RESOLUTIONS OF TEMPERED REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

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Abstract. Let G be a reductive group over a non-archimedean local field and let $\mathcal{S}(G)$ be its Schwartz algebra. We compare Ext-groups of tempered G-representations in several module categories: smooth G-representations, algebraic $\mathcal{S}(G)$-modules, bornological $\mathcal{S}(G)$-modules and an exact category of $\mathcal{S}(G)$-modules on LF-spaces, which contains all admissible $\mathcal{S}(G)$-modules. We simplify the proofs of known comparison theorems for these Ext-groups, due to Meyer and Schneider–Zink. Our method is based on the Bruhat–Tits building of G and on analytic properties of the Schneider–Stuhler resolutions.

Contents

Introduction

Let G be the group of $\mathbb{F}$-rational points of a connected reductive linear algebraic group defined over a non-archemedean local field $\mathbb{F}$ of arbitrary characteristic. Let $\mathcal{H}(G)$ denote the Hecke algebra of locally constant compactly supported complex functions on G, and $\mathcal{S}(G)$ the Harish-Chandra Schwartz algebra of G. The abelian category $\mathrm{Mod}(G)$ of complex smooth representations of G is equivalent to the category $\mathrm{Mod}(\mathcal{H}(G))$ of nondegenerate $\mathcal{H}(G)$-modules. By [ScSt2, Proposition 1] an admissible representation V of G is tempered if and only if it extends to a module of $\mathcal{S}(G)$, and then such an extension is unique. Let V, W be $\mathcal{S}(G)$-modules, with V admissible. A profound theorem due to Schneider and Zink [ScZi2] (based on work of R. Meyer) states that for all $n \in \mathbb{Z}_{\geq 0}$:

$$\mathrm{Ext}^n_{\mathcal{H}(G)}(V,W) = \mathrm{Ext}^n_{\mathcal{S}(G)}(V,W). \tag{1}$$

If W is also admissible, then both V and W admit a canonical structure as LF-spaces such that they become complete topological modules over the LF-algebra $\mathcal{S}(G)$. We

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introduce an exact category $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ of certain LF-modules over $\mathcal{S}(G)$, whose exact sequences are split as LF-spaces, and then one also has

$$\mathrm{Ext}^n_{\mathcal{S}(G)}(V,W) = \mathrm{Ext}^n_{\mathrm{Mod}_{LF}(\mathcal{S}(G))}(V,W). \tag{2}$$

One can choose a good compact open subgroup K such that $V = \mathcal{H}(G)V^K$ and $W = \mathcal{H}(G)W^K$. Now V^K and W^K are finite dimensional modules over the K-spherical Hecke algebra $\mathcal{H}(G,K) := e_K\mathcal{H}(G)e_K$ which uniquely extend to topological modules over the Fréchet algebra $\mathcal{S}(G,K) := e_K\mathcal{S}(G)e_K$. In that context we have

$$\mathrm{Ext}^n_{\mathcal{H}(G,K)}(V^K,W^K) = \mathrm{Ext}^n_{\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))}(V^K,W^K), \tag{3}$$

where Mod_{Fr} denotes the exact category of Fréchet modules with linearly split exact sequences.

These are powerful statements wich provide a link between harmonic analysis and homological properties of admissible smooth representations of G. For example it follows that a discrete series representation of G is a projective module in the full subcategory of $\mathcal{H}(G)$-modules which are restrictions of $\mathcal{S}(G)$-modules. The identities (1), (2) and (3) were used in [OpSo2] to explicitly compute the spaces $\mathrm{Ext}^i_{\mathcal{H}(G)}(V,W)$ for irreducible tempered admissible representations of G in terms of analytic R-groups. As a further consequence, we proved the Kazhdan orthogonality relations for admissible characters of G directly from the Plancherel isomorphism for $\mathcal{S}(G)$. These applications motivated us to revisit the proofs of the results of Meyer and the subsequent results of Schneider, Stuhler and Zink discussed above.

Equation (1) is somewhat unexpected since $\mathcal{S}(G)$ is not a flat ring over $\mathcal{H}(G)$. Meyer's proof of these type of results [Mey2] relies in an essential way on the machinery of bornological vector spaces. In the present paper we prove the results in a way which is intuitively more clear and which reveals their geometric origin. The methods we are using are similar to those used in [OpSo1] for the analogous statements for tempered modules over an affine Hecke algebra. The pleasant surprise is that such an explicit construction of a continuous contraction of the Schneider–Stuhler resolutions is still possible in this more complicated context, and the computations are not too unpleasant.

First recall the construction of Schneider and Stuhler of a functorial projective resolution $C_*(\mathcal{B}(G),V)$ of V by G-equivariant sheaves on the Bruhat–Tits building $\mathcal{B}(G)$. We start by constructing a functorial contraction of the K-invariant part $V^K \leftarrow C_*(\mathcal{B}(G),V)^K$ of this resolution of V, where K runs over a neighborhood basis of G consisting of good compact open subgroups. This is a projective resolution of V^K as a $\mathcal{H}(G,K)$-module. The construction of these contractions reflects the contractibility of the affine building $\mathcal{B}(G)$. Next we show directly that these contractions extend continuously to the natural Fréchet completion $C^t_*(\mathcal{B}(G),V)^K$ of this resolution of V^K. This shows that the Fréchet completion of the resolution is an admissible projective resolution of the $\mathcal{S}(G,K)$-module V^K in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$, and leads to (3).

