# Resolution of Ties in Parametric Quadratic Programming 

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#### Abstract

We consider the convex parametric quadratic programming problem when the end of the parametric interval is caused by a multiplicity of possibilities ("ties"). In such cases, there is no clear way for the proper active set to be determined for the parametric analysis to continue. In this thesis, we show that the proper active set may be determined in general by solving a certain nonparametric quadratic programming problem. We simplify the parametric quadratic programming problem with a parameter both in the linear part of the objective function and in the right-hand side of the constraints to a quadratic programming without a parameter. We break the analysis into three parts. We first study the parametric quadratic programming problem with a parameter only in the linear part of the objective function, and then a parameter only in the right-hand side of the constraints. Each of these special cases is transformed into a quadratic programming problem having no parameters. A similar approach is then applied to the parametric quadratic programming problem having a parameter both in the linear part of the objective function and in the right-hand side of the constraints.


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## Chapter 1

## Introduction

The general parametric quadratic programming (PQP) problem is

$$
\begin{equation*}
\min \left\{\left.(c+t q)^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b+t p\right\} \tag{1.1}
\end{equation*}
$$

where $c$ and $q$ are given $n$-vectors, $b$ and $p$ are given $m$-vectors, $C$ is a given $(n, n)$ symmetric positive semi-definite matrix, $A=\left[a_{1}, \ldots, a_{m}\right]^{\prime}$ is a given $(m, n)$ matrix, where $a_{i}$ is an $n$-vector, $i=1, \ldots, m$, and $x$ is an $n$-vector whose optimal value is to be determined. Throughout this thesis, prime( ${ }^{\prime}$ ) denotes transposition. All vectors are column vectors unless primed. For quick reading, the end of a proof will be denoted by a hollow box $(\square)$ and the end of an example will be denoted by a diamond $(\diamond)$.

The optimality conditions [1] for (1.1) are

$$
\begin{gather*}
A x \leq b+t p,  \tag{1.2}\\
-(c+t q)-C x=A^{\prime} u, u \geq 0,  \tag{1.3}\\
u^{\prime}(A x-b-t p)=0 . \tag{1.4}
\end{gather*}
$$

We refer to (1.2), (1.3) and (1.4) as primal feasibility, dual feasibility and complementary slackness, respectively. For a convex parametric QP problem, these conditions are necessary and sufficient for optimality.

It is known [1] that both the optimal solution and the associated multiplier vector for (1.1) are piecewise linear functions of $t$ in a finite set of intervals $t_{0} \leq t \leq t_{1}, t_{1} \leq t \leq t_{2}, \ldots, t_{v-1} \leq t \leq t_{v}$, and $t_{0}<t_{1}<t_{2}<\ldots<t_{v}$. Each interval corresponds to a different set of the active constraints. At the end of each interval, either some previously inactive constraints become active, or some previously active constraints become inactive, or both. Each $t_{i}$ corresponds to a "corner" point. The optimal solution and the associated multiplier vector are of the form

$$
x(t)= \begin{cases}h_{10}+t h_{20} & 0 \leq t \leq t_{1} \\ h_{11}+t h_{21} & t_{1} \leq t \leq t_{2} \\ \ldots & \cdots \\ h_{1 j}+t h_{2 j} & t_{j} \leq t \leq t_{j+1} \\ \cdots & \cdots \\ h_{1, v-1}+t h_{2, v-1} & t_{v-1} \leq t \leq t_{v}\end{cases}
$$

and

$$
u(t)= \begin{cases}u_{10}+t u_{20} & 0 \leq t \leq t_{1} \\ u_{11}+t u_{21} & t_{1} \leq t \leq t_{2} \\ \ldots & \cdots \\ u_{1 j}+t u_{2 j} & t_{j} \leq t \leq t_{j+1} \\ \ldots & \ldots \\ u_{1, v-1}+t u_{2, v-1} & t_{v-1} \leq t \leq t_{v}\end{cases}
$$

where $h_{1 j}$ and $h_{2 j}(j=0, \ldots, v-1)$ are $n$-vectors, $u_{1 j}$ and $u_{2 j}(j=0, \ldots, v-1)$ are $m$-vectors.

In each interval $j, x(t)$ and $u(t)$ must satisfy the primal and dual feasibility. The first restriction
that $x(t)$ is feasible implies $t_{j+1} \leq \hat{t}_{j+1}$, where

$$
\begin{align*}
\hat{t}_{j+1} & =\min \left\{\left.\frac{b_{i}-a_{i}^{\prime} h_{1 j}}{a_{i}^{\prime} h_{2 j}-p_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with } a_{i}^{\prime} h_{2 j}>p_{i}\right\}, \\
& =\frac{b_{l}-a_{l}^{\prime} h_{1 j}}{a_{l}^{\prime} h_{2 j}-p_{l}} \tag{1.5}
\end{align*}
$$

The second restriction that $u(t)$ is non-negative implies $t_{j+1} \leq \tilde{t}_{j+1}$, where

$$
\begin{align*}
\tilde{t}_{j+1} & =\min \left\{\left.\frac{-\left(u_{1 j}\right)_{i}}{\left(u_{2 j}\right)_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with }\left(u_{2 j}\right)_{i}<0\right\}, \\
& =-\frac{\left(u_{1 j}\right)_{k}}{\left(u_{2 j}\right)_{k}} \tag{1.6}
\end{align*}
$$

We have used $\left(u_{1 j}\right)_{i}$ to denote the $i$-th component of $u_{1 j}$, and $\left(u_{2 j}\right)_{i}$ to denote the $i$-th component of $u_{2 j}$. We use the convention $\hat{t}_{j+1}=+\infty$ to mean that $a_{i}^{\prime} h_{2 j} \leq p_{i}$ for all $i=1, \ldots, m$. These two restrictions above give the upperbound of the interval; i.e., $t_{j+1}=\min \left\{\hat{t}_{j+1}, \tilde{t}_{j+1}\right\}$.

## Definition 1.1

(a) The problem (1.1) has primal ties at the corner point $t_{j+1}$, if $\hat{t}_{j+1}<\tilde{t}_{j+1}$ and the minimum in (1.5) is obtained for at least two distinct indices.
(b) The problem (1.1) has dual ties at the corner point $t_{j+1}$, if $\tilde{t}_{j+1}<\hat{t}_{j+1}$ and the minimum in (1.6) is obtained for at least two distinct indices.
(c) The problem (1.1) has primal-dual ties at the corner point $t_{j+1}$ if $\hat{t}_{j+1}=\tilde{t}_{j+1}$.
(d) The problem has ties at the corner point $t_{j+1}$ if it has primal ties, dual ties or primal-dual ties.

If there are no ties at all the corner points, the PQP problem (1.1) can be solved by using Best's method [2].

For the remainder of this chapter, we will present some basic properties of the PQP plus a number of examples which illustrate the types of problems which can arise when ties do occur. We begin with an example of a PQP which has no ties at the corner point.

## Example 1.1

$$
\begin{align*}
\operatorname{minimize}: \quad(-2+t) x_{1} & -2 x_{2}+x_{1}^{2}+\frac{1}{2} x_{2}^{2} \\
\text { subject to : } x_{1} & \leq 1, \quad(1) \\
x_{2} & \leq 1, \quad(2) \\
x_{1} & \geq 0, \quad(3)  \tag{3}\\
x_{2} & \geq 0 \tag{4}
\end{align*}
$$

For every $t$ with $t \leq 0$, the optimal solution is $x(t)=(1,1)^{\prime}$. There are no ties at $t=0$. The first two constraints are active at $t=0$. The first constraint becomes inactive and the second constraint remains active when $t$ increases a small amount from 0 . The optimal solution is

$$
x(t)=\left[\begin{array}{c}
1-\frac{1}{2} t \\
1
\end{array}\right],
$$

for every t with $0 \leq t \leq 2$.


Figure 1.1: An example of no ties with $t$ in the linear part of the objective function.

The geometry of this example is illustrated in Figure 1.1, where the feasible region is shaded.

The objective function is an ellipse, its center moves along the line $x_{2}=2$ from the point $(1,2)^{\prime}$ to the point $(0,2)^{\prime}$, as $t$ increases from 0 to 2 . The optimal solution moves along the line $x_{2}=1$ from the point $(1,1)^{\prime}$ to the point $(0,1)^{\prime}$. For every $t$ with $t \geq 2$, the optimal solution is $(0,1)^{\prime}$.

Best [2] solves the PQP problem under the assumption that ties do not occur at the corner points. He gives an algorithm which requires solving linear equations with the coefficient matrix

$$
H_{j}=\left[\begin{array}{cc}
C & A_{j}^{\prime} \\
A_{j} & 0
\end{array}\right],
$$

where $A_{j}^{\prime}$ is the matrix of gradients of all the constraints active at iteration $j$.

Best proves one of the properties of $H(A)$ as follows [2]:
Let $H(A)=\left[\begin{array}{cc}C & A^{\prime} \\ A & 0\end{array}\right]$, where $A$ is any $(m, n)$ matrix. Suppose $A$ has full row rank. Then $H(A)$ is nonsingular only if $s^{\prime} C s>0$ for all non-zero $s$ such that $A s=0$.

Suppose (1.1) has an optimal solution $x\left(t_{j}\right)$ for $t=t_{j}$. Suppose $A_{j}^{\prime}$ is the matrix of gradients of all the constraints active at $x\left(t_{j}\right) ; b_{j}$ and $p_{j}$ are vectors whose components are associated with the rows of $A_{j}$, respectively. Assume $A_{j}$ has full row rank and $s^{\prime} C s>0$ for all $s \neq 0$ with $A_{j} s=0$. Then, $H_{j}=\left[\begin{array}{cc}C & A_{j}^{\prime} \\ A_{j} & 0\end{array}\right]$ is non-singular. The optimality conditions assert that the optimal solution $x\left(t_{j}\right)$ and the associated multiplier vector $v\left(t_{j}\right)$ are uniquely determined by the linear equations

$$
H_{j}\left[\begin{array}{l}
x\left(t_{j}\right) \\
v\left(t_{j}\right)
\end{array}\right]=\left[\begin{array}{c}
-c \\
b_{j}
\end{array}\right]+t_{j}\left[\begin{array}{c}
-q \\
p_{j}
\end{array}\right] .
$$

The full (m-dimensional) vector of multipliers, $u\left(t_{j}\right)$, is obtained from $v\left(t_{j}\right)$ by assigning zero to those components of $u\left(t_{j}\right)$ associated with constraints inactive at $x\left(t_{j}\right)$, and the appropriately indexed components of $v\left(t_{j}\right)$, otherwise.

Now suppose $t$ increases from $t_{j}$. Let $x(t)$ denote the optimal solution and $v(t)$ denote the multiplier vector whose components are associated with the active constraints as functions of the parameter $t$. Provided there are no changes in the active set, $x(t)$ and $v(t)$ are uniquely determined
by the linear equations

$$
H_{j}\left[\begin{array}{l}
x(t) \\
v(t)
\end{array}\right]=\left[\begin{array}{c}
-c \\
b_{j}
\end{array}\right]+t\left[\begin{array}{c}
-q \\
p_{j}
\end{array}\right]
$$

The solution can conveniently be obtained by solving two sets of linear equations:

$$
\begin{align*}
& H_{j}\left[\begin{array}{l}
h_{1 j} \\
v_{1 j}
\end{array}\right]=\left[\begin{array}{c}
-c \\
b_{j}
\end{array}\right],  \tag{1.7}\\
& H_{j}\left[\begin{array}{l}
h_{2 j} \\
v_{2 j}
\end{array}\right]=\left[\begin{array}{c}
-q \\
p_{j}
\end{array}\right], \tag{1.8}
\end{align*}
$$

which both have the coefficient matrix $H_{j}$. Having solved these for $h_{1 j}, h_{2 j}, v_{1 j}$ and $v_{2 j}$, the optimal solution for (1.1) is

$$
\begin{equation*}
x(t)=h_{1 j}+t h_{2 j}, \tag{1.9}
\end{equation*}
$$

and the associated multiplier vector is

$$
\begin{equation*}
v(t)=v_{1 j}+t v_{2 j} . \tag{1.10}
\end{equation*}
$$

The full vector of multipliers $u(t)$ may be obtained from $v(t)$ and the set of the constraints inactive at $x\left(t_{j}\right)$. We write $u(t)$ as

$$
\begin{equation*}
u(t)=u_{1 j}+t u_{2 j} . \tag{1.11}
\end{equation*}
$$

We set $t_{j+1}=\min \left\{\hat{t}_{j+1}, \tilde{t}_{j+1}\right\}$, where

$$
\begin{align*}
\hat{t}_{j+1} & =\min \left\{\left.\frac{b_{i}-a_{i}^{\prime} h_{1 j}}{a_{i}^{\prime} h_{2 j}-p_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with } a_{i}^{\prime} h_{2 j}>p_{i}\right\}, \\
& =\frac{b_{l}-a_{l}^{\prime} h_{1 j}}{a_{l}^{\prime} h_{2 j}-p_{l}},  \tag{1.12}\\
\tilde{t}_{j+1} & =\min \left\{\left.\frac{-\left(u_{1 j}\right)_{i}}{\left(u_{2 j}\right)_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with }\left(u_{2 j}\right)_{i}<0\right\}, \\
& =-\frac{\left(u_{1 j}\right)_{k}}{\left(u_{2 j}\right)_{k}} . \tag{1.13}
\end{align*}
$$

Then the optimal solution is given by (1.9) and the associated multiplier vector is given by (1.11), for every $t$ with $t_{j} \leq t \leq t_{j+1}$.

The algorithm leaves a specified method by which the linear equations (1.7) and (1.8) are solved. Possible ways of solving them are factorizations of $H_{j}$, submatrices of $H_{j}$, or partitions of $H_{j}^{-1}$. It shows updating formulae for the new factors when $A_{j}$ is modified by the addition, deletion, or exchange of a row. At the end of each parametric interval, the active set changes by either adding, deleting or exchanging a constraint, with the assumption that no ties occur. The method terminates when either the optimal solution has been obtained for all values of the parameter, or, a further increase in the parameter results in either the feasible region being null or the objective function being unbounded from below. It uses the linear equation solving method associated with a particular quadratic programming algorithm to provide a natural extension of that method for the solution of the PQP problem (1.1).

Ritter [2] gives a more general method for the PQP problem with ties. In his method, he solves the similar linear equations with the same coefficient matrix $H_{j}$ in each iteration $j$. If at a corner point $t_{j}$, there are ties, then the method chooses an $\varepsilon>0$ sufficiently small, and solves the problem from $t=t_{j}+\varepsilon \leq t_{j+1}$, which has no ties. The difficulty of this approach is that $t_{j+1}$ is not known before we determined the optimal solution for the interval $j$. Therefore we do not know how small to make $\varepsilon$ such that $t \leq t_{j+1}$.

Perold [4] does not consider the possibility of "ties" when describing a parametric algorithm for large-scale mean-variance portfolio optimization problems.

In practice, PQP problems can be quite large. For example, portfolio optimization problems may have many thousands of variables. There may exist ties at the corner points. If there are ties, it is not easy to decide which constraints become active and which constraints become inactive in the next interval of $t$.

Suppose $t=t_{j}$ is a corner point, and assume that there are ties at $t_{j}$. There may be many subsets of linearly independent gradients of the active constraints, and it is hard to find which subset will remain active when $t$ increases a small amount from $t_{j}$.

Arseneau [5] develops an PQP algorithm in which "ties" may occur. The algorithm solves a convex quadratic programming problem where both the objective function and the constraints involve a small and positive scalar $\varepsilon$ if the corner point $t_{j}$ has "ties". However, $\varepsilon$ is only used to symbolically determine the current active set. A numerical value for $\varepsilon$ is never used. The algorithm is modified from algorithm in [2]. This modified QP algorithm give an efficient way to solve the PQP problem with "ties", however, it is complex and needs several assumptions.

Berkelaar, Roos and Terlaky [8] introduce an algorithm for solving a parametric QP with the perturbation either in the linear part of the objective function or in the right-hand side of the constraints. They use the optimal set and optimal partition approach to solve the problem when degeneracy occurs. It is an algorithm using primal and dual optimal solutions. Terlaky and his students Hadigheh, Romanko [9] extended the algorithm for solving the convex quadratic optimization with the perturbation with in the linear part of the objective function and in the right-hand side of the constraints.

The convex parametric quadratic programming problem can also be solved by a different method of solving a parametric LCP (linear complementarity programming) problem [10]. In order to solve the parametric PQP problem of the following form:

$$
\begin{array}{cc}
\operatorname{minimize}: & \left(c+\lambda c^{*}\right)^{\prime} x+\frac{1}{2} x^{\prime} D x \\
\text { subject to : } & A x \geq b+\lambda b^{*}, \\
x \geq 0,
\end{array}
$$

where $D$ is a symmetric positive semi-definite matrix and $\lambda$ is the parameter, we can solve a parametric LCP problem

$$
\begin{aligned}
w-M z & =q+\lambda q^{*} \\
w, z & \geq 0
\end{aligned}
$$

where

$$
M=\left[\begin{array}{cc}
D & -A^{\prime} \\
A & 0
\end{array}\right], q=\left[\begin{array}{c}
c \\
-b
\end{array}\right], q^{*}=\left[\begin{array}{c}
c^{*} \\
-b^{*}
\end{array}\right] .
$$

Since $M$ is a positive semi-definite matrix, this parametric LCP can be solved by the algorithm given in Murty's book [10].

The contribution of this thesis is to provide solutions for (1.1) in the presence of ties, by simplifying the parametric QP problem into a related QP problem without the parameter. The results depend on a number of special cases which will be analyzed separately. In the rest of this chapter, we will give a series of numerical examples which will illustrate the nature of the problem.

The following is an example of a PQP problem having a tie at the corner point with the parameter $t$ only in the linear part of the objective function. The problem will be solved in Section 2.1 by the method proposed in this thesis.

## Example 1.2

$$
\begin{align*}
& \operatorname{minimize}:-\frac{10}{3} x_{1}+\left(-\frac{8}{3}+t\right) x_{2}+\frac{1}{2} x_{1}^{2}+x_{2}^{2} \\
& \text { subject to : } x_{1}+2 x_{2} \leq 2,  \tag{1}\\
& 2 x_{1}+x_{2} \leq 2,  \tag{2}\\
& x_{1}+x_{2} \leq \frac{4}{3}  \tag{3}\\
& \geq 0  \tag{4}\\
& x_{1}(3)  \tag{5}\\
& x_{2} \geq 0
\end{align*}
$$

When $-\frac{4}{3} \leq t \leq 0$, the optimal solution is $x_{0}=\left(\frac{2}{3}, \frac{2}{3}\right)^{\prime}$, the first three constraints are active at $x_{0}$ and their gradients are linearly dependent. The multiplier vector for the first three constraints is

$$
u=\left[\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right]=\left[\begin{array}{c}
-t \\
\frac{4}{3}+t \\
-t
\end{array}\right],
$$

for every $t$ with $-\frac{4}{3} \leq t \leq 0$. When $t$ increases from negative to zero, $u_{1}$ and $u_{3}$ become zero simultaneously, therefore, there are dual ties at the corner point $t=0$. When $t$ increases a small
amount from zero, both the first and the third constraints become inactive and only the second constraint remains active. The optimal solution is

$$
x(t)=\left[\begin{array}{l}
\frac{2}{3}+\frac{2}{9} t \\
\frac{2}{3}-\frac{4}{9} t
\end{array}\right],
$$

for every $t$ with $0 \leq t \leq \frac{3}{2}$.


