \\ \vdots & & & & \\ 0 & \cdots & -1 & \gamma & -1 \\ 0 & \cdots & 0 & -1 & \gamma \end{bmatrix}$$

be $n \times n$. Let $D_n = \det M_n$. Then $D_n = G_{n+1}$.

Proof: Expanding $\det M_n$ along the first row, we see that D_n satisfies the recurrence relation

$$D_n = D_{n-1} - D_{n-2},$$

with the initial conditions, $D_1 = \gamma, D_2 = \gamma^2 - 1$. Solving the recurrence in the standard fashion, we see that

$$D_n = \delta_1 \left(\frac{\gamma + \sqrt{\gamma^2 - 4}}{2} \right)^n + \delta_2 \left(\frac{\gamma - \sqrt{\gamma^2 - 4}}{2} \right)^n,$$

for some δ_1 and δ_2 . Using the initial conditions it follows that $D_n = G_{n+1}$. ■

Theorem 5 *Let $n \geq 3$ be a positive integer. The following results hold for W_n :*

1. *The resistance distance between vertex $n + 1$ and vertex i , $i \in \{1, 2, \dots, n\}$, is $r_{n+1,i} = r_{i,n+1} = \frac{G_n^2}{G_{2n} - 2G_n}$*
2. *The resistance distance between vertices $i, j \in \{1, 2, \dots, n\}$ is*

$$r_{ij} = \frac{G_n^2}{G_{2n} - 2G_n} \left[2 - \frac{G_{2k}}{G_k} \right] + G_k,$$

where

$$k = \begin{cases} |j - i| & \text{if } |j - i| \leq \lfloor \frac{n}{2} \rfloor. \\ n - |j - i| & \text{if } |j - i| > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Proof: The Laplacian of W_n is the $(n + 1) \times (n + 1)$ matrix

$$L = \begin{bmatrix} \gamma & -1 & 0 & 0 & \cdots & 0 & 0 & -1 & -\frac{1}{\alpha} \\ -1 & \gamma & -1 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{\alpha} \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & \gamma & -1 & -\frac{1}{\alpha} \\ -1 & 0 & 0 & \cdots & 0 & 0 & -1 & \gamma & -\frac{1}{\alpha} \\ -\frac{1}{\alpha} & -\frac{1}{\alpha} & -\frac{1}{\alpha} & -\frac{1}{\alpha} & \cdots & -\frac{1}{\alpha} & -\frac{1}{\alpha} & -\frac{1}{\alpha} & \frac{n}{\alpha} \end{bmatrix} \quad (1)$$

Let

$$H = \begin{bmatrix} L(n + 1 | n + 1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (2)$$

Then H is a generalized inverse of L . By Lemma 3, the resistance distance r_{ij} between vertices i and j is given by

$$r_{ij} = h_{ii} + h_{jj} - h_{ij} - h_{ji}. \quad (3)$$

Note that

$$L(n+1|n+1) = \gamma I - P - P^{n-1}, \quad (4)$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 \cdots & 0 & 0 \end{bmatrix}. \quad (5)$$

The inverse of $L(n+1|n+1)$ must be a polynomial in P of the following form. If $n = 2m+1$, then

$$\begin{aligned} L(n+1|n+1)^{-1} &= \beta_0 I + \beta_1 (P + P^{2m}) + \beta_2 (P^2 + P^{2m-1}) + \cdots \\ &+ \beta_r (P^r + P^{2m+1-r}) + \cdots + \beta_{m-1} (P^{m-1} + P^{m+2}) + \beta_m (P^m + P^{m+1}), \end{aligned} \quad (6)$$

and if $n = 2m$, then

$$\begin{aligned} L(n+1|n+1)^{-1} &= \beta_0 I + \beta_1 (P + P^{2m-1}) + \beta_2 (P^2 + P^{2m-2}) + \cdots \\ &+ \beta_r (P^r + P^{2m-r}) + \cdots + \beta_{m-1} (P^{m-1} + P^{m+1}) + \beta_m (P^m). \end{aligned} \quad (7)$$

Using $L(n+1|n+1)L(n+1|n+1)^{-1} = I$ and $P^n = I$, and comparing the coefficients of P , we get,

$$\gamma\beta_0 - 2\beta_1 = 1 \quad (8)$$

and

$$\gamma\beta_k - \beta_{k-1} - \beta_{k+1} = 0, k \in \{1, 2, \dots, m+1\}, \quad (9)$$

where $\beta_{m+1} = \beta_m$ if $n = 2m+1$ and $\beta_{m+1} = \beta_{m-1}$ if $n = 2m$.

