Resolution of singularities of pairs preserving semi-simple normal crossings

by

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Abstract

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Let X denote a reduced algebraic variety and D a Weil divisor on X. The pair (X, D) is said to be *semi-simple normal crossings* (semi-snc) at $a \in X$ if X is simple normal crossings at a (i.e., a simple normal crossings hypersurface, with respect to a local embedding in a smooth ambient variety), and D is induced by the restriction to X of a hypersurface that is simple normal crossings with respect to X. For a pair (X, D), over a field of characteristic zero, we construct a composition of blowings-up $f : \widetilde{X} \to X$ such that the transformed pair $(\widetilde{X}, \widetilde{D})$ is everywhere semi-simple normal crossings, and f is an isomorphism over the semi-simple normal crossings locus of (X, D). The result answers a question of Kollár.

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Contents

Chapter 1

Introduction

The subject of this thesis is partial desingularization of a pair (X, D), where X is a reduced algebraic variety defined over a field of characteristic zero and D is a Weil $\mathbb{Q}$-divisor on X.

The purpose of partial desingularization is to provide representatives of a birational equivalence class that have mild singularities — almost as good as smooth — which have to be admitted in natural situations. For example, in order to simultaneously resolve the singularities of curves in a parametrized family, one needs to allow special fibers that have simple normal crossings singularities. Likewise, log resolution of singularities of a divisor produces a divisor with simple normal crossings. For these reasons, it is natural to consider simple normal crossings singularities as acceptable from the start, and to seek a partial desingularization which is an isomorphism over the simple normal crossings locus.

Our main theorem (Theorem 1.2) is a solution of a problem of János Kollár [Kol08, Problem 19] on resolution of singularities of pairs (X, D) except for *semi-simple normal crossings (semi-snc)* singularities.

Definition 1.1 (Definition semi-snc)**.** Following Kollár, we say that (X, D) is *semi-snc* at a point $a \in X$ if X has a neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular local coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $a = 0$ in which

U is defined by a monomial equation

$$x_1 \cdots x_p = 0 \tag{1.1}$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{Q}. \tag{1.2}$$

We say that (X, D) is *semi-snc* if it is semi-snc at every point of X.

According to Definition 1.1, the support, $\operatorname{Supp} D|_U$, of $D|_U$ as a subset of Y is defined by a pair of monomial equations

$$x_1 \cdots x_p = 0, \quad y_{i_1} \cdots y_{i_q} = 0. \tag{1.3}$$

Let $f : \widetilde{X} \to X$ be a birational mapping. Denote by $Ex(f)$ the exceptional set of f (i.e. the set of points where f is not a local isomorphism). Assuming that $Ex(f)$ is a divisor we define $\widetilde{D} := D' + \mathrm{Ex}(f)$, where D' is the birational transform of D by f^{-1}. We call $(\widetilde{X}, \widetilde{D})$ the *(total) transform* of (X, D) by f.

Theorem 1.2 (Main theorem). *Let X denote a reduced algebraic variety over a field of characteristic zero, and D a Weil $\mathbb{Q}$-divisor on X. Let $U \subset X$ be the largest open subset such that $(U, D|_U)$ is semi-snc. Then there is a morphism $f : \widetilde{X} \to X$ given by a composite of blowings-up with smooth (admissible) centers, such that*

1. *$(\widetilde{X}, \widetilde{D})$ is semi-snc;*

2. *f is an isomorphism over U.*

Remarks 1.3. (1) We say that a blowing-up (or its center) is *admissible* if its center is smooth and has simple normal crossings with respect to the exceptional divisor.

(2) In the special case that X is smooth, we say that D is a *simple normal crossings* or *snc* divisor on X if (X, D) is semi-snc (i.e., Definition 1.1 is satisfied with $p = 1$ at every point of X). This means that the irreducible components of D are smooth and

intersect as coordinate hyperplanes. Theorem 1.2, in this case, will be called *snc-strict log resolution* — this means *log resolution of singularities* of D by a morphism that is an isomorphism over the snc locus (see Theorem 2.14 below). The latter is proved in [BM11, Thm. 3.1]. Earlier versions can be found in [Sza94], [BM97, Sec. 12] and [Kol08].

Theorem 1.2, in the special case that $D = 0$, also follows from the earlier results; see Theorem 2.14 below. Both Theorems 2.13 and 2.14 are important ingredients in the proof of Theorem 1.2. Theorem 2.13 is used to reduce Theorem 1.2 to the case that X has only snc singularities. When X has only snc singularities Theorem 2.14 is used to begin an induction on the number of components of X.

(3) The desingularization morphism of Theorem 1.2 is functorial in the category of algebraic varieties over a field of characteristic zero with a fixed ordering on its irreducible components and with respect to étale morphisms that preserve the number of irreducible components —both of X and D—passing through every point. See Section 4.7. Note that a desingularization that avoids semi-snc and in particular snc points cannot be functorial with respect to all *étale* morphisms in general (as is the case for functorial resolution of singularities), because a normal crossings point becomes snc after an *étale* morphism; see Definitions 2.2. (Non-simple normal crossings are to be eliminated while simple normal crossings are to be preserved.) Therefore we must restrict functoriality to a smaller class of morphisms.

(4) Theorem 1.2 holds also with the following stronger version of condition 2: The morphism f is a composite $\sigma_1 \circ \ldots \circ \sigma_t$ of blowings-up σ_i, where each σ_i is an isomorphism over the semi-snc locus of the transform of (X, D) by $\sigma_1 \circ \ldots \circ \sigma_{i-1}$. Our proof provides this stronger statement, by using a stronger version of log resolution, where every blowing up is an isomorphism over the snc locus of the preceding transform of D. The latter strong version of log resolution has been proved in [BDVP11].

Our approach to partial resolution of singularities is based on the idea developed in [BM11] and [BLM11] that the desingularization invariant of [BM97] together with natural geometric information can be used to characterize and compute local normal forms of mild singularities. The local normal forms in the latter involve monomials in exceptional divisors that can be simplified or *cleaned* by desingularization of invariantly defined monomial marked ideals. These ideas are used in [BM11] and [BDVP11] in the proofs of log resolution by a morphism which is an isomorphism over the snc locus, and are also used in [BM11] to treat other problems stated in [Kol08], where one wants to find a class of singularities that have to be admitted if *normal crossings* singularities in a weaker local analytic or formal sense are to be preserved, see Definition 2.3.

In [BM11] and [BDVP11], the mild singularities (for example, simple normal crossings singularities) are all *hypersurface* singularities (see Definition 2.1). The desingularization invariant for a hypersurface is simpler than for general varieties because it begins with the *order* at a point, rather than with the *Hilbert-Samuel function*, as in the general case. Semi-simple normal crossings singularities (Definition 1.1) cannot be described as singularities of a hypersurface in an ambient smooth variety. An essential feature of this thesis is our use of the Hilbert-Samuel function and the desingularization invariant based on it to characterize semi-snc singularities. The idea of using the desingularization invariant to find local normal forms appears below in Section 4.2.

Chapter 2

Preliminaries

Definition 2.1. We say that X is a *hypersurface* at a point a if, locally at a, X can be defined by a principal ideal on a smooth variety.

Definitions 2.2 (cf. Remark 1.3(1))**.** Let X be an algebraic variety over a field of characteristic zero, and D a Weil $\mathbb{Q}$-divisor on X. The pair (X, D) is said to be *simple normal crossings (snc)* at a closed point $a \in X$ if X is smooth at a and there is a regular coordinate neighborhood U of a with a system of coordinates $(x_1, x_2, \ldots, x_n)$ such that $\operatorname{Supp} D|_U = (x_1 x_2 \ldots x_k = 0)$, for some $k \leq n$ (or perhaps $\operatorname{Supp} D|_U = \emptyset$). Clearly, the set of snc points is open in X. The *snc locus* of (X, D) is the largest subset of X on which (X, D) is snc. The pair (X, D) is *snc* if it is snc at every point of X.

Likewise, we will say that an algebraic variety X is *simple normal crossings (snc)* at $a \in X$ if there is a neighbourhood U of a in X and a local embedding $X|_U \overset{\iota}{\hookrightarrow} Y$, where Y is a smooth variety, such that $(Y, X|_U)$ is simple normal crossings at $\iota(a)$. (Thus, if X is snc at a, then X is a *hypersurface* at a.)

Definition 2.3. The pair (X, D) is called *normal crossings (nc)* at $a \in X$ if there is an étale morphism $f : U \to X$ and a point $b \in U$ such that $a = f(b)$ and $(U, f^*(D))$ is snc at the point b.

Definition 2.4. If $D = \sum a_i D_i$, where D_i are prime divisors, then D_{red} denotes $\sum D_i$, i.e. D_{red} is $\operatorname{Supp} D$ considered as a divisor.

Example 2.5. The curve $X := (y^2 + x^2 + x^3 = 0) \subset \mathbb{A}^2$ is nc but not snc at 0. It is not snc because it has only one irreducible component which is not smooth at 0. But X is nc at 0 because X has two analytic branches at 0 which intersect transversely.

It is important to distinguish between nc and snc. For example, the analogues for nc of log resolution preserving the nc locus or of Theorem 1.2 are false:

Example 2.6. Consider the pair $(\mathbb{C}^3, D)$, where $D = (x^2 - yz^2 = 0)$. The singularity at 0 is called a *pinch point*. The pair is nc at every point except the origin. The analogue of Theorem 1.2 for nc fails in this example because we cannot get rid of the pinch point without blowing up the y-axis, according to the following argument of Kollár (see [Kol08, Ex. 8]). The hypersurface D has two sheets over every non-zero point of $(z = 0)$. Going around the origin in $(z = x = 0)$ permutes the sheets, and this phenomenon persists after any birational morphism which is an isomorphim over the generic point of $(z = x = 0)$.

Definitions 2.7. If $f : X \to Y$ is a rational mapping and $Z \subset X$ is a subvariety such that f is defined in a dense subset Z_0, then we define the *birational transform* $f_*(Z)$ of Z as the closure of $f(Z_0)$ in Y. In the case that f is birational, then we have the notion of $f_*^{-1}(Z)$ for subvarieties $Z \subset Y$ such that f^{-1} is defined in a dense subset of Z. For a divisor $D = \sum \alpha_i D_i$, where the D_i are prime divisors, we define $f_*^{-1}(D) := \sum \alpha_i f_*^{-1}(D_i)$.

If $f : X \to Y$ is a birational mapping, we let $\operatorname{Ex}(f)$ denote the set of points $a \in X$ where f is not biregular; i.e., f^{-1} is not a morphism at $f(a)$. We consider $\operatorname{Ex}(f)$ with the structure of a reduced subvariety of X.

As before, consider (X, D), where X is an algebraic variety X over a field of characteristic zero and D is a Weil divisor. Let $f : \widetilde{X} \to X$ be a proper birational map and *assume that $Ex(f)$ is a divisor*. Then we define

$$D' := f_*^{-1}(D) \quad \text{and} \quad \widetilde{D} := D' + \operatorname{Ex}(f).$$

We call D' the *strict* or *birational transform* of D by f, and we call $\widetilde{D}$ the *total transform* of D. We also call $(\widetilde{X}, \widetilde{D})$ the *(total) transform* of (X, D) by f.

Remark 2.8. It will be convenient to treat D' and $\mathrm{Ex}(f)$ separately in our proof of Theorem 1.2 — we need to count the components of D' rather than those of $\widetilde{D}$. For this reason, we will work with data given by a triple (X, D, E), where initially (X, D) is the given pair and $E = \emptyset$. After a blowing-up $f : X' \to X$, we will consider the transformed data given by $(X', D', \tilde{E})$, where $D' := f_*^{-1}(D)$ as above and $\widetilde{E} := f_*^{-1}(E) + Ex(f)$.

We will write $f : (X', D') \to (X, D)$ to mean that $f : X' \to X$ is birational and D' is the strict transform of D by f.

Definition 2.9. We say that a triple (X, D, E), where D and E are both divisors on X, is *semi-snc* at $a \in X$ if $(X, D + E)$ is semi-snc at a (see Definition 1.1).

For economy of notation when there is no possibility of confusion, we will sometimes denote the transform of (X, D, E) by a sequence of blowings-up still simply as (X, D, E). Other constructions depending on X and D are also denoted by symbols that will be preserved after transformation by blowings-up. This convention is convenient for the purpose of describing an algorithm, and imitates computer programs written in *imperative languages*, where the state of a variable may change while preserving its name.

Definition 2.10. Let $\sigma : X' \to X$ be a birational morphism such that $Ex(\sigma)$ is a divisor on X'. We say that the "total transform" of X by σ is snc at $a \in X'$ if $(X', Ex(\sigma))$ is semi-snc at a.

Assume X is embedded in the smooth variety Y and $\sigma : Y' \to Y$ is a composition of blowings-up with smooth centers. Let X' be the strict transform of X by σ and assume that $Ex(\sigma)|_{X'}$ is a divisor on X'. Then the total transform $\sigma^{-1}(X)$ is snc at $a \in X'$ according to Definition 2.2 if and only if the total transform of X by $\sigma|_{X'} : X' \to X$ is snc at a according to Definition 2.10. Therefore Definition 2.10 is just extending a terminology that is usually used in the case of an embedded variety to the non-embedded case.

Example 2.11. Consider (X, D), where $X = (x_1^2 - x_2^2 x_3 = 0) \subset \mathbb{A}^3$ and $D = (x_1 = x_3 = 0)$. Let f denote the blowing-up of $\mathbb{A}^3$ with centre the x_3-axis. Then, the strict transform $X' = \widetilde{X}$ of X by f (i.e., the blowing-up of X with centre the x_3-axis) lies in one chart of f (the "x_2-chart") with coordinates (y_1, y_2, y_3) in which f is given by

$$x_1 = y_1 y_2, \quad x_2 = y_2, \quad x_3 = y_3.$$

Therefore we have $\widetilde{X} = (y_1^2 - y_3 = 0)$ and $\widetilde{D} = f_*^{-1}(D) + E$, where E is the exceptional divisor; $E = (y_1^2 - y_3 = y_2 = 0)$. Then

$$\begin{aligned}\tilde{D} =& (y_1 = y_3 = 0) + (y_1^2 - y_3 = y_2 = 0) \\ =& (y_1 = y_1^2 - y_3 = 0) + (y_1^2 - y_3 = y_2 = 0).\end{aligned}$$

We see that, at the origin in the system of coordinates $z_1 := y_1$, $z_2 := y_2$, $z_3 := y_3 - y_1^2$, the pair $(\widetilde{X}, \widetilde{D})$ is given by $\widetilde{X} = (z_3 = 0)$, $\widetilde{D} = (z_3 = y_1 = 0) + (z_3 = y_2 = 0)$, and is therefore snc.

Example 2.12. If $X = (xy = 0) \subset Y := \mathbb{A}^3$ and $D = a_1 D_1 + a_2 D_2$, where $D_1 = (x = z = 0)$ and $D_2 = (y = z = 0)$, then the pair (X, D) is semi-snc if and only if $a_1 = a_2$.

At a semi-snc point, the local picture is that X is a snc hypersurface in a smooth variety Y, and D is given by the intersection of X with a snc divisor H in Y which is snc together with X (in Example 2.12, $H = (z = 0)$). For this reason, we should have the same multiplicities when one component of H intersects different components of X.

2.1 Structure of the proof

The desingularization morphism from Theorem 1.2 is a composition of blowings-up with smooth centers. In the sequel, (X, D) will always denote a pair satisfying the assumptions of Theorem 1.2. Our proof of the theorem involves an algorithm for successively choosing the centers of blowings-up, that will be described precisely in section 4.3. We will give an idea of the main ingredients in the current subsection. As noted in Remark 1.3 (2), the following two theorems are previously known special cases of our main result that are used in its proof.

Theorem 2.13 (snc-strict desingularization)**.** *Let X denote a reduced scheme of finite type over a field of characteristic zero. Then, there is a finite sequence of blowings-up with smooth centers*

$$X := X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} \dots \xleftarrow{\sigma_t} X_t =: X', \tag{2.1}$$

such that, if D' denotes the exceptional divisor of (2.1)*, then (X', D') is semi-snc and $(X, 0) \leftarrow (X', D')$ is an isomorphism over the snc-locus, X^{snc}, of X.*

Theorem 2.13 can be strengthened so that, not only is $X' \to X$ an isomorphism over the snc locus of X but also σ_k is an isomorphism over the snc points of the total transform of X by $\sigma_1 \circ \dots \circ \sigma_{k-1}$, for every $k = 1, \dots, t$. (See [BDVP11]; cf. Remarks 1.3(4); see also Definition 2.10).

Theorem 2.14 (snc-strict log-resolution [BM11, Thm. 3.1])**.** *Consider a pair (X, D), as in Theorem 1.2. Assume that X is smooth. Then there is a finite sequence of blowings-up with smooth centers over the support of D (or its strict transforms)*

$$X := X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} \dots \xleftarrow{\sigma_t} X_t =: X',$$

such that the (reduced) total transform of D is snc and $X \leftarrow X'$ is an isomorphism over the snc locus of (X, D).

Remark 2.15. Theorems 2.13 and 2.14 are both functorial in the sense of Remark 1.3(3). Theorem 2.13 follows from functoriality in Theorem 2.14.

Proof of Theorem 2.13. We can first reduce Theorem 2.13 to the case that X is a hypersurface: If X is of pure dimension, this reduction follows simply from the strong desingularization algorithm of [BM97, BM08]. The algorithm involves blowing up with smooth centers in the maximum strata of the Hilbert-Samuel function $H_{X,a}$, see Chapter 3. The latter determines the local embedding dimension $e_X(a) := H_{X,a}(1) - 1$, see Lemma 3.22, so the algorithm first eliminates points of embedding codimension > 1 without modifying nc points.