Figure 1.2: An example of a tie with $t$ in the linear part of the objective function.

The geometry of this example is illustrated in Figure 1.2, where the feasible region is shaded. The objective function is an ellipse, its center moves down the line $x_{1}=\frac{10}{3}$ from the point $\left(\frac{10}{3}, \frac{4}{3}\right)^{\prime}$, as $t$ increases from 0 . The optimal solution moves along the line $2 x_{1}+x_{2}=2$ from the point $\left(\frac{2}{3}, \frac{2}{3}\right)^{\prime}$ to the point $(1,0)^{\prime}$ as $t$ increases from 0 to $\frac{3}{2}$. For every $t$ with $t \geq \frac{3}{2}$, the optimal solution is $(1,0)^{\prime}$, and is a constant.

The following example is a special case of several constraints becoming active simultaneously (ties). This problem will be solved in Section 2.3 by the method proposed in this thesis.

## Example 1.3

$$
\begin{equation*}
\min \left\{t q^{\prime} x \mid A x \leq b\right\} \tag{1.14}
\end{equation*}
$$

Assume the feasible region for (1.14) is non-null. The geometry of this type of problem is illustrated in Figure 1.3, where the feasible region is shaded. When $t=0$, (1.14) becomes

$$
\min \{0 \mid A x \leq b\}
$$

The phenomenon being illustrated here is alternate optimal solutions. When $t=0$, any feasible solution is also optimal. Let $y^{-}$denote the optimal solution for (1.14) when $t<0$, and let $y^{+}$ denote the optimal solution for (1.14) when $t>0$. Let $x_{0}$ be an interior point in the feasible region, then $x_{0}$ is optimal for (1.14) for $t=0$.


Figure 1.3: A special case of parametric programming problem.
In this example, when $t<0$, constraints (1) and (4) in Figure 1.3 are active at the optimal solution $y^{-}$; when $t=0$, there are no constraints active at the optimal solution $x_{0}$; when $t>0$, constraints (2) and (3) are active at the optimal solution $y^{+}$. There are ties at the corner point $t=0$ because when $t$ increases from negative to zero, the multipliers corresponding to constraints (1) and (4) become zero simultaneously. This example also illustrates that $x_{0}$ and $y^{+}$cannot be connected with a linear function of $t$.

The following is an example of a PQP problem having a tie at a corner point with the parameter $t$ only in the right-hand side of the constraints. The problem will be solved in Section 3.1 by the method proposed in this thesis.

## Example 1.4

$$
\begin{align*}
& \text { minimize : } \quad-2 x_{1}-2 x_{2}+\frac{1}{2} x_{1}^{2}+x_{2}^{2} \\
& \text { subject to : } x_{1} \leq 1,  \tag{1}\\
& x_{2} \leq 1,  \tag{2}\\
& x_{1}+x_{2} \leq 2-t,  \tag{3}\\
& x_{1}+2 x_{2} \leq 3-\frac{1}{2} t,  \tag{4}\\
& x_{1} \quad \geq 0,  \tag{5}\\
& x_{2} \geq 0 . \tag{6}
\end{align*}
$$

When $t \leq 0$, the optimal solution is $x(t)=(1,1)^{\prime}$. The first two constraints are active at $x_{0}$ when $t<0$. The third and the fourth constraints become active simultaneously when $t=0$, so there are primal ties at $t=0$. When $t$ increases a small amount from zero, the second and the fourth constraints become inactive, and both the first and the third constraints remain active. The optimal solution is

$$
x(t)=\left[\begin{array}{c}
1 \\
1-t
\end{array}\right],
$$

for every $t$ with $0 \leq t \leq \frac{1}{2}$.

The geometry of this example is illustrated in Figure 1.4, where the feasible region for $t=0$ is shaded. The objective function is an ellipse. The optimal solution moves along the line $x_{1}=1$ from the point $(1,1)^{\prime}$ to the point $\left(1, \frac{1}{2}\right)^{\prime}$, as $t$ increases from 0 to $\frac{1}{2}$.

For every $t$ with $\frac{1}{2} \leq t \leq 2$, the optimal solution is $x(t)=\left[\begin{array}{c}\frac{2}{3}(2-t) \\ \frac{1}{3}(2-t)\end{array}\right]$. For every $t$ with $t>2$, the problem is infeasible.


Figure 1.4: An example of a tie with $t$ in the right-hand side of the constraints.

The following is an example of a PQP problem having a tie at a corner point with the parameter $t$ both in the linear part of the objective function and in the right-hand side of the constraints. The problem will be solved in Section 4.1 by the method proposed in this thesis.

## Example 1.5

$$
\begin{align*}
\operatorname{minimize}: & -2 x_{1}+(-2+t) x_{2}+\frac{1}{2} x_{1}^{2}+x_{2}^{2} \\
\text { subject to : } x_{1} & \leq 1-t,  \tag{1}\\
x_{2} & \leq 1,  \tag{2}\\
x_{1}+\quad x_{2} & \leq 2-t  \tag{3}\\
x_{1}+2 x_{2} & \leq 3-\frac{1}{2} t,  \tag{4}\\
& \geq 0  \tag{5}\\
x_{1} & (4)  \tag{6}\\
x_{2} & \geq 0
\end{align*}
$$

For every $t$ with $-\frac{1}{5} \leq t \leq 0$, the optimal solution is $x(t)=\left(1-t, 1+\frac{1}{4} t\right)^{\prime}$. The first and the fourth constraints are active when $-\frac{1}{5}<t \leq 0$. The second and the third constraints become active simultaneously when $t=0$, so there are primal ties at $t=0$. When $t$ increases a small amount from zero, the second, the third and the fourth constraints all become inactive and only the first constraint remains active. The optimal solution is

$$
x(t)=\left[\begin{array}{c}
1-t \\
1-\frac{1}{2} t
\end{array}\right],
$$

for every $t$ with $0 \leq t \leq 1$.

The geometry of this example is illustrated in Figure 1.5, where the feasible region for $t=0$ is shaded. The objective function is an ellipse, its center moves down along the line $x_{1}=2$ from the point $(2,1)^{\prime}$, as $t$ increases from zero. The optimal solution moves along the line $x_{1}-2 x_{2}=-1$ from the point $(1,1)^{\prime}$ to the point $\left(0, \frac{1}{2}\right)^{\prime}$, as $t$ increases from 0 to 1 . When $t>1$, the problem is infeasible.


Figure 1.5: An example of a tie with $t$ both in linear part of the objective function and in the right-hand side of the constraints.

In the following part, some definitions, notations and lemmas will be introduced. They will be used in the later chapters.

Notation 1.1 Let $x_{0}$ be an optimal solution and $u_{0}$ be an associated multiplier vector for (1.1) for $t=0$. Let $A_{0}^{\prime}$ be the matrix of gradients of all the constraints active at $x_{0}$, let $b_{0}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{0} ;$ i.e., $A_{0} x_{0}=b_{0}$.

Definition 1.2 The optimal solution for (1.1) for some $t>0, x(t)$, is a diminishment of $x_{0}$ if the set of the constraints active at $x(t)$ is a subset of or equals to the set of those constraints active at $x_{0}$.

The following result shows that the optimal solution and the associated multiplier vector for a general parametric QP problem are linear functions of the parameter provided that the active constraints remain unchanged.

Lemma 1.1 Let $x_{1}$ and $x_{2}$ be optimal solutions for (1.1) for $t=t_{1}$ and $t=t_{2}$, respectively, and suppose $t_{1}<t_{2}$. Let $u_{1}$ and $u_{2}$ be associated multiplier vectors for $x_{1}$ and $x_{2}$, respectively. Assume that $x_{1}$ and $x_{2}$ have the same active constraints. Let $A_{0}^{\prime}$ be the matrix of gradients of all the constraints active at $x_{1}$, let $b_{0}$ and $p_{0}$ be the vectors whose components are those $b_{i}$ and $p_{i}$ associated with the rows of $A_{0}$, respectively. So, $A_{0} x_{1}=b_{0}+t_{1} p_{0}, A_{0} x_{2}=b_{0}+t_{2} p_{0}$. Then

$$
x^{*}(t)=x_{1}+\frac{x_{2}-x_{1}}{t_{2}-t_{1}}\left(t-t_{1}\right)
$$

is an optimal solution for (1.1) with an associated multiplier vector

$$
u^{*}(t)=u_{1}+\frac{u_{2}-u_{1}}{t_{2}-t_{1}}\left(t-t_{1}\right),
$$

for every $t$ with $t_{1} \leq t \leq t_{2}$.

## Proof

Since $u_{1}$ and $u_{2}$ are multiplier vectors, we have $u_{1} \geq 0, u_{2} \geq 0$. It is easy to see that $u^{*}(t) \geq 0$ for every $t$ with $t_{1} \leq t \leq t_{2}$. Let $A_{1}^{\prime}$ be the matrix of gradients of all the constraints inactive at $x_{1}$, and let $b_{1}$ and $p_{1}$ be the vectors whose components are those $b_{i}$ and $p_{i}$ associated with the rows of $A_{1}$. Then $A_{1} x_{1}<b_{1}+t_{1} p_{1}, A_{1} x_{2}<b_{1}+t_{2} p_{1}$. We have

$$
\begin{align*}
& A_{0} x^{*}(t)=A_{0} x_{1}+A_{0}\left(x_{2}-x_{1}\right) \frac{t-t_{1}}{t_{2}-t_{1}}=A_{0} x_{1} \frac{t_{2}-t}{t_{2}-t_{1}}+A_{0} x_{2} \frac{t-t_{1}}{t_{2}-t_{1}}=b_{0}+t p_{0}  \tag{1.15}\\
& A_{1} x^{*}(t)=A_{1} x_{1}+A_{1}\left(x_{2}-x_{1}\right) \frac{t-t_{1}}{t_{2}-t_{1}}=A_{1} x_{1} \frac{t_{2}-t}{t_{2}-t_{1}}+A_{1} x_{2} \frac{t-t_{1}}{t_{2}-t_{1}}<b_{1}+t p_{1} \tag{1.16}
\end{align*}
$$

So $A_{0}^{\prime}$ is the matrix of gradients of all the constraints active at $x^{*}(t)$, for every $t$ with $t_{1} \leq t \leq t_{2}$. For $t=t_{1}$ and $t=t_{2}$, the optimality conditions for (1.1) assert that

$$
\begin{align*}
& -c-t_{1} q-C x_{1}=A_{0}^{\prime} u_{1},  \tag{1.17}\\
& -c-t_{2} q-C x_{2}=A_{0}^{\prime} u_{2} . \tag{1.18}
\end{align*}
$$

Subtract (1.17) from (1.18), then multiply both sides by $\frac{t-t_{1}}{t_{2}-t_{1}}$,

$$
\begin{equation*}
-\left(t-t_{1}\right) q-\frac{t-t_{1}}{t_{2}-t_{1}} C\left(x_{2}-x_{1}\right)=\frac{t-t_{1}}{t_{2}-t_{1}} A_{0}^{\prime}\left(u_{2}-u_{1}\right) . \tag{1.19}
\end{equation*}
$$

Add (1.17) to (1.19),

$$
-c-t q-C x_{1}-\frac{t-t_{1}}{t_{2}-t_{1}} C\left(x_{2}-x_{1}\right)=A_{0}^{\prime} u_{1}+\frac{t-t_{1}}{t_{2}-t_{1}} A_{0}^{\prime}\left(u_{2}-u_{1}\right) .
$$

That is,

$$
-c-t q-C x^{*}(t)=A_{0}^{\prime} u^{*}(t) .
$$

From (1.15) and (1.16),

$$
A x^{*}(t) \leq b+t p,
$$

for every $t$ with $t_{1} \leq t \leq t_{2}$. Thus,

$$
\begin{aligned}
& A x^{*}(t) \leq b+t p, \\
& -c-t q-C x^{*}(t)=A_{0}^{\prime} u^{*}(t), u^{*}(t) \geq 0, \\
& A_{0} x^{*}(t)=b_{0}+t p_{0} .
\end{aligned}
$$

So, $x^{*}(t)$ and $u^{*}(t)$ satisfy the optimality conditions for (1.1). Therefore, $x^{*}(t)$ is an optimal solution for (1.1) and $u^{*}(t)$ is an associated multiplier vector, for every $t$ with $t_{1} \leq t \leq t_{2}$.

From Lemma 1.1, we know that if the active constraints for the optimal solutions for some $t_{1}$ and $t_{2}$ are coincident, the optimal solution is a linear function of t , for every $t$ with $t_{1} \leq t \leq t_{2}$. In fact, when $t$ changes a small amount, the active constraints of the optimal solutions may not remain coincident. When $t$ increases, sometimes there are originally inactive constraints becoming active; sometimes all the inactive constraints remains inactive and there may be some active constraints becoming inactive. In this chapter, we mainly study the latter case: there are no inactive constraints becoming active, but there may be some active constraints becoming inactive. In this case, the optimal solution (when $t$ only increases a small amount) is also a linear function of $t$.

Instead of studying $t$ from $t_{j}$ to $t_{j+1}$, in this thesis, we always let $t$ begin from $t_{0}=0$. We can do that because at each interval $t_{j} \leq t \leq t_{j+1}$, we can let $t=t-t_{j}$, then t begins from zero.

For $t$ beginning at 0 , we have an $n$-vector $h_{0}$ such that $x(t)=x_{0}+t h_{0}$ is an optimal solution for (1.1), for every $t$ with $0 \leq t \leq \bar{t}$, for some $\bar{t}>0$.

Lemma 1.2 Let $x(t)=x_{0}+t h_{0}$ be an optimal solution for (1.1) and $u(t)=\bar{u}_{0}+t u_{1}$ be an associated multiplier vector, for every $t$ with $0<t \leq \bar{t}$, where $\bar{t}$ is some positive number. Suppose that $x(t)$ is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$. Then $\bar{u}_{0}$ is an associated multiplier vector for $x_{0}$ for (1.1).

## Proof

Let $t^{0}$ satisfy $0<t^{0}<\bar{t}$.
When $t=t^{0}, x\left(t^{0}\right)=x_{0}+t^{0} h_{0}, u\left(t^{0}\right)=\bar{u}_{0}+t^{0} u_{1}$, from the optimality conditions,

$$
\begin{equation*}
-c-t^{0} q-C x_{0}-t^{0} C h_{0}=A_{0}^{\prime} \bar{u}_{0}+t^{0} A_{0}^{\prime} u_{1} . \tag{1.20}
\end{equation*}
$$

When $t=\frac{t^{0}}{2}<\bar{t}, x\left(\frac{t^{0}}{2}\right)=x_{0}+\frac{t^{0}}{2} h_{0}, u\left(\frac{t^{0}}{2}\right)=\bar{u}_{0}+\frac{t^{0}}{2} u_{1}$, from the optimality conditions,

$$
\begin{equation*}
-c-\frac{t^{0}}{2} q-C x_{0}-\frac{t^{0}}{2} C h_{0}=A_{0}^{\prime} \bar{u}_{0}+\frac{t^{0}}{2} A_{0}^{\prime} u_{1} \tag{1.21}
\end{equation*}
$$

Multiply (1.21) by 2 ,

$$
\begin{equation*}
-2 c-t^{0} q-2 C x_{0}-t^{0} C h_{0}=2 A_{0}^{\prime} \bar{u}_{0}+t^{0} A_{0}^{\prime} u_{1} . \tag{1.22}
\end{equation*}
$$

Subtracting (1.20) from (1.22) gives

$$
-c-C x_{0}=A_{0}^{\prime} \bar{u}_{0} .
$$

Since $u(t)=\bar{u}_{0}+t u_{1} \geq 0$, for every $t$ with $0<t \leq \bar{t}$, we have $\bar{u}_{0} \geq 0$. Thus,

$$
\begin{align*}
& A x_{0} \leq b, \\
& -c-C x_{0}=A_{0}^{\prime} \bar{u}_{0}, \bar{u}_{0} \geq 0  \tag{J}\\
& A_{0} x_{0}=b_{0}
\end{align*}
$$

Thus, $\bar{u}_{0}$ is an associated multiplier vector for $x_{0}$ for (1.1) as required.

From the above analysis, we can introduce the following notation.

Notation 1.2 Let $x(t)=x_{0}+t h_{0}$ denote an optimal solution for (1.1) with an associated multiplier vector $u(t)=u_{0}+t u_{1}$, for every $t$ with $0 \leq t \leq \bar{t}$, where $x_{0}$ is an optimal solution for (1.1) for $t_{0}=0$, and $u_{0}$ is an associated multiplier vector for $x_{0}$.

Lemma 1.3 Assume that (1.1) has optimal solutions, for every $t$ with $0 \leq t \leq \hat{t}$. There exists an optimal solution $x(t)=x_{0}+$ th $h_{0}$ for (1.1) being a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$, where $0<\bar{t} \leq \hat{t}$.

## Proof

Let $H$ be the set of all the indices of the gradients of the constraints inactive at $x_{0}$. For every
$i \in H, a_{i}^{\prime} x_{0}<b_{i}$. If $a_{i}^{\prime} h_{0} \leq p_{i}$, for every $t$ with $0 \leq t<\hat{t}$,

$$
a_{i}^{\prime} x(t)=a_{i}^{\prime}\left(x_{0}+t h_{0}\right)=a_{i}^{\prime} x_{0}+t a_{i}^{\prime} h_{0}<b_{i}+t p_{i} .
$$

If $a_{i}^{\prime} h_{0}>p_{i}$, let $c_{i}=b_{i}-a_{i}^{\prime} x_{0}>0$, there always exists a $\bar{t}_{i}>0$ such that $\bar{t}_{i}\left(a_{i}^{\prime} h_{0}-p_{i}\right) \leq c_{i}$, then for every $t$ with $0 \leq t<\min \left\{\bar{t}_{i}, \hat{t}\right\}$,

$$
a_{i}^{\prime} x(t)=a_{i}^{\prime}\left(x_{0}+t h_{0}\right)<a_{i}^{\prime} x_{0}+\bar{t}_{i} a_{i}^{\prime} h_{0}=b_{i}-c_{i}+t a_{i}^{\prime} h_{0} \leq b_{i}+t p_{i} .
$$

Let $\bar{t}=\min \left\{\hat{t}, \min \left\{\bar{t}_{i} \mid a_{i}^{\prime} h_{0}>0\right\}\right\}>0$. Then for every $t$ with $0 \leq t<\bar{t}$ and every $i \in H$, we have $a_{i}^{\prime} x(t)<b_{i}+t p_{i}$, and this completes the proof.