Solving (9) as a recurrence relation we get

$$\beta_k = C_1 \left(\frac{\gamma + \sqrt{\gamma^2 - 4}}{2} \right)^k + C_2 \left(\frac{\gamma - \sqrt{\gamma^2 - 4}}{2} \right)^k. \quad (10)$$

Putting $k = 0, 1$ in (10) and using (8) we get

$$\beta_0 = C_1 + C_2 \quad (11)$$

and

$$\beta_1 = \frac{\gamma(C_1 + C_2)}{2} + \frac{\sqrt{\gamma^2 - 4}(C_1 - C_2)}{2}. \quad (12)$$

Solving (11) and (12) we get

$$C_1 = \frac{1}{\sqrt{\gamma^2 - 4}} \left(\frac{\beta_0 \sqrt{\gamma^2 - 4} - 1}{2} \right), \quad C_2 = \frac{1}{\sqrt{\gamma^2 - 4}} \left(\frac{\beta_0 \sqrt{\gamma^2 - 4} + 1}{2} \right).$$

Hence

$$\begin{aligned} \beta_k &= \frac{1}{\sqrt{\gamma^2 - 4}} \left[\left(\frac{\beta_0 \sqrt{\gamma^2 - 4} - 1}{2} \right) \left(\frac{\gamma + \sqrt{\gamma^2 - 4}}{2} \right)^k + \left(\frac{\beta_0 \sqrt{\gamma^2 - 4} + 1}{2} \right) \left(\frac{\gamma - \sqrt{\gamma^2 - 4}}{2} \right)^k \right] \\ &= \frac{1}{\sqrt{\gamma^2 - 4}} \left[\left(\frac{\beta_0 \sqrt{\gamma^2 - 4}}{2} (\mu^k + \nu^k) \right) - \frac{1}{2\sqrt{\gamma^2 - 4}} (\mu^k - \nu^k) \right] \\ &= \frac{\beta_0}{2} (\mu^k + \nu^k) - \frac{1}{2\sqrt{\gamma^2 - 4}} (\mu^k - \nu^k) \\ &= \frac{\beta_0 G_{2k}}{2 G_k} - \frac{G_k}{2}. \end{aligned}$$

It remains to find β_0 , and this is achieved using the fact that β_0 is the $(1, 1)$ -element of $L(n+1|n+1)^{-1}$. Since $\det L(1, n+1|1, n+1) = D_{n-1}$ and

$$\det L(n+1|n+1) = \gamma D_{n-1} - 2D_{n-2} - 2,$$

$$\beta_0 = \frac{D_{n-1}}{\det(L(n+1|n+1))}, \text{ which, on simplification gives } \beta_0 = \frac{G_n^2}{G_{2n-2}G_n}.$$

Recall from (6), 7 and 2

$$h_{i,j} = 0 \text{ if } i = n+1 \text{ or } j = n+1$$

$$h_{i,j} = \beta_k \text{ otherwise}$$

where

$$k = \begin{cases} |j-i| & \text{if } |j-i| \leq \lfloor \frac{n}{2} \rfloor. \\ n - |j-i| & \text{if } |j-i| > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Also from (3)

$$r_{ij} = h_{ii} + h_{jj} - h_{ij} - h_{ji}.$$

Clearly, the resistance distance across the spoke $(i, n+1)$, $i \in \{1, 2, \dots, n\}$, is β_0 , since $h_{i,n+1} = h_{n+1,i} = 0$ for all $i \in \{1, 2, \dots, n\}$. It is also easy to see that the resistance between two vertices i and j on the circumference of the wheel, where the numbers are assigned to vertices in a cyclic order, is $r_{ij} = 2(\beta_0 - \beta_k)$, where

$$k = \begin{cases} |j-i| & \text{if } |j-i| \leq \lfloor \frac{n}{2} \rfloor. \\ n - |j-i| & \text{if } |j-i| > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

This completes the proof. ■

Corollary 6 Let $n \geq 3$ be a positive integer and let W_n be the unweighted wheel on $n + 1$ vertices. Then the following assertions hold:

1. The resistance distance between vertex $n + 1$ and vertex $i \in \{1, 2, \dots, n\}$ is $r_{n+1,i} = r_{i,n+1} = \frac{F_{2n}^2}{F_{4n} - 2F_{2n}}$
2. The resistance distance between vertices $i, j \in \{1, 2, \dots, n\}$ is

$$r_{ij} = \frac{F_{2n}^2}{F_{4n} - 2F_{2n}} \left[2 - \frac{F_{4k}}{F_{2k}} \right] + F_{2k},$$

$$\text{where } k = \begin{cases} |j - i| & \text{if } |j - i| \leq \lfloor \frac{n}{2} \rfloor \\ n - |j - i| & \text{if } |j - i| > \lfloor \frac{n}{2} \rfloor \end{cases}$$

and where F_k is the k^{th} Fibonacci number.