When X is not of pure dimension the desingularization algorithm [BM11, BM08] may involve blowing up hypersurface singularities in higher dimensional components of X before X becomes a hypersurface everywhere. This problem can be corrected by a modification of the desingularization invariant described in [BMTnt]:

Let $\#(a)$ denote the number of different dimensions of irreducible components of X at $a \in X$. Let $q(a)$ be the smallest dimension of an irreducible component of X at a and set $d := dim(X)$. Then, instead of using the Hilbert-Samuel function as first entry of the invariant, we use the pair $\phi(a) := (\#(a), H_{X\times\mathbb{A}^{d-q(a)},(a,0)})$.

The original and modified invariants admit the same local presentations (in the sense of [BM97]). This implies that every component of a constant locus of one of the invariant is also a component of a constant locus of the other. The modification ensures that the irreducible components of the maximal locus of the usual invariant are blown up in a convenient order rather that at the same time. Since the modified invariant begins with $\#(a)$, points where there are components of different dimensions will be blown up first. Points with $\#(a) > 1$ are not hypersurface points.

If $\#(a) = \#(b) = 1$ and $q(a) < q(b)$, then the adjusted Hilbert-Samuel function guarantees that the point with larger value of

$$H_{X\times\mathbb{A}^{d-q(a)}}(1) = e(\cdot) + 1 + d - q(\cdot),$$

where $e = e_X$, will be blown up first. In particular, non-hypersurface singularities (where $e(\cdot) - q(\cdot) > 1$) will be blown up before hypersurface singularities (where $e(\cdot) - q(\cdot) \leq 1$).

We can thus reduce to the case in which X is everywhere a hypersurface. Then X locally admits a codimension one embedding in a smooth variety. For each local embedding we can apply Theorem 2.14. Functoriality in Theorem 2.14 guarantees that local desingularizations glue together to define global centers of blowing up for X. $\square$

We now outline the proof of the main theorem. First, we can use Theorem 2.13 to reduce to the case that X is snc; see Section 4.3, Step 1. Moreover, there is a simple combinatorial argument to reduce to the case that D is a *reduced* divisor (i.e., each $\alpha_i = 1$ in Definition 1.1); see Section 4.3, Step 4 and Section 4.6.

So we can assume that X snc and D reduced. We now argue by induction on the number of components of X.

To begin the induction (Section 4.3, Step 3), we use Theorem 2.14 to transform the first component of X together with the components of D lying in it, into a semi-snc pair. By induction, we can assume that the pair given by X minus its last component, together with the corresponding restriction of D, is semi-snc. (By *restriction* we mean the divisorial part of the restriction of D). To complete the inductive step, we then have to describe further blowings-up to remove the unwanted singularities in the last component of X. These blowings-up are separated into blocks which resolve the non-semi-snc singularities in a sequence of strata that exhaust the variety. Definition 2.16 below describes these strata.

Note first that, in the special case that X is snc, each component of D either lies in precisely one component of X (as, for example, if (X, D) is semi-snc) or it is a component of the intersection of a pair of components of X (e.g., if $X := (xy = 0) \subset \mathbb{A}^2$ and $D = (x = y = 0)$). We can reduce to the case that each component of D lies in precisely one component of X by blowing up to eliminate components of D that are contained in the singular locus of X (see Section 4.3, Step 2). Except for this step, our algorithm never involves blowing up with centers of codimension one in X.

Definition 2.16. Assume that D has no components in the singular locus of X. We define $\Sigma_{p,q} = \Sigma_{p,q}(X, D)$ as the set of points $a \in X$ such that a lies in exactly p components of X, and q is the minimum number of components of D at a which lie in any component of X. In the case of a triple (X, D, E), we write $\Sigma_{p,q} = \Sigma_{p,q}(X, D, E)$ to denote $\Sigma_{p,q}(X, D)$ (so the strata $\Sigma_{p,q}$ depend on X and D but not on E).

For example, if $X := (x_1x_2 = 0)$ and $D = (x_1 = y_1 = 0) + (x_2 = y_1y_2 = 0)$, then the origin is in $\Sigma_{2,1}$.

We remove non-semi-snc singularities iteratively in the strata $\Sigma_{p,q}$, for decreasing values of (p, q). The cases $p = 1$, $p = 2$ and $p \geq 3$ are treated differently.

In the case $p = 1$ the notions of snc and semi-snc coincide, so again we use snc-strict log resolution (Theorem 2.14). The cases $p = 2$ and $p \geq 3$ will be treated in sections 4.5 and 4.4, respectively. All of these cases are part of Step 3 in Section 4.3.

As remarked in Section 1, our approach is based on the idea that the desingularization invariant of [BM97] together with natural geometric information can be used to characterize mild singularities. For snc singularities, it is enough to use the desingularization invariant for a hypersurface together with the number of irreducible components at a point [BM11, §3].

In this thesis, the main object is a pair (X, D). If X is locally embedded as a hypersurface in a smooth variety Y (for example, if X is snc), then (the support of) D is of codimension two in Y. We will need the desingularization invariant for the support of D. The first entry in this invariant is the *Hilbert-Samuel function* of the local ring of $\operatorname{Supp} D$ at a point (see Chapter 3 and Section 4.1 below). Information coming from the Hilbert-Samuel function will be used to identify non-semi-snc singularities.

Chapter 3

Computation of the Hilbert-Samuel function

We will need to compute and compare the Hilbert-Samuel function for some singularities. In this chapter we summarize the notions allowing this computation. The *diagram of initial exponents* and *Hironaka division algorithm* are the tools for this computation. These are presented in the next two sections before we give the definition of the Hilbert-Samuel function in the last section of this chapter.

3.1 The diagram of initial exponents

Definition 3.1 (Partial order in $\mathbb{N}^n \times \{1, \dots, q\}$)**.** If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, put $|\beta| := \sum \beta_i$. We order the $(n+2)$-tuples $(|\beta|, j, \beta_1, \dots, \beta_n)$, where $(\beta, j) \in \mathbb{N}^n \times \{1, \dots, q\}$ lexicographically. This induces a total ordering of $\mathbb{N}^n \times \{1, \dots, q\}$.

Definition 3.2 (Lattice of exponents of formal power series)**.** Let $k[\![y]\!] = k[\![y_1, \dots, y_n]\!]$ be the ring of formal power series in n variables. For $G \in k[\![y]\!]^q$, $G = (G_1, \dots, G_q)$ write $G_j = \sum_{\beta \in \mathbb{N}^n} g_{\beta,j} y^\beta$, $j = 1, \dots, q$, where $g_{\beta,j} \in k$ and y^β denotes $y_1^{\beta_1} \cdots y_n^{\beta_n}$. We also let $y^{\beta,j}$ denote the q-tuple $(0, \dots, y^\beta, \dots, 0)$ with y^β in the j-th position and zeros elsewhere, so that $G = \sum_{\beta,j} g_{\beta,j} y^{\beta,j}$.

Definition 3.3 (Support and initial exponent)**.** Let the *support* of a power series $G \in k[\![y]\!]^q$ be $supp(G) := \{(\beta, j) \in \mathbb{N}^n \times \{1, \ldots, q\} : \; g_{\beta,j} \neq 0\}$ and $\nu(G)$ to be the smallest element of $supp(G)$ and $in(G)$ to be $g_{\nu(G)} y^{\nu(G)}$. We call $\nu(G)$ the *initial exponent* of G.

Definition 3.4 (Diagram of initial exponents)**.** Let R be a submodule of $k[\![y]\!]^q$. The *Diagram of initial exponents*, $\mathcal{N}(R)$, is defined to be $\{\nu(G) : \; G \in R\}$.

Definition 3.5 (The set of all diagrams)**.** Clearly, $\mathcal{N}(R) + \mathbb{N}^n = \mathcal{N}(R)$, where addition is defined by $(\beta, j) + \gamma := (\beta + \gamma, j)$, for $(\beta, j) \in \mathbb{N}^n \times \{1, \ldots, q\}, \; \gamma \in \mathbb{N}^n$. Define, for each positive integers n and q, $\mathcal{D}(n, q) := \{\mathcal{N} \subset \mathbb{N}^n \times \{1, \ldots, q\} : \; \mathcal{N} + \mathbb{N}^n = \mathcal{N}\}$.

Lemma 3.6. *Let $\mathcal{N} \in \mathcal{D}(n, q)$. Then there is a smallest finite subset, $\mathcal{B} = \mathcal{B}(\mathcal{N}) \subset \mathcal{N}$ such that $\mathcal{N} = \mathcal{B} + \mathbb{N}^n$.*

Proof. It is enough to prove this lemma for $q = 1$. Assume $\mathcal{N} = B_1 + \mathbb{N}^n = B_2 + \mathbb{N}^n$ and B_1, B_2 are minimal, by inclusion, satisfying that condition. Then for $a \in B_1$ there is $b_a \in B_2$ such that $a - b_a \in \mathbb{N}^n$. It also should happen that there is $a_{b_a} \in B_1$ such that $b_a - a_{b_a} \in \mathbb{N}^n$. From this we get that $(a - b_a) + (b_a - a_{b_a}) = a - a_{b_a} \in \mathbb{N}^n$. This means that $a = b_a = a_{b_a}$ since otherwise you would be able to have $\mathcal{N} = (B_1 - \{a\}) + \mathbb{N}^n$ as a can be generated using a_{b_a}. This means that actually $B_1 = B_2$ and there is really *a smallest* B satisfying $\mathcal{N} = B + \mathbb{N}^n$, which is also contained in any other set of generators of $\mathcal{N}$.

Call $B = B(\mathcal{N})$ to that minimum set of generators. If $n = 1$ the claim of the lemma is clear. The finitude of B follows from the well-order of $\mathcal{N}$. Assume that the lemma is true in dimension $n - 1$ and call $\pi : \mathcal{N}^n \to \mathcal{N}^{n-1}$ to the projection onto the first $n - 1$ components. Let $b_1, b_2, \ldots$ be the elements of B taken in an increasing sequence in the ordering defined at the beginning of the section. Clearly $\pi(\mathcal{N}) = \pi(B) + \mathbb{N}^{n-1}$. This means that the minimum set of generators of $\pi(\mathcal{N})$ is a subset of $\pi(B)$. Therefore there is N such that $C = \{\pi(b_1), \ldots, \pi(b_N)\}$ generates $\pi(N)$. We claim that $\mathcal{N} = \{b_1, \ldots, b_N\} + \mathbb{N}$, i.e. $\{b_1, \ldots, b_N\}$ generates $\mathcal{N}$ and since B is a subset of any set of generators it should be finite. In fact, take $b = b_i$ with $i > N$ and call $\pi^{-1}(a) := (a, 0)$ for $a \in \mathbb{N}^{n-1}$. Observe that $\pi(b) = \pi(b_j) + a$ with $j \leq N$ and $a \in \mathbb{N}^{n-1}$. Then $b = b_j + \pi^{-1}(a) + (0, \ldots, 0, (b)_{N+1})$, where $(b)_{N+1}$ denotes the last components of b. □

Definition 3.7 (Vertices of a diagram)**.** We call the minimum set $B = B(\mathcal{N})$ of generators the *set of vertices* of $\mathcal{N}$, see Lemma 3.6.

Definition 3.8 (Order in the set of diagrams)**.** The set $\mathcal{D}(n, q)$ is totally ordered as follows: Let $\mathcal{N}^1, \mathcal{N}^2 \in \mathcal{D}(n, q)$. For each $i = 1, 2$, let (β_k^i, j_k^i), $k = 1, \ldots, t_i$, denote the vertices of $\mathcal{N}^1$ and $\mathcal{N}^2$ indexed in ascending order. After perhaps interchanging $\mathcal{N}^1$ and $\mathcal{N}^2$, there exists $t \in \mathbb{N}$ such that $(\beta_k^1, j_k^1) = (\beta_k^2, j_k^2)$, $k = 1, \ldots, t$, and either

1. $t_1 = t = t_2$,

2. $t_1 > t = t_2$ or

3. $t_1, t_2 > t$ and $(\beta_{k+1}^1, j_{k+1}^1) < (\beta_{k+1}^2, j_{k+1}^2)$.

in case (1), $\mathcal{N}^1 = \mathcal{N}^2$. In case (2) and (3) we say that $\mathcal{N}^1 < \mathcal{N}^2$.

Remark 3.9. Clearly, if $\mathcal{N}^1 \supset \mathcal{N}^2$ then $\mathcal{N}^1 < \mathcal{N}^2$.

3.2 Hironaka division algorithm

The following theorem of Hironaka [Hir64] is a generalization of the Euclidean division algorithm for polynomials. As before, let $G := (G_1, \ldots, G_q) \in k[\![y]\!]^q$ such that $G_i \neq 0$ for $i = 1, \ldots, q$, $\alpha_i := \nu(G_i)$, for $i = 1, \ldots, q$.

Definition 3.10 (Decomposition of a diagram)**.** Using α_i, it can be constructed the following decomposition of $\mathbb{N}^n$. Let

$$\Delta_i := (\alpha_i + \mathbb{N}^n) - \bigcup_{j=1}^{i-1} \Delta_j, \text{ for } i = 1, \dots, q, \tag{3.1}$$

$$\square_0 := \mathbb{N}^n - \bigcup_{i=1}^{q} \Delta_i, \tag{3.2}$$

$$\tag{3.3}$$

and define $\square_i \subset \mathbb{N}^n$ by $\Delta_i = \alpha_i + \square_i, \ i = 1, \dots, q$.

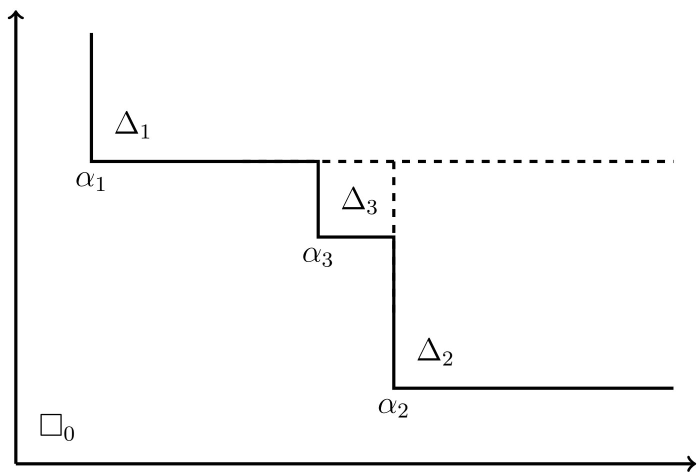

Figure 3.1: Decomposition of the diagram assuming $\alpha_1 < \alpha_2 < \alpha_3$.

Theorem 3.11 (Hironaka division algorithm)**.** *Given $F \in k[\![y]\!]$, there are unique $Q_i \in k[\![y]\!]$ and $R \in k[\![y]\!]$ such that supp$(Q_i) \subset \square_i$, supp$(R) \subset \square_0$ and $F = \sum_{i=1}^{q} Q_i G_i + R$.*

Proof. To proof *uniqueness* notice that $\nu(Q_i G_i) = \nu(Q_i) + \nu(G_i) \in \Delta_i$ and $\nu(R) \in \square_0$. Since the initial exponents lie in disjoints regions of $\mathbb{N}^n$, if $F = 0$ then necessarily $R = 0$ and $Q_i = 0$.

The *existence* will be proven by constructing, algorithmically the Q_i and R. There exist $Q_i^0 \in k[\![y]\!]$, $i = 1, \ldots, q$, and $R^0 \in k[\![y]\!]$ such that

$$F = \sum_{i=1}^{q} Q_i^0 y^{\alpha_i} + R^0,$$

$$\alpha_i + supp(Q_i^0) \subset \Delta_i, \ i = 1, \ldots, q$$

$$supp(R^0) \subset \Delta_0.$$

Write $G_i = \sum_\beta g_{\beta,i} y^\beta$ and define

$$Q_i(F) := (g_{\alpha_i,i})^{-1} Q_i^0 \in k[\![y]\!],$$

$$R(F) := R^0 \in k[\![y]\!].$$

Observe that $\nu(Q_i(F) g_{\alpha_i,i} y^{\alpha_i}) \geq \nu(F)$ and $\nu(R(F)) \geq \nu(F)$. Let

$$\begin{aligned} E(F) &:= F - \sum_{i=1}^{q} Q_i(F) G_i - R(F) \\ &= \sum_{i=1}^{q} Q_i(F)(g_{\alpha_i,i} y^{\alpha_i} - G_i) \end{aligned}$$

We have the following relation of initial exponents

$$\begin{aligned} \nu(E(F)) &= min_i(\nu(Q_i(F)(g_{\alpha_i,i} y^{\alpha_i}))) \\ &= \nu(Q_i(F)) + \nu(g_{\alpha_i,i} y^{\alpha_i} - G_i) \\ &> \alpha_i + \nu(Q_i(F)) \\ &\geq \nu(F). \end{aligned}$$

Define

$$Q_i := \sum_{k=0}^{\infty} Q_i(E^k(F))$$

$$R := \sum_{k=0}^{\infty} R(E^k(F)),$$

where $E^0(F) := F$ and $E^k(F) := E(E^{k-1}(F))$, $k \geq 1$. Then the series above converge in the Krull topology in $k[\![y]\!]$. Since for all $l \in \mathbb{N}$

$$F - \sum_{i=1}^{q} \sum_{k=0}^{l} Q_i(E^k(F)) G_i - \sum_{k=0}^{l} R(E^k(F)) = E^{l+1}(F),$$

then $F = \sum_{i=1}^{q} Q_i G_i + R$. $\square$

Remark 3.12. Let m denote the maximal ideal of $k[\![y]\!]$. In theorem 3.11, if $n \in \mathcal{N}$ and $F \in m^n$, then $R \in m^n$ and each $Q_i \in m^{n-|\alpha_i|}$, where $m^l := k[\![y]\!]$ if $l \leq 0$.