Definition 1.3 We call $\left(h_{0}, \bar{t}\right)$ an optimal continuation of $x_{0}$ for (1.1), where $\bar{t}>0$, if $x(t)=$ $x_{0}+t h_{0}$ is optimal for (1.1) with an associated multiplier vector $u(t)=u_{0}+t u_{1}$ for every $t$ with $0 \leq t<\bar{t}$.

Remark 1.1 An optimal continuation depends on a specified active set. For some active sets, an optimal continuation may not exist.

From Best's algorithm, $\bar{t}=\min \{\hat{t}, \tilde{t}\}$, where

$$
\begin{align*}
\hat{t} & =\min \left\{\left.\frac{b_{i}-a_{i}^{\prime} x_{0}}{a_{i}^{\prime} h_{0}-p_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with } a_{i}^{\prime} h_{0}>p_{i}\right\}, \\
& =\frac{b_{l}-a_{l}^{\prime} x_{0}}{a_{l}^{\prime} h_{0}-p_{l}},  \tag{1.23}\\
\tilde{t} & =\min \left\{\left.\frac{-\left(u_{0}\right)_{i}}{\left(u_{1}\right)_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with }\left(u_{1}\right)_{i}<0\right\}, \\
& =-\frac{\left(u_{0}\right)_{k}}{\left(u_{1}\right)_{k}} . \tag{1.24}
\end{align*}
$$

Remark 1.2 For $t=\bar{t}, x(\bar{t})=x_{0}+\bar{t} h_{0}$ is also an optimal solution for (1.1). However, we do not consider it in the optimal continuation because $x(\bar{t})$ is not a diminishment of $x_{0}$. In the proofs of this thesis, we mainly base on the "diminishment".

Many proofs in the next three chapters involve the following property of the convex quadratic programming problem.

Lemma 1.4 If the convex quadratic programming problem

$$
\begin{equation*}
\min \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b\right\} \tag{1.25}
\end{equation*}
$$

is unbounded from below, then for any feasible solution $x_{1}$ for (1.25), there exists a vector $s$ such that $x_{1}-\sigma s$ is feasible for (1.25), for every positive scalar $\sigma$, and $s^{\prime} C s=0, c^{\prime} s>0$.

## Proof

Since (1.25) is unbounded from below, for any feasible solution $x_{1}$, there exists a vector $s$ such that $x_{1}-\sigma s$ is also feasible for (1.25), for every positive scalar $\sigma$, and

$$
c^{\prime}\left(x_{1}-\sigma s\right)+\frac{1}{2}\left(x_{1}-\sigma s\right)^{\prime} C\left(x_{1}-\sigma s\right) \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty .
$$

The objective function for (1.25) for $x_{1}-\sigma s$ is

$$
\begin{equation*}
c^{\prime}\left(x_{1}-\sigma s\right)+\frac{1}{2}\left(x_{1}-\sigma s\right)^{\prime} C\left(x_{1}-\sigma s\right)=c^{\prime} x_{1}+\frac{1}{2} x_{1}^{\prime} C x_{1}-\sigma c^{\prime} s-\sigma x_{1}^{\prime} C s+\frac{1}{2} \sigma^{2} s^{\prime} C s \tag{1.26}
\end{equation*}
$$

If $s^{\prime} C s \neq 0$, then $s^{\prime} C s>0$, since $C$ is positive semidefinite. Then (1.26) is bounded from below, for $\sigma>0$, which is a contradiction. Therefore, $s^{\prime} C s=0$, and this implies $C s=0$ since $C$ is positive semidefinite. So the right-hand side of (1.26) becomes

$$
c^{\prime} x_{1}+\frac{1}{2} x_{1}^{\prime} C x_{1}-\sigma c^{\prime} s
$$

Since this must decrease to negative infinity as $\sigma$ increases to positive infinity. Thus, $c^{\prime} s>0$, and it completes the proof.

## Chapter 2

## A Parameter only in the Objective

## Function

Before studying (1.1), we analyze a simpler case in which the parameter is only in the linear part of the objective function. We solve it by solving a related QP problem which has no parameter.

### 2.1 Solution of the PQP Problem with a Parameter in the Linear Part of the Objective Function by Solving a Related QP Problem without the Parameter

Consider the following PQP problem

$$
\begin{equation*}
\min \left\{\left.(c+t q)^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b\right\} \tag{2.1}
\end{equation*}
$$

Assumption 2.1 There exists a $\hat{t}>0$ such that (2.1) has an optimal solution for every $t$ with
$0 \leq t<\hat{t}$.

Before introducing the theorems, we need the following lemma first.

Lemma 2.1 If there is a $t_{1}>0$ such that (2.1) has no optimal solution, for every $t$ with $0<t \leq t_{1}$, then (2.1) is unbounded from below, for every $t$ with $t>0$.

## Proof

Since the feasible region of (2.1) doesn't change as $t$ changes, (2.1) is always feasible. Since (2.1) has no optimal solution for every $t$ with $0<t \leq t_{1}$, (2.1) is unbounded from below for every $t$ with $0<t \leq t_{1}$. Let $f_{t}(x)$ denote the objective function of (2.1) with the subscript $t$ denoting explicit dependence on $t$. Then for a feasible solution $x_{1}$ for (2.1), there exists an $n$-vector $s$ such that $x_{1}-\sigma s$ is feasible for (2.1), for every positive scalar $\sigma$, and the objective function $f_{t}(x)$ satisfies

$$
f_{t}\left(x_{1}-\sigma s\right) \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty
$$

for every $t$ with $0<t \leq t_{1}$. Therefore, $s$ satisfies $\left(c+t_{1} q\right)^{\prime} s>0$ and $s^{\prime} C s=0$. Combining these with the feasibility restriction, $s$ needs to satisfy

$$
A s \geq 0, s^{\prime} C s=0, t_{1} q^{\prime} s>-c^{\prime} s
$$

Since (2.1) has an optimal solution for $t=0$, we have $c^{\prime} s \leq 0$. So, $t_{1} q^{\prime} s>-c^{\prime} s \geq 0$. Then for every $t$ with $t>t_{1}$, we have $A\left(x_{1}-\sigma s\right) \leq b, C s=0$ and $(c+t q)^{\prime} s>0$, for every positive scalar $\sigma$. Therefore,
$(c+t q)^{\prime}\left(x_{1}-\sigma s\right)+\frac{1}{2}\left(x_{1}-\sigma s\right)^{\prime} C\left(x_{1}-\sigma s\right)=(c+t q)^{\prime} x_{1}+\frac{1}{2} x_{1}^{\prime} C x_{1}-\sigma(c+t q)^{\prime} s \rightarrow-\infty$, as $\sigma \rightarrow+\infty$.
Thus, (2.1) is unbounded from below, for every $t$ with $t>0$.

Remark 2.1 If there are alternate optimal solutions for (2.1) when $t=0$, then $x_{0}$ may have no optimal continuation.

In order to illustrate Remark 2.1, recall (1.14) in Example 1.3 that we introduced in Chapter 1. In (1.14), $x_{0}$ is an optimal solution for $t=0$, the optimal solution "jumps" to $y^{+}$from $x_{0}$ as $t$ increases from zero to positive, and $x_{0}$ and $y^{+}$cannot be connected with a linear function of $t$. Thus, $x_{0}$ has no optimal continuation in this case. However, there does exist an optimal solution for $t=0$ for which there is an optimal continuation, namely $y^{+}$.

The following two theorems show how to get $h_{0}^{*}$ in the optimal continuation of $x_{0}$ for (2.1).

Theorem 2.1 Let Assumption 2.1 be satisfied. Suppose $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (2.1). In addition, suppose the optimal solution $x(t)=x_{0}+t h_{0}^{*}$ is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$. Then $h_{0}^{*}$ is an optimal solution for

$$
\begin{equation*}
\min \left\{\left.q^{\prime} h_{0}+\frac{1}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}=0\right\} \tag{2.2}
\end{equation*}
$$

## Proof

The optimality conditions for (2.1) when $t=0$ assert that

$$
\left.\begin{array}{l}
A_{0} x_{0}=b_{0},  \tag{2.3}\\
-c-C x_{0}=A_{0}^{\prime} u_{0}, \quad u_{0} \geq 0,
\end{array}\right\}
$$

where $u_{0}$ is a multiplier vector for $x_{0}$ whose components are associated with the rows of $A_{0}$. The optimality conditions when $0<t<\bar{t}$ assert that

$$
\left.\begin{array}{l}
A x(t) \leq b,  \tag{2.4}\\
-c-t q-C x(t)=A^{\prime} u, \quad u \geq 0 \\
u^{\prime}(A x(t)-b)=0
\end{array}\right\}
$$

Since the optimal solution $x(t)$ is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$, the matrix of gradients of all the constraints active at $x(t)$ is a submatrix of $A_{0}^{\prime}$. We can simplify (2.4) to

$$
\left.\begin{array}{l}
A_{0} x(t) \leq b_{0}  \tag{2.5}\\
-c-t q-C x(t)=A_{0}^{\prime} u, \quad u \geq 0 \\
u^{\prime}\left(A_{0} x(t)-b_{0}\right)=0,
\end{array}\right\}
$$

where $u=u(t)$ is a multiplier vector for $x(t)$ whose components are those $u_{i}$ associated with the rows of $A_{0}$. Some components of $u$ may be zero corresponding to constraints active at $x_{0}$ but inactive at $x(t)$. Substitute $x(t)=x_{0}+t h_{0}^{*}$ and $u=u_{0}+t u_{1}$ into (2.5), and with (2.3), we have

$$
\begin{align*}
& A_{0} h_{0}^{*} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}^{*}=0 \\
& -c-t q-C x_{0}-t C h_{0}^{*}=A_{0}^{\prime} u, \quad u \geq 0  \tag{2.6}\\
& u^{\prime} A_{0} h_{0}^{*}=0
\end{align*}
$$

The optimality conditions for the problem

$$
\begin{equation*}
\min \left\{\left.\left(c+C x_{0}\right)^{\prime} h_{0}+t q^{\prime} h_{0}+\frac{t}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}=0\right\} \tag{2.7}
\end{equation*}
$$

are

$$
\begin{align*}
& A_{0} h_{0} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}=0 \\
& -c-t q-C x_{0}-t C h_{0}=A_{0}^{\prime} v+\left(c+C x_{0}\right) w, \quad v \geq 0  \tag{2.8}\\
& v^{\prime} A_{0} h_{0}=0
\end{align*}
$$

Define $h_{0}=h_{0}^{*}, v=u$, and $w=0$. Then from (2.6), $h_{0}, v$ and $w$ satisfy (2.8). So, $h_{0}=h_{0}^{*}$ is optimal for (2.7). Using the primal constraint $\left(c+C x_{0}\right)^{\prime} h_{0}=0$, the objective function for (2.7) can be simplified. Thus $h_{0}=h_{0}^{*}$ is also optimal for

$$
\min \left\{\left.t q^{\prime} h_{0}+\frac{t}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}=0\right\} .
$$

For $t>0$, we have shown that $h_{0}=h_{0}^{*}$ is optimal for

$$
\min \left\{\left.q^{\prime} h_{0}+\frac{1}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}=0\right\}
$$

as required.

The importance of the optimal problem (2.2) is illustrated in the following theorem.

Theorem 2.2 Let Assumption 2.1 be satisfied. Suppose $h_{0}^{*}$ is an optimal solution for (2.2), and suppose that $w_{1}$ and $w_{2}$ are multipliers associated with the constraints $A_{0} h_{0} \leq 0$ and $\left(c+C x_{0}\right)^{\prime} h_{0}=0$ in (2.2), respectively. Then $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (2.1), and $v(t)=u_{0}+$ $t\left(w_{1}-w_{2} u_{0}\right)$ is a multiplier vector for $x(t)=x_{0}+t_{0}^{*}$ whose components are associated with the rows of $A_{0}$, for every $t$ with $0 \leq t<\bar{t}$, where $\bar{t}=\min \{\hat{t}, \tilde{t}\}>0$, and

$$
\begin{gather*}
\hat{t}=\min \left\{\left.\frac{b_{i}-a_{i}^{\prime} x_{0}}{a_{i}^{\prime} h_{0}^{*}} \right\rvert\, \text { all } i=1, \ldots, m \text { with } a_{i}^{\prime} h_{0}^{*}>0\right\},  \tag{2.9}\\
\tilde{t}=\min \left\{\left.\frac{-\left(u_{0}\right)_{i}}{\left(w_{1}-w_{2} u_{0}\right)_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with }\left(w_{1}-w_{2} u_{0}\right)_{i}<0\right\} . \tag{2.10}
\end{gather*}
$$

The full (m-dimensional) vector of multipliers, $u(t)$, is obtained from $v(t)$ by assigning zero to those components of $u(t)$ associated with constraints inactive at $x_{0}$ and the appropriately indexed components of $v(t)$, otherwise.

## Proof

Let $A_{1}^{\prime}$ be the matrix of gradients of all the constraints inactive at $x_{0}$, let $b_{1}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{1}$; i.e., $A_{1} x_{0}<b_{1}$. Similar to the proof of Lemma 1.3, there exists a $\bar{t}_{1}>0$ such that $A_{1}\left(x_{0}+t h_{0}^{*}\right)<b_{1}$, for every $t$ with $0<t<\bar{t}_{1}$.

Since $h_{0}^{*}$ is optimal for (2.2), $h_{0}^{*}$ and the associated multipliers $w_{1}$ and $w_{2}$ satisfy the optimality conditions

$$
\begin{array}{r}
A_{0} h_{0}^{*} \leq 0,\left(c+C x_{0}\right)^{\prime} h_{0}^{*}=0, \\
-q-C h_{0}^{*}=A_{0}^{\prime} w_{1}+\left(c+C x_{0}\right) w_{2}, w_{1} \geq 0, \\
w_{1}^{\prime} A_{0} h_{0}^{*}=0, \tag{2.13}
\end{array}
$$

where $w_{2}$ is a scalar. For $t>0,(2.12)$ can also be written as

$$
\begin{equation*}
-t q-t C h_{0}^{*}=A_{0}^{\prime}\left(t w_{1}\right)+\left(c+C x_{0}\right)\left(t w_{2}\right), t w_{1} \geq 0 \tag{2.14}
\end{equation*}
$$

or

$$
\begin{equation*}
-c-C x_{0}-t q-t C h_{0}^{*}=A_{0}^{\prime}\left(t w_{1}\right)+\left(c+C x_{0}\right)\left(t w_{2}-1\right), t w_{1} \geq 0 \tag{2.15}
\end{equation*}
$$

Since $x_{0}$ is an optimal solution for (2.1) for $t=0$ and $A_{0}^{\prime}$ is the matrix of gradients of all the constraints active at $x_{0}$, there exists an multiplier vector $u_{0}$ whose components are associated with the constraints active at $x_{0}$, satisfying

$$
\begin{equation*}
-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \tag{2.16}
\end{equation*}
$$

From (2.15) and (2.16),

$$
\begin{align*}
-\left(c+C x_{0}\right)-t q-t C h_{0}^{*} & =A_{0}^{\prime}\left(t w_{1}\right)-\left(t w_{2}-1\right) A_{0}^{\prime} u_{0}, \\
& =A_{0}^{\prime}\left[t w_{1}-\left(t w_{2}-1\right) u_{0}\right] . \tag{2.17}
\end{align*}
$$

There exists a $\bar{t}_{2}>0$ such that $t w_{2} \leq 1$; i.e., $t w_{2}-1 \leq 0$, for every $t$ with $0<t<\bar{t}_{2}$. Since $\left(t w_{1}\right) \geq 0, u_{0} \geq 0$ and $t w_{2}-1 \leq 0$, then for every $t$ with $0<t \leq \bar{t}_{2}$,

$$
\begin{equation*}
t w_{1}-\left(t w_{2}-1\right) u_{0} \geq 0 \tag{2.18}
\end{equation*}
$$

From (2.16) and the second constraint in (2.2), we have

$$
u_{0}^{\prime} A_{0} h_{0}^{*}=-\left(c+C x_{0}\right)^{\prime} h_{0}^{*}=0
$$

together with (2.13), it follows

$$
\begin{equation*}
\left[t w_{1}-\left(t w_{2}-1\right) u_{0}\right]^{\prime} A_{0} h_{0}^{*}=0 \tag{2.19}
\end{equation*}
$$

Then from (2.17), (2.18) and (2.19), we have

$$
\left.\begin{array}{l}
-c-C x_{0}-t q-t C h_{0}^{*}=A_{0}^{\prime}\left[t w_{1}-\left(t w_{2}-1\right) u_{0}\right], \quad t w_{1}-\left(t w_{2}-1\right) u_{0} \geq 0  \tag{2.20}\\
\left(t w_{1}-\left(t w_{2}-1\right) u_{0}\right)^{\prime} A_{0} h_{0}^{*}=0
\end{array}\right\}
$$

Let $v=t w_{1}-\left(t w_{2}-1\right) u_{0}$, combine (2.11) and (2.20),

$$
\left.\begin{array}{l}
A_{0} h_{0}^{*} \leq 0  \tag{2.21}\\
-c-t q-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime} v, \quad v \geq 0 \\
v^{\prime}\left(A_{0} t h_{0}^{*}\right)=0
\end{array}\right\}
$$

Let $\bar{t}=\min \left\{\bar{t}_{1}, \bar{t}_{2}\right\}>0$. Since $A_{0} x_{0}=b_{0}$ and $A_{1}\left(x_{0}+t h_{0}^{*}\right)<b_{1}$, for every $t$ with $0<t<\bar{t}_{1}$, it follows that for every $t$ with $0<t<\bar{t}$,

$$
\left.\begin{array}{l}
A\left(x_{0}+t h_{0}^{*}\right) \leq b,  \tag{2.22}\\
-c-t q-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime} v, \quad v \geq 0 \\
v^{\prime}\left(A_{0}\left(x_{0}+t h_{0}^{*}\right)-b_{0}\right)=0 .
\end{array}\right\}
$$

Thus, $x(t)=x_{0}+t h_{0}^{*}$ and the multiplier vector $v=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)$ whose components are associated with the rows of $A_{0}$ satisfy the optimality conditions for (2.1), for every $t$ with $0 \leq t<\bar{t}$. So $x(t)=x_{0}+t h_{0}^{*}$ is optimal for (2.1), for every $t$ with $0 \leq t<\bar{t}$. Therefore, $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (2.1) as required.