Proof: Clearly $G_k = F_{2k}$ when $\alpha = 1$ i.e., $\gamma = 2 + \frac{1}{\alpha} = 3$. The result follows from Theorem 5. ■

We now turn to the fan graph Fan_n defined earlier.

Theorem 7 Let $n \geq 1$ be a positive integer. Then for the fan graph Fan_n , the following results hold:

1. The resistance distance between vertex $n + 1$ and vertex $i \in \{1, 2, \dots, n\}$ is $\frac{F_{2(n-i)+1}F_{2i-1}}{F_{2n}}$.
2. The resistance distance between vertices $i, j \in \{1, 2, \dots, n\}, i < j$ is

$$\frac{F_{2(n-j)+1}(F_{2j-1} - F_{2i-1}) + F_{2i-1}(F_{2(n-i)+1} - F_{2(n-j)+1})}{F_{2n}}.$$

Proof: The result can be proved either by evaluating determinants and using Lemma 1, or by combinatorial arguments. We prove the first statement by evaluating determinants.

The Laplacian of Fan_n is the $(n+1) \times (n+1)$ matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & -1 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & n+1 \end{bmatrix} \quad (13)$$

By Lemma 1,

$$r_{n+1,1} = \frac{\det L(n+1, 1|n+1, 1)}{\det L(n+1|n+1)} \quad (14)$$

We continue to use the notation D_n introduced in Lemma 4. However we keep in mind that now $\gamma = 3$.

By Lemma 4,

$$\det L(n+1|n+1) = F_{2n}.$$

Thus the number of spanning trees in Fan_n is F_{2n} . This result is in fact well-known (see [8], p.362).

Using Laplace expansion and Lemma 4 we get

$$\begin{aligned} \det L(n+1, 1|n+1, 1) &= 2\det L(n+1, n, 1|n+1, n, 1) \\ &\quad - \det L(n+1, n, 1, n-1|n+1, 1, n, n-1) \\ &= 2D_{n-2} - D_{n-3} \\ &= 2F_{2n-2} - F_{2n-4} \\ &= F_{2n-2} + F_{2n-3} + F_{2n-4} - F_{2n-4} \\ &= F_{2n-1}. \end{aligned}$$

Hence $r_{n+1,1} = \frac{F_{2n-1}}{F_{2n}}$. It can be similarly shown that

$$\begin{aligned} \det(L(n+1, i|n+1, i)) &= \det L[1, \dots, i-1|1, \dots, i-1] \\ &\quad \times \det L[i+1, \dots, n|i+1, \dots, n] \\ &= F_{2i-1} \times F_{2(n-i+1)-1}. \end{aligned}$$

Therefore the first statement is true.

We shall give a combinatorial proof of the second statement. Let $i, j \in \{1, \dots, n\}, i < j$ be fixed. We must calculate the number of spanning 2-trees separating i and j . First consider the number of spanning 2-trees with components T_i and T_j , where T_i contains i and $n + 1$. Then T_j must be a path containing j . Let $i \leq k \leq j - 1$ be the vertex between i and j such that k is in T_i and $k + 1$ is in T_j . Let $j \leq \ell \leq n$ be the vertex between j and n such that $\ell - 1$ is in T_j and ℓ in tree T_i . Then T_i consists of a spanning tree of the fan with vertices $1, 2 \dots k$ and $n + 1$, along with a spanning tree of the fan with vertices ℓ, \dots, n and $n + 1$ or none of these vertices. Since the number of spanning trees in a fan with $m + 1$ vertices is F_{2m} , therefore the number ways in which T_i can be selected is

$$\left(\sum_{k=i}^{j-1} F_{2k} \right) \left(\left(\sum_{l=j+1}^n F_{2(n-l+1)} \right) + 1 \right).$$

Now let us find the number of spanning 2-trees with two components such that i and j are in the two components T_i and T_j respectively, and $n + 1$ is in T_j . This case is similar to the previous case. Relabel the fan Fan_n such that the vertex corresponding to K_1 remains labeled as $n + 1$, but the vertex i in Fan_n corresponding to the i^{th} vertex in the path P_n is labeled as $n - i + 1$. In this new labeling, the vertex formerly labeled j is labeled $n - j + 1$, and vertex formerly labeled i is labeled $n - i + 1$. Thus $n - j + 1 < n - i + 1$. Using similar technique as before, we get the number of spanning 2-trees such that i is in one components, and j and $n + 1$ are in the other component as

$$\left(\sum_{k=n-j+1}^{n-i} F_{2k} \right) \left(\left(\sum_{l=n-i+2}^n F_{2(n-l+1)} \right) + 1 \right).$$