In analogy to the Gröbner basis for ideals in polynomial rings, we have the notion, in the ring of formal power series, of *standard basis*.

Corollary 3.13 (Standard basis). *Let M be a submodule of $k[\![y]\!]^q$, $\mathcal{N} := \mathcal{N}(M)$ its diagram of initial exponents, and (α_i, j_i), $i = 1, \ldots, t$ the vertices of $\mathcal{N}$ taken in increasing order. Choose $G_i \in M$ such that $\nu(G_i) = (\alpha_i, j_i)$, $i = 1, \ldots, t$ and let $\{\Delta_i, \Box_0\}$ be the decomposition of $\mathcal{N}^n \times \{1, \ldots, q\}$ determined by the vertices of $\mathcal{N}$, see Definition 3.10. Then:*

1. *$\mathcal{N} = \bigcup_{i=1}^{t} \Delta_i$, and the G_i generate M.*

2. *There is a unique set of generators F_i, $i = 1, \ldots, t$, of M such that for each each i, $in(F_i) = y^{\alpha_i, j_i}$ and $supp(F_i - y^{\alpha_i, j_i}) \subset \Box_0$.*

The set of generators $F_1, \ldots, F_t$ is called the *standard basis* of M.

Proof. The first part of Item (1) is clear from its definition. If $F \in M$ then its remainder R in the division by $(G_1, \ldots, G_t)$ must be zero. In fact, $supp(R) \cap \mathcal{N} = \emptyset$ while $F \in M$ implies $\nu(R) \in \mathcal{N}$.

To prove Item (2), for each $i = 1, \ldots, t$ consider the division of y^{α_i, j_i} by $(G_1, \ldots, G_t)$. We can write

$$y^{\alpha_i, j_i} = \sum_{k=1}^{t} Q_{k,i} G_k + R_i.$$

Let $F_i := y^{\alpha_i, j_i} - R_i$. It is clear that $F_i \in M$, and $in(F_i) = y^{\alpha_i, j_i}$ and $supp(F_i - y^{\alpha_i, j_i}) \subset \Box_0$. Uniqueness is clear, since any set other set of generators of M with these properties must differ from (F_i) in an element supported in the complement of $\mathcal{N}$ and therefore the difference must be zero. $\Box$

3.3 Hilbert-Samuel function

We begin with the definition of the Hilbert-Samuel function and its relationship with the diagram of initial exponents (cf. [BM89]).

Definition 3.14 (Hilbert-Samuel function)**.** Let A denote a Noetherian local ring A with maximal ideal $\mathfrak{m}$. The *Hilbert-Samuel function* $H_A \in \mathbb{N}^{\mathbb{N}}$ of A is defined by

$$H_A(k) := \text{length}\,\frac{A}{\mathfrak{m}^{k+1}}, \quad k \in \mathbb{N}.$$

If $I \subset A$ is an ideal, we sometimes write $H_I := H_{A/I}$. If X is an algebraic variety and $a \in X$ is a closed point, we define $H_{X,a} := H_{\mathcal{O}_{X,a}}$, where $\mathcal{O}_{X,a}$ denotes the local ring of X at a.

Definition 3.15 (Order in the set of Hilbert-Samuel functions)**.** Let $f, g \in \mathbb{N}^{\mathbb{N}}$. We say that $f > g$ if $f(n) \geq g(n)$, for every n, and $f(m) > g(m)$, for some m. This relation induces a partial order on the set of all possible values for the Hilbert-Samuel functions of local rings.

Note that $f \nleq g$ if and only if either $f > g$ or f is incomparable with g.

Let $\widehat{A}$ denote the completion of A with respect to $\mathfrak{m}$. Then $H_A = H_{\hat{A}}$, see [Mat80, §24.D]. If A is regular, then we can identify $\widehat{A}$ with a ring of formal power series, $\mathbb{K}[\![x]\!]$, where $x = (x_1, \ldots, x_n)$, see [Eis95, Theorem 7.7]. Then $\mathfrak{n} := (x_1, \ldots, x_n)$ is the maximal ideal of $\mathbb{K}[\![x]\!]$. If $I \subset \mathbb{K}[\![x]\!]$ is an ideal, then

$$H_I(k) := \dim_{\mathbb{K}} \frac{\mathbb{K}[\![x]\!]}{I + \mathfrak{n}^{k+1}}.$$

Definition 3.16. . Consider an ideal $I \subset K[\![x]\!]$. The *initial monomial ideal* $in(I)$ of I denotes the ideal generated by $\{in(f) : \; f \in I\}$. The *diagram of initial exponents* $\mathcal{N}(I) \subset \mathbb{N}^n$ is defined as

$$\mathcal{N}(I) := \{\nu(f) : f \in I \setminus \{0\}\}.$$

Proposition 3.17. *For every $k \in \mathbb{N}$, $H_I(k) = H_{\mathrm{mon}(I)}(k)$ is the number of elements $\alpha \in \mathbb{N}^n$ such that $\alpha \notin \mathcal{N}(I)$ and $|\alpha| \leq k$.*

Proof. From Theorem 3.11 each $F \in \mathbb{K}[\![x]\!]$ has a unique remainder R_F in the Hironaka division by the standard basis of I. This remainder is supported in the complement of $\mathcal{N}(I)$. From Remark 3.12, $\dim_{\mathbb{K}} \frac{\mathbb{K}[\![x]\!]}{I+\mathfrak{n}^{k+1}}$ is equal to the dimension of the subespace of elements of $\mathbb{K}[\![x]\!]$ supported in $\{\alpha \in \mathbb{N}^n : \ \alpha \notin \mathcal{N}(I) \text{ and } |\alpha| \leq k\}$.

The equality $H_I(k) = H_{in(I)}(k)$ follows from the above since $\mathcal{N}(I) = \mathcal{N}(in(I))$ □

Definition 3.18 (Hilbert-Samuel function of a diagram)**.** Let $\mathcal{N} \in \mathcal{D}(n, 1)$. The previous proposition justifies calling Hilbert-Samuel function of the diagram $\mathcal{N}$, $H_{\mathcal{N}}(k)$ to the number of $\alpha \in \mathbb{N}^n$ such that $\alpha \notin \mathcal{N}$ and $|\alpha| \leq k$.

Definition 3.19. Let $H_{p,q} = H_{p,q,n}$ denote the Hilbert-Samuel function of the ideal $(x_1 \cdots x_p, y_1 \cdots y_q)$ in a ring of formal power series $\mathbb{K}[\![x_1, \ldots, x_p, y_1, \ldots, y_{n-p}]\!]$, where $p+q \leq n$.

We will omit the n since it will be fixed throughout the arguments using $H_{p,q}$.

Proposition 3.17 allows us to compute Hilbert-Samuel functions.

Example 3.20. Assume that in $\mathbb{A}^n$, with coordinates $x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_{n-p-q}$ we have $X := (x_1 \cdots x_p = y_1 \cdots y_q = 0)$. Without loss of generality let's assume that $p \leq q$. Let a be the origin. Then

$$H_{X,a}(r) = \begin{cases} \binom{n+r}{n}, \text{ for } r < p \\ \binom{n+r}{n} - \binom{n+r-p}{n}, \text{ for } p \leq r < q \\ \binom{n+r}{n} - \binom{n+r-p}{n} - \binom{n+r-q}{n}, \text{ for } q \leq r < p+q \\ \binom{n+r}{n} - \binom{n+r-p}{n} - \binom{n+r-q}{n} + \binom{n+r-p-q}{n}, \text{ for } p+q \leq r \end{cases}$$

Notice how the Hilbert-Samuel function detects, not only p, which is the order of the ideal, but also q since $p+q$ is the minimum s such that the Hilbert-Samuel function is equal to a polynomial for all $r \geq s$.

Definition 3.21 (Minimal embedding dimension)**.** Let X be an algebraic variety and $a \in$. We say that the germ X_a of X at a has minimal embedding dimension $e_{X,a} \in \mathbb{N}$ if this number is the minimum d such that there is a closed embedding $X_a \hookrightarrow \mathbb{A}^d$.

Lemma 3.22. *Let X be an algebraic variety and $a \in X$. Then, $H_{X,a}(1) - 1$ is equal to the minimal embedding dimension of the germ of X_a.*

Proof. Let $e = H_{X,a}(1) - 1$. By definition $H_{X,a}(1) = 1 + \dim_k \frac{m}{m^2}$, where m is the maximal ideal of $\mathcal{O}_{X,a}$ and $k := \mathcal{O}_{X,a}/m$. Let $y_1, \ldots, y_e$ be elements of m with projections to m/m^2 giving a basis of it. Mapping $x_i \mapsto y_i$, for $i = 1, \ldots, e$ we get $k[x_1, \ldots, x_e] \to \mathcal{O}_{X,a}$ which induces an embedding $X_a \hookrightarrow \mathbb{A}^e$. This embedding is minimal because the dimension of the cotangent space m/m^2 of X_a at the origin is equal to e. □

Chapter 4

Desingularization preserving semi-snc

4.1 The Hilbert-Samuel function controling the geometry of the divisor

Lemma 4.4 of this section will play an important part in our use of the Hilbert-Samuel function to characterize semi-snc points, in Section 4.2. See Chapter 3 for the definition and ways to compute the Hilbert-Samuel function. Recall Definition 3.19 for the definition of $H_{p,q}$. The $H_{p,q}$ are precisely the values that the Hilbert-Samuel function of $\operatorname{Supp} D$ can take at semi-snc points.

Definition 4.1. We can use the partial ordering of the set of all Hilbert-Samuel functions to also order the strata $\Sigma_{p,q}$ (see Definition 2.16). We say that Σ_{p_1,q_1} *precedes* Σ_{p_2,q_2} if $(\delta(p_1), H_{p_1,q_1}) > (\delta(p_2), H_{p_2,q_2})$ in the lexicographic order, where

$$\delta(p) = \begin{cases} 3, \text{ if } p \geq 3 \\ p \text{ otherwise.} \end{cases}$$

This order corresponds to the order in which we are going to attempt removing the non-semi-snc from these strata.

The following two examples illustrate the kind of information we can expect to get from the Hilbert-Samuel function.

Example 4.2. Let $X := X_1 \cup X_2$, where $X_1 := (x_1 = 0)$, $X_2 := (x_2 = 0) \subset \mathbb{A}^4_{(x_1,x_2,y,z)}$. Note that, if (X, D) is semi-snc, then $\operatorname{Supp} D|_{X_1} \cap \operatorname{Supp} D|_{X_2}$ has codimension 2 in X. Consider $D := (x_1 = y = 0) + (x_2 = z = 0)$. Then, the origin is not semi-snc. In fact, $\operatorname{Supp} D|_{X_1} \cap \operatorname{Supp} D|_{X_2} = (x_1 = x_2 = y = z = 0)$, which has codimension 3 in X. The Hilbert-Samuel function of $\operatorname{Supp} D$ at the origin detects such an anomaly in codimension at a point in a given stratum $\Sigma_{p,q}$ (see Remark 4.7 and Lemma 4.8). In the preceding example the origin here belongs to $\Sigma_{2,1}$ but the Hilbert-Samuel function is not equal to $H_{2,1}$. In fact, the ideal of $\operatorname{Supp} D$ (as a subvariety of $\mathbb{A}^4$) is $(x_1, y) \cap (x_2, z) = (x_1, y) \cdot (x_2, z)$, which has order 2 while (x_1x_2, y), which is the ideal of the support of D at a semi-snc point in $\Sigma_{2,1}$, is of order 1. The Hilbert-Samuel function determines the order and therefore differs in these two examples.

Example 4.3. This example will show that, nevertheless, the Hilbert-Samuel function together with the number of components of X and D does not suffice to characterize semi-snc. Consider $X := (x_1x_2 = 0) \subset \mathbb{A}^4_{(x_1,x_2,y,z)}$ and $D := D_1 + D_2 := (x_1 = y = 0) + (x_2 = x_1 + yz = 0)$. Again the origin is not semi-snc, since the intersection of D_1 with $X_2 := (x_2 = 0)$ and of D_2 with $X_1 := (x_1 = 0)$ are not the same (as they should be at semi-snc points). On the other hand, the Hilbert-Samuel function does not detect the non-semi-snc singularity, since it is the same for the ideals $(x_1, y) \cap (x_2, x_1 + yz)$ and (x_1x_2, y). In fact, the Hilbert-Samuel function is determined by the initial monomial ideal of $\operatorname{Supp} D$. Since $(x_1, y) \cap (x_2, x_1 + yz) = (x_1x_2, x_2y, x_1 + yz)$, we compute its initial monomial ideal as (x_1, x_2y). The latter has the same Hilbert-Samuel function as (x_1x_2, y). This example motivates definition 4.15, which is the final ingredient in our characterization of the semi-snc singularities (Lemma 4.16).

In Example 4.3, although the intersections of D_1 with X_2 and of D_2 with X_1 are not the same, the intersection $D_2 \cap X_1$ has the same components as $D_1 \cap X_2$ plus some extra components (precisely, plus one extra component $(x_1 = x_2 = z = 0)$). The following lemma shows that this is the worst that can happen when we have the correct value $H_{p,q}$ of the Hilbert-Samuel function in $\Sigma_{p,q}$.

Lemma 4.4. *Assume that (X, D) is locally embedded in a coordinate chart of a smooth variety Y with a system of coordinates $(x_1, \dots, x_p, y_1, \dots, y_q, w_1, \dots, w_{n-p-q})$. Assume $X = (x_1 \cdots x_p = 0)$. Suppose that D is a reduced divisor (so we view it as a subvariety), with no components in the singular locus of X, given by an ideal I_D at $a = 0$ of the form*

$$I_D = (x_1 \cdots x_{p-1}, y_1 \cdots y_r) \cap (x_p, f). \tag{4.1}$$

Consider $a \in \Sigma_{p,q}$, where $p \geq 2$. (In particularq is the minimum of r and the number of irreducible factors of $f|_{(x_p=0)}$.) Let H_D denote the Hilbert-Samuel function of I_D.

Then $H_D = H_{p,q}$ if and only if we can choose f so that $\mathrm{ord}(f) = q$, $r = q$ and $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_r, x_p)$. Moreover, if either $f \notin (x_1 \cdots x_{p-1}, y_1 \cdots y_r, x_p)$, $\mathrm{ord}(f) > q$ or $r > q$ then $H_D \nleq H_{p,q}$ (see Definition 3.15 ff.).

Remark 4.5. It follows immediately from the conclusion of the lemma that $H_D \not< H_{p,q}$ at a point in $\Sigma_{p,q}$.

Proof of Lemma 4.4. First we will give a more precise description of the ideal I_D. Let $I \subset \{1, 2, \dots, p-1\} \times \{1, 2, \dots, r\}$ denote the set of all (i, j) such that $(x_p, f) + (x_i, y_j)$ defines a subvariety of codimension 3 in the ambient variety Y (i.e. a subvariety of codimension 2 in X). For such (i, j), any element in (x_p, f) belongs to the ideal (x_p, x_i, y_j). Set $G := \bigcap_{(i,j)\in I}(x_i, y_j)$ and $H := \bigcap_{(i,j)\notin I}(x_i, y_j)$; note that these are the prime decompositions. Then any element of (x_p, f) then belongs to $\bigcap_{(i,j)\in I}(x_p, x_i, y_j) = (x_p) + G$. Therefore we can take $f \in G$. Observe that we still have $f \notin (x_i, y_j)$ for $(i, j) \notin I$.

We claim that

$$G \cap (x_p, f) = (x_p) \cdot G + (f). \tag{4.2}$$

To prove (4.2): The inclusion $G \cap (x_p, f) \supset (x_p) \cdot G + (f)$ is clear since $f \in G$. To prove the other inclusion, consider $a \in G \cap (x_p, f)$. Write $a = fg_1 + x_p g_2$. Then $x_p g_2 \in G = \bigcap_{(i,j)\in I}(x_i, y_j)$. Since $x_p \notin (x_i, y_j)$, for every $(i, j) \in G$, we have $g_2 \in G$. It follows that $a = x_p g_2 + fg_1 \in (x_p) \cdot G + (f)$, as required.

We now claim that

$$H \cap [G \cdot (x_p) + (f)] = (x_p) \cdot [G \cap H] + H \cap (f) : \tag{4.3}$$

As in the previous claim, the inclusion $H \cap [G \cdot (x_p) + (f)] \supset (x_p) \cdot [G \cap H] + H \cap (f)$ is clear. To prove the other inclusion, consider $a \in H \cap [G \cdot (x_p) + (f)]$. Then $a = fg_1 + x_p g \in H$, where $g \in G$. This implies that $fg_1 \in (x_p) + H = \bigcap_{(i,j)\notin I}(x_p, x_i, y_j)$. Consider $(i, j) \notin I$. Assume that $f \in (x_p, x_i, y_j)$. Then there is an irreducible factor f_0 of f, such that $f_0 \in (x_p, x_i, y_j)$. If $f_0 = x_p h_1 + x_i h_2 + y_j h_3$ with $h_3 \neq 0$, then $(x_p, f) + (x_i, y_j) = (x_p, x_i, y_j)$, which contradicts $(i, j) \notin I$. Now, if $h_3 = 0$, then $f_0 = x_p h_1 + x_i h_2 \in (x_p, x_i)$, which implies $f \in (x_p, x_i)$, contradicting the assumption that D has no component in the singular locus of X. Thus $f \notin (x_p, x_i, y_j)$. Since (x_p, x_i, y_j) is prime, it follows that $g_1 \in (x_p) + H$ and $g_1 = x_p g_{11} + h$, where $h \in H$. Thus $a = fh + x_p(fg_{11} + g)$ and therefore $x_p(fg_{11} + g) \in H$. Since x_p is not in any of the prime factors of H, it follows that $fg_{11} + g \in H$. Thus $a \in (x_p) \cdot [G \cap H] + H \cap (f)$.