Since $x(t)=x_{0}+t h_{0}^{*}$ is an optimal solution for (2.1), if $a_{i}^{\prime} h_{0}^{*}>0$, then $a_{i}^{\prime} x_{0}<b_{i}$. From (2.9), $\hat{t}>0$. Since $v=u_{0}+t\left(w_{1}-w_{2} u_{0}\right) \geq 0$, if $\left(w_{1}-w_{2} u_{0}\right)_{i}<0$, then $\left(u_{0}\right)_{i}>0$. Thus, from (2.10), $\tilde{t}>0$. Therefore, $\bar{t}=\min \{\hat{t}, \tilde{t}\}>0$.

Note: If the constraint $a_{i}^{\prime} x \leq b_{i}$ active at $x_{0}$ remains active at $x(t)$ for every $t$ with $0<t<\bar{t}$, then $a_{i}^{\prime} h_{0}=0$.

We illustrate Theorems 2.1 and 2.2 by applying them to Example 1.2. In Example 1.2, $c=$ $\left(-\frac{10}{3},-\frac{8}{3}\right)^{\prime}, q=(0,1)^{\prime}, C=\left[\begin{array}{ll}1 & 0 \\ 0 & 2\end{array}\right], x_{0}=\left(\frac{2}{3}, \frac{2}{3}\right)^{\prime}$, and $c+C x_{0}=\left(-\frac{8}{3},-\frac{4}{3}\right)^{\prime}$. Since the first three constraints are active at $x_{0}$, we have $A_{0}=\left[\begin{array}{ll}1 & 2 \\ 2 & 1 \\ 1 & 1\end{array}\right]$. The optimal problem (2.2), in this case
becomes

$$
\begin{array}{lrl}
\operatorname{minimize}: & h_{2}+\frac{1}{2} h_{1}^{2}+h_{2}^{2} \\
\text { subject to : } & h_{1}+2 h_{2} & \leq 0, \\
& 2 h_{1}+h_{2} \leq 0,  \tag{2.23}\\
& h_{1}+h_{1} \leq 0, \\
& -\frac{8}{3} h_{1}-\frac{4}{3} h_{2} & =0 .
\end{array}
$$

The optimal solution for (2.23) is

$$
h^{*}=\left[\begin{array}{c}
\frac{2}{9} \\
-\frac{4}{9}
\end{array}\right] .
$$

The geometry of the problem (2.23) is shown in Figure 2.1. The half-line beginning at $\alpha$ and going towards $\beta$ is the feasible region of (2.23). The level sets of the objective function are ellipses centered at $\left(0,-\frac{1}{2}\right)^{\prime}$.


Figure 2.1: The related QP problem for Example 1.2.

Let $w_{1}=\left(w_{11}, w_{12}, w_{13}\right)^{\prime}$ be a multiplier vector for the first three constraints, and $w_{2}$ be a multiplier for the fourth constraint in (2.23). Then, we have

$$
w_{11}=w_{13}=0,
$$

and $w_{12}, w_{2}$ satisfy

$$
w_{12}-\frac{4}{3} w_{2}=-\frac{1}{9} .
$$

Since $u_{0}=\left(0, \frac{4}{3}, 0\right)^{\prime}$, it follows that

$$
w_{1}-u_{0} w_{2}=\left[\begin{array}{c}
0 \\
-\frac{1}{9} \\
0
\end{array}\right],
$$

thus

$$
v(t)=\left[\begin{array}{l}
0 \\
\frac{4}{3} \\
0
\end{array}\right]+t\left[\begin{array}{c}
0 \\
-\frac{1}{9} \\
0
\end{array}\right] .
$$

From Theorem 2.2,

$$
x(t)=x_{0}+t h_{0}^{*}=\left[\begin{array}{l}
\frac{2}{3}+\frac{2}{9} t \\
\frac{2}{3}-\frac{4}{9} t
\end{array}\right],
$$

is an optimal solution for the problem in Example 1.2, for every $t$ with $0<t \leq \bar{t}$, where $\bar{t}$ is solved as following by applying (2.9) and (2.10):

$$
\begin{gathered}
\hat{t}_{1}=\min \left\{-,-,-,-, \frac{\frac{2}{4}}{9}\right\}=\frac{3}{2}, \\
\tilde{t}_{1}=\min \left\{-, \frac{-\frac{4}{3}}{-\frac{1}{9}},-,-,-\right\}=12,
\end{gathered}
$$

from which

$$
\bar{t}=\min \left\{\frac{3}{2}, 12\right\}=\frac{3}{2} .
$$

One might wonder in (2.2) if the constraint $\left(c+C x_{0}\right)^{\prime} h=0$ is really necessary. If we remove it from the previous problem, (2.23) becomes

$$
\begin{array}{ll}
\operatorname{minimize}: & h_{2}+\frac{1}{2} h_{1}^{2}+h_{2}^{2} \\
\text { subject to : } & h_{1}+2 h_{2} \leq 0  \tag{2.24}\\
& 2 h_{1}+h_{2} \leq 0 \\
& h_{1}+h_{1} \geq 0
\end{array}
$$

The optimal solution is

$$
h^{*}=\left[\begin{array}{c}
0 \\
-\frac{1}{2}
\end{array}\right] .
$$

Because the objective function for the problem in Example 1.2 is strictly convex, the optimal solution for it is uniquely determined. Then $h_{0}^{*}$ for the present problem is different from what obtained from (2.23), and is therefore incorrect. Thus, the constraint $\left(c+C x_{0}\right)^{\prime} h=0$ is essential in (2.2).

### 2.2 The Boundedness of the Problem (2.2) in Theorem 2.1

It is possible that when $t=0$ the optimal solutions for (2.1) are not unique. In this case, if we cannot choose a proper $x_{0}$, the problem (2.2) may be unbounded from below. (The problem (2.2) is always feasible because $h_{0}=0$ is its feasible solution.) Refer back to the discussion and Figure 1.3 following Example 1.3.

What is the implication of (2.2) being unbounded from below? Theorem 2.3 below will give the answer. Before introducing the theorem, we first need two lemmas.

Lemma 2.2 If (2.2) is unbounded from below, then the following problem

$$
\begin{equation*}
\min \left\{-q^{\prime} s \mid C s=0, c^{\prime} s=0, A_{0} s \geq 0\right\} \tag{2.25}
\end{equation*}
$$

is unbounded from below.

## Proof

Since (2.2) is unbounded from below, then for a feasible solution $h_{1}$ for (2.2), there exists an $s$ such that $h_{1}-\sigma s$ is also feasible for (2.2), for any positive scalar $\sigma$, and $s$ satisfies $q^{\prime} s>0$ and $s^{\prime} C s=0$. From the feasibility of $h_{1}-\sigma s$, we can get $A_{0} s \geq 0$ and $\left(c+C x_{0}\right)^{\prime} s=0$. Because $C$ is positive semi-definite, $s^{\prime} C s=0$ implies $C s=0$. Furthermore, $C s=0$ and $\left(c+C x_{0}\right)^{\prime} s=0$ imply $c^{\prime} s=0$. Thus, $s$ satisfies $-q^{\prime} s<0, C s=0, c^{\prime} s=0, A_{0} s \geq 0$.

Therefore, $s$ is feasible for (2.25) and $-q^{\prime} s<0$. For every positive scalar $\sigma, \sigma s$ is also feasible for (2.25), and

$$
-q^{\prime}(\sigma s) \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty
$$

Thus (2.25) is unbounded from below.

Lemma 2.3 If (2.25) has an optimal solution $s^{*}=0$, then (2.2) is bounded from below and thus has an optimal solution.

## Proof

Assume on the contrary that (2.2) is unbounded from below. Then from Lemma 2.2, (2.25) is also unbounded from below. This contradicts that (2.25) has an optimal solution $s^{*}=0$. This contradiction establishes that (2.2) is indeed bounded from below.

Consider the problem

$$
\begin{equation*}
\min \left\{-q^{\prime} s \mid c^{\prime} s=0, C s=0, A_{0} s \geq 0, A s \geq A x_{0}-b\right\} \tag{2.26}
\end{equation*}
$$

It is feasible, because $s=0$ is its feasible solution.

Theorem 2.3 Assume (2.2) is unbounded from below. Assume (2.26) is bounded from below and has an optimal solution $s$. Then, $s \neq 0$. Let $x_{1}=x_{0}-s$. Let $A_{1}^{\prime}$ be the matrix of gradients of all the constraints active at $x_{1}$ in (2.1), and let $b_{1}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{1}$; i.e., $A_{1} x_{1}=b_{1}$. Then $x_{1}$ is also optimal for (2.1) for $t=0$, and moreover, the problem

$$
\begin{equation*}
\min \left\{\left.q^{\prime} h_{1}+\frac{1}{2} h_{1}^{\prime} C h_{1} \right\rvert\, A_{1} h_{1} \leq 0,\left(c+C x_{1}\right)^{\prime} h_{1}=0\right\} \tag{2.27}
\end{equation*}
$$

has a finite optimal solution.

## Proof

We first show that if (2.26) has an optimal solution $s$, then

$$
\begin{equation*}
s \neq 0 \tag{2.28}
\end{equation*}
$$

Otherwise, if $s=0$ is an optimal solution for (2.26), the optimality conditions assert

$$
\begin{gather*}
q=c u_{1}+C u_{2}-A_{0}^{\prime} u_{3}-A^{\prime} u_{4}, u_{3}, u_{4} \geq 0  \tag{2.29}\\
u_{4}^{\prime}\left(A x_{0}-b\right)=0 \tag{2.30}
\end{gather*}
$$

Since $A_{0}^{\prime}$ is the matrix of gradients of all the constraints active at $x_{0},(2.29)$ and $(2.30)$ can be simplified to

$$
\begin{gather*}
q=c u_{1}+C u_{2}-A_{0}^{\prime} u_{3}-A_{0}^{\prime} \bar{u}_{4}=c u_{1}+C u_{2}-A_{0}^{\prime}\left(u_{3}+\bar{u}_{4}\right), u_{3}, \bar{u}_{4} \geq 0  \tag{2.31}\\
A_{0} x_{0}-b_{0}=0 \tag{2.32}
\end{gather*}
$$

where $\bar{u}_{4}$ is the multiplier vector whose components are those $\left(u_{4}\right)_{i}$ associated with the rows of $A_{0}$. Then $s=0, u_{1}, u_{2}$ and $u_{3}+\bar{u}_{4}$ satisfy the optimality conditions for $(2.25)$, which are

$$
\begin{aligned}
& C s=0, c^{\prime} s=0, A_{0} s \geq 0 \\
& q=c u_{1}+C u_{2}-A_{0}^{\prime}\left(u_{3}+\bar{u}_{4}\right), u_{3}+\bar{u}_{4} \geq 0 \\
& \left(u_{3}+\bar{u}_{4}\right)^{\prime} A_{0} s=0
\end{aligned}
$$

Thus, $s=0$ being an optimal solution for (2.25), together with Lemma 2.3, contradicts that (2.2) is unbounded from below. Thus, if (2.26) has an optimal solution $s$, then $s \neq 0$, which verifies (2.28).

Now we will prove that $x_{1}$ is also optimal for (2.1) for $t=0$, and (2.27) has a finite optimal solution. From the fourth constraint of $(2.26), A s \geq A x_{0}-b$, we have

$$
A\left(x_{0}-s\right) \leq b
$$

which means

$$
A x_{1} \leq b
$$

From the first and second constraints of $(2.26), c^{\prime} s=0, C s=0$, the objective function for $x=x_{1}$ is

$$
c^{\prime} x_{1}+\frac{1}{2} x_{1}^{\prime} C x_{1}=c^{\prime}\left(x_{0}-s\right)+\frac{1}{2}\left(x_{0}-s\right)^{\prime} C\left(x_{0}-s\right)=c^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0}
$$

Thus, $x_{1}$ is also an optimal solution for (2.1) for $t=0$.

Since $s$ is an optimal solution for (2.26), the optimality conditions give us

$$
\left.\begin{array}{l}
q=C u+c v-A_{0}^{\prime} w_{0}-A^{\prime} w_{1}, w_{0}, w_{1} \geq 0  \tag{2.33}\\
w_{0}^{\prime} A_{0} s=0, w_{1}^{\prime}\left(A x_{0}-b-A s\right)=0
\end{array}\right\}
$$

Since $A_{1}$ is the matrix of gradients of all the constraints active at $x_{1}, A_{1} x_{1}=b_{1} ; i . e ., A_{1}\left(x_{0}-s\right)=b_{1}$, (2.33) can be simplified to

$$
\left.\begin{array}{l}
q=C u+c v-A_{0}^{\prime} w_{0}-A_{1}^{\prime} \bar{w}_{1}, w_{0}, \bar{w}_{1} \geq 0  \tag{2.34}\\
w_{0}^{\prime} A_{0} s=0, A_{1} s=A_{1} x_{0}-b_{1}
\end{array}\right\}
$$

where $\bar{w}_{1}$ is the multiplier vector whose components are those $\left(w_{1}\right)_{i}$ associated with the rows of $A_{1}$. From $w_{0}^{\prime} A_{0} s=0$, we know that if $a_{i}^{\prime} s=\left(A_{0} s\right)_{i} \neq 0$, then $\left(w_{0}\right)_{i}=0$. Let $A_{2}^{\prime}$ be the matrix of all the $a_{i}$ in $A_{0}$ satisfying $a_{i}^{\prime} s=0$; i.e., $A_{2} s=0$. Let $b_{2}$ be the vector whose components are those $b_{i}$
associated with the rows of $A_{2}$. Since $A_{2}$ is a submatrix of $A_{0}, A_{2} x_{0}=b_{2}$. We have $A_{2}\left(x_{0}-s\right)=b_{2}$; i.e., $A_{2} x_{1}=b_{2}$. Thus, $A_{2}$ is also a submatrix of $A_{1}$. So, (2.34) is equivalent to

$$
\begin{aligned}
& q=C u+c v-A_{2}^{\prime} \bar{w}_{0}-A_{1}^{\prime} \bar{w}_{1}=C u+c v-A_{1}^{\prime} w, \quad \bar{w}_{0}, \bar{w}_{1}, w \geq 0 \\
& A_{2} s=0, A_{1} s=A_{1} x_{0}-b_{1}
\end{aligned}
$$

where $w$ is a vector whose components are associated with the rows of $A_{1}$. Therefore, $s_{1}=0$ and $u, v, w$ satisfy

$$
\begin{aligned}
& C s_{1}=0, c^{\prime} s_{1}=0, A_{1} s_{1} \geq 0 \\
& q=C u+c v-A_{1}^{\prime} w, w \geq 0 \\
& w^{\prime} A_{1} s_{1}=0
\end{aligned}
$$

Thus, $s_{1}=0$ is optimal for (2.35). Therefore from Lemma 2.3, (2.27) has a finite optimal solution.

Theorem 2.4 If (2.26) is unbounded from below, then (2.1) is also unbounded from below, for every $t$ with $t>0$.

## Proof

If (2.26) is unbounded from below, then for a feasible solution $s$ for (2.26), there exists a vector $d$ such that $s-\sigma d$ is feasible for (2.26), for every positive scalar $\sigma$, and $q^{\prime} d<0$. That is, $q^{\prime} d<$ $0, c^{\prime} d=0, C d=0$, and $A d \leq 0$.

Then, $x_{0}+\sigma d$ satisfies

$$
A\left(x_{0}+\sigma d\right)=A x_{0}+\sigma A d \leq b
$$

and
$(c+t q)^{\prime}\left(x_{0}+\sigma d\right)+\frac{1}{2}\left(x_{0}+\sigma d\right)^{\prime} C\left(x_{0}+\sigma d\right)=(c+t q)^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0}+\sigma t q^{\prime} d \rightarrow-\infty$, as $\sigma \rightarrow+\infty$,
for every $t$ with $t>0$. Thus, (2.1) is unbounded from below, for every $t$ with $t>0$.

### 2.3 Example 1.3 Continued

Consider (1.14) in Example 1.3. If we begin with $t=0$ and an interior point optimal solution $x_{0}$, (2.2) in this problem becomes

$$
\min \left\{q^{\prime} h_{0} \mid h_{0} \text { has no constraints }\right\}
$$

It is unbounded if $q \neq 0$. Then we consider (2.26), which in this problem is

$$
\begin{equation*}
\min \left\{-q^{\prime} s \mid A s \geq A x_{0}-b\right\} \tag{2.36}
\end{equation*}
$$

Its optimal solution is same as

$$
\min \left\{q^{\prime}\left(x_{0}-s\right) \mid A\left(x_{0}-s\right) \leq b\right\}
$$

Suppose $s_{0}$ is an optimal solution for (2.36), and let $x_{1}=x_{0}-s_{0}$. Then, $x_{1}=x_{0}-s_{0}$ is optimal for

$$
\begin{equation*}
\min \left\{q^{\prime} x_{1} \mid A x_{1} \leq b\right\} \tag{2.37}
\end{equation*}
$$

So for every $t$ with $t>0, x_{1}$ is also optimal for

$$
\min \left\{t q^{\prime} x_{1} \mid A x_{1} \leq b\right\}
$$

which is precisely (1.14), and $x_{1}$ is precisely the point $y^{+}$in Figure 1.3.

Assume $A_{1}^{\prime}$ is the matrix of the gradients of all the constraints active at $x_{1}$. From the optimality conditions for $(2.37)$, we have $-q=A_{1}^{\prime} u_{1}, u_{1} \geq 0$. In this example, (2.27) becomes

$$
\begin{equation*}
\min \left\{q^{\prime} h_{1} \mid A_{1} h_{1} \leq 0\right\} \tag{2.38}
\end{equation*}
$$

Since $-q=A_{1}^{\prime} u_{1}$ and $u_{1} \geq 0$, it follows that $h_{1}=0$ satisfies the optimality conditions for (2.38), which are

$$
\left.\begin{array}{l}
A_{1} h_{1} \leq 0 \\
-q=A_{1}^{\prime} u_{1}, u_{1} \geq 0 \\
u_{1}^{\prime} A_{1} h_{1}=0
\end{array}\right\}
$$

Thus, $h_{1}=0$ is an optimal solution for (2.38), the multiplier vector $u_{1}$ satisfies $-q=A_{1}^{\prime} u_{1}$. When $t=0$, the multiplier for $x_{1}=x_{0}-s_{0}$ is $u_{0}=0$. When $t>0, x_{1}$ and $t u_{1}$ satisfy $-q=$ $A_{1}^{\prime}\left(t u_{1}\right),\left(t u_{1}\right) \geq 0$. Thus, $v(t)=u_{0}+t u_{1}=t u_{1}$ is the multiplier for $x(t)=x_{1}$, for every $t$ with $t>0$.