Hence the total number of spanning 2-trees of Fan_n with separating i and j is

$$\left(\sum_{k=i}^{j-1} F_{2k} \right) \left(\left(\sum_{l=j+1}^n F_{2(n-l+1)} \right) + 1 \right) + \left(\sum_{k=n-j+1}^{n-i} F_{2k} \right) \left(\left(\sum_{l=n-i+2}^n F_{2(n-l+1)} \right) + 1 \right),$$

By using the fact that $\sum_{i=1}^r F_{2i} = F_{2i+1} - 1$, The above expression reduces to:

$$F_{2(n-j)+1} \times (F_{2j-1} - F_{2i-1}) + F_{2i-1} \times (F_{2(n-i)+1} - F_{2(n-j)+1}). \quad (15)$$

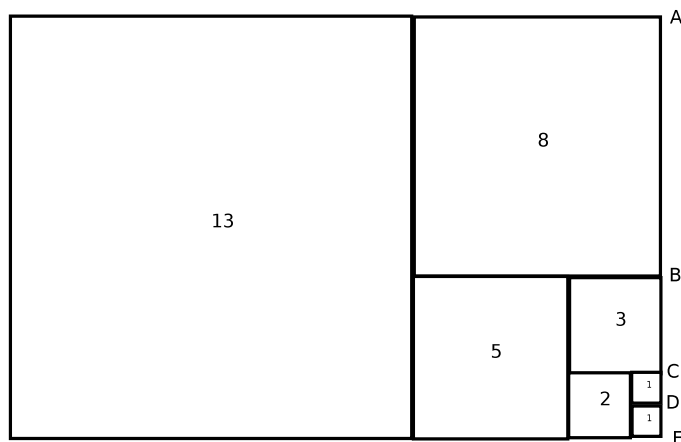


Figure 1: Fibonacci tiling of a rectangle with sides 13 x 21

Dividing (15) by F_{2n} , the number of spanning forests in Fan_n , (ii) is proved.

■

3 Fibonacci Tilings

We describe as *Fibonacci Tiling* as a tiling of a rectangle of side F_n and F_{n-1} with $n-1$ squares of lengths F_1, F_2, \dots, F_{n-1} . This tiling has a very interesting relation with the squaring a rectangle problem and Fan graphs. Squaring a rectangle problem as described by Brooks et. al. [5] is the problem of dividing a rectangle into a finite number of non-overlapping squares, no two of which are equal. Note the Fibonacci tiling does not exactly fit into this description since two squares have same lengths namely F_1 and F_2 , but this defect can be easily corrected by tiling one of the squares with another known tiling for a square (To know more see[9]).

Now let us describe the relation of Fan graphs to Fibonacci tilings. We apply the general approach for converting a squared rectangle to an electrical network as described in [3]. Consider the squared rectangle shown in Fig. 1. Let us cut this rectangle out of a sheet of nichrome or any other material with low conductivity and let us put rods made of silver or some material with high

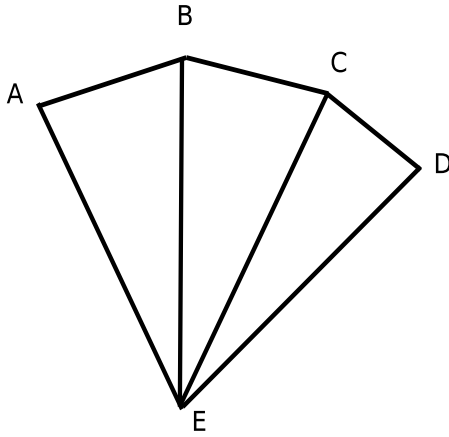


Figure 2: Electrical network corresponding to the Fibonacci tiling in Fig. 1

conductivity at the top and bottom. Now if we ensure that the silver rod at the top is at 13 volts and the rod at the bottom is kept at 0 volts, the current will flow from top to bottom and the potential at a point at height x will be x . Since no current flows horizontally across the rectangle, the current will not change if (1) we place silver rods on the horizontal sides of the squares and (2) cut narrow slits along the vertical sides.

Now since silver is a very good conductor, the points of each silver rod have been shortened, so can be identified (as A,B,C,D,E in Fig. 1). Thus as an electric conductor, the whole rectangle behaves like the *Fan* shown in Fig. 2. Now it is clear that the resistance distance between node A and E in Fig. 2 is nothing but the resistance of the rectangular conductor between rods A and E in Fig. 1 i.e., $\frac{13}{21}$. Also, it is interesting to note that Fibonacci tiling gives a geometrical proof of the sum of Fibonacci numbers, sum of odd/even indexed Fibonacci numbers and sum of squares of Fibonacci numbers.

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