By (4.2) and (4.3),

$$\begin{aligned} I_D &= G \cap H \cap (x_p, f) \\ &= H \cap [G \cdot (x_p) + (f)] \\ &= (x_p) \cdot [H \cap G] + H \cap (f). \end{aligned} \tag{4.4}$$

We are allowed to pass to the completion of the local ring of Y at a with respect to its maximal ideal. So we can assume we are working in a formal power series ring where $(x_1, \ldots, x_p, y_1, \ldots, y_{n-p})$ are the indeterminates. We can pass to the completion because this doesn't change the Hilbert-Samuel function, the order of f or ideal membership.

For simplicity, we use the same notation for ideals and their generators before and after completion.

We can compute the Hilbert-Samuel function H_D using the diagram of initial exponents of our ideal I_D, see Proposition 3.17. This diagram should be compared to the diagram of the ideal $(x_1 \cdots x_p, y_1 \cdots y_q)$, which has exactly two vertices, in degrees p and q.

All elements of $H \cap (f) = H \cdot (f)$ have order strictly greater than $\mathrm{ord}(f)$ (which is $\geq q$), unless $H = (1)$ and $\mathrm{ord}(f) = q$. Moreover, all elements of

$$(x_p) \cdot [G \cap H] = (x_1 x_2 \cdots x_{p-1} x_p,\ x_p y_1 y_2 \cdots y_r)$$

of order less than $q + 1$ have initial monomial divisible by $x_1 x_2 \cdots x_p$.

It follows that, if $f \notin (x_1 \cdots x_{p-1}, y_1 \cdots y_r)$ i.e. if $H \neq (1)$, then $H_D \nleq H_{p,q}$. To see this, first assume that $p \geq q+1$. Then all elements of the ideal $I_D = (x_p) \cap [H \cap G] + H \cap (f)$ have order $\geq q + 1$, but $(x_1 \cdots x_p, y_1 \cdots y_q)$ contains an element of order q. Therefore $H_D \nleq H_{p,q}$ (obvious from the diagram of initial exponents). Now suppose that $p < q + 1$. All elements of $(x_p) \cap [H \cap G]$ of order less than $q + 1$ have initial monomials divisible by $x_1 \cdots x_p$, while $y_1 \cdots y_q \in (x_1 \cdots x_p, y_1 \cdots y_q)$ has order $q < q + 1$ but its initial monomial is not divisible by $x_1 \cdots x_p$. Therefore we again get $H_D \nleq H_{p,q}$.

Assume that $ord(f) > q$. We have just seen that every element of $(x_p) \cap [H \cap G]$ of order $< q + 1$ has initial monomial divisible by $x_1 \cdots x_{p-1}$. Therefore every element of $I_D = (x_p) \cap [H \cap G] + H \cdot (f)$ of order $< q + 1$ has initial monomial divisible by $x_1 \cdots x_{p-1}$. But, in $(x_1 \cdots x_p, y_1 \cdots y_q)$, the element $y_1 \cdots y_q$ has order $q < q + 1$ but is not divisible by $x_1 \cdots x_{p-1}$. Therefore $H_D \nleq H_{p,q}$.

If $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_r)$, $\mathrm{ord}(f) = q$ but $r > q$, then the initial monomial of f is divisible by $x_1 \cdots x_{p-1}$. A simple computation shows that the ideal of initial monomials of I_D is

$$(x_1 \cdots x_p, x_p y_1 \cdots y_r, \mathrm{mon}(f)).$$

This follows from the fact that canceling the initial monomial of f using $x_1 \cdots x_p$ or

$x_py_1\cdots y_q$ leads to a function whose initial monomial is already in $(x_1\cdots x_p, x_py_1\cdots y_q)$. For convenience, write $a := x_1\cdots x_p$, $b := \mathrm{mon}(f)$ and $c := x_py_1\cdots y_q$. From the diagram it follows that $H_D \nleq H_{p,q}$ because the monomials that are multiples of both a and of b are not only those that are multiples of ab and therefore $H_D(q+1) > H_{p,q}(q+1)$.

It remains to show that if $f \in (x_1\cdots x_{p-1}, y_1\cdots y_r)$ (i.e., $H = (1)$), $r = q$ and that $\mathrm{ord}(f) = q$ then $H_D = H_{p,q}$. Assume that $H = (1)$ and that $\mathrm{ord}(f) = q$. The first assumption implies that

$$I_D = (x_1\cdots x_p, x_py_1\cdots y_q, f). \tag{4.5}$$

We consider two cases: (1) $p \le q$. Since $H = (1)$, $f \in G = I_D$. Therefore, we have one of the following options for the initial monomial of f.

$$\mathrm{mon}(f) = \begin{cases} y_1y_2\cdots y_q \\ x_1\cdots x_{p-1}\overline{y} \\ x_1\cdots x_{p-1}\overline{y}z \\ x1\cdots x_{p-1}z, \end{cases} \tag{4.6}$$

where $\overline{y}$ is a product of some of the y_j and z is a product of some of the remaining coordinates (possibly including some of the x_i). (In every case, the degree of the monomial is q.)

In each case in (4.6) we can compute the ideal of initial monomials of I_D is $(x_1\cdots x_p,$ $x_py_1\cdots y_q, \mathrm{mon}(f))$.

We want to prove now that, in all cases in (4.6), $H_{\mathrm{mon}(I_D)} = H_{p,q}$. For convenience, write $a := x_1\cdots x_p$, $b := \mathrm{mon}(f)$ and $c := x_py_1\cdots y_q$. In the first case of (4.6), the equality is precisely the definition of $H_{p,q}$. Note that, in the remaining cases, the Hilbert-Samuel function of the ideal (a, b) is larger than $H_{p,q}$ because the monomials that are multiples of both a and of b are not only those that are multiples of ab. For example, in the second case (i.e., $\mathrm{mon}(f) = x_1\cdots x_{p-1}\overline{y}$), such monomials are those of the form $a\overline{y}m = bx_pm$ where $m \notin (x_1\cdots x_{p-1})$. When $\deg(m) = d$, these terms have degree $q + d + 1$, but the monomial $x_p y_1 \cdots y_q m \in \mathrm{mon}(I_D)$ (of the same degree) does not belong to the ideal (a, b). This implies that the diagrams of initial exponents of the ideals I_D and $(x_1 \cdots x_p, y_1 \cdots y_q)$ have the same number of points in each degree. Therefore $H_D = H_{\mathrm{mon}(I_D)} = H_{p,q}$.

Case (2) $q < p$. Then (from (4.5)), the options for the initial monomial of f are:

$$\mathrm{mon}(f) = \begin{cases} y_1 \cdots y_q, & q < p - 1, \\ x_1 \cdots x_{p-1}, & q = p - 1. \end{cases}$$

In each of these cases, we can compute the initial monomial ideal of I_D. In the first case, $\mathrm{mon}(I_D) = (x_1 \cdots x_p, y_1 \cdots y_q)$. In the second case, $\mathrm{mon}(I_D) = (x_p y_1 \cdots y_q, x_1 \cdots x_{p-1})$. In both cases, $H_D = H_{p,q}$. This completes the proof of the lemma. □

Corollary 4.6. *In the settings of Lemma 4.4, if there are p', q' such that $H_{p',q'} \geq H_D$ at $a \in \Sigma_{p,q}$ then $H_{p',q'} \geq H_{p,q}$.*

Proof. Without loss of generality we can assume that $p' \leq q'$. As in the proof of Lemma 4.4 we pass to the completion of the local ring at a in Y. We also have that

$$I_D = (x_1 \cdots x_p, x_p y_1 \cdots y_r) + (f) \cap H.$$

Recall that $r \geq q$ and $\mathrm{ord}(f) \geq q$. If $p > q$ then $\mathrm{ord}(I_D) \geq q$. Since $H_{p',q'} \geq H_D$ we must have $p', q' \geq q$ and then $H_{p',q'} \geq H_{p,q}$. If $p \leq q$ then $\mathrm{ord}(I_D) = p$. Since $H_{p',q'} \geq H_D$ we must have $min(p', q') = p' \geq p = min(p, q)$. Any element of I_D of order $< q + 1$ has initial monomial divisible by $x_1 \cdots x_p$, therefore the inequality $H_{p',q'} \geq H_D$ is not possible if $q' < q$. Hence $p' \geq p$ and $q' \geq q$, i.e. $H_{p',q'} \geq H_{p,q}$. □

Remark 4.7. Lemma 4.4 is at the core of our proof of Theorem 1.2. The lemma describes the ideal of the support of D at a point $a \in \Sigma_{p,q}$, under the following assumptions:

1. X is snc at a and, after removing its last component, the resulting pair (with D) is semi-snc at a.

2. No component of D at a lies in the singular locus of X.

3. $H_{\operatorname{Supp} D,a} = H_{p,q}$.

Under these assumptions we see that

$$(x_p = 0) \cap (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) \subset (x_1 \cdots x_{p-1} = 0) \cap (x_p = f = 0)$$

(as in Example 4.3). Also

$$(x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) \cap (x_p = f = 0) = (x_p = x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0),$$

i.e. the intersection of $D^{p-1} := (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0)$ and $D_p := (x_p = f = 0)$ has only components of codimension 2 in X.

The previous statement is in fact true without the assumption (1) of Remark 4.7. It is possible that this stronger version is one of the steps needed to prove Theorem 1.2 without using an ordering of the components of X. The stronger lemma can be stated as follows.

Lemma 4.8. *Assume that X is snc and no component of D lies in the singular locus of X. Let X_i, $i = 1, \ldots, n$, be the irreducible components of X at a, and let D_i be the divisorial part of $D|_{X_i}$. If $a \in X$ belongs to the stratum $\Sigma_{p,q}$ and $H_{\operatorname{Supp} D,a} = H_{p,q}$, then, for every i, j, the irreducible components of the intersection $D_i \cap D_j$ are all of codimension 2 in X.*

In the proof of this lemma we will use repeatedly the following simple observations for elements f, g, x and y of a complete regular local ring.

Claim 4.9. *If $\operatorname{ord}(f) = \operatorname{ord}(g)$, $\operatorname{ord}(x) = 1$, $g \in (x, f)$ and $in(g) \notin (x)$, then $(x, f) = (x, g)$.*

Proof. We have that $g = xh_1 + fh_2$. Since $in(g) \notin (x)$, then $in(g) = in(fh_2)$. From $\operatorname{ord}(f) = \operatorname{ord}(g)$ we get that $\operatorname{ord}(h_2) = 0$. Hence $(x, f) = (x, fh) = (x, xh_1 + fh_2) = (x, g)$. □

Claim 4.10. *Assume that $f \notin (x, y)$, where x, y is part of a regular system of parameters. Then $(x, f) \cap (y) = (x, f) \cdot (y)$*

Proof. Let $a \in (x, f) \cap (y)$. Then we can write $a = xh_1 + fh_2$ and $a \in (y)$. Hence $fh_2 \in (x, y)$. Since $f \notin (x, y)$ then $h_2 \in (x, y)$. Hence we can write $a = xh_1 + fxh_3 + fyh_4$. Therefore $x(h_1 + fh_3) \in (y)$. Since $x \notin (y)$, then $h_1 + fh_2 \in (y)$. Hence $a \in (x, f) \cdot (y)$. □

Claim 4.11. *If x, y are part of a regular system of parameters, $f \notin (x, y, g)$ and $g \notin (x, y, f)$, and $\mathrm{ord}(f), \mathrm{ord}(g) \geq q$, then any element of $(x, f) \cap (y, g)$ of order $\leq q$ has initial monomial divisible by xy.*

Proof. Let $h \in (x, f) \cap (y, g)$, with $\mathrm{ord}(h) \leq q$. We can write $h = xa + fb$. If the initial monomial of h is not divisible by x then it is a monomial in fb. This implies that $q \geq \mathrm{ord}(h) \geq \mathrm{ord}(fb) \geq q$. Therefore b is a unit. This gives a contradiction because it implies that $f = hb^{-1} - xab^{-1} \in (x, y, g)$. □

Remark 4.12. Consider the diagram of initial exponents for an ideal $I_{a,b} := (a, b)$ generated by two monomials that do not divide each other. It consists of the union of two quadrants with vertices on the two vertices of the diagram given by the exponents of its two generators. Observe that, if a and b are relatively primes, then any point in the diagram lying in the intersection of the two quadrants corresponds to a monomial that is divisible by ab. This is not the case if a and b are not relatively primes.

If the two monomials were not relatively primes, then the two quadrants in the diagram would have a larger intersection. This is manifested in the Hilbert-Samuel function of the diagram, see Definition 3.18. In fact, if $J := (a', b')$ is an ideal generated by two monomials that are relatively primes and such that $\mathrm{ord}(a) = \mathrm{ord}(a')$ and $\mathrm{ord}(b) = \mathrm{ord}(b')$, then $H_I(k) = H_j(k)$, for $k < \mathrm{ord}(\mathrm{lcm}(a, b))$ but $H_I(k) > H_J(k)$, for $k \geq \mathrm{ord}(\mathrm{lcm}(a, b))$. This is because the point in the diagram corresponding to the $\mathrm{lcm}(a, b)$ has smaller degree if a and b are not relatively primes.

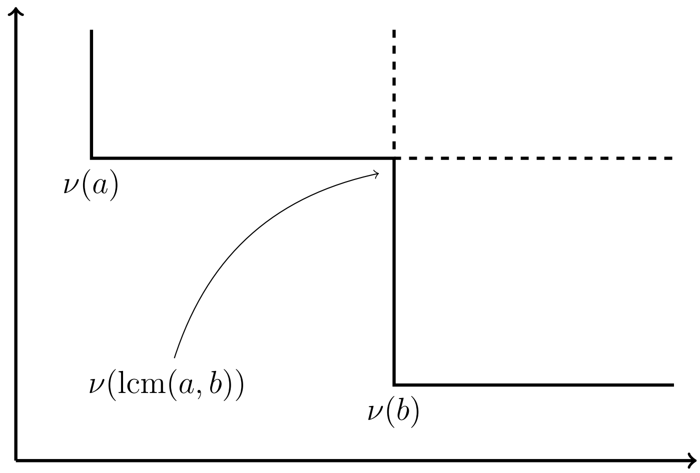

Figure 4.1: Diagram of $I = (a, b)$, for monomials a, b that do not divide each other.

Proof of Lemma 4.8. From the hypothesis we can locally embed X in a smooth variety Y, with a system of coordinates $x_1, \ldots, x_n$, with $n \geq p+q$, such that $X = (x_1 \cdots x_p = 0)$ and $D_i = (x_i = f_i = 0)$ for $i = 1, \ldots, p$ and some $f_i \in \mathcal{O}_{Y,a}$. We can pass to the completion of $\mathcal{O}_{Y,a}$ as this doesn't change the Hilbert-Samuel function or the dimensions of the intersections of the different D_i. Therefore for the remainder of the proof we will assume that the f_i are formal power series in $(x_1, \ldots, x_n)$. We can assume that f_i is intependent of x_i. Since no component of D lies in the singular locus of X, in particular $f_i \notin (x_j)$ for every $1 \leq i, j \leq p$.

Let

$$I_p := \bigcap_{i=1}^{p} (x_i, f_i).$$

In order to get a contradiction let us assume that $H_{I_p} = H_{p,q}$ and that the intersection of D_1 and D_2 has components of codimension in X different from 2. This implies that $f_1 \notin (x_1, x_2, f_2)$ and $f_2 \notin (x_1, x_2, f_1)$. From the assumption that $a \in \Sigma_{p,q}$ we get that for every i, $\mathrm{ord}(f_i) \geq q$. Therefore from Claim 4.11 and any element of $I_{D_1 \cap D_2}$ of order $\leq q$ has initial monomial divisible by x_1x_2. As a consequence, any element of I_p of order $\leq q$ has initial monomial divisible by x_1x_2.

Let $\alpha_1 < \ldots < \alpha_m$ be the vertices of the diagram of initial exponents of I_p, see

Definition 3.7. We must have $|\alpha_1| = min(p,q)$ and $|\alpha_2| = max(p,q)$. We also have that $x_1 \cdots x_p \in I_p$. If we assume $p \leq q$, then $\nu(x_1 \cdots x_p) = \alpha_1$, otherwise $\nu(x_1 \cdots x_p) = \alpha_2$. From now on, we will assume that $p \leq q$. To get the argument for $p > q$ it is enough to swap the names of α_1 and α_2 in the remainder of the proof.

Define $g_1 := x_1 \cdots x_p$. We will be choosing $g_i \in I_p$, $i = 1, \ldots, m$ such that $\nu(g_i) = \alpha_i$. By definition of α_i as vertices of the diagram of initial exponents of I_p, these g_i must exist. Since $|\alpha_2| = q$ then $\mathrm{ord}(g_2) = q$. Observe that $g_2 \in (x_1, f_1) \cap (x_2, f_2)$. This implies, by Claim 4.11, that $in(g_2) \in (x_1 x_2)$.

Let $K_2 := \{1 \leq i \leq p : \ in(g_2) \in (x_i)\}$ and observe that $1, 2 \in K_2$. We must also have $K_2 \neq \{1, \ldots, p\}$ since α_2 should be a vertex of the diagram of initial exponents of I_p, otherwise $in(g_2) \in (x_1 \cdots x_p)$, i.e. α_2 would be in the diagram who's vertex is α_1. By Claim 4.9, $(x_i, f_i) = (x_i, g_2)$ for every $i \notin K_2$. Therefore

$$\begin{aligned} I_p &= (\prod_{i \notin K_2} x_i) \cap \bigcap_{i \in K_2} (x_i, f_i) + (g_2) \\ &= (\prod_{i \notin K_2} x_i) \cdot \bigcap_{i \in K_2} (x_i, f_i) + (g_2), \qquad \text{by Claim 4.10.} \end{aligned}$$

Let $p_2 := p - |K_2|$ and denote $I_{p_2} := \cap_{i \in K_2} (x_i, f_i)$.