## Chapter 3

## A Parameter only in the Constraints

In this chapter, we will study another simple case - the PQP problem with the parameter $t$ only in the right-hand side of the constraints.

Consider the following PQP problem

$$
\begin{equation*}
\min \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b+t p\right\} . \tag{3.1}
\end{equation*}
$$

From the QP duality, the dual of (3.1) is

$$
\begin{equation*}
\max \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x+u^{\prime}(A x-b-t p) \right\rvert\, C x+A^{\prime} u=-c, u \geq 0\right\} \tag{3.2}
\end{equation*}
$$

It is a PQP problem with the parameter $t$ only in the linear part of the objective function, thus we could solve it using the method introduced in Chapter 2. Assume $\left(x^{*}(t), u^{*}(t)\right)^{\prime}$ is an optimal solution for (3.2), for every $t$ with $t \in G$, where $G$ is the region of $t$ on which (3.2) has optimal solutions. If $C$ is positive definite, then (3.1) is strictly convex. From Strict Converse Duality Theorem[1], $x^{*}(t)$ is an optimal solution for (3.1), for every $t$ with $t \in G$. However, if $C$ is positive semi-definite, that is, (3.1) is not strictly convex, then $x^{*}(t)$ may not be an optimal solution for (3.1), and it is hard to find an optimal solution for (3.1) from its dual.

In this chapter, we will develop another way to solve the PQP problem with the parameter only in the right-hand side of the constraints, with $C$ being positive semi-definite.

### 3.1 Solution of the PQP Problem with a Parameter in the RightHand Side of the Constraints by Solving a Related QP Problem without the Parameter

Assumption 3.1 There exists a $\hat{t}>0$ such that (3.1) has an optimal solution for every $t$ with $0 \leq t<\hat{t}$.

To determine the feasibility of (3.1) for $t>0$, we can solve the $(n+1)$-variable linear programming problem

$$
\begin{equation*}
\max \{t \mid A x-t p \leq b\} \tag{3.3}
\end{equation*}
$$

where both $x$ and $t$ are variables in the LP. If the optimal solution for (3.3) is zero, then there exists no $t>0$ such that $A x \leq b+t p$ has a solution, therefore, (3.1) is infeasible for every $t$ with $t>0$. If (3.3) has an optimal solution $\hat{t}>0$, then (3.1) is feasible for every $t$ with $0 \leq t \leq \hat{t}$, and infeasible for every $t$ with $t>\hat{t}$. If (3.3) is unbounded from above, then (3.1) is feasible for every $t$ with $t \geq 0$.

The following two theorems show how to obtain $h_{0}^{*}$ in the optimal continuation of $x_{0}$ for (3.1).

Recall the notation in Chapter 1. $x_{0}$ is an optimal solution for (3.1) for $t=0 . A_{0}^{\prime}$ is the matrix of gradients of all the constraints active at $x_{0}$. Let $p_{0}$ be the vector whose components are those $p_{i}$ associated with the rows of $A_{0}$.

Theorem 3.1 Let Assumption 3.1 be satisfied. Suppose $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$
for (3.1). In addition, suppose the optimal solution $x(t)=x_{0}+t h_{0}^{*}$ for (3.1) is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$. Then there exists a vector $u_{0}$ such that $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for the problem

$$
\begin{equation*}
\min \left\{\left.-p_{0}^{\prime} u_{0}+\frac{1}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\} \tag{3.4}
\end{equation*}
$$

## Proof

The outline of the proof is as follows. We first prove that $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (3.10) (see below). Then we change (3.10) to its equivalent form (3.12) (see below), and using the optimality conditions for (3.12), prove that $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (3.4).

Since $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (3.1); i.e., $x(t)=x_{0}+t h_{0}^{*}$ is an optimal solution for (3.1), for every $t$ with $0 \leq t<\bar{t}$. The optimality conditions for (3.1) assert

$$
\begin{aligned}
& A\left(x_{0}+t h_{0}^{*}\right) \leq b+t p, \\
& -c-C\left(x_{0}+t h_{0}^{*}\right)=A^{\prime} u, u \geq 0, \\
& u^{\prime}\left[A\left(x_{0}+t h_{0}^{*}\right)-(b+t p)\right]=0,
\end{aligned}
$$

where $u=u(t)$. These are equivalent to

$$
\left.\begin{array}{l}
A\left(x_{0}+t h_{0}^{*}\right) \leq b+t p,  \tag{3.5}\\
-c-C x_{0}-t C h_{0}^{*}=A^{\prime} u, u \geq 0 \\
u^{\prime}\left[\left(A x_{0}-b\right)+t\left(A h_{0}^{*}-p\right)\right]=0
\end{array}\right\}
$$

Since $x(t)$ is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$, all the constraints active at $x(t)$ are also active at $x_{0}$. Consequently, the matrix of the gradients of all the constraints active at $x(t)$ is a submatrix of $A_{0}^{\prime}$. Thus, (3.5) can be simplified to

$$
\left.\begin{array}{l}
A_{0} h_{0}^{*} \leq p_{0}  \tag{3.6}\\
-c-C x_{0}-t C h_{0}^{*}=A_{0}^{\prime} u, u \geq 0 \\
u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0,
\end{array}\right\}
$$

where $u=u_{0}+t u_{1}$ is a multiplier vector whose components are associated with the rows of $A_{0}$, and $u_{0}$ is a multiplier vector for $x_{0}$ whose components are associated with the rows of $A_{0}$, so $u_{0}$ satisfies the optimality conditions for $t=0$,

$$
\begin{equation*}
-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \tag{3.7}
\end{equation*}
$$

Since $\left(u_{0}+t u_{1}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$, for every $t$ with $0<t<\bar{t}, u_{0}$ must satisfy

$$
\begin{equation*}
u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 \tag{3.8}
\end{equation*}
$$

Because $-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$ can also be written as

$$
\begin{equation*}
\left(c+C x_{0}\right)^{\prime} h_{0}^{*}+p_{0}^{\prime} u_{0}=0 . \tag{3.9}
\end{equation*}
$$

Combining (3.6), (3.7) and (3.9), we get

$$
\begin{aligned}
& A_{0} h_{0}^{*} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}^{*}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& -c-C x_{0}-t C h_{0}^{*}=A_{0}^{\prime} u, u \geq 0 \\
& u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0
\end{aligned}
$$

Therefore, $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for the problem

$$
\begin{equation*}
\min \left\{\left.\left(c+C x_{0}\right)^{\prime} h_{0}+\frac{1}{2} t h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\} \tag{3.10}
\end{equation*}
$$

because $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]=\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ and $v_{1}=u, v_{2}, v_{3}, v_{4}=0$ satisfy the optimality conditions

$$
\begin{align*}
& A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& {\left[\begin{array}{c}
-\left(c+C x_{0}\right)-t C h_{0} \\
0
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] v_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] v_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] v_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] v_{4}, v_{1}, v_{4} \geq 0} \\
& v_{1}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0 \\
& v_{4}^{\prime} u_{0}=0 \tag{3.11}
\end{align*}
$$

From the constraint $\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0$ in (3.10), (3.10) is equivalent to

$$
\begin{equation*}
\min \left\{\left.-p_{0}^{\prime} u_{0}+\frac{1}{2} t h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\} \tag{3.12}
\end{equation*}
$$

The optimality conditions for (3.12) are

$$
\begin{align*}
& A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& {\left[\begin{array}{c}
-t C h_{0} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] w_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] w_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] w_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] w_{4}, w_{1}, w_{4} \geq 0}  \tag{3.13}\\
& w_{1}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0 \\
& w_{4}^{\prime} u_{0}=0
\end{align*}
$$

From (3.11), it follows that $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]=\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ and $w=\left[\begin{array}{l}w_{1} \\ w_{2} \\ w_{3} \\ w_{4}\end{array}\right]=\left[\begin{array}{l}u \\ 1 \\ 0 \\ 0\end{array}\right]$ satisfy (3.13). Therefore, $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ satisfies

$$
\left[\begin{array}{c}
-C h_{0}^{*} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] \frac{u}{t}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] \frac{1}{t}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] \alpha_{1}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] \alpha_{2},
$$

where

$$
\alpha_{1}=\left(1-\frac{1}{t}\right) h_{0}^{*},
$$

and

$$
\alpha_{2}=\left(1-\frac{1}{t}\right)\left(A_{0} h_{0}^{*}-p_{0}\right) .
$$

Since $A_{0} h_{0}^{*} \leq p_{0}$, and for every $t$ with $0<t<1,1-\frac{1}{t}<0$, thus,

$$
\alpha_{2}=\left(1-\frac{1}{t}\right)\left(A_{0} h_{0}^{*}-p_{0}\right) \geq 0 .
$$

Since $u \geq 0, t \geq 0$,

$$
\frac{u}{t} \geq 0
$$

Since $u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$, it follows that

$$
\left(\frac{u}{t}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 .
$$

Since we have $u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$ from (3.8),

$$
\alpha_{2}^{\prime} u_{0}=\left(1-\frac{1}{t}\right)\left(A_{0} h_{0}^{*}-p_{0}\right)^{\prime} u_{0}=0 .
$$

Thus, $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]=\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ and $w_{1}=\frac{u}{t}, w_{2}=\frac{1}{t}, w_{3}=\alpha_{1}, w_{4}=\alpha_{2}$ satisfy

$$
\begin{aligned}
& A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& {\left[\begin{array}{c}
-C h_{0} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] w_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] w_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] w_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] w_{4}, w_{1}, w_{4} \geq 0} \\
& w_{1}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0 \\
& w_{4}^{\prime} u_{0}=0
\end{aligned}
$$

which are precisely the optimality conditions for (3.4). Thus, we have $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (3.4) as required.

The importance of the optimal problem (3.4) is illustrated in the following theorem.

Theorem 3.2 Let Assumption 3.1 be satisfied. Suppose $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (3.4), and suppose that $w_{1}, w_{2}, w_{3}$ and $w_{4}$ are multipliers associated with the constraints $A_{0} h_{0} \leq p_{0}$, $\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}$ and $u_{0} \geq 0$, respectively. Then $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (3.1), and $v(t)=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)$ is an associated multiplier vector for $x(t)=x_{0}+t h_{0}^{*}$, for every $t$ with $0 \leq t<\bar{t}$, where $\bar{t}=\min \{\hat{t}, \tilde{t}\}>0$, and

$$
\begin{gather*}
\hat{t}=\min \left\{\left.\frac{b_{i}-a_{i}^{\prime} x_{0}}{a_{i}^{\prime} h_{0}^{*}-p_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with } a_{i}^{\prime} h_{0}^{*}>p_{i}\right\},  \tag{3.14}\\
\tilde{t}=\min \left\{\left.\frac{-\left(u_{0}\right)_{i}}{\left(w_{1}-w_{2} u_{0}\right)_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with }\left(w_{1}-w_{2} u_{0}\right)_{i}<0\right\} . \tag{3.15}
\end{gather*}
$$

The full (m-dimensional) vector of multipliers, $u(t)$, is obtained from $v(t)$ by assigning zero to those components of $u(t)$ associated with constraints inactive at $x_{0}$ and the appropriately indexed components of $v(t)$, otherwise.

## Proof

Let $A_{1}^{\prime}$ be the matrix of the gradients of all the constraints inactive at $x_{0}$ for (3.1), let $b_{1}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{1}$. Then $A_{1} x_{0}<b_{1}$. Similar to the proof of Lemma 1.3, there is a $\bar{t}_{1}>0$, such that $A_{1}\left(x_{0}+t h_{0}^{*}\right)<b_{1}+t p_{1}$ for every $t$ with $0 \leq t<\bar{t}_{1}$.

Since $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (3.4), the optimality conditions assert that

$$
\begin{gather*}
{\left[\begin{array}{c}
-C h_{0}^{*} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] w_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] w_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] w_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] w_{4}, \quad w_{1}, w_{4} \geq 0}  \tag{3.16}\\
w_{1}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 \tag{3.17}
\end{gather*}
$$

Multiplying both sides of (3.16) and (3.17) by $t$ gives

$$
\begin{gather*}
{\left[\begin{array}{c}
-t C h_{0}^{*} \\
t p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right]\left(t w_{1}\right)+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right]\left(t w_{2}\right)+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right]\left(t w_{3}\right)+\left[\begin{array}{c}
0 \\
-I
\end{array}\right]\left(t w_{4}\right), \quad\left(t w_{1}\right),\left(t w_{4}\right) \geq 0}  \tag{3.18}\\
\left(t w_{1}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 \tag{3.19}
\end{gather*}
$$

From (3.18), it follows

$$
-t C h_{0}^{*}=A_{0}^{\prime}\left(t w_{1}\right)+\left(c+C x_{0}\right)\left(t w_{2}\right),
$$

and this is equivalent to

$$
-\left(c+C x_{0}\right)-t C h_{0}^{*}=A_{0}^{\prime}\left(t w_{1}\right)+\left(c+C x_{0}\right)\left(t w_{2}-1\right) .
$$

From the optimality conditions for (3.1) when $t=0,-\left(c+C x_{0}\right)=A_{0}^{\prime} u_{0}$, so

$$
-\left(c+C x_{0}\right)-t C h_{0}^{*}=A_{0}^{\prime}\left(t w_{1}-\left(t w_{2}-1\right) u_{0}\right)
$$

The second and the third constraints of (3.4) give

$$
u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0,
$$

together with (3.19), we have

$$
\left(t w_{1}-\left(t w_{2}-1\right) u_{0}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0
$$

Let $v(t)=t w_{1}-\left(t w_{2}-1\right) u_{0}$. For $\bar{t}_{2}>0$ given small enough, $t w_{2} \leq 1$; i.e., $t w_{2}-1 \leq 0$, for every $t$ with $0 \leq t \leq \bar{t}_{2}$. Thus, $v(t) \geq 0$ since $w_{1}, u_{0} \geq 0$. Since $A_{0} x_{0}=b_{0}$, we have

$$
\begin{aligned}
& t A_{0} h_{0}^{*} \leq t p_{0}, \\
& -c-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime} v(t), v(t) \geq 0, \\
& v(t)^{\prime}\left(\left(A_{0} x_{0}-b_{0}\right)+t\left(A h_{0}^{*}-p_{0}\right)\right)=0,
\end{aligned}
$$

and since $A_{1}\left(x_{0}+t h_{0}^{*}\right)<b_{1}+t p_{1}$, it follows

$$
\begin{aligned}
& A\left(x_{0}+t h_{0}^{*}\right) \leq b+t p \\
& -c-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime} v(t), v(t) \geq 0 \\
& v(t)^{\prime}\left(A_{0}\left(x_{0}+t h_{0}^{*}\right)-\left(b_{0}+t p_{0}\right)\right)=0 .
\end{aligned}
$$

Let $\bar{t}=\min \left\{\bar{t}_{1}, \bar{t}_{2}\right\}>0$. Then $x(t)=x_{0}+t h_{0}^{*}$ and the associated multiplier $v(t)=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)$ satisfy the optimality conditions for (3.1), for every $t$ with $0 \leq t<\bar{t}$. Thus $x(t)=x_{0}+t h_{0}^{*}$ is optimal for (3.1), for every $t$ with $0 \leq t<\bar{t}$. Therefore, $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (3.1) as required.

Since $x(t)=x_{0}+t h_{0}^{*}$ is an optimal solution for (3.1), if $a_{i}^{\prime} h_{0}^{*}>p_{i}$, then $a_{i}^{\prime} x_{0}<b_{i}$. From (3.14), $\hat{t}>0$. Since $v=u_{0}+t\left(w_{1}-w_{2} u_{0}\right) \geq 0$, if $\left(w_{1}-w_{2} u_{0}\right)_{i}<0$, then $\left(u_{0}\right)_{i}>0$. From (3.15), $\tilde{t}>0$. Therefore, $\bar{t}=\min \{\hat{t}, \tilde{t}\}>0$.

Recall Example 1.4 in Chapter 1. The first four constraints are active at $x_{0}=(1,1)^{\prime}$. Let $u_{0}=\left(v_{1}, v_{2}, v_{3}, v_{4}\right)^{\prime}$ be an multiplier vector for $x_{0}$ whose components are associated with the first four constraints. Then we can get an optimal continuation $h_{0}^{*}=\left[\begin{array}{l}h_{1} \\ h_{2}\end{array}\right]$ of $x_{0}$ by solving (3.4), which
in this problem is

$$
\begin{aligned}
& \text { minimize : } \quad \frac{1}{2} h_{1}^{2}+h_{2}^{2}+v_{3}+\frac{1}{2} v_{4} \\
& \text { subject to : } h_{1} \leq 0, \\
& h_{2} \quad \leq 0, \\
& h_{1}+h_{2} \quad \leq-1, \\
& h_{1}+2 h_{2} \quad \leq-\frac{1}{2}, \\
& -h_{1}-v_{3}-\frac{1}{2} v_{4}=0, \\
& v_{1}+v_{3}+v_{4}=1, \\
& v_{2}+v_{3}+2 v_{4}=0, \\
& v_{1} \quad \geq 0, \\
& v_{2} \quad \geq 0, \\
& v_{3} \quad \geq 0, \\
& v_{4} \geq 0 .
\end{aligned}
$$

The optimal solution is

$$
h_{0}^{*}=\left[\begin{array}{c}
0 \\
-1
\end{array}\right],
$$

and

$$
u_{0}=\left[\begin{array}{l}
v_{1} \\
v_{2} \\
v_{3} \\
v_{4}
\end{array}\right]=\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right]
$$

From Theorem 3.2, the optimal solution for the problem of Example 1.4 is

$$
x(t)=x_{0} t h_{0}^{*}=\left[\begin{array}{c}
1 \\
1-t
\end{array}\right],
$$

with the multiplier vector

$$
v(t)=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)=\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right]+t\left[\begin{array}{c}
-2 \\
0 \\
2 \\
0
\end{array}\right]
$$

whose components are associated with the first four constraints, for every $t$ with $0<t<\bar{t}$.

Again from Theorem 3.2, the upper limit $\bar{t}$ is determined by applying (3.14) and (3.15):

$$
\begin{aligned}
& \hat{t}_{1}=\min \left\{-,-,-,-,-, \frac{1}{1}\right\}=1 \\
& \tilde{t}_{1}=\min \left\{-\frac{1}{-2},-,-,-\right\}=\frac{1}{2}
\end{aligned}
$$

from which

$$
\bar{t}=\min \left\{1, \frac{1}{2}\right\}=\frac{1}{2} .
$$

Therefore,

$$
x(t)=\left[\begin{array}{c}
1 \\
1-t
\end{array}\right]
$$

is optimal for the problem, for every $t$ with $0 \leq t \leq \frac{1}{2}$, in agreement with our geometric determination of the optimal solution in Example 1.4.