In the ideal $I_{p,q} := (x_1 \cdots x_p, x_{p+1} \cdots x_{p+q})$, the elements $x_1 \cdots x_p$, and $x_{p+1} \cdots x_{p+q}$, who's exponents give the vertices of the diagram of initial exponents, are relatively prime. To have $H_{I_p} = H_{p,q}$, since $\gcd(in(g_1), in(g_2)) = \prod_{i \in K_2} x_i$ is not equal to one, we must have $m \geq 3$ and $|\alpha_3| = |\nu(\mathrm{lcm}(in(g_1), in(g_2)))| = p + q - |K_2|$. Otherwise, $H_{I_p}(p + q - |K_2|) > H_{p,q}(p + q - |K_2|)$. Without loss of generality, we can assume that $g_3 \in (\prod_{i \notin K_2} x_i) \cdot \bigcap_{i \in K_2} (x_i, f_i)$. Therefore, there is $h_3 \in I_{p_2}$ such that $g_3 = h_3 \prod_{i \notin K_2} x_i$. Observe that $\mathrm{ord}(h_3) = q$. Therefore, by Claim 4.11, $in(h_3) \in (x_1, x_2)$.

Let $K_3 = \{i \in K_2 : \ in(h_3) \in (x_i)\}$ and observe that $1, 2 \in K_3$. We must also have $K_3 \neq K_2$ since α_3 is a vertex of the diagram different from α_1. By Claim 4.9,

$(x_i, f_i) = (x_i, h_3)$ for every $i \notin K_3$. Therefore

$$I_p = (\prod_{i \notin K_3} x_i) \cdot \bigcap_{i \in K_3} (x_i, f_i) + (\prod_{i \notin K_2} x_i) \cdot (h_3) + (g_2)$$
$$= (\prod_{i \notin K_3} x_i) \cdot \bigcap_{i \in K_3} (x_i, f_i) + (g_3) + (g_2)$$

Let $p_3 := p - |K_3|$ and denote $I_{p_3} := \cap_{i \in K_3}(x_i, f_i)$. To have $H_{I_p} = H_{p,q}$ we must have $m \geq 4$ and α_4 such that $|\alpha_4| = \min(|\nu(\mathrm{lcm}(in(g_1), in(g_3)))|, |\nu(\mathrm{lcm}(in(g_2, in(g_3))))|)$. We can compute that $|\nu(\mathrm{lcm}(in(g_1), in(g_3)))| = p + q - |K_3|$ while $|\nu(\mathrm{lcm}(in(g_2, in(g_3))))| \geq p + q$. Therefore $|\alpha_4| = p + q - |K_3|$.

We can assume that $g_4 \in (\prod_{i \notin K_3} x_i) \cdot I_{p_3}$ and let h_4 be such that $g_4 = h_4 \prod_{i \notin K_3} x_i$. Therefore $\mathrm{ord}(h_4) = q$ and $h_4 \in I_{p_3}$. By Claim 4.11, $in(h_4) \in (x_1, x_2)$.

Let $K_4 = \{i \in K_3 : \ in(h_4) \in (x_i)\}$ and observe that $1, 2 \in K_4$ and $K_4 \neq K_3$. By Claim 4.9, $(x_i, f_i) = (x_i, h_4)$ for every $i \notin K_4$. Therefore,

$$I_p = (\prod_{i \notin K_4} x_i) \cdot \bigcap_{i \in K_4} (x_i, f_i) + (\prod_{i \notin K_3} x_i) \cdot (h_4) + (g_3) + (g_2)$$
$$= (\prod_{i \notin K_4} x_i) \cdot \bigcap_{i \in K_4} (x_i, f_i) + (g_4) + (g_3) + (g_2)$$

Let $p_4 := p - |K_4|$ and denote $I_{p_4} := \cap_{i \in K_4}(x_i, f_i)$. To have $H_{I_p} = H_{p,q}$ we must have $m \geq 5$ and α_5 such that $|\alpha_5|$ is equal to the minimum of the orders of the least common multiples of the initial monomials of g_1, g_2, g_3 and g_4. This minimum is given by the order of $\mathrm{lcm}(in(g_1), in(g_4))$. Since $in(g_4) \in (x_i)$, for $i \in K_4$ or $i \notin K_3$, we get that $|\nu(\mathrm{lcm}(in(g_1), in(g_4)))| = p + q + |K_4|$.

We can continue this process producing the elements g_i, for $i = 1, \ldots, m$. Such that $g_i \in (\prod_{i \notin K_{i-1}} x_i) \cdot I_{p_{i-1}}$ and $\mathrm{ord}(g_i) = p + q - |K_{i-1}|$, while $K_i \supsetneq K_{i-1}$. Since K_i is a strictly decreasing sequence of subsets of $\{1, \ldots, p\}$ such that all contain $\{1, 2\}$ we must have $m \leq p - 2$ and $\mathrm{ord}(g_m) = p + q - |K_{m-1}|$.

Notice that as long as h_i, defined such that $g_i = h_i \prod_{i \notin K_{i-1}} x_i$, is divisible by some x_i this will force the existence of another vertex α_{i+1} of the diagram in order to have

$H_{I_p} = H_{p,q}$. This is because the new vertices are needed to compensate from the loss of points in the diagram produced by having vertices comming from generators g_i, who's initial monomials are not relatively prime.

For the last term g_m we still have $h_m \in (x_1, f_1) \cap (x_2, f_2)$ and $\mathrm{ord}(h_m) = q$. From Claim 4.11 $in(g_m) \in (x_1, x_2)$. Therefore in order to have $H_{I_p} = H_{p,q}$ an extra vertex is needed. This gives a contradiction proving the Lemma. □

4.2 Characterization of semi-snc

In this section we characterize semi-snc points using the Hilbert-Samuel function, or the desingularization invariant of [BM97] in general, together with simple geometric data. Lemma 4.4 provides some initial control over the divisor D at a point of $\Sigma_{p,q}$ or $\Sigma_{q,p}$ where the Hilbert-Samuel function has the *correct* value $H_{p,q}$, provided that $p \geq 2$. When $p = 1$, the point lies in a single component of X, so that semi-snc just means snc. A characterization of snc points using the desingularization invariant is given in [BM11, Lemma 3.5].

Remark 4.13 (Characterization of snc singularities). Let D be a reduced Weil divisor on a smooth variety X. Assume that $a \in \mathrm{Supp}\,(D)$ lies in exactly q irreducible components of D. Then D is snc at a if and only if the value of the desingularization invariant is $(q, 0, 1, 0, \ldots, 1, 0, \infty)$, where there are $q - 1$ pairs $(1, 0)$. (This is in "year zero" — before any blowings-up given by the desingularization algorithm.)

The first entry of the invariant at a point a of a hypersurface D in a smooth variety is the order q of D at a. For a subvariety in general, the Hilbert-Samuel function is the first entry of the invariant. (In the case of a hypersurface, the order and the Hilbert-Samuel function each determine the other; see [BM97, Remark 1.3] and Section 4.1.)

In Example 4.3 we saw that Hilbert-Samuel function $= H_{2,1}$ at a point of $\Sigma_{2,1}$ is not enough to ensure semi-snc. Additional geometric data is needed. This will be given using an ideal sheaf that is the final obstruction to semi-snc. Blowing up to remove this obstruction include transformations analogous to the cleaning procedure of [BM11, Section 2], see Proposition 4.33.

Definition 4.14. Consider a pair (X, D), where X is an algebraic variety, and D denotes a Weil divisor on X. Let $X_1, \ldots, X_m$ denote the irreducible components of X, with a given ordering. Let $X^i := X_1 \cup \ldots \cup X_i$, $1 \leq i \leq m$. Let D^i denote the sum of all components of D lying in X^i; i.e. D^i is the divisorial part of the restriction of D to X^i. We will sometimes write $D^i = D|_{X^i}$.

Definition 4.15. Consider a pair (X, D) as in Definition 4.14, where X is (locally) an embedded hypersurface in a smooth variety Y. Assume that $m \geq 2$, (X^{m-1}, D^{m-1}) is semi-snc, and D is reduced. Let $J = J(X, D)$ denote the quotient ideal

$$J = J(X, D) := [I_{D_m} + I_{X^{m-1}} : I_{D^{m-1}} + I_{X_m}],$$

where I_{D_m}, $I_{X^{m-1}}$, $I_{D^{m-1}}$ and I_{X_m} are the defining ideal sheaves of D_m, X^{m-1}, D^{m-1} and X_m (respectively) on Y.

Lemma 4.16 (Characterization of semi-snc points.)**.** *Consider a pair (X, D), where X is (locally) an embedded hypersurface in a smooth variety Y. Assume that X is snc, D is reduced and none of the components of D lie in the intersection of a pair of components of X. Let $a \in X$ be a point lying in at least two components of X. Then (X, D) is semi-snc at a if and only if*

1. *(X^{m-1}, D^{m-1}) is semi-snc at a.*
2. *There exist p and q such that $a \in \Sigma_{p,q}$ and $H_{\operatorname{Supp} D, a} = H_{p,q}$.*
3. *$J_a = \mathcal{O}_{Y,a}$.*

Remark 4.17. If a lies in a single component of X, then condition (1) is vacuous and J is not defined. In this case, Remark 4.13 replaces Lemma 4.16.

Proof of Lemma 4.16. The assertion is trivial at a point in $X \setminus X_m$, so we assume that $a \in X_m$.

At a semi-snc point a of the pair (X, D) the conditions are clearly satisfied. In fact, the ideal of D is of the form $(x_1 \cdots x_p, y_1 \cdots y_q)$ in a system of coordinates for Y at $a = 0$ (recall that D is reduced). We can then compute

$$J_a = [(x_p,\, x_1 \cdots x_{p-1},\, y_1 \cdots y_q) : (x_p,\, x_1 \cdots x_{p-1},\, y_1 \cdots y_q)] = \mathcal{O}_{Y,a}.$$

Assume the conditions (1)–(3). By (1), there is system of coordinates $(x_1, \ldots, x_p,$ $y_1, \ldots, y_q,\, z_1, \ldots, z_{n-p-q})$ for Y at a, in which $X_m = (x_p = 0)$ and D is of the form

$$D = (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) + (x_p = f = 0).$$

By Condition (2) and Lemma 4.4, we can choose $f \in (x_1 \cdots x_{p-1},\, y_1 \cdots y_q,\, x_p)$ and, therefore, we can choose $f \in (x_1 \cdots x_{p-1},\, y_1 \cdots y_q)$. Write f in the form $f = x_1 \cdots x_{p-1} g_1 +$ $y_1 \cdots y_q g_2$. Then

$$\begin{aligned} J_a &= [(x_p,\, x_1 \cdots x_{p-1},\, f) : (x_p,\, x_1 \cdots x_{p-1},\, y_1 \cdots y_q)] \\ &= [(x_p,\, x_1 \cdots x_{p-1},\, y_1 \cdots y_q g_2) : (x_p,\, x_1 \cdots x_{p-1},\, y_1 \cdots y_q)] \\ &= (x_p,\, x_1 \cdots x_{p-1},\, g_2). \end{aligned} \tag{4.7}$$

The condition $J_a = \mathcal{O}_{Y,a}$ means that g_2 is a unit. Then

$$\begin{aligned} D &= (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) + (x_p = f = 0) \\ &= (x_1 \cdots x_{p-1} = y_1 \cdots y_q g_2 = 0) + (x_p = f = 0) \\ &= (x_1 \cdots x_{p-1} = x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2 = 0) + (x_p = f = 0) \\ &= (x_1 \cdots x_p = f = 0). \end{aligned}$$

By Lemma 4.4, since $a \in \Sigma_{p,q}$, $\mathrm{ord}(f) = q$. It follows that $f|_{(x_p=0)}$ is a product $f_1 \cdots f_q$ of q irreducible factors each of order one. For each f_i set $I_i := \{(j,k) : f_i \in (x_j, y_k)|_{x_p=0},\ j \le p-1,\ k \le q\}$ then $f_i \in \cap_{(j,k)\in I_i} (x_j, y_k)|_{(x_p=0)}$, where the intersection is understood to be the whole local ring if I_i is empty. Note that $\cup_i I_i = \{(j,k) : \ j \leq p-1, \ k \leq q\}$, since $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q)$.

We will extend each f_i to a regular function on Y (still denoted f_i) preserving this condition, i.e. such that $f_i \in \cap_{(j,k)\in I_i}(x_j, y_k)$. In fact, $\cap_{(j,k)\in I_i}(x_j, y_k)|_{(x_p=0)}$ is generated by a finite set of monomials $\{m_r\}$ in the $x_j|_{(x_p=0)}$ and $y_k|_{(x_p=0)}$. Then f_i is a combination, $\sum m_r a_r$, of these monomials. So we can get an extension of f_i as desired, using arbitrary extensions of the a_r to regular functions on Y. This means we can assume that $f = f_1 \cdots f_q \in (x_1 \cdots x_{p-1}, \ y_1 \cdots y_q)$ (using the extended f_i).

Since $f|_{(x_1=\cdots=x_p=0)} = y_1 \ldots y_q g_2$ where g_2 is a unit, it follows that $f = y_1 \ldots y_q g_2$ mod $(x_1, \ldots, x_p)$, where g_2 is a unit. Because $D = (x_1 \cdots x_p, f)$, it remains only to check that $x_1, \ldots, x_p, f_1, \ldots, f_q$ are part of a system of coordinates. We can pass to the completion of the ring with respect to its maximal ideal, which we can identify with a ring of formal power series in variables including $x_1, \ldots, x_p$, $y_1, \ldots, y_q$. It is enough to prove that the images of the f_i and x_i in $\hat{m}/\hat{m}^2$ are linearly independent, where $\hat{m}$ is the maximal ideal of the completion of the local ring $\mathcal{O}_{X,a}$. If we put $x_1 = \cdots = x_p = 0$ in the power series representing each f_i we get

$$(f_1 \cdots f_q)|_{(x_1=\ldots=x_p)} = y_1 \cdots y_q.$$

This means that, after a reordering the f_i, each $f_i|_{(x_1=\ldots=x_p)} \in (y_i)$, and the desired conclusion follows. $\square$

4.3 Algorithm

In this section we prove Theorem 1.2. We divide the proof into several steps or subroutines each of which specify certain blowings-up.

Step 1: *Make X snc.* This can be done simply by applying Theorem 2.13 to $(X, 0)$. The blowings-up involved preserve snc singularities of X and therefore also preserve the semi-snc singularities of (X, D). After Step 1 we can therefore assume that X is everywhere snc.

Step 2: *Remove components of D lying inside the singular locus of X.* Consider the union Z of the supports of the components of D lying in the singular locus of X. Blowings-up as needed can simply be given by the usual desingularization of Z, followed by blowing up the final strict transform.

The point is that, locally, there is a smooth ambient variety, with coordinates $(x_1, \ldots, x_p, \ldots, x_n)$ in which each component of Z is of the form $(x_i = x_j = 0)$, $i < j \leq p$. Let C denote the set of irreducible components of intersections of arbitrary subsets of components of Z. Elements of C are partially ordered by inclusion. Desingularization of Z involves blowing up elements of C starting with the smallest, until all components of Z are separated. Then blowing up the final (smooth) strict transform removes all components of Z.

After Step 2 we can therefore assume that no component of D lies in the singular locus of X.

Step 3: *Make (X, D_{red}) semi-snc.* (I.e., transform (X, D) by the blowings-up needed to make (X, D_{red}) semi-snc.) The algorithm for Step 3 is given following Step 4 below.

We can now therefore assume that X is snc, D has no components in the singular locus of X and (X, D_{red}) is semi-snc.

Step 4: *Make (X, D) semi-snc.* A simple combinatorial argument for Step 4 will be given in Section 4.6. This finishes the algorithm.

Algorithm for Step 3: The input is (X, D), where X is snc, D is reduced and no component of D lies in the singular locus of X. We will argue by induction on the number of components of X. It will be convenient to formulate the inductive assumption in terms of triples rather than pairs.

Definition 4.18. Consider a triple (X, D, E), where X is an algebraic variety, and D, E

are Weil divisors on X. Let $X_1, \ldots, X_m$ denote the irreducible components of X with a given ordering. We use the notation of Definition 4.14. Define

$$E^i := E|_{X^i} + (X - X^i)|_{X^i},$$
$$(X, D, E)^i := (X^i, D^i, E^i),$$

where $(X - X^i)|_{X^i}$ is viewed as a divisor on X^i.

Recall Definitions 2.7, 2.9 and Remark 2.8.

Theorem 4.19. *Assume that X is snc, D is a reduced Weil divisor on X with no component in the singular locus of X, and E is a Weil divisor on X such that (X, E) is semi-snc. Then there is a composite of blowings-up with smooth centers $f : X' \to X$, such that:*

1. *Each blowing-up is an isomorphism over the semi-snc points of its target triple.*
2. *The transform $(X', D', \tilde{E})$ of the (X, D, E) by f is semi-snc.*

Proof. The proof is by induction on the number of components m of X.

Case $m = 1$. Since $m = 1$, then $(X, D + E)$ is semi-snc if and only if $(X, D + E)$ is snc. This case therefore follows from Theorem 2.14 applied to $(X, D + E)$.

General case. The sequence of blowings-up will depend on the ordering of the components X_i of X. We will use the notation of Definitions 4.14, 4.18. Since X is snc and no component of D lies in the singular locus of X, it follows that every component of D lies inside exactly one component of X.

By induction, we can assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc. We want to make (X^m, D^m, E^m) semi-snc. For this purpose, we only have to remove the unwanted singularities from the last component X_m of $X = X^m$.

Recall that X is partitioned by the sets $\Sigma_{p,q} = \Sigma_{p,q}(X, D)$; see Definition 2.16. Clearly for all p and q, the closure $\overline{\Sigma}_{p,q}$ of $\Sigma_{p,q}$ has the property

$$\overline{\Sigma}_{p,q} \subset \bigcup_{p' \geq p,\, q' \geq q} \Sigma_{p',q'}.$$

We will construct sequences of blowings-up $X' \to X$ such that X' is semi-snc on certain strata $\Sigma_{p,q}(X', D')$, and then iterate the process. The following definitions are convenient to describe the process precisely.

Definitions 4.20. Consider the partial order on $\mathbb{N}^2$ induced by the order on the set $\{\Sigma_{p,q}\}$, see Definition 4.1. For $I \subset \mathbb{N}^2$, define the *monotone closure* $\overline{I}$ of I as $\overline{I} := \{x \in \mathbb{N}^2 :\ \exists y \in I,\ x \geq y\}$. We say that $I \subset \mathbb{N}^2$ is *monotone* if $\overline{I} = I$. The set of monotone subsets of $\mathbb{N}^2$ is partially ordered by inclusion, and has the property that any increasing sequence stabilizes. Given a monotone I and a pair (X, D), set

$$\Sigma_I(X, D) = \bigcup_{(p,q)\in I} \Sigma_{p,q}(X, D).$$

Then $\Sigma_I(X, D)$ is closed. In fact, if I is monotone then $\Sigma_I(X, D) = \bigcup_{(p,q)\in I} \overline{\Sigma}_{p,q}$.

Definition 4.21. Given (X, D) and monotone I, let $K(X, D, I)$ denote the set of maximal elements of $\{(p, q) \in \mathbb{N}^2 \backslash I :\ \Sigma_{p,q}(X, D) \neq \emptyset\}$. Also set $K(X, D) := K(X, D, \emptyset)$. Note that $K(X, D, I)$ consists only of incomparable pairs (p, q) and that it does not simultaneusly contain strata $\Sigma_{p,q}$ with $p \geq 3$, $p = 2$ and $p = 1$.

Case A: *We first deal with the case in which $K(X, D)$ contains strata $\Sigma_{p,q}$ with $p \geq 3$.*

We start with the variety $W_0 := X^m$ and the divisors $F_0 := D^m$, $G_0 = E^m$, and we define I_0 as the monotone closure of

$$\{\text{maximal elements of } \{(p, q) \in \mathbb{N}^2 :\ \Sigma_{p,q}(W_0, F_0) \neq \emptyset\}\}.$$

Put $j_0 = 0$. Inductively, for $k \geq 0$, we will construct admissible blowings-up

$$W_{j_k} \leftarrow \cdots \leftarrow W_{j'_k} \leftarrow \cdots \leftarrow W_{j_{k+1}} \tag{4.8}$$

such that, if $(W_{j_{k+1}}, F_{j_{k+1}}, G_{j_{k+1}})$ denotes the transform of the triple $(W_{j_k}, F_{j_k}, G_{j_k})$, then $(W_{j_{k+1}}, F_{j_{k+1}})$ semi-snc on $\Sigma_{I_k}(W_{j_{k+1}}, F_{j_{k+1}})$. Then we define

$$I_{k+1} := \overline{I_k \cup K(W_{j_{k+1}}, F_{j_{k+1}}, I_k)}.$$

We have $I_{k+1} \supset I_k$, with equality only if $\Sigma_{I_k}(W_{j_k}, F_{j_k}) = W_{j_k}$.

In this way we define a sequence $I_0 \subset I_1 \subset \ldots$. Since this sequence stabilizes, there is t such that $\Sigma_{I_t}(W_{j_t}, F_{j_t}) = W_t$. By construction, W_{j_t} is semi-snc on $\Sigma_{I_t}(W_{j_t}, F_{j_t})$, so that (W_{j_t}, F_{j_t}) is everywhere semi-snc.

The blowing-up sequence (4.8) will be described in two steps. The first provides a sequence of admissible blowings-up $W_{j_k} \leftarrow \ldots \leftarrow W_{j'_k}$ for the purpose of making the Hilbert-Samuel function equal to $H_{p,q}$ on $\Sigma_{p,q}$, for each $(p, q) \in K(W_{j'_k}, F_{j'_k})$. The second step provides a sequence of admissible blowings-up $W_{j'_k} \leftarrow \ldots \leftarrow W_{j_{k+1}}$ that finally removes the non-semi-snc points from the $\Sigma_{p,q}$, where $(p, q) \in K(W_{j_{k+1}}, F_{j_{k+1}})$.

Step A.1: We can assume that, locally, $X + E$ is embedded as an snc hypersurface in a smooth variety Z. We consider the embedded desingularization algorithm applied to $\operatorname{Supp} D$ with the divisor $X + E$ in Z. We will blow up certain components of the centers of blowing up involved. These centers are the maximum loci of the desingularization invariant, which decreases after each blowing-up. Our purpose is to decrease the Hilbert-Samuel function, which is the first entry of the invariant. During the desingularization process, some components of $X + E$ may be moved away from $\operatorname{Supp} D$ before $\operatorname{Supp} D$ becomes smooth. We will only use centers from the desingularization algorithm that contain no semi-snc points. By assumption, all non-semi-snc points lie in X_m, so that all centers we will consider are inside D_m. Therefore X_m (which is a component of $X + E$) is not moved away before D_m becomes smooth.

We are interested in the maximum locus of the invariant on the complement U_k of $\Sigma_{I_k}(W_{j_k}, F_{j_k})$ in W_{j_k}. The corresponding blowings-up are used to decrease the maximal values of the Hilbert-Samuel function.

Lemma 4.22. *Let C be an irreducible smooth subvariety of* $\operatorname{Supp} D$*. Assume that the Hilbert-Samuel function equals $H_{p,q}$ (for given p, q) at every point of C. If $C \cap \Sigma_{p,q} \neq \emptyset$, then $C \subset \Sigma_{p,q}$.*

Proof. Let $a \in C \cap \Sigma_{p,q}$. Since the Hilbert-Samuel function of $\operatorname{Supp} D$ is constant on C, then a has a neighborhood $U \subset C$, each point of which lies in precisely those components of D at a. Therefore, $U \subset \Sigma_{p,q}$. Since the closure of $\Sigma_{p,q}$ lies in the union of the $\Sigma_{p',q'}$ with $p' \geq p$, $q' \geq q$, any $b \in C \setminus U$ belongs to $\Sigma_{p',q'}$, for some $p' \geq p$, $q' \geq q$. Thus $H_{\operatorname{Supp} D,b} = H_{p,q} \leq H_{p',q'}$. But, by Lemma 4.4, the Hilbert-Samuel function cannot be $< H_{p',q'}$ on $\Sigma_{p',q'}$. Therefore $b \in \Sigma_{p,q}$. □

We write the maximum locus of the invariant in U_k as a disjoint union $A \cup B$ in the following way: A is the union of those components of the maximum locus containing no semi-snc points, and B is the union of the remaining components. Thus B is the union of those components of the maximum locus of the invariant with generic point semi-snc. Each component of B has Hilbert-Samuel function $H_{p,q}$, for some p, q, and lies in the corresponding $\Sigma_{p,q}$ by Lemma 4.22. On the other hand, any component C of the maximum locus of the invariant where either the invariant does not begin with $H_{p,q}$, for some p, q, or the invariant begins with some $H_{p,q}$ but no point of C belongs to $\Sigma_{p,q}$, is a component of A.

Both A and B are closed in the open set $U_k \subset W_{j_k}$. B is not necessarily closed in W_{j_k}. But all points in the complement of U_k are semi-snc, and the semi-snc points are open. Since no points of A are semi-snc, A has no limit points in the complement of U_k. Thus A is closed in W_{j_k}.

We blow up with center A. Then the invariant decreases in the preimage of A. Recall that a A and B depend on (X, D). We use the same notation A and B to denote the sets with the same meaning as above, after blowing up. So we can continue to blow up until $A = \emptyset$. Say we are now in year j_k'.

Claim 4.23. *If $(p, q) \in K(W_{j_k'}, F_{j_k'})$ (so that $A = \emptyset$), then the Hilbert-Samuel function equals $H_{p,q}$ at every point of $\Sigma_{p,q}$.*

Proof. Let $a \in \Sigma_{p,q}$, where $(p, q) \in K(W_{j_k'}, F_{j_k'})$. Assume that the Hilbert-Samuel function

H at a is not equal to $H_{p,q}$. Recall that every point of B has Hilbert-Samuel function of the form $H_{p',q'}$ for some p', q', and belongs to $\Sigma_{p',q'}$. Therefore $a \notin B$, so the invariant at a is not maximal. Thus there is $b \in B$ where the Hilbert-Samuel function is $H_{p',q'} > H$ for some p', q' and $b \in \Sigma_{p',q'}$. By Corollary 4.6, $H_{p',q'} > H_{p,q}$. This means that $\Sigma_{p',q'} > \Sigma_{p,q}$. Since $(p, q) \in K(W_{j'_k}, F_{j'_k})$ then $(p', q') \in I_k$. We have reached a contradiction because $b \in B$ and B lies in the complement of $\Sigma_{I_k}(W_{j'_k}, F_{j'_k})$. □

The claim 4.23 shows that when $A = \emptyset$ we have achieved the goal of Step A.1, i.e., the Hilbert-Samuel function equals $H_{p,q}$ at every point of $\Sigma_{p,q}$, where $(p, q) \in K(W_{j'_k}, F_{j'_k})$.

Step A.2: We now describe blowings-up that eliminate non-semi-snc points from the strata $\Sigma_{p,q}$, with $(p, q) \in K(W_{j'_k}, F_{j'_k})$. Note this does not mean that all the points in the preimage of these strata will be semi-snc. Only the points of the strata $\Sigma_{p,q}$, for the transformed (X, D, E), for $(p, q) \in K(W_{j'_k}, F_{j'_k})$, will necessarily be semi-snc. The remaining points of the preimages will belong to new strata $\Sigma_{p',q'}$, where $p' < p$ or $q' < q$ and therefore will be treated in further iterations of Steps 3.1, 3.2.

We are assuming that $K(W_{j'_k}, F_{j'_k})$ contains some stratum $\Sigma_{p,q}$ with $p \geq 3$. Hence, by Definition 4.1, all strata in $K(W_{j'_k}, F_{j'_k})$ is of the form $\Sigma_{p,q}$ with $p \geq 3$. Therefore this case follows from Proposition 4.28 applied to $(X, D, E)|_U$, where U is the complement of $\Sigma_{I_k}(W_{j'_k}, F_{j'_k})$ in $W_{j'_k}$. Observe that the center of the blowing-up involved never intersects a stratum $\Sigma_{p,q}$ with $p \leq 2$.

Case B: *Assume that $K(W_{j'_k}, F_{j'_k})$ contains a stratum $\Sigma_{2,q}$.* In particular this means that it doesn't contain any stratum $\Sigma_{1,q}$ or $\Sigma_{p,q}$ with $p \geq 3$. This case follows from Proposition 4.33 applied to $(X, D, E)|_U$, where U is the complement of $\Sigma_{I_k}(W_{j'_k}, F_{j'_k})$. Observe that the centers involved never intersect a stratum $\Sigma_{p,q}$ with $p \neq 2$.

Case C: *Finally, assume that (X, D, E) is semi-snc at every point in $\Sigma_{p,q}$ for $p \geq 2$.* Recall that if X has only one component (and is therefore smooth), then semi-snc is the same as snc. Hence this case follows from Theorem 2.14 applied to the pair $(X^m, D^m + E^m)|_U$, where U is the complement of the union of all $\Sigma_{p,q}$ with $p \geq 1$. $\square$

Remark 4.24. The centers of blowing up used in Proposition 4.28 (Case A), as well as in Proposition 4.33 (Case B) and also in Theorem 2.14 (Case C) are closed in U and contain only non-semi-snc points. Since (X, D, E) is semi-snc on $\Sigma_{I_k}(W_{j'_k}, F_{j'_k})$, and therefore in a neighborhood of the latter, we see that these centers are also closed in $W_{j'_k}$.

4.4 The case of more than two components

In this section, we show that the unwanted singularities in the strata $\Sigma_{p,q}(X, D)$, with $p \geq 3$, can be eliminated by a single blowing-up.

Throughout the section, (X, D, E) denotes a triple as in Definition 4.18, and we use the notation of the latter. As in Theorem 4.19, we assume that X is snc, D is reduced and has no component in the singular locus of X, and (X, E) is semi-snc. We consider $K(X, D)$ as in Definition 4.21.

Lemma 4.25. *Assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc and let $(p, q) \in K(X, D)$. Define*

$$C_{p,q} := X_m \cap \Sigma_{p-1,q}(X^{m-1}, D^{m-1}). \tag{4.9}$$

Then:

1. *$C_{p,q}$ is smooth;*

2. *$\Sigma_{p,q}(X, D) \subset C_{p,q} \subset \bigcup_{q' \leq q} \Sigma_{p,q'}(X, D)$.*

Lemma 4.26. *Assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc and let $(p, q) \in K(X, D)$. Assume that $p \geq 3$ and that the Hilbert-Samuel function equals $H_{p,q}$, at every point of $\Sigma_{p,q} = \Sigma_{p,q}(X, D)$. Then:*

1. *Every irreducible component of $C_{p,q}$ which contains a non-semi-snc point of $\Sigma_{p,q}$ consists entirely of non-semi-snc points.*

2. *Every irreducible component of* $\Sigma_{p,q}$ *consists entirely either of semi-snc points or non-semi-snc points.*

Definition 4.27. Assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc and that, for all $(p,q) \in K(X,D)$, where $p \geq 3$, the Hilbert-Samuel function equals $H_{p,q}$, at every point of $\Sigma_{p,q}$. Let C denote the union over all $(p,q) \in K(X,D)$, $p \geq 3$, of the union of all components of $C_{p,q}$ which contain non-semi-snc points of $\Sigma_{p,q}$.

Proposition 4.28. *Under the assumptions of Definition 4.27, let* $\sigma : X' \to X$ *denote the blowing-up with center* C *defined above. Then:*

1. *The transform* $(X', D', \tilde{E})$ *of* (X, D, E) *is semi-snc on the stratum* $\Sigma_{p,q}(X', D')$*, for all* $(p,q) \in K(X,D)$ *with* $p \geq 3$.

2. *Let* $a \in \Sigma_{p,q}$*, where* $(p,q) \in K(X,D)$ *and* $p \geq 3$*. If* $a \in C$ *and* $a' \in \sigma^{-1}(a)$*, then* $a' \in \Sigma_{p',q'}(X', D')$*, where* $p' \leq p$*,* $q' \leq q$*, and at least one of these inequalities is strict.*

Proof of Lemma 4.25. This is immediate from the definitions of $\Sigma_{p,q} = \Sigma_{p,q}(X,D)$, $K(X,D)$ and $C_{p,q}$. □

Proof of Lemma 4.26. Let $a \in \Sigma_{p,q}$ be a non-semi-snc point, and let S be the irreducible component of $\Sigma_{p,q}$ containing a. Let C_0 denote the component of $C_{p,q}$ containing S. We will prove that all points of C_0 are non-semi-snc, as required for (1). In particular, all points in S are non-semi-snc and (2) follows.

By Lemma 4.4, X is embedded locally at a in a smooth variety Y with a system of coordinates $x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_{n-p-q}$ in a neighborhood U of $a = 0$, in which we can write:

$$X_m = (x_p = 0),$$

$$X = (x_1 \cdots x_p = 0),$$

$$D = D^{m-1} + D_m,$$

where

$$D^{m-1} := (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0),$$

$$D_m := (x_p = x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2 = 0).$$

Since (X, D, E) is not semi-snc at a then g_2 is not a unit (see Lemma 4.16(3) and (4.7)). In fact, the ideal $J(X, D)$ (see Definition 4.15) is given at a by $(x_p, x_1 \cdots x_{p-1}, g_2)$; the latter coincides with the local ring of Y at a if and only if g_2 is a unit. In the given coordinates,

$$C_0 = (x_1 = \ldots = x_p = y_1 = \ldots = y_q = 0). \tag{4.10}$$

To show that all the points in C_0 are non-semi-snc, it is enough to show that g_2 is in the ideal $(x_1, \ldots, x_p, y_1, \ldots, y_q)$. In fact, the latter implies that g_2 is not a unit, and therefore that $J(X, D) = (x_p, x_1 \cdots x_{p-1}, g_2)$ is a proper ideal at every point of $C_0 \cap U$. Since C_0 is irreducible, $C_0 \cap U$ is dense in C_0. But the set of semi-snc is open, so it follows that all points in C_0 are non-semi-snc.

Proposition 4.29 below shows that if g_2 is not a unit, then $g_2 \in (x_1, \ldots, x_p, y_1, \ldots, y_q)$, concluding the proof of Lemma 4.26. □

Proof of Propostion 4.28. With reference to the preceding proof, it is clear from (4.10) that blowing up C_0, either p or q decreases in the preimage. This implies (2) in the proposition. It also implies that, after the blowing-up σ of C, all points in the preimage of $\Sigma_{p,q}(X, D)$ which belong to $\Sigma_{p,q}(X', D')$ are semi-snc. This establishes (1). □

Proposition 4.29. *Let f denote an element of a regular local ring. Assume that f has q irreducible factors, each of order 1, that $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q)$, where $p \geq 3$ and the x_i, y_i form part of a regular system of parameters, and that $f = x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2$, where g_2 is not a unit. Then $g_2 \in (x_1, \ldots, x_{p-1}, y_1, \ldots, y_q)$.*

Remark 4.30. The condition $p \geq 3$ is crucial, as can be seen from Example 4.3. In the latter, we have $D = (x_1 = y_1 = 0) + (x_2 = x_1 + y_1 z = 0)$, so that $f = x_1 + y_1 z$ and $g_2 = z \notin (x_1, x_2, y_1)$.