### 3.2 Reduction of Theorem 3.2 to the "No Ties" Case

In this section, we will show that (3.4) can be simplified to known results for the "no ties" case.

Consider the problem

$$
\begin{equation*}
\min \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b+t p\right\} \tag{3.20}
\end{equation*}
$$

Let $x_{0}$ be an optimal solution for (3.20) for $t=0$, let $A_{0}^{\prime}$ be the matrix of gradients of all the constraints active at $x_{0}$, and let $b_{0}$ and $p_{0}$ be the vectors whose components are those $b_{i}$ and $p_{i}$
associated with the rows of $A_{0}$, respectively. Suppose that there exists a $\bar{t}>0$ such that an optimal solution $x(t)=x_{0}+t h_{0}^{*}$ for (3.20) has the same active constraints as those for $x_{0}$, for every $t$ with $0<t<\bar{t}$, and $u=u_{0}+t u_{1}$ is an associated multiplier vector. Assume $A_{0}$ has full row rank and $H_{0}=\left[\begin{array}{cc}C & A_{0}^{\prime} \\ A_{0} & 0\end{array}\right]$ is nonsingular. Then, $s^{\prime} C s>0$ for all $s \neq 0, A_{0} s=0$.

From the optimality conditions for (3.20) for $t=0$,

$$
\left[\begin{array}{cc}
C & A_{0}^{\prime}  \tag{3.21}\\
A_{0} & 0
\end{array}\right]\left[\begin{array}{l}
x_{0} \\
u_{0}
\end{array}\right]=\left[\begin{array}{c}
-c \\
b_{0}
\end{array}\right] .
$$

From the optimality conditions for (3.20) for $t>0$, we have

$$
A_{0}\left(x_{0}+t h_{0}^{*}\right)=b_{0}+t p_{0} \Rightarrow A_{0} h_{0}^{*}=p_{0}
$$

and

$$
\begin{equation*}
-c-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime}\left(u_{0}+t u_{1}\right) . \tag{3.22}
\end{equation*}
$$

From (3.21), we have $-c-C x_{0}=A_{0}^{\prime} u_{0}$. Thus (3.22) implies $-C h_{0}^{*}=A_{0}^{\prime} u_{1}$. So we get

$$
\left[\begin{array}{cc}
C & A_{0}^{\prime}  \tag{3.23}\\
A_{0} & 0
\end{array}\right]\left[\begin{array}{l}
h_{0}^{*} \\
u_{1}
\end{array}\right]=\left[\begin{array}{c}
0 \\
p_{0}
\end{array}\right] .
$$

Since $\left[\begin{array}{cc}C & A_{0}^{\prime} \\ A_{0} & 0\end{array}\right]$ is nonsingular, $\left[\begin{array}{l}h_{0}^{*} \\ u_{1}\end{array}\right]$ and $\left[\begin{array}{l}x_{0} \\ u_{0}\end{array}\right]$ are uniquely determined by (3.23) and (3.21). Indeed, this is the identical solution obtained by Best in the "no ties" case.

Under the same "no ties" assumption, (3.4) can be simplified. Since $A_{0}$ has full row rank, which means that the active constraints at $x_{0}$ are linear independent, we know that $u_{0}$ is unique. So we can take out the third and the fourth constraints without changing the problem. The second constraint $\left(c+C x_{0}\right)^{\prime} h_{0}+u_{0}^{\prime} p_{0}=0$ can be written as $u_{0}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0$. Also because of the uniqueness of $u_{0}$, the term $-p_{0}^{\prime} u_{0}$ in the objective function is a constant. Thus, the optimal solution $h_{0}^{*}$ for (3.4) is also optimal for

$$
\begin{equation*}
\min \left\{\left.\frac{1}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0}, u_{0}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0\right\} . \tag{3.24}
\end{equation*}
$$

From the optimality conditions, the optimal solution $h_{0}^{*}$ for (3.24) satisfies

$$
\begin{gathered}
-C h_{0}^{*}=A_{0}^{\prime} v_{1}+A_{0}^{\prime} u_{0} v_{2}, v_{1} \geq 0, \\
v_{1}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0,
\end{gathered}
$$

where $v_{2}$ is a scalar. Let $v=v_{1}+v_{2} u_{0}$. Since $u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$, we have

$$
\begin{gather*}
-C h_{0}^{*}=A_{0}^{\prime} v,  \tag{3.25}\\
v^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 . \tag{3.26}
\end{gather*}
$$

Since $\left[\begin{array}{cc}C & A_{0}^{\prime} \\ A_{0} & 0\end{array}\right]$ is nonsingular, we can get a unique solution $\left[\begin{array}{c}h_{0} \\ v\end{array}\right]$ from

$$
\left[\begin{array}{cc}
C & A_{0}^{\prime}  \tag{3.27}\\
A_{0} & 0
\end{array}\right]\left[\begin{array}{c}
h_{0} \\
v
\end{array}\right]=\left[\begin{array}{c}
0 \\
p_{0}
\end{array}\right] .
$$

The solution $\left[\begin{array}{c}h_{0} \\ v\end{array}\right]$ for (3.27) satisfies (3.25) and (3.26), so it is an optimal solution for (3.24). Thus the optimal solution that has same active constraints as $x_{0}$ is uniquely determined by (3.27). This verifies that we will get the correct optimal solution using the result in Theorem 3.2 in Section 3.1 for the "no ties" case.

### 3.3 Feasibility and Boundedness of the Problem (3.4) in Theorem 3.1

Assume (3.1) is feasible for every $t$ with $0<t \leq \bar{t}$ throughout this section. The critical problem (3.4) may in general be infeasible, feasible and bounded, or feasible and unbounded. In this section, we will show that it is always feasible and bounded.

Consider the feasible region of (3.4), namely
$S \equiv\left\{\left.\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right] \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\}$,
$=\left\{\left.\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right] \right\rvert\, A_{0} h_{0} \leq p_{0}, u_{0}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\}$.

The optimality conditions for the problem

$$
\begin{equation*}
\min \left\{\left(c+C x_{0}\right)^{\prime} h_{0} \mid A_{0} h_{0} \leq p_{0}\right\} \tag{3.28}
\end{equation*}
$$

imply that if (3.28) has an optimal solution, then the set $S$ is not empty.

Lemma 3.1 (3.28) is feasible.

## Proof

We know that $x_{0}$ is optimal for $\min \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b\right\}$ and $A_{0} x_{0}=b_{0}$. Assumption 3.1 implies that (3.1) is feasible for every $t$ with $0<t<\bar{t}$. Then, there exists an $x_{1}$ such that $A_{0} x_{1} \leq b_{0}+t p_{0}$, for some $t_{0}$ satisfying $0<t_{0}<\bar{t}$. Then,

$$
\begin{aligned}
& A_{0}\left(x_{1}-x_{0}\right) \leq b_{0}+t_{0} p_{0}-b_{0} \\
& \Rightarrow A_{0}\left(x_{1}-x_{0}\right) \leq t_{0} p_{0} \\
& \quad \Rightarrow A_{0} \frac{x_{1}-x_{0}}{t_{0}} \leq p_{0}
\end{aligned}
$$

Thus, $\frac{x_{1}-x_{0}}{t_{0}}$ is a feasible solution for (3.28). So (3.28) is feasible.

Lemma 3.2 (3.28) is bounded.

## Proof

Assume on the contrary that (3.28) is unbounded. Then for a feasible solution $h_{1}$ for (3.28), there exists an $s_{1}$ such that $h_{1}-\sigma s_{1}$ is feasible, for every positive scalar $\sigma$, and

$$
\left(c+C x_{0}\right)^{\prime}\left(h_{1}-\sigma s_{1}\right) \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty
$$

So we have

$$
\left.\begin{array}{l}
\left(c+C x_{0}\right)^{\prime} s_{1}>0  \tag{3.29}\\
A_{0} s_{1} \geq 0
\end{array}\right\}
$$

From the optimality conditions for the original problem (3.1), when $t=0$, we have

$$
-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 .
$$

Together with (3.29), it follows

$$
\left(c+C x_{0}\right)^{\prime} s_{1}=-\left(A_{0}^{\prime} u_{0}\right)^{\prime} s_{1}=-u_{0}^{\prime}\left(A_{0} s_{1}\right) \leq 0
$$

This is in contradiction to $\left(c+C x_{0}\right)^{\prime} s_{1}>0$. So, (3.28) is bounded.

Theorem 3.3 (3.4) is feasible.

From Lemma 3.1 and Lemma 3.2, (3.28) is feasible and bounded, which means (3.28) has an optimal solution. So the set $S$ is not empty. Therefore, (3.4) is feasible.

To study the boundedness of (3.4), we rewrite (3.4) as

$$
\begin{align*}
& \operatorname{minimize}: \quad\left[\begin{array}{ll}
0 & -p_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{ll}
h_{0}^{\prime} & u_{0}^{\prime}
\end{array}\right]\left[\begin{array}{ll}
C & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right] \\
& \text { subject to : }\left[\begin{array}{ll}
A_{0} & 0
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right] \leq p_{0}, \\
& {\left[\begin{array}{ll}
0 & A_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right]=-c-C x_{0},}  \tag{3.30}\\
& {\left[\begin{array}{ll}
\left(c+C x_{0}\right)^{\prime} & p_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right]=0,} \\
& \\
& {\left[\begin{array}{ll}
0 & -I
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right] \leq 0 .}
\end{align*}
$$

Theorem 3.4 (3.30); i.e., (3.4) is bounded.

## Proof

Assume on the contrary that (3.30) is unbounded. Then for a feasible solution $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]$ for (3.30), such that there exists a vector $\left[\begin{array}{l}s_{1} \\ s_{2}\end{array}\right]$, satisfying

$$
\begin{gathered}
{\left[\begin{array}{ll}
0 & -p_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
s_{1} \\
s_{2}
\end{array}\right]>0,} \\
{\left[\begin{array}{ll}
s_{1}^{\prime} & s_{2}^{\prime}
\end{array}\right]\left[\begin{array}{ll}
C & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
s_{1} \\
s_{2}
\end{array}\right]=0,}
\end{gathered}
$$

and $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]-\sigma\left[\begin{array}{l}s_{1} \\ s_{2}\end{array}\right]$ is feasible, for every positive scalar $\sigma$. Thus we have

$$
\begin{aligned}
& -p_{0}^{\prime} s_{2}>0, \\
& s_{1}^{\prime} C s_{1}=0 \Rightarrow C s_{1}=0, \\
& A_{0} s_{1} \geq 0 \\
& A_{0}^{\prime} s_{2}=0, \\
& \left(c+C x_{0}\right)^{\prime} s_{1}+p_{0}^{\prime} s_{2}=0 \Rightarrow c^{\prime} s_{1}+p_{0}^{\prime} s_{2}=0, \\
& s_{2} \leq 0 .
\end{aligned}
$$

From $-p_{0}^{\prime} s_{2}>0$ and $c^{\prime} s_{1}+p_{0}^{\prime} s_{2}=0$, we get $c^{\prime} s_{1}>0$. Since $A_{0} s_{1} \geq 0, A\left(x_{0}-\sigma s_{1}\right) \leq b$, for $\sigma$ small and positive. we have

$$
\begin{aligned}
& c^{\prime}\left(x_{0}-\sigma s_{1}\right)+\frac{1}{2}\left(x_{0}-\sigma s_{1}\right)^{\prime} C\left(x_{0}-\sigma s_{1}\right) \\
= & c^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0}-\sigma c^{\prime} s_{1}<c^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0} .
\end{aligned}
$$

This is in contradiction to $x_{0}$ being an optimal solution for (3.1) for $t=0$. Therefore, (3.30) is bounded.

### 3.4 The Boundedness of the Original Problem (3.1)

Lemma 3.3 If (3.1) has an optimal solution $x_{0}$ when $t=0$, and it is feasible for every $t$ with $t>0$, then it is also bounded from below for every $t$ with $t>0$.

## Proof

Assume on the contrary that (3.1) is unbounded for some $t=t_{1}>0$. Then for a feasible solution $x_{1}$ for $t=t_{1}$, there exists a vector $s$ such that $x_{1}-\sigma s$ is feasible for (3.1) for $t=t_{1}$, for every
positive scalar $\sigma$, and $c^{\prime} s>0, s^{\prime} C s=0$. From the feasibility of $x_{1}$ and $x_{1}-\sigma s$, we have $A s \geq 0$. Let $x_{2}(\sigma)=x_{0}-\sigma s$. Then $x_{2}(\sigma)$ is feasible for (3.1) for $t=0$, for every positive $\sigma$. The objective function

$$
c^{\prime} x_{2}(\sigma)+\frac{1}{2} x_{2}(\sigma)^{\prime} C x_{2}(\sigma)=c^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0}-\sigma c^{\prime} s \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty .
$$

This contradicts that (3.1) has an optimal solution $x_{0}$ when $t=0$. Thus we get the result as required.

## Chapter 4

## The General Parametric QP Problem

In this chapter, we will study the general parametric QP problem with a parameter both in the linear part of the objective function and in the right-hand side of the constraints.

### 4.1 Solution of the General PQP Problem by Solving a Related QP Problem without the Parameter

Consider the following PQP problem

$$
\begin{equation*}
\min \left\{\left.(c+t q)^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b+t p\right\} . \tag{4.1}
\end{equation*}
$$

Assumption 4.1 There exists $a \hat{t}>0$ such that (4.1) has an optimal solution for every $t$ with $0 \leq t<\hat{t}$.

Recall the notation in Chapter 1. $x_{0}$ is an optimal solution for (4.1) for $t=0 . A_{0}^{\prime}$ is the matrix of the gradients of all the constraints active at $x_{0}$. Let $p_{0}$ be the vector whose components are
those $p_{i}$ associated with the rows of $A_{0}$.

Theorem 4.1 Let Assumption 4.1 be satisfied. Suppose $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (4.1). In addition, suppose the optimal solution $x(t)=x_{0}+t h_{0}^{*}$ is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$. Then there exists a vector $u_{0}$ such that $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for the problem

$$
\begin{equation*}
\min \left\{\left.-p_{0}^{\prime} u_{0}+q^{\prime} h_{0}+\frac{1}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\} \tag{4.2}
\end{equation*}
$$

The proof of the theorem is similar to the proof of Theorem 3.1.

## Proof

Since $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (4.1), $x(t)=x_{0}+t h_{0}^{*}$ is an optimal solution for (4.1), for every $t$ with $0<t<\bar{t}$. The optimality conditions for (4.1) assert,

$$
\left.\begin{array}{l}
A\left(x_{0}+t h_{0}^{*}\right) \leq b+t p, \\
-c-t q-C\left(x_{0}+t h_{0}^{*}\right)=A^{\prime} u, \quad u \geq 0, \\
u^{\prime}\left[A\left(x_{0}+t h_{0}^{*}\right)-(b+t p)\right]=0,
\end{array}\right\}
$$

where $u=u(t)$. These are equivalent to

$$
\left.\begin{array}{l}
A\left(x_{0}+t h_{0}^{*}\right) \leq b+t p,  \tag{4.3}\\
-c-t q-C x_{0}-t C h_{0}^{*}=A^{\prime} u, \quad u \geq 0, \\
u^{\prime}\left[\left(A x_{0}-b\right)+t\left(A h_{0}^{*}-p\right)\right]=0 .
\end{array}\right\}
$$

Since $x(t)$ is a diminishment of $x_{0}$, for every $t$ with $0<t<\bar{t}$, all the constraints active at $x(t)$ are also active at $x_{0}$. So the matrix of the gradients of all the constraints active at $x(t)$ is a submatrix
of $A_{0}^{\prime}$. Thus, (4.3) can be simplified to

$$
\begin{align*}
& A_{0} h_{0}^{*} \leq p_{0} \\
& -c-C x_{0}-t q-t C h_{0}^{*}=A_{0}^{\prime} u, \quad u \geq 0  \tag{4.4}\\
& u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0
\end{align*}
$$

where $u=u_{0}+t u_{1}$ is a multiplier vector for $x(t)$ whose components are associated with the rows of $A_{0}$, and $u_{0}$ is a multiplier vector for $x_{0}$ whose components are also associated with the rows of $A_{0}$. So $u_{0}$ satisfies the optimality conditions for $t=0$, which are

$$
\begin{equation*}
-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \tag{4.5}
\end{equation*}
$$

From (4.5), it follows that

$$
\begin{equation*}
\left(c+C x_{0}\right)^{\prime} h_{0}^{*}+p_{0}^{\prime} u_{0}=0 \tag{4.6}
\end{equation*}
$$

Combining (4.4), (4.5) and (4.6), we get

$$
\begin{aligned}
& A_{0} h_{0}^{*} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}^{*}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& -c-C x_{0}-t q-t C h_{0}^{*}=A_{0}^{\prime} u, u \geq 0 \\
& u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0
\end{aligned}
$$

Therefore, $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for the problem
$\min \left\{\left.\left(c+C x_{0}\right)^{\prime} h_{0}+t q^{\prime} h_{0}+\frac{1}{2} t h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\}$,
because $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]=\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ and $v_{1}=u, v_{2}, v_{3}, v_{4}=0$ satisfy the optimality conditions for (4.7), which are

$$
\begin{align*}
& A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& {\left[\begin{array}{c}
-\left(c+C x_{0}\right)-t\left(q+C h_{0}\right) \\
0
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] v_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] v_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] v_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] v_{4}, v_{1}, v_{4} \geq 0,} \\
& v_{1}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0 \\
& v_{4}^{\prime} u_{0}=0 \tag{4.8}
\end{align*}
$$

From the second constraint of (4.7), (4.7) is equivalent to
$\min \left\{\left.-p_{0}^{\prime} u_{0}+t q^{\prime} h_{0}+\frac{1}{2} t h_{0}^{\prime} C h_{0} \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\}$.