To prove Proposition 4.29 we will use the following lemma.

Lemma 4.31. *Let $p \geq 3$ and $s \geq 0$ be integers. Consider*

$$f = (x_1 m_1 + a_1) \cdots (x_{p-1} m_{p-1} + a_{p-1})(y_{r_1} n_1 + b_1) \cdots (y_{r_s} n_s + b_s) g, \tag{4.11}$$

where the x_i, y_i, a_i, b_i, m_i, n_i and g are elements of a regular local ring with $x_1, \ldots, x_{p-1}$, $y_1, \ldots, y_q$ part of a regular system of parameters, and $1 \leq r_1 < \cdots < r_s \leq q$. Assume that, for every $i = 1, \ldots, p-1$ and $j = 1, \ldots, q$,

$$\text{if } a_i \notin (y_j), \text{ then } y_j = y_{r_k} \text{ and } b_k \in (x_i), \text{ for some } k. \tag{4.12}$$

Then, after expanding the right hand side of (4.11), all the monomials (in the elements above) appearing in the expression are in either the ideal $(x_1 \cdots x_{p-1})$ or the ideal $(y_1 \cdots y_q) \cdot (x_1, \ldots, x_{p-1}, y_1, \ldots, y_q)$.

Remark 4.32. The conclusion of the lemma implies that f can be written as $x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2$ with $g_2 \in (x_1, \ldots, x_{p-1}, y_1, \ldots, y_q)$. This is precisely what we need for Proposition 4.29.

Proof of Lemma 4.31. First consider $s = 0$. Then (4.12) implies that each a_i is in the ideal $(y_1 \cdots y_q)$. The expansion of

$$(x_1 m_1 + a_1) \cdots (x_{p-1} m_{p-1} + a_{p-1}),$$

includes the monomial $x_1 \cdots x_{p-1} m_1 \ldots m_{p-1}$, which belongs to the ideal $(x_1 \cdots x_{p-1})$. Each of the remaining monomials is a multiple of some $x_i a_j$ or of some $a_i a_j$, and therefore belongs to $(y_1 \cdots y_q) \cdot (x_1, \ldots, x_{p-1}, y_1, \ldots, y_q)$.

By induction, assume the lemma for $p, s-1$, where $s \geq 1$. Consider f as in the lemma (for p, s). Then $f/(y_{r_s} n_s + b_s)$ satisfies the hypothesis of the lemma (with $s-1$) when y_{r_s} is deleted from the given elements of the ring. (Note that the lemma also depends on q. Here we are using it for $s-1$ and $q-1$.) Then, by induction, all the terms appearing after expanding $f/(y_{r_s}n_s + b_s)$ are either in the ideal $(x_1 \cdots x_{p-1})$ or in the ideal

$$\left(\frac{y_1 \cdots y_q}{y_{r_s}}\right) \cdot (x_1, \ldots, x_{p-1}, y_1, \ldots, y_q). \tag{4.13}$$

Assume there is a term ξ appearing after expanding (4.11) which is not in $(x_1 \cdots x_{p-1})$. Then there is x_k such that $\xi \notin (x_k)$. Then ξ is divisible by a_k, according to (4.11), and ξ belongs to the ideal (4.13).

If $a_k \in (y_{r_s})$, we are done. By (4.11), ξ is a multiple either of $y_{r_s}n_s$ or b_s. If $a_k \notin (y_{r_s})$, and if we assume that ξ was obtained by multiplying by b_s rather than by $y_{r_s}n_s$, then ξ is divisible by x_k, which is a contradiction. $\square$

Proof of Proposition 4.29. To prove this proposition it is enough to show that f can be written as a product as in the previous lemma. To begin with, $f = h_1 \cdots h_q \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q) = \cap(x_i, y_j)$. Since each (x_i, y_j) is prime, it follows that, for each $i = 1, \ldots, p-1$ and $j = 1, \ldots, q$, there is a k such that $h_k \in (x_i, y_j)$. If there is a unit u such that $h_k = y_j u + a$, where $\mathrm{ord}(a) \geq 2$, then we say that h_k is *associated to* y_j; otherwise we say that h_k is *associated to* x_i. There may be h_k that belong to no (x_i, y_j) and are, therefore, not associated to any x_i or y_j.

By definition, any $h = h_k$ cannot be associated to some x_i and y_j at the same time. Let us prove that h can be associated to at most one x_i. Assume that h is associated to x_{i_1} and x_{i_2}, where $i_1 \neq i_2$. Then $h \in (x_{i_1}, y_{j_1}) \cap (x_{i_2}, y_{j_2})$, for some j_1 and j_2. If $j_1 \neq j_2$, then h cannot be of order 1, since $(x_{i_1}, y_{j_1}) \cap (x_{i_2}, y_{j_2})$ only contains elements of order ≥ 2. If $j_1 = j_2$ then $(x_{i_1}, y_{j_1}) \cap (x_{i_2}, y_{j_2}) = (x_{i_1}x_{i_2}, y_{j_1})$, but this would mean that h is associated to y_{j_1}, and therefore not to x_{i_1} or x_{i_2}.

An analogous argument shows that an h cannot be associated to two different y_j. Therefore, the collection of h_k is partitioned into those associated to a unique x_i, those associated to a unique y_j and those associated to neither some x_i nor some y_j.

We now show that, for each $i = 1, \ldots, p-1$, there exists $h = h_k$ associated to x_i. Assume there is an x_i (say x_1) with no associated h. For each $j = 1, \ldots,$, there exists k_j

such that $h_{k_j} \in (x_1, y_j)$. Then h_{k_j} is associated to y_j. It follows that each k_j corresponds to a unique j. Thus, after reordering the h_k, we have h_i is associated to y_i, for each $i = 1, \ldots, q$. This means that $h_i = y_i u_i + a_i$, where u_i is a unit and $\operatorname{ord} a_i \geq 2$. This contradicts the assumption that g_2 is not a unit. Therefore, for each $i = 1, \ldots, p-1$, there exists h_k associated to x_i.

We take the product of all members of each set in the partition above. The product of all h_k associated to x_i can be written as $x_i m_i + a_i$, and it satisfies the property that

$$x_i m_i + a_i \notin (x_\alpha, y_\beta) \quad \text{unless } \alpha = i. \tag{4.14}$$

In fact, if $x_i m_i + a_i \in (x_\alpha, y_\beta)$ then there exists $h = h_k$ associated to x_i such that $h \in (x_\alpha, y_\beta)$. But then h is associated to either y_β or x_α, which contradicts the condition that h is associated to x_i, where $i \neq \alpha$.

In the same way, write the product of all h_k associated to y_{r_i} as $y_{r_i} m_i + b_i$. Then

$$y_{r_i} m_i + b_i \notin (x_\alpha, y_\beta) \quad \text{unless } \beta = r_i. \tag{4.15}$$

Also write the product of all h_k not associated to any x_i or y_j as g. We get the expression

$$f = (x_1 m_1 + a_1) \cdots (x_{p-1} m_{p-1} + a_{p-1})(y_{r_1} n_1 + b_1) \cdots (y_{r_s} n_s + b_s) g, \tag{4.16}$$

but (4.16) does not *a priori* satisfy the hypotheses of Lemma 4.31.

We will use the properties (4.14) and (4.15) above to modify the elements $m_.$, $a_.$, $n_.$ and $b_.$ in (4.16) to get the hypotheses of the lemma.

We will check whether (4.12) is satisfied, for all $i = 1, \ldots, p-1$ and $j = 1, \ldots, q$. Order the pairs (i, j) reverse-lexicographically (or, in fact, in any way). Given (i, j), assume, by induction, that (4.12) is satisfied for all $(i', j') < (i, j)$. Suppose that (4.12) is not satisfied for (i, j). Then we will modify $m_.$, $a_.$, $b_.$ and $n_.$ so that (4.12) will be satisfied for all $(i', j') \leq (i, j)$. We consider the following cases.

1. $j \neq r_k$, for any k. Then, if $a_i \in (y_j)$, there is nothing to do. If $a_i \notin (y_j)$, we can modify a_i and m_i so that the new a_i will satisfy $a_i \in (y_j)$, and (4.12) will still

be satisfied for $(i', j') < (i, j)$: Since $f \in (x_i, y_j)$ and, for every k, $y_j \neq y_{r_k}$, then $a_i \in (x_i, y_j)$. Write $a_i = ya$, where y is a monomial in the y_ℓ and a is divisible by no y_ℓ. Then $a \in (x_i, y_j)$ and we can write $a = x_i g_1 + y_j g_2$, $x_i m_i + a_i = x_i(m_i + y g_1) + y y_j g_2$. Relabel $m_i + y g_1$ and $y_j y g_2$ as our new m_i and a_i, respectively. Then $a_i \in (y_j)$, and clearly (4.12) is still satisfied for $(i', j') < (i, j)$.

2. $j = r_k$, for some k. Since $f \in (x_i, y_j)$, then $a_i b_k \in (x_i, y_j)$. Since (x_i, y_j) is prime, either $a_i \in (x_i, y_j)$ (in which case we proceed as before), or $b_k \in (x_i, y_j)$. Consider the latter case. If $b_k \in (x_i)$, there is nothing to do. Assume $b_k \notin (x_i)$. Write $b_k = xb$, where x is a monomial in the x_ℓ and b is divisible by no x_ℓ. Then $b \in (x_i, y_j)$. Thus we can write $b = x_i g_1 + y_j g_2$ and $y_j m_k + b_k = y_j(m_k + x g_2) + x_i x g_1$. Relabel $m_k + x g_2$ and $x_i x g_1$ as our new n_k and b_k, respectively. Then $b_k \in (x_i)$, and (4.12) is still satisfied for $(i', j') < (i, j)$.

We thus modify the $m_., n_., a_., b_.$ in (4.16) to get the hypotheses of Lemma 4.31. □

4.5 The case of two components

In this section, we show how to eliminate non-semi-snc singularities from the strata $\Sigma_{2,q}$.

Again, (X, D, E) denotes a triple as in Definition 4.18, and we use the notation of the latter. As in Theorem 4.19, we assume that X is snc, D is reduced and has no component in the singular locus of X, and (X, E) is semi-snc.

Proposition 4.33. *Assume that every point of X lies in at most two components of X and that (X^1, D^1, E^1) is semi-snc. Then there is a sequence of blowings-up with smooth admissible centers such that:*

1. *Each center of blowing-up consists of only non-semi-snc points.*

2. *For each blowing-up, the preimage of $\Sigma_{2,q}$, for any q, lies in the union of the $\Sigma_{2,r}$ $(r \leq q)$ and the $\Sigma_{1,s}$.*

3. In the final transform of (X, D, E), all points of $\Sigma_{2,q}$ are semi-snc, for every q.

The proof will involve some lemmas. First we show how to blow-up to make $J_a = \mathcal{O}_{X,a}$ at every point a. We will use the assumptions of Proposition 4.33 throughout the section. Consider $a \in X$. Then X is embedded locally at a in a smooth variety Y with a system of coordinates $x_1, x_2, y_1, \ldots, y_q, z_1, \ldots, z_{n-q-2}$ in a neighborhood U of $a = 0$, in which we can write:

$$X = X_1 \cup X_2,$$

$$D = D_1 + D_2,$$

where $X_1 = (x_1 = 0)$, $X_2 = (x_2 = 0)$, $D_1 = (x_1 = y_1 \cdots y_q = 0)$ and $D_2 = (x_2 = f = 0)$, for some $f \in \mathcal{O}_{Y,a}$.

Recall the ideal $J = J(X, D)$ (Definition 4.15) that captures one obstruction to semi-snc, see Lemma 4.16; J is the quotient of the ideals of $D_2 \cap X_1$ and $D_1 \cap X_2$ in $\mathcal{O}_Y$.

Consider $V(J)$ as a hypersurface in $X_1 \cap X_2$, and the divisor $D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$. We will blow up to make $J = \mathcal{O}_Y$, using desingularization of $(V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})$; i.e., using the desingularization algorithm for the hypersurface $V(J)$ embedded in the smooth variety $X_1 \cap X_2$, with exceptional divisor $D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$. The resolution algorithm gives a sequence of blowings-up that makes the strict transform of $V(J)$ smooth and snc with respect to the exceptional divisor; we include a final blowing-up of the smooth hypersurface to make the strict transform empty ("principalization" of the ideal J). Observe that it is not necessarily true that $J(X, D)' = J(X', D')$. Therefore, after applying the blowings-up corresponding to the desingularization of $(V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})$ we don't necessarily have $J(X', D') = \mathcal{O}_{Y'}$. Additional "cleaning" blowings-up will be needed afterwards.

Example 4.3 gives a simple illustration of the problem we resolve in this section. In the example, $V(J) = (x_1 = x_2 = z = 0)$, and our plan is to blow-up with the latter as center C to resolve J. In the example this blowing-up is enough to make (X, D) semi-snc.

Lemma 4.34. *If $x_1, x_2, y_1, \ldots, y_q$ are part of regular system of parameters in a regular local ring R and $f \in R$. Then, we can write $f = x_1 g_1 + x_2 g_2 + y_{i_1} \cdots y_{i_t} g_3$ with $\{i_1 < \ldots < i_t\}$ maximum by inclusion among the subsets of $\{1, \ldots, q\}$. Moreover*

$$[(x_1, x_2, f) : (y_1 \cdots y_q)] = (x_1, x_2, g_3).$$

Proof. Let $f = x_1 g_1 + x_2 g_2 + y_{i_1} \cdots y_{i_t} g_3$ with $\{i_1, \ldots, i_t\}$ maximal, by inclusion, among the subsets of $\{1, \ldots, q\}$. Assume $f = x_1 \tilde{g}_1 + x_2 \tilde{g}_2 + y_j \tilde{g}_3$ with $j \notin \{i_1, \ldots, i_t\}$. Then $y_{i_1} \cdots y_{i_t} g_3 \in (x_1, x_3, y_j)$. Since (x_1, x_2, y_j) is prime, $x_1, x_2, y_1, \ldots, y_q$ are coordinates and the assumption on j, we must have $g_3 \in (x_1, x_2, y_j)$. From this it follows that there are $\hat{g}_1, \hat{g}_2, \hat{g}_3$ such that $f = x_1 \hat{g}_1 + x_2 \hat{g}_2 + y_j y_{i_1} \cdots y_{i_t} \hat{g}_3$. Contradicting the maximality of $\{i_1, \ldots, i_t\}$. Therefore this set is actually maximum.

Now, we prove the second part of the lemma. Observe that the inclusion $[(x_1, x_2, f) : (y_1 \cdots y_q)] \supset (x_1, x_2, g_3)$ is clear from the definition of quotient of ideals. Assume h is in the left hand side. Then $y_1 \cdots y_q h \in (x_1, x_2, f)$. From this it follows that there is $c \in R$ such that $y_1 \cdots y_q y - y_{i_1} \cdots y_{i_t} g_3 c \in (x_1, x_2)$. Since $(x_, x_1)$ is prime we must have $y_{j_1} \cdots y_{j_{q-t}} h - g_3 c \in (x_1, x_2)$ where $\{j_1, \ldots, j_{q-t}\} = \{1, \ldots, q\} \setminus \{i_1, \ldots, i_t\}$. Then $g_3 c \in (x_1, x_2, y_{j_k})$ for every $k = 1, \ldots, q-t$.

We must have $g_3 \notin (x_1, x_2, y_{j_k})$. In fact, if $g_3 \in (x_1, x_2, y_{j_k})$, then there is $\tilde{g}_3$ such that $f - y_{j_k} y_{i_1} \cdots y_{i_t} \tilde{g}_3 \in (x_1, x_2)$ contradicting that $\{i_1, \ldots, i_t\}$ is maximum.

If $c \in (x_1, x_2, y_{j_k})$ then there is $\tilde{c}$ such that $h y_{j_1} \cdots \hat{y_{j_k}} \cdots y_{j_{q-t}} - g_3 \tilde{c} \in (x_1, x_2)$, where $\hat{y_{j_k}}$ means that the term is omitted. Repeating this argument for every $k \in \{j_1, j_{q-t}\}$ we get that $h - g_3 \tilde{c} \in (x_1, x_2)$, for some $\tilde{c}$. This implies that $h \in (x_1, x_2, g_3)$, proving that $[(x_1, x_2, f) : (y_1 \cdots y_q)] \subset (x_1, x_2, g_3)$. □

Given a smooth variety W and a blowing-up $\sigma : W' \to W$ with smooth center $C \subset W$, we denote by I' the strict transform by σ of an ideal $I \subset \mathcal{O}_W$, and by Z' the strict transform of a subvariety $Z \subset W$. (We sometimes use the same notation for the strict transform by a sequence of blowings-up.) We also denote by f' the "strict transform" of a function $f \in \mathcal{O}_{W,a}$, where $a \in W$. The latter is defined up to an invertible factor at a point $a' \in \sigma^{-1}(a)$; $f' := u^{-d} \cdot f \circ \sigma$, where $(u = 0)$ defines $\sigma^{-1}(C)$ at a' and d is the maximum such that $f \circ \sigma \in (u^d)$ at a'.

Lemma 4.35. *Consider any sequence of blowings-up $\sigma : Y' \to Y$ which is admissible for $(V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})$; i.e., with center in $\operatorname{Supp} \mathcal{O}/J$ and snc with respect to $D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$. Then*

$$J(X', D') \subset J(X, D)'. \tag{4.17}$$

Moreover, if $J(X, D)' = \mathcal{O}_{Y'}$, then $J(X', D')_a = (x_1, x_2, u^\alpha)$ for every $a \in X_1 \cap X_2$ in coordinates as at the beginning of the section, and where u^α is a monomial in the generators of the ideals of the components of the exceptional divisor of σ.