The optimality conditions for (4.9) are

$$
\begin{align*}
& A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& {\left[\begin{array}{c}
-t\left(q+C h_{0}\right) \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] w_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] w_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] w_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] w_{4}, w_{1}, w_{4} \geq 0}  \tag{4.10}\\
& w_{1}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0 \\
& w_{4}^{\prime} u_{0}=0
\end{align*}
$$

From (4.8), it follows that $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]=\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ and $w=\left[\begin{array}{l}w_{1} \\ w_{2} \\ w_{3} \\ w_{4}\end{array}\right]=\left[\begin{array}{l}u \\ 1 \\ 0 \\ 0\end{array}\right]$ satisfy (4.10).
Therefore, $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ satisfies

$$
\left[\begin{array}{c}
-q-C h_{0}^{*} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] \frac{u}{t}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] \frac{1}{t}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] \alpha_{1}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] \alpha_{2},
$$

where

$$
\alpha_{1}=\left(1-\frac{1}{t}\right) h_{0}^{*},
$$

and

$$
\alpha_{2}=\left(1-\frac{1}{t}\right)\left(A_{0} h_{0}^{*}-p_{0}\right) .
$$

Since $A_{0} h_{0}^{*} \leq p_{0}$, and for every $t$ with $0<t<1,1-\frac{1}{t}<0$. Thus,

$$
\alpha_{2}=\left(1-\frac{1}{t}\right)\left(A_{0} h_{0}^{*}-p_{0}\right) \geq 0
$$

and

$$
\frac{u}{t} \geq 0
$$

Since $u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$,

$$
\left(\frac{u}{t}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 .
$$

Since $u=u_{0}+t u_{1}$, we can write the third equation in (4.4), $u^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$, as $\left(u_{0}+t u_{1}\right)^{\prime}\left(A_{0} h_{0}^{*}-\right.$ $\left.p_{0}\right)=0$, for every $t$ with $0<t<\bar{t}$. It follows that $u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0$. So we have

$$
\alpha_{2}^{\prime} u_{0}=\left(1-\frac{1}{t}\right)\left(A_{0} h_{0}^{*}-p_{0}\right)^{\prime} u_{0}=0
$$

Thus, $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]=\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ and $w_{1}=\frac{u}{t}, w_{2}=\frac{1}{t}, w_{3}=\alpha_{1}, w_{4}=\alpha_{2}$ satisfy

$$
\begin{aligned}
& A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0 \\
& {\left[\begin{array}{c}
-q-C h_{0} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] w_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] w_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] w_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] w_{4}, w_{1}, w_{4} \geq 0,} \\
& w_{1}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0 \\
& w_{4}^{\prime} u_{0}=0
\end{aligned}
$$

which are precisely the optimality conditions for (4.2). Thus, $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (4.2) as required.

The importance of the optimal problem (4.2) is illustrated in the following theorem.

Theorem 4.2 Let Assumption 4.1 be satisfied. Suppose $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (4.2), and suppose that $w_{1}, w_{2}, w_{3}$ and $w_{4}$ are multipliers associated with the constraints $A_{0} h_{0} \leq p_{0}$, $\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}$ and $u_{0} \geq 0$, respectively. Then $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (4.1), and $v(t)=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)$ is an associated multiplier vector for $x(t)=x_{0}+t h_{0}^{*}$, for every $t$ with $0 \leq t<\bar{t}$, where $\bar{t}=\min \{\hat{t}, \tilde{t}\}>0$, and

$$
\begin{gather*}
\hat{t}=\min \left\{\left.\frac{b_{i}-a_{i}^{\prime} x_{0}}{a_{i}^{\prime} h_{0}^{*}-p_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with } a_{i}^{\prime} h_{0}^{*}>p_{i}\right\},  \tag{4.11}\\
\tilde{t}=\min \left\{\left.\frac{-\left(u_{0}\right)_{i}}{\left(w_{1}-w_{2} u_{0}\right)_{i}} \right\rvert\, \text { all } i=1, \ldots, m \text { with }\left(w_{1}-w_{2} u_{0}\right)_{i}<0\right\} . \tag{4.12}
\end{gather*}
$$

The full (m-dimensional) vector of multipliers, $u(t)$, is obtained from $v(t)$ by assigning zero to those components of $u(t)$ associated with constraints inactive at $x_{0}$ and the appropriately indexed components of $v(t)$, otherwise.

## Proof

Let $A_{1}^{\prime}$ be the matrix of the gradients of all the constraints inactive at $x_{0}$ for (4.1), let $b_{1}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{1}$. Then $A_{1} x_{0}<b_{1}$. Similar to the proof of Lemma 1.3, there exists a $\bar{t}_{1}>0$, such that $A_{1}\left(x_{0}+t h_{0}^{*}\right)<b_{1}+t p_{1}$, for every $t$ with $0 \leq t<\bar{t}_{1}$.

Since $\left[\begin{array}{l}h_{0}^{*} \\ u_{0}\end{array}\right]$ is an optimal solution for (4.2), the optimality conditions assert that

$$
\begin{gather*}
{\left[\begin{array}{c}
-q-C h_{0}^{*} \\
p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right] w_{1}+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right] w_{2}+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right] w_{3}+\left[\begin{array}{c}
0 \\
-I
\end{array}\right] w_{4}, w_{1}, w_{4} \geq 0}  \tag{4.13}\\
w_{1}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 \tag{4.14}
\end{gather*}
$$

Multiply both sides of (4.13) and (4.14) by $t$,

$$
\begin{gather*}
{\left[\begin{array}{c}
-t q-t C h_{0}^{*} \\
t p_{0}
\end{array}\right]=\left[\begin{array}{c}
A_{0}^{\prime} \\
0
\end{array}\right]\left(t w_{1}\right)+\left[\begin{array}{c}
c+C x_{0} \\
p_{0}
\end{array}\right]\left(t w_{2}\right)+\left[\begin{array}{c}
0 \\
A_{0}
\end{array}\right]\left(t w_{3}\right)+\left[\begin{array}{c}
0 \\
-I
\end{array}\right]\left(t w_{4}\right), \quad\left(t w_{1}\right),\left(t w_{4}\right) \geq 0}  \tag{4.15}\\
\left(t w_{1}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 \tag{4.16}
\end{gather*}
$$

From (4.15), we have

$$
-t q-t C h_{0}^{*}=A_{0}^{\prime}\left(t w_{1}\right)+\left(c+C x_{0}\right)\left(t w_{2}\right),
$$

and this is equivalent to

$$
-\left(c+C x_{0}\right)-t\left(q+C h_{0}^{*}\right)=A_{0}^{\prime} t w_{1}+\left(c+C x_{0}\right)\left(t w_{2}-1\right) .
$$

From the optimality conditions for (4.1) when $t=0$,

$$
-\left(c+C x_{0}\right)=A_{0}^{\prime} u_{0}, u_{0} \geq 0
$$

we have,

$$
-\left(c+C x_{0}\right)-t\left(q+C h_{0}^{*}\right)=A_{0}^{\prime}\left(t w_{1}-u_{0}\left(t w_{2}-1\right)\right)
$$

The second and the third constraints of (4.2) give

$$
u_{0}^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0
$$

together with (4.16), we have

$$
\left(t w_{1}-\left(t w_{2}-1\right) u_{0}\right)^{\prime}\left(A_{0} h_{0}^{*}-p_{0}\right)=0 .
$$

Let $v(t)=t w_{1}-u_{0}\left(t w_{2}-1\right)$. For $\bar{t}_{2}$ is given small enough, we have $t w_{2} \leq 1 ; i . e ., t w_{2}-1 \leq 0$, for every $t$ with $0 \leq t \leq \bar{t}_{2}$. Thus, $v(t) \geq 0$. Since $A_{0} x_{0}=b_{0}$, it follows from above that

$$
\begin{aligned}
& t A_{0} h_{0}^{*} \leq t p_{0} \\
& -c-t q-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime} v(t), v(t) \geq 0 \\
& v(t)^{\prime}\left(\left(A_{0} x_{0}-b_{0}\right)+t\left(A h_{0}^{*}-p_{0}\right)\right)=0
\end{aligned}
$$

Furthermore, since $A_{1}\left(x_{0}+t h_{0}^{*}\right)<b_{1}+t p_{1}$, for every $t$ with $0 \leq t<\bar{t}_{1}$, we have

$$
\begin{aligned}
& A\left(x_{0}+t h_{0}^{*}\right) \leq b+t p, \\
& -c-t q-C\left(x_{0}+t h_{0}^{*}\right)=A_{0}^{\prime} v(t), v(t) \geq 0, \\
& v(t)^{\prime}\left(A_{0}\left(x_{0}+t h_{0}^{*}\right)-\left(b_{0}+t p_{0}\right)\right)=0 .
\end{aligned}
$$

Let $\bar{t}=\min \left\{\bar{t}_{1}, \bar{t}_{2}\right\}>0$. Then $x(t)=x_{0}+t h_{0}^{*}$ and the associated multiplier $v(t)=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)$ satisfy the optimality conditions for (4.1), for every $t$ with $0 \leq t<\bar{t}$. Thus, $x(t)=x_{0}+t h_{0}^{*}$ is optimal for (4.1), for every $t$ with $0 \leq t<\bar{t}$. Therefore, we have $\left(h_{0}^{*}, \bar{t}\right)$ is an optimal continuation of $x_{0}$ for (4.1) as required.

Since $x(t)=x_{0}+t h_{0}^{*}$ is an optimal solution for (4.1), if $a_{i}^{\prime} h_{0}^{*}>p_{i}$, then $a_{i}^{\prime} x_{0}<b_{i}$. From (4.11), $\hat{t}>0$. Since $v=u_{0}+t\left(w_{1}-w_{2} u_{0}\right) \geq 0$, if $\left(w_{1}-w_{2} u_{0}\right)_{i}<0$, then $\left(u_{0}\right)_{i}>0$. From (4.12), $\tilde{t}>0$. Therefore, $\bar{t}=\min \{\hat{t}, \tilde{t}\}>0$.

Recall Example 1.5 in Chapter 1. In the problem of Example 1.5, the first four constraints are active at $x_{0}=(1,1)^{\prime}$. Let $u_{0}=\left(v_{1}, v_{2}, v_{3}, v_{4}\right)^{\prime}$ be an associated multiplier vector for $x_{0}$ whose components are associated with the first four constraints. Then, we can get an optimal continuation $h_{0}^{*}=\left[\begin{array}{l}h_{1} \\ h_{2}\end{array}\right]$ of $x_{0}$ by solving (4.2), which in this problem is

$$
\begin{aligned}
& \text { minimize: } \quad h_{2}+\frac{1}{2} h_{1}^{2}+h_{2}^{2}+v_{1}+v_{3}+\frac{1}{2} v_{4} \\
& \text { subject to : } h_{1} \quad \leq-1 \text {, } \\
& h_{2} \quad \leq 0, \\
& h_{1}+h_{2} \leq-1, \\
& h_{1}+2 h_{2} \leq-\frac{1}{2}, \\
& -h_{1}-v_{1} \quad-v_{3}-\frac{1}{2} v_{4}=0, \\
& v_{1}+v_{3}+v_{4}=1, \\
& v_{2}+v_{3}+2 v_{4}=0, \\
& v_{1} \quad \geq 0, \\
& v_{2} \quad \geq 0, \\
& v_{3} \quad \geq 0, \\
& v_{4} \geq 0 .
\end{aligned}
$$

The optimal solution is

$$
h_{0}^{*}=\left[\begin{array}{l}
-1 \\
-\frac{1}{2}
\end{array}\right]
$$

and

$$
u_{0}=\left[\begin{array}{l}
v_{1} \\
v_{2} \\
v_{3} \\
v_{4}
\end{array}\right]=\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right] .
$$

From Theorem 4.2, the optimal solution for the problem of Example 1.5 is

$$
x(t)=x_{0}+t h_{0}^{*}=\left[\begin{array}{c}
1-t \\
1-\frac{1}{2} t
\end{array}\right],
$$

with the multiplier vector

$$
v(t)=u_{0}+t\left(w_{1}-w_{2} u_{0}\right)=\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right]+t\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right]
$$

whose components are associated with the first four constraints, for every $t$ with $0<t<\bar{t}$. From Theorem 4.2, the upper limit $\bar{t}$ is determined by applying (4.11) and (4.12):

$$
\begin{gathered}
\hat{t}_{1}=\min \left\{-,-,-,-, \frac{1}{1}, \frac{1}{\frac{1}{2}}\right\}=1, \\
\tilde{t}_{1}=\min \{-,-,-,-\}=+\infty,
\end{gathered}
$$

from which

$$
\bar{t}=\min \{1,+\infty\}=1 .
$$

Therefore,

$$
x(t)=\left[\begin{array}{c}
1-t \\
1-\frac{1}{2} t
\end{array}\right]
$$

is optimal for the problem, for every $t$ with $0 \leq t \leq 1$, in agreement with our geometric determination of the optimal solution in Example 1.5.

### 4.2 Feasibility of the Problem (4.2) in Theorem 4.1

In this section, we will show that the critical problem (4.2) is feasible. Let $S$ be the feasible region of (4.2), namely,
$S \equiv\left\{\left.\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right] \right\rvert\, A_{0} h_{0} \leq p_{0},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{0}^{\prime} u_{0}=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\}$,
$=\left\{\left.\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right] \right\rvert\, A_{0} h_{0} \leq p_{0}, u_{0}^{\prime}\left(A_{0} h_{0}-p_{0}\right)=0,-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0\right\}$.
The optimality conditions for the problem

$$
\begin{equation*}
\min \left\{\left(c+C x_{0}\right)^{\prime} h_{0} \mid A_{0} h_{0} \leq p_{0}\right\} \tag{4.17}
\end{equation*}
$$

imply that if (4.17) has an optimal solution, then the set $S$ is not empty.

Theorem 4.3 Assume that (4.1) is feasible for every $t$ with $0<t \leq \bar{t}$. Then $S$ is not empty. So (4.2) is feasible.

## Proof

From the analysis of above, we only need to show that (4.17) has an optimal solution, that is, (4.17) is feasible and bounded.

We know that $x_{0}$ is optimal for $\min \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b\right\}$ and $A_{0} x_{0}=b_{0}$. Assumption 4.1 implies that (4.1) is feasible, for every $t$ with $0<t<\hat{t}$. Let $x_{1}$ be a feasible solution for (4.1) for $t=t_{0}$, where $t_{0}$ satisfies $0<t_{0}<\hat{t}$. Then, $A_{0} x_{1} \leq b_{0}+t_{0} p_{0}$. It follows that

$$
\begin{aligned}
& A_{0}\left(x_{1}-x_{0}\right) \leq b_{0}+t_{0} p_{0}-b_{0}, \\
& \quad \Rightarrow A_{0}\left(x_{1}-x_{0}\right) \leq t_{0} p_{0} \\
& \quad \Rightarrow A_{0} \frac{x_{1}-x_{0}}{t_{0}} \leq p_{0}
\end{aligned}
$$

Thus, $\frac{x_{1}-x_{0}}{t_{0}}$ is a feasible solution for (4.17), so (4.17) is feasible.

Assume on the contrary that (4.17) is unbounded. Then for a feasible solution $h_{1}$ for (4.17), there exists an $s_{1}$ such that $h_{1}-\sigma s_{1}$ is feasible, for every positive scalar $\sigma$, and

$$
\left(c+C x_{0}\right)^{\prime}\left(h_{1}-\sigma s_{1}\right) \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty
$$

Thus, we have

$$
\left.\begin{array}{l}
\left(c+C x_{0}\right)^{\prime} s_{1}>0  \tag{4.18}\\
A_{0} s_{1} \geq 0
\end{array}\right\}
$$

In the original problem (4.1), when $t=0$, the optimality conditions assert that

$$
-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0
$$

Together with (4.18), it follows that

$$
\left(c+C x_{0}\right)^{\prime} s_{1}=-\left(A_{0}^{\prime} u_{0}\right)^{\prime} s_{1}=-u_{0}^{\prime}\left(A_{0} s_{1}\right) \leq 0 .
$$

This contradicts $\left(c+C x_{0}\right)^{\prime} s_{1}>0$. So, (4.17) is bounded.

Therefore, (4.17) is feasible and bounded, thus has an optimal solution, then we have (4.2) is feasible as required.

### 4.3 The Boundedness of the Problem (4.2) in Theorem 4.1

The problem (4.2) maybe unbounded and have no optimal solution, which means that $x_{0}$ has no optimal continuation. Then we want to find another optimal solution $x_{1}$ for (4.1) for $t=0$, such that $x_{1}$ has an optimal continuation.

In this section, we show how to decide whether (4.2) unbounded or not, and prove that such $x_{1}$ above always exists if Assumption 4.1 satisfies, and also give a way to find $x_{1}$.

The following theorem gives to a way to check the boundedness of (4.2) by checking a simpler optimal problem.

Theorem 4.4 Let Assumption 4.1 be satisfied. Then, (4.2) is unbounded if and only if the problem

$$
\begin{equation*}
\min \left\{-q^{\prime} s_{1} \mid C s_{1}=0, c^{\prime} s_{1}=0, A_{0} s_{1} \geq 0\right\} \tag{4.19}
\end{equation*}
$$

is unbounded from below.

## Proof

Rewrite (4.2) as

$$
\begin{aligned}
& \text { minimize : }\left[\begin{array}{ll}
q^{\prime} & -p_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right]+\frac{1}{2}\left[\begin{array}{ll}
h_{0}^{\prime} & u_{0}^{\prime}
\end{array}\right]\left[\begin{array}{ll}
C & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right] \\
& \text { subject to : }\left[\begin{array}{ll}
A_{0} & 0
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right] \leq p_{0} \\
& {\left[\begin{array}{ll}
0 & A_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right]=-c-C x_{0}} \\
& {\left[\begin{array}{ll}
\left(c+C x_{0}\right)^{\prime} & \left.p_{0}^{\prime}\right]
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right]=0} \\
& {\left[\begin{array}{ll}
0 & -I
\end{array}\right]\left[\begin{array}{l}
h_{0} \\
u_{0}
\end{array}\right] \leq 0}
\end{aligned}
$$

If (4.2) is unbounded, then there exists a vector $\left[\begin{array}{l}s_{1} \\ s_{2}\end{array}\right]$, such that

$$
\begin{gathered}
{\left[\begin{array}{ll}
q^{\prime} & -p_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
s_{1} \\
s_{2}
\end{array}\right]>0,} \\
{\left[\begin{array}{ll}
s_{1}^{\prime} & s_{2}^{\prime}
\end{array}\right]\left[\begin{array}{ll}
C & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
s_{1} \\
s_{2}
\end{array}\right]=0,}
\end{gathered}
$$

and $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]-\sigma\left[\begin{array}{l}s_{1} \\ s_{2}\end{array}\right]$ is feasible, for every positive scalar $\sigma$. Thus, we have

$$
\begin{aligned}
& q^{\prime} s_{1}-p_{0}^{\prime} s_{2}>0 \\
& s_{1}^{\prime} C s_{1}=0 \Rightarrow C s_{1}=0 \\
& A_{0} s_{1} \geq 0 \\
& A_{0}^{\prime} s_{2}=0 \\
& \left(c+C x_{0}\right)^{\prime} s_{1}+p_{0}^{\prime} s_{2}=0 \Rightarrow c^{\prime} s_{1}+p_{0}^{\prime} s_{2}=0, \\
& s_{2} \leq 0
\end{aligned}
$$

The optimal conditions for (4.1) when $t=0$ assert that $-c-C x_{0}=A_{0}^{\prime} u_{0}, u_{0} \geq 0$. Together with $A_{0} s_{1} \geq 0, C s_{1}=0$, we have

$$
c^{\prime} s_{1}=\left(c+C x_{0}\right)^{\prime} s_{1}=\left(-A_{0}^{\prime} u_{0}\right)^{\prime} s_{1}=-u_{0}^{\prime}\left(A_{0} s_{1}\right) \leq 0 .
$$

That is, $c^{\prime} s_{1} \leq 0$.