Remark 4.36. From (4.17) we get that, $J(X', D') \neq \mathcal{O}_{Y'}$ whenever $J(X, D)' \neq \mathcal{O}_{Y'}$. By Lemma 4.16, this implies that, while desingularizing $J(X, D)$, we never blow-up semi-snc points of the transforms of (X, D).

Proof. Let I_{X_1}, I_{X_2}, I_{D_1} and I_{D_2} denote the ideals in Y of X_1, X_2, D_1 and D_2 respectively. Locally at $a \in X_1 \cap X_2$ we have $I_{X_1} = (x_1)$, $I_{X_2} = (x_2)$, $I_{D_1} = (x_1, y_1 \cdots y_q)$ and $I_{D_2} = (x_2, f)$. Then

$$\begin{aligned} J(X, D) &= [(I_{X_1} + I_{D_2}) : (I_{X_2} + I_{D_1})] \\ &= [(x_1, x_2, f) : (x_1, x_2, y_1 \cdots y_q)] \\ &= [(x_1, x_2, f) : (y_1 \cdots y_q)]. \end{aligned}$$

Where the last equality follows from the definition of the quotient of ideals and the fact that $x_1, x_2 \in (x_1, x_2, f)$. Now at $a' \in \sigma^{-1}(a)$, with $a' \in X_1' \cap X_2'$,

$$\begin{aligned} J(X', D') &= [(I'_{X_1} + I'_{D_2}) : (I'_{X_2} + I'_{D_1})] \\ &= [((x_1') + (x_2, f)') : (x_1', x_2', y_1' \cdots y_q')] \\ &= [((x_1') + (x_2, f)') : (y_1' \cdots y_q')]. \end{aligned}$$

In general, $(I+K)' \supset I' + K'$ for any ideals I, K. Also if $I \supset K$ then $[I : L] \supset [K : L]$ for any ideals I, K, L. Then,

$$\begin{aligned} J(X,D)' &= [(x_1, x_2, f)' : (y_1' \cdots y_q')] \\ &= [((x_1) + (x_2, f))' : (y_1' \cdots y_q')] \\ &\supset [((x_1)' + (x_2, f)') : (y_1' \cdots y_q')] \\ &= J(X', D'). \end{aligned}$$

Assume now that $J(X,D)' = \mathcal{O}_{Y'}$. Write $f = x_1 g_1 + x_2 g_2 + y_{i_1} \cdots y_{i_t} g_3$ as in Lemma 4.34. The center of the blowings-up are in $\operatorname{Supp} \mathcal{O}_Y / J$ and in particular in $X_1 \cap X_2$. They are also normal crossings to $D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$. Therefore we must have $I'_{X_1} + I'_{D_2} = (x_1', x_2', u^\alpha y_{i_1}' \cdots y_{i_t}' g_3')$ for $u^\alpha = u_1^{\alpha_1} \cdots u_t^{\alpha_t}$ a monomial in the generators of the ideals of the components of the exceptional divisor. From this we can compute that

$$J(X', D') = [(I'_{X_1} + I'_{D_2}) : (y_1' \cdots y_q')] \tag{4.18}$$

$$= [(x_1', x_2', u^\alpha y_{i_1}' \cdots y_{i_t}' g_3') : (y_1' \cdots y_q')]. \tag{4.19}$$

But

$$\begin{aligned} \mathcal{O}_{Y',a'} &= J(X,D)' \\ &= [(I_{X_1} + I_{D_2}) : (y_1 \cdots y_q)]' \\ &= [(x_1, x_2, y_{i_1} \cdots y_{i_t} g_3) : (y_1 \cdots y_q)]' \\ &= (x_1, x_2, g_3)'. \end{aligned}$$

Since $a' \in X_1 \cap X_2$ we must have that g_3' is a unit. Therefore applying Lemma 4.34 to Equation (4.19), we get the last part of the lemma. □

Lemma 4.37. *Consider the transform $(X', D', \tilde{E})$ of (X, D, E) by the desingularization of $(V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})$. Then:*

1. *For every q, $\Sigma_{2,q}(X', D')$ lies in the inverse image of $\Sigma_{2,q}(X, D)$.*

2. *Let $a' \in X'$. Then the ideal $J(X', D')_{a'}$ is of the form (x_1, x_2, u^α), where $X'_1 = (x_1 = 0)$, $X'_2 = (x_2 = 0)$ and $u = u_1^{\alpha_1} \cdots u_t^{\alpha_t}$ is a monomial in the generators u_i of the ideals of the components of $\tilde{E}$. This means that $V(J(X', D'))$ consists of some components of $X_1 \cap X_2 \cap E$.*

3. *After a finite number of blowings-up with centers on the components of $V(J(X', D'))$ and its successive strict transforms, the transform (X'', D'') of (X, D) satisfies $J(X'', D'') = \mathcal{O}_{Y''}$. For functoriality the components to be blown up are taken according to the corresponding order on the components of E.*

Proof. Item (1) is clear and is independent of the hypothesis. Item (2) follows from the second part of Lemma 4.35.

Consider the intersection of X_1, X_2 and the component H_1 of the exceptional divisor defined by $(u_1 = 0)$. We blow-up the irreducible components of this intersection lying inside $\operatorname{Supp} \mathcal{O}/J$. Locally, $X_1 \cap X_2 \cap H_1$ is defined by $(x_1 = x_2 = u_1 = 0)$. In the u_1-chart, $D'_2 = (x'_2 = f' = 0)$. Since $(x_1, x_2, u^\alpha) = J(X, D) = [(x_1, x_2, f) : (y_1 \cdots y_q)]$, we can write $f = x_1 g_0 + x_2 g_1 + y u^\alpha$ with $y = y_{i_1} \cdots y_{i_t}$ as in Lemma 4.34. Therefore, after the blowing-up, $J(X', D') = (x'_1 x'_2, u_1^{\beta_1} u_2^{\alpha_1} \cdots u_t^{\alpha_t})$ with $\beta_1 < \alpha_1$ in the u_1-chart. In the x_1 and x_2-charts, X_1 and X_2 are moved apart; i.e. we have only strata $\Sigma_{1,k}$ (for certain k). After a finite number of such blowings-up, we get $J(X', D') = \mathcal{O}_{Y'}$ as wanted. □

Proof of Proposition 4.33. To prove this proposition we will:

(1) Use Lemma 4.37 to reduce to the case $J = \mathcal{O}_Y$.

(2) Let $r := r(X, D)$ denote the maximum number of components of D_1 passing through a non-semi-snc point in $X_1 \cap X_2$. Then we will make a single blowing-up to reduce r. As a result J becomes a monomial ideal as in Lemma 4.37.(2). Therefore we can:

(3) Proceed as in Lemma 4.37.(3) to reduce again to $J = \mathcal{O}_Y$.

Steps (2) and (3) are repeated until the set of non-semi-snc points in $X_1 \cap X_2$ is empty. This occurs after finitely many iterations, since r can not decrease indefinitely.

We begin by applying Lemma 4.37 to make $J = \mathcal{O}_Y$. Once this is the case, let $a \in X$. We use a local embedding of X and notation, as in the beginning of this section to write

$$X = X_1 \cup X_2$$

$$D = D_1 + D_2,$$

where $X_1 = (x_1 = 0)$, $X_2 = (x_2 = 0)$, $D_1 = (x_1 = y_1 \cdots y_q = 0)$, $D_2 = (x_2 = f = 0)$ for some $f \in \mathcal{O}_Y$. Since $J = \mathcal{O}_Y$, after re-indexing the y_i, we must have, by Lemma 4.34, that $f = x_1 g_0 + x_2 g_1 + y_1 \cdots y_s$, for some $s \leq q$. Write $f|_{(x_2=0)} = f_1 \cdots f_\ell$, where each f_i is irreducible. We must have $\ell \leq \mathrm{ord}_a(f) \leq s \leq q$ and therefore $a \in \Sigma_{2,\ell}$. By Lemma 4.4, $H_{\mathrm{Supp}\, D,a} = H_{p,\ell}$ if and only if $\ell = q$. Therefore, by Lemma 4.16, (X, D, E) is semi-snc at a if and only if $\ell = q$. The idea is to blow-up a center that locally is described as $(x_1 = x_2 = y_1 = \ldots = y_q = 0)$.

Let $r := r(X, D)$ denote the maximum number of components of D_1 passing through a non-semi-snc point in $X_1 \cap X_2$. Define

$$C_r := \Sigma_{1,r}(X_1, D_1) \cap X_2.$$

Consider a component Q of C_r that intersects the non-semi-snc points of (X, D, E) in $X_1 \cap X_2$. We will prove that Q is closed and consists only of non-semi-snc points of (X, D, E). We will blow-up the union C off all such components of C_r.

Since the set of semi-snc points is open then an open set of Q consists of semi-snc points. At a non-semi-snc point a in Q we have a local embedding and coordinates as above in which we can write

$$D_1 = (x_1 = y_1 \cdots y_r = 0)$$

$$D_2 = (x_2 = f = 0).$$

With $f|_{(x_2=0)}$ factoring into $\ell < r$ irreducible factors, i.e. D_2 having k irreducible components passing through a. In this neighborhood of a in S all points of S are non-semi-snc. Therefore the set of non-semi-snc points is also open in S. Since Q is irreducible this implies that it only contains non-semi-snc points. At a point of $\overline{S} \setminus S$ the number of components of D_1 passing through a point can only be strictly larger than r. Since r was maximum over the non-semi-snc points, it can only be that such a limit point is semi-snc. This is a contradiction with the fact that S contains only non-semi-snc points. Therefore S is closed.

Thus C is closed and consists only of non-semi-snc points. We can compute locally what is the effect of blowing-up C. In the x_1 and x_2-charts the preimages of the point a lie in only one component of X. In the y_i-chart, we can compute

$$D_1' = (x_1 = y_1 \cdots \hat{y}_i \cdots y_r = 0)$$
$$D_2' = (x_2 = x_1 y_i^{j_1} g_0' + x_2 y_i^{j_2} g_1' + y_1 \cdots y_i^{j_3} \cdots y_s = 0),$$

where y_i is now a generator of the ideal of a component of the exceptional divisor, $\hat{y}_i$ means that the factor is missing from the product and at least one of j_1, j_2, j_3 is equal to zero. As a result, $r(X', D') < r(X, D)$. It also happens that $J(X', D')$ is no longer equal to $\mathcal{O}_{Y'}$ but we can compute that $J(X', D') = (x_1, x_2, y_i^{\alpha})$.

We apply again Lemma 4.37. It is clear that the blowings-up needed are those of the last part of this lemma. In fact, we only need to blow up, a finite number of times, the intersection of $X_1 \cap X_2$ with the component of the exceptional divisor defined by $(y_i = 0)$. These blowings-up do not increase $r(X, D)$. Therefore after a finite number of iterations every point lying in two components of X is semi-snc. $\square$

4.6 Non-reduced case

The previous sections establish Theorem 1.2 in the case that D is reduced. In this section we describe the blowings-up necessary to deduce the non reduced case. In other words, we assume that (X, D_{red}) is semi-snc, and we will prove Theorem 1.2 under this assumption.

The assumption implies that, for every $a \in X$, there is a local embedding in a smooth variety Y with coordinates $x_1, \ldots, x_p, y_1, \ldots, y_q, z_1 \ldots, z_{n-p-q}$ in which $a = 0$ and

$$X = (x_1 \cdots x_p = 0),$$
$$D = \sum_{(i,j)} a_{ij}(x_i = y_j = 0),$$

for some $a_{ij} \in \mathbb{Q}$. Since the reduced pair is semi-snc we know that for every i, j, $a_{ij} \neq 0$. Nevertheless, the procedure below also works if we allow the possibility of some a_{ij} being equal to zero.

We have that the pair (X, D) is semi-snc at a if and only if $a_{ij} = a_{i'j}$ for all i, i', j; see Example 2.12. In this section we continue to transform D by taking only its strict transform D', see Definition 2.7. We can forget about the exceptional divisor since, for blowings-up $\sigma : X' \to X$ with smooth centers simultaneously normal crossings to X and $\operatorname{Supp} D$, if (X, D_{red}) is semi-snc then $(X', D'_{red} + Ex(\sigma))$ is also semi-snc. Therefore, since all the components of $Ex(f)$ appear with multiplicity one, if we make (X', D') semi-snc then $(X', D' + Ex(\sigma))$ is semi-snc as well.

Definition 4.38. At each point $a \in X$ we define the following equivalence relation of components of D passing through a. We say that two components of D passing through a, say D_1 and D_2, are *equivalent* (at a) if either $D_1 = D_2$ or the irreducible component of $D_1 \cap D_2$ containing a has codimension 2 in X. It is clear that this irreducible component is smooth and that D_1 and D_2 are equivalent at any of its points.

To check that this is a transitive relation let $D_1 = (x_{i_1} = y_{j_1} = 0)$, $D_2 = (x_{i_2} = y_{j_2} = 0)$ and $D_3 = (x_{i_3} = y_{j_3} = 0)$ in coordinates as before. If D_1 is equivalent to D_2 at the origin, then $j_1 = j_2$. If D_2 is equivalent to D_3 at the origin, then $j_2 = j_3$ and therefore D_3 is equivalent to D_1 at the origin. Reflexivity and symmetry are clear from the definition.

Using this equivalence relation we define $\iota : X \to \mathbb{N}^2$, $\iota(a) = (p(a), q(a))$. For $a \in X$, let $p(a)$ be the number of components of X passing through a and $q(a)$ be the number of equivalence classes in the set of components of D passing through a. In local coordinates as before $q(a)$ is the total number of j for which there is one $a_{i,j} \neq 0$. If $\mathbb{N}^2$ is endowed with the partial order in which $(p_1, q_1) \geq (p_2, q_2)$ if and only if $p_1 \geq p_2$ and $q_1 \geq q_2$, then ι is upper semi-continuous. This implies that the maximal locus of ι is a closed set.

Observe that (X, D) is semi-snc at a if and only if $a_{i,j}$ is constant on each equivalence class of the set of components of D passing through a. Consider the maximal locus of ι. Each irreducible component of the maximal locus of ι consists only of semi-snc points or of non-semi-snc points. This is because all points in one of these irreducible components are contained in the same irreducible components of D. We blow up with center the union of those components of the maximal locus of ι that contain only non-semi-snc points. In the preimage of the center ι decreases. In fact, at a point, in local coordinates as before we are simply blowing up with center

$$C = (x_1 = \ldots = x_{p(a)} = y_1 = \ldots = y_{q(a)} = 0).$$

Therefore, either one component of X is moved away or all components of D in one equivalence class are moved away.

Let W be the union of those components of the maximal locus consisting of semi-snc points. The previous blowing-up is an isomorphism on W. We have that (X', D') is semi-snc on W' and therefore on a neighborhood of it. For this reason, if we consider the union of the components of the maximal locus of ι on $X' \setminus W'$ that only contain non-semi-snc points this will be a closed set in X'. Therefore we can repeat the procedure in $X' \setminus W'$.

$\mathbb{N}^2$ with the given order is well founded. After the previous blowing-up the maximal values of ι on the set of non-semi-snc points of (X, D) decrease. Therefore after a finite number of iterations of the previous procedure, the set of non-semi-snc becomes empty.

Remark 4.39. If (X, D_{red}) is semi-snc, i.e. all $a_{ij} \neq 0$ at every point, in local coordinates as in the beginning of the section. The the blowing-up sequence in this section is simply to follow the desingularization algorithm for Supp D, but blowing up only those components of the maximal locus of the invariant at non-semi-snc points.

4.7 Functoriality

In this section we prove and make precise the statement of Remark 1.3.(3).

We say that a morphism $f : Y \to X$ preserves the number of irreducible components at every point if for every $b \in Y$ the number of irreducible components of Y at b is equal to the number of components of X at $f(b)$.

The Hilbert-Samuel function, and in fact the whole desingularization invariant of which this is its first entry, is an invariant with respect to étale morphisms, see [BM97, Remark 1.5]. To show that the desingularization sequence of Theorem 1.2 is functorial with respect ot étale morphism that preserve tha number of irreducible components we just need to show that each blowing-up constructed is defined using only the desingularization invariant and the number of components of X and D passing through a point. We can recapitulate each step of the algorithm in Section 4.3.

Step 1, is an application of Theorem 2.14. That the sequence of blowings-up coming from this theorem is functorial is proved in the reference [BM11] and [BDVP11]. Step 2, is the desingularization of [BM97] applied to those components of D lying in the intersection of pairs of components of X. This sequence is functorial with respect to étale morphisms in general. Step 3, Case A, gives a sequence completely determined by the Hilbert-Samuel function and the strata $\Sigma_{p,q}$ for $p \geq 3$. The strata $\Sigma_{p,q}$ is defined in terms of the number of components of X and D passing through a point. Step 3, Case B, gives a sequence of blowing-up determined by the desingularization of the hypersurface $V(J)$ and the number $\mathbf{r}(X, D)$ defined in terms of number of components of D, see Proposition 4.33. Step 3, Case C, is again a use of Theorem 2.14. Finally the blowings-up of Step 4 are determined by the number of components of D passing through a point and the equivalence relation defined in Section 4.6 on the components of D passing through a point, see Definition 4.38. From its definition we see that this equivalence relation is preserved by étale morphisms.

It is not possible to drop the condition on the preservation of the number of components in the functoriality statement of any desingularization that preserves snc points. In fact, Assume that X is nc at a but not snc, e.g. $X = (y^2 - x^3 - x^2 = 0) \subset \mathbb{A}^2$ at the origin. By definition There is an étale morphism $f : Y \to X$ such that Y is snc at b and $f(b) = a$. The snc-strict desingularization must modify X at a. It is not possible to pull back this desingularization to Y and still get a snc-strict desingularization because this must be an isomorphism at b.

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