If $c^{\prime} s_{1}<0$, from $c^{\prime} s_{1}+p_{0}^{\prime} s_{2}=0$, we have $p_{0}^{\prime} s_{2}>0$. Since Assumption 4.1 satisfies, (4.1) has optimal solutions when $0 \leq t<\hat{t}$. Let $x\left(t_{0}\right)=x_{0}+t_{0} h_{0}$ be an optimal solution for (4.1), for $t=t_{0}$ with $0 \leq t_{0}<\hat{t}$, then $x\left(t_{0}\right)=x_{0}+t_{0} h_{0}$ is also an feasible solution for (4.1) for $t=t_{0}$; i.e., $A_{0}\left(x_{0}+t_{0} h_{0}\right) \leq b_{0}+t_{0} p_{0}$. So $A_{0} x_{0}=b_{0}$ implies $A_{0} h_{0} \leq p_{0}$. Furthermore, since $s_{2} \leq 0$, we have

$$
\begin{equation*}
h_{0}^{\prime} A_{0}^{\prime} s_{2} \geq p_{0}^{\prime} s_{2}>0 \tag{4.21}
\end{equation*}
$$

But since $A_{0}^{\prime} s_{2}=0$, the left-hand side of (4.21) equals to zero. It is a contradiction. Thus, we have $c^{\prime} s_{1}=0$ and $p_{0}^{\prime} s_{2}=0$.

Since $p_{0}^{\prime} s_{2}=0$ and $q^{\prime} s_{1}-p_{0}^{\prime} s_{2}>0$, it follows that $q^{\prime} s_{1}>0$; i.e., $-q^{\prime} s_{1}<0$. Since $s_{1}$ satisfies $C s_{1}=0, c^{\prime} s_{1}=0, A_{0} s_{1} \geq 0$ and $-q^{\prime} s_{1}<0$. For every positive scalar $\sigma, \sigma s_{1}$ also satisfies $C\left(\sigma s_{1}\right)=0, c^{\prime}\left(\sigma s_{1}\right)=0, A_{0}\left(\sigma s_{1}\right) \geq 0$, and

$$
-q^{\prime}\left(\sigma s_{1}\right) \rightarrow-\infty, \text { as } \sigma \rightarrow+\infty
$$

Thus (4.19) is unbounded.

On the other hand, if (4.19) is unbounded from below, then there exists an $s_{1}$ such that

$$
q^{\prime} s_{1}>0, C s_{1}=0, c^{\prime} s_{1}=0, A_{0} s_{1} \geq 0
$$

Then let $s_{2}=0$, and deserve that

$$
\left[\begin{array}{ll}
q^{\prime} & -p_{0}^{\prime}
\end{array}\right]\left[\begin{array}{l}
s_{1} \\
s_{2}
\end{array}\right]>0,
$$

and

$$
\left[\begin{array}{ll}
s_{1}^{\prime} & s_{2}^{\prime}
\end{array}\right]\left[\begin{array}{ll}
C & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
s_{1} \\
s_{2}
\end{array}\right]=0
$$

Furthermore, $\left[\begin{array}{l}h_{0} \\ u_{0}\end{array}\right]-\sigma\left[\begin{array}{l}s_{1} \\ s_{2}\end{array}\right]$ is feasible, for every positive scalar $\sigma>0$. Thus, (4.2) is unbounded from below.

From the theorem above, it is straightforward to deduce the following lemma.

Lemma 4.1 If (4.19) has an optimal solution $s_{1}^{*}=0$, then (4.2) is bounded and thus has an optimal solution.

Consider the problem

$$
\begin{equation*}
\min \left\{-q^{\prime} s \mid c^{\prime} s=0, C s=0, A_{0} s \geq 0, A s \geq A x_{0}-b\right\} \tag{4.22}
\end{equation*}
$$

It is feasible since $s_{0}=0$ is a feasible solution.

Theorem 4.5 Suppose that (4.2) is unbounded from below. Assume (4.22) has an optimal solution s. Then $s \neq 0$. Let $x_{1}=x_{0}-s$. Let $A_{1}^{\prime}$ be the matrix of gradients of all the constraints active at $x_{1}$, let $b_{1}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{1}$; i.e., $A_{1} x_{1}=b_{1}$. Let $p_{1}$ be the vector whose components are those $p_{i}$ associated with the rows of $A_{1}$. Then $x_{1}$ is also an optimal solution for problem (4.1) when $t=0$, and moreover, the problem

$$
\begin{equation*}
\min \left\{\left.-p_{1}^{\prime} u_{0}+q^{\prime} h_{0}+\frac{1}{2} h_{0}^{\prime} C h_{0} \right\rvert\, A_{1} h_{0} \leq p_{1},\left(c+C x_{0}\right)^{\prime} h_{0}+p_{1}^{\prime} u_{0}=0,-c-C x_{1}=A_{1}^{\prime} u_{0}, u_{0} \geq 0\right\} \tag{4.23}
\end{equation*}
$$

has a finite optimal solution.

## Proof

We first show that if (4.22) has an optimal solution $s$, then

$$
\begin{equation*}
s \neq 0 \tag{4.24}
\end{equation*}
$$

Otherwise, if $s=0$ is an optimal solution for (4.22), the optimality conditions assert that

$$
\begin{equation*}
q=c u_{1}+C u_{2}-A_{0}^{\prime} u_{3}-A^{\prime} u_{4}, \quad u_{3}, u_{4} \geq 0 \tag{4.25}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{4}^{\prime}\left(A x_{0}-b\right)=0 \tag{4.26}
\end{equation*}
$$

Since $A_{0}^{\prime}$ is the matrix of gradients of all the constraints active at $x_{0}$, (4.25) and (4.26) can be simplified to

$$
\begin{gather*}
q=c u_{1}+C u_{2}-A_{0}^{\prime} u_{3}-A_{0}^{\prime} \bar{u}_{4}=c u_{1}+C u_{2}-A_{0}^{\prime}\left(u_{3}+\bar{u}_{4}\right), \quad u_{3}, \bar{u}_{4} \geq 0,  \tag{4.27}\\
A_{0} x_{0}-b_{0}=0, \tag{4.28}
\end{gather*}
$$

where $\bar{u}_{4}$ is the multiplier vector whose components are those $\left(u_{4}\right)_{i}$ associated with the rows of $A_{0}$. Then $s=0, u_{1}, u_{2}$ and $u_{3}+\bar{u}_{4}$ satisfy the optimality conditions for (4.19), which are

$$
\left.\begin{array}{l}
C s=0, c^{\prime} s=0, A_{0} s \geq 0 \\
q=c u_{1}+C u_{2}-A_{0}^{\prime}\left(u_{3}+\bar{u}_{4}\right), u_{3}+\bar{u}_{4} \geq 0 \\
\left(u_{3}+\bar{u}_{4}\right)^{\prime} A_{0} s=0 .
\end{array}\right\}
$$

Thus, $s=0$ being an optimal solution for (4.19), together with Lemma 4.1, contradicts that (4.2) is unbounded from below. Thus, if (4.22) has an optimal solution $s$, then $s \neq 0$, which verifies (4.24).

Now we will prove that $x_{1}$ is also optimal for (4.1) for $t=0$, and (4.23) has a finite optimal solution. From the fourth constraint of (4.22), $A s \geq A x_{0}-b$, we have $A\left(x_{0}-s\right) \leq b$, which means

$$
A x_{1} \leq b
$$

From the first and second constraints of (4.22), $c^{\prime} s=0, C s=0$, the objective function for $x_{1}$ is

$$
c^{\prime} x_{1}+\frac{1}{2} x_{1}^{\prime} C x_{1}=c^{\prime}\left(x_{0}-s\right)+\frac{1}{2}\left(x_{0}-s\right)^{\prime} C\left(x_{0}-s\right)=c^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0}
$$

Thus, $x_{1}$ is also optimal for (4.1) for $t=0$.

Since $s$ is an optimal solution for (4.22), the optimality conditions give us:

$$
\left.\begin{array}{l}
q=C u+c v-A_{0}^{\prime} w_{0}-A^{\prime} w_{1}, w_{0}, w_{1} \geq 0  \tag{4.29}\\
w_{0}^{\prime} A_{0} s=0, w_{1}^{\prime}\left(A x_{0}-b-A s\right)=0
\end{array}\right\}
$$

Since $A_{1}$ is the matrix of gradients of all the constraints active at $x_{1}, A_{1} x_{1}=b_{1} ; i . e ., A_{1}\left(x_{0}-s\right)=b_{1}$, (4.29) can be simplified to

$$
\left.\begin{array}{l}
q=C u+c v-A_{0}^{\prime} w_{0}-A_{1}^{\prime} \bar{w}_{1}, \quad w_{0}, \bar{w}_{1} \geq 0 \\
w_{0}^{\prime} A_{0} s=0, A_{1} s=A_{1} x_{0}-b_{1},
\end{array}\right\}
$$

where $\bar{w}_{1}$ is a multiplier vector whose components are associated with the rows of $A_{1}$. From $w_{0}^{\prime} A_{0} s=0$, we know that if $a_{i}^{\prime} s=\left(A_{0} s\right)_{i} \neq 0$, then $\left(w_{0}\right)_{i}=0$. Let $A_{2}^{\prime}$ be the matrix of all the $a_{i}$ in $A_{0}$ such that $a_{i}^{\prime} s=0 ;$ i.e., $A_{2} s=0$. Let $b_{2}$ be the vector whose components are associated with the rows of $A_{2}$. Since $A_{2}$ is a submatrix of $A_{0}$, we have $A_{2} x_{0}=b_{2}$. Then, $A_{2}\left(x_{0}-s\right)=b_{2}$; i.e., $A_{2} x_{1}=b_{2}$. Thus, $A_{2}$ is also a submatrix of $A_{1}$. So,

$$
\left.\begin{array}{l}
q=C u+c v-A_{2}^{\prime} \bar{w}_{0}-A_{1}^{\prime} \bar{w}_{1}=C u+c u-A_{1}^{\prime} w, \quad \bar{w}_{0}, \bar{w}_{1}, w \geq 0 \\
A_{2} s=0, A_{1} s=A_{1} x_{0}-b_{1}
\end{array}\right\}
$$

where $w$ is a vector whose components are associated with the rows of $A_{1}$. Therefore, $s_{1}=0$ and $w$ satisfy

$$
\begin{aligned}
& C s_{1}=0, c^{\prime} s_{1}=0, A_{1} s_{1} \geq 0 \\
& q=C u+c v-A_{1}^{\prime} w, w \geq 0 \\
& w^{\prime} A_{1} s_{1}=0
\end{aligned}
$$

which are the optimality conditions for

$$
\begin{equation*}
\min \left\{-q^{\prime} s_{1} \mid C s_{1}=0, c^{\prime} s_{1}=0, A_{1} s_{1} \geq 0\right\} \tag{4.30}
\end{equation*}
$$

So $s_{1}=0$ is optimal for (4.30).

Then, from Lemma 4.1, (4.23) has a finite optimal solution.

Theorem 4.6 If (4.22) is unbounded from below, then (4.1) is either infeasible or unbounded from below, for every $t$ with $t>0$.

## Proof

If (4.22) is unbounded from below, then for a feasible solution $s$ for (4.22), there exists a vector $d$ such that $s-\sigma d$ feasible for (4.22), for every positive scalar $\sigma$, and $q^{\prime} d<0$. So d satisfies

$$
q^{\prime} d<0, c^{\prime} d=0, C d=0, A d \leq 0
$$

If for a $t>0,(4.1)$ is feasible. Let $\bar{x}(t)$ be a feasible solution for it. Then $A \bar{x}(t) \leq b+t p$. Since $A d \leq 0$, we have

$$
A(\bar{x}(t)+\sigma d)=A \bar{x}(t)+\sigma A d \leq b+t p .
$$

Furthermore,
$(c+t q)^{\prime}(\bar{x}(t)+\sigma d)+\frac{1}{2}(\bar{x}(t)+\sigma d)^{\prime} C(\bar{x}(t)+\sigma d)=(c+t q)^{\prime} \bar{x}(t)+\frac{1}{2} \bar{x}(t)^{\prime} C \bar{x}(t)+\sigma t q^{\prime} d \rightarrow-\infty$, as $\sigma \rightarrow+\infty$, since $q^{\prime} d<0$. Thus, (4.2) is either infeasible or unbounded from below for every $t$ with $t>0$.

## Chapter 5

## Concluding Remarks

We want to solve the general parametric quadratic programming problem (4.1). Assume it has an optimal solution $x_{0}$ for $t=0$. First we study the feasibility of (4.1) for $t>0$ by checking the optimal solution for (3.3). If (3.3) has an optimal solution $\hat{t}>0$, then (4.1) is feasible for every $t$ with $0 \leq t \leq \hat{t}$. Then we solve a non-parametric quadratic programming problem (4.2). We prove that (4.2) is feasible. If (4.2) is bounded and thus has an optimal solution $h_{0}^{*}$, then $x(t)=x_{0}+t h_{0}^{*}$ is an optimal solution for (4.1), for every $t$ with $0 \leq t<\bar{t}$, where $\bar{t}$ can be solved from (4.11) and (4.12), and $\bar{t}$ is a "corner" point for the parametric QP. If (4.2) is unbounded from below, then we consider the LP problem (4.22). If (4.22) has an optimal solution $s$, then let $x_{1}=x_{0}-s$, and solve (4.23) for $h_{0}^{*}$. Then $x(t)=x_{1}+t h_{0}^{*}$ is an optimal solution for (4.1), for every $0 \leq t \leq \bar{t}$. If (4.22) is unbounded from below, then (4.1) is unbounded from below, for every $t$ with $0<t \leq \hat{t}$.

## Appendix

Throughout this thesis, we have shown that difficulties arising from ties in a PQP can be resolved by solving an appropriate QP. It is possible that the resulting QP may have degenerate points, thus creating further difficulties. However, we argue here that such degenerate points are a consequence of the linear constraints in the model problem and can be resolved by solving an LP. The use of Bland's rules in solving the LP [11] guarantees that the LP and thus the QP can be solved in a finite number of steps.

Consider general convex QP problem

$$
\begin{equation*}
\min \left\{\left.c^{\prime} x+\frac{1}{2} x^{\prime} C x \right\rvert\, A x \leq b\right\} \tag{1}
\end{equation*}
$$

Let $f(x)=c^{\prime} x+\frac{1}{2} x^{\prime} C x$. Suppose $x_{0}$ is a quasi-stationary point determined by an algorithm. Suppose that $x_{0}$ is degenerate; i.e., the gradients of those constraints active at $x_{0}$ are linearly dependent. Let $A_{0}^{\prime}$ be the matrix of gradients of all the constraints active at $x_{0}$ and let $b_{0}$ be the vector whose components are those $b_{i}$ associated with the rows of $A_{0}$. We can consider the following LP problem

$$
\begin{equation*}
\min \left\{-\left(c+C x_{0}\right)^{\prime} s_{0} \mid A_{0} s_{0} \geq 0\right\} \tag{2}
\end{equation*}
$$

Theorem 1 The problem (2) either has an optimal solution $s_{0}=0$ or is unbounded from below. If (2) has an optimal solution then $x_{0}$ is optimal for the original $Q P$ problem. If (2) is unbounded from below, let $s_{0}$ be a feasible solution such that $\left(c+C x_{0}\right)^{\prime} s_{0}>0$ and let
$\hat{\sigma}=\max \left\{\sigma \mid A\left(x_{0}-\sigma s_{0}\right) \leq b\right\}$,
$\tilde{\sigma}= \begin{cases}\left(s_{0}^{\prime} C s_{0}\right)^{-1}\left(c+C x_{0}\right)^{\prime} s_{0}, & s_{0}^{\prime} C s_{0}>0, \\ +\infty, & s_{0}^{\prime} C s_{0}=0,\end{cases}$
and $\sigma=\min \{\hat{\sigma}, \tilde{\sigma}\}$.
Then $\sigma>0$. If $\sigma=+\infty$, then (1) is unbounded from below. If $\sigma<+\infty$, then $x_{0}-\sigma s_{0}$ is feasible for (1) and $f\left(x_{0}-\sigma s_{0}\right)<f\left(x_{0}\right)$.

## Proof

The problem (2) is feasible since $s_{0}=0$ is a feasible solution. If (2) has an optimal solution, then the optimal solution is $s_{0}=0$, otherwise (2) is unbounded from below. From the optimality conditions for (2), there exists a $u_{0} \geq 0$ such that $c+C x_{0}=-A_{0}^{\prime} s_{0}$, thus $x_{0}$ is optimal for (1).

If (2) is unbounded from below, then there exist a feasible solution $s_{0}$ such that $\left(c+C x_{0}\right)^{\prime} s_{0}>0$. If $\hat{\sigma}=+\infty$, and $s_{0}^{\prime} C s_{0}=0$; i.e., $C s_{0}=0$, then $c^{\prime} s_{0}>0$. Thus, $f\left(x_{0}-\sigma s_{0}\right)=c^{\prime}\left(x_{0}-\sigma s_{0}\right)+\frac{1}{2}\left(x_{0}-\sigma s_{0}\right)^{\prime} C\left(x_{0}-\sigma s_{0}\right)=c^{\prime} x_{0}+\frac{1}{2} x_{0}^{\prime} C x_{0}-\sigma c^{\prime} s_{0} \rightarrow-\infty$, as $\sigma \rightarrow+\infty$.

If $\hat{\sigma}<+\infty$ and $\tilde{\sigma}=+\infty$, then

$$
f\left(x_{0}-\sigma s_{0}\right)-f\left(x_{0}\right)=-\sigma c^{\prime} s_{0}<0 .
$$

If $\tilde{\sigma}<+\infty$, then
$f\left(x_{0}-\sigma s_{0}\right)-f\left(x_{0}\right)=-\sigma\left(c+C x_{0}\right) s_{0}+\frac{\sigma^{2}}{2} s_{0}^{\prime} C s_{0} \leq-\sigma\left(c+C x_{0}\right)^{\prime} s+\frac{\sigma}{2}\left(c+C x_{0}\right)^{\prime} s_{0}=-\frac{\sigma}{2}\left(c+C x_{0}\right)^{\prime} s_{0}<0$.
Therefore, $f\left(x_{0}-\sigma s_{0}\right)<f\left(x_{0}\right)$.

Theorem 1 shows that when a degenerate quasi stationary point is determined by an active set QP algorithm, solving the indicated LP using Bland's rules will determine in a finite number of steps that either the current point is optimal or will construct a search direction which will give a strict decrease in the objective function.

Theorem 1 is apparently well known and was communicated to the author by M. J. Best [1].

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