Given a good maximal compact subgroup K, we denote by $\mathrm{Mod}(\mathcal{H}(G),K)$ the full subcategory of $\mathcal{H}(G)$-modules V such that $V = \mathcal{H}(G)V^K$. By well known results of Bernstein the functor from $\mathrm{Mod}(\mathcal{H}(G),K)$ to $\mathrm{Mod}(\mathcal{H}(G,K))$ given by $V \to V^K$ is an equivalence of categories, and by results of Schneider and Zink [ScZi1] a similar statement holds for modules V over $\mathcal{S}(G)$ satisfying $V = \mathcal{S}(G)V^K$. If we take K sufficiently small, such that V^K generates V as a G-module and W^K generates W

as a G-module, then one derives (1) and (2) from (3) using these equivalences.

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1. The differential complex

Let $\mathbb{F}$ be a non-archimedean local field of arbitrary characteristic. Let $\mathcal{G}$ be a connected reductive algebraic group defined over $\mathbb{F}$ and let $G = \mathcal{G}(\mathbb{F})$ be its $\mathbb{F}$-rational points. We briefly call G a reductive p-adic group. Let $Z(\mathcal{G})$ be the centre of $\mathcal{G}$ and denote by $X_*(H)$ the set of $\mathbb{F}$-algebraic cocharacters of an $\mathbb{F}$-group H.

The (enlarged) Bruhat–Tits building of G is

$$\mathcal{B}(G) = \mathcal{B}(\mathcal{G}, \mathbb{F}) = \mathcal{B}(\mathcal{G}/Z(\mathcal{G}), \mathbb{F}) \times X_*\big(Z(\mathcal{G}(\mathbb{F}))\big) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{4}$$

Recall that $\mathcal{B}(\mathcal{G}/Z(\mathcal{G}), \mathbb{F})$ is in a natural way a polysimplicial complex with a G-action. The choice of a basis of the lattice $X_*(Z(G))$ induces a polysimplicial structure on $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$, isomorphic to a direct product of some copies of $\mathbb{R}$ with the intervals $[n, n+1]$ as 1-simplices. The resulting polysimplicial structure on $\mathcal{B}(G)$ is G-stable because G acts on $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$ via translation over $X_*(Z(G))$.

A crucial role will be played by a system of compact open subgroups of G introduced by Schneider and Stuhler [ScSt1]. The group associated to a given polysimplex $\sigma \subset \mathcal{B}(G)$ and a natural number e is denoted $U_\sigma^{(e)}$. We will need the following properties, which can be found in [ScSt1, Chapter 1] and in [MeSo2, Theorem 5.5].

Proposition 1.1. (*a*) *$U_\sigma^{(e)}$ is an open pro-p subgroup of the stabilizer G_σ of σ.*
(*b*) *The collection $\{U_\sigma^{(e)} \mid e \in \mathbb{N}\}$ is a neighborhood basis of 1 in G.*
(*c*) *$U_\sigma^{(e)}$ depends only on the projection of σ on $\mathcal{B}(\mathcal{G}/Z(\mathcal{G}), \mathbb{F})$.*
(*d*) *$gU_\sigma^{(e)}g^{-1} = U_{g\sigma}^{(e)}$ for all $g \in G$, in particular $U_\sigma^{(e)}$ is a normal in G_σ.*
(*e*) *$U_\sigma^{(e)}$ fixes the star of σ in $\mathcal{B}(G)$ pointwise.*
(*f*) *$U_\sigma^{(e)}$ is the product (in any order) of the groups $U_x^{(e)}$, where x runs over the vertices of σ.*
(*g*) *If σ_1, σ_2 and σ_3 are polysimplices of $\mathcal{B}(G)$ such that σ_2 lies in every apartment of $\mathcal{B}(G)$ that contains $\sigma_1 \cup \sigma_3$, then $U_{\sigma_2}^{(e)} \subset U_{\sigma_1}^{(e)} U_{\sigma_3}^{(e)}$.*

Let $\mathcal{O}$ be the ring of integers of $\mathbb{F}$ and let π be a uniformizer of $\mathcal{O}$. Let p be the characteristic of the residue field $\mathcal{O}/\pi\mathcal{O}$. We recall the main result of [MeSo1], which works in the generality of modules over a commutative unital ring R in which p is invertible. Let $C_c^\infty(G, R)$ be the R-module of locally constant, compactly supported functions $G \to R$. Since G is locally a pro-p group, there exists a Haar measure on G such that all pro-p subgroups of G have volume in $p^{\mathbb{Z}}$. We fix such a measure once and for all. Thus we obtain a convolution product on $C_c^\infty(G; R)$, which makes it into an R-algebra denoted $\mathcal{H}(G; R)$. Let $\mathrm{Mod}(\mathcal{H}(G; R))$ be the category of $\mathcal{H}(G; R)$-modules V with $\mathcal{H}(G; R)V = V$. It is naturally equivalent to the category $\mathrm{Mod}_R(G)$ of smooth G-representations on R-modules.

Now we describe how the above objects can be used to construct resolutions of certain modules. Given any polysimplex σ, let $e_\sigma = e_{U_\sigma^{(e)}}$ be the corresponding idempotent of $\mathcal{H}(G; R)$; it exists because the volume of $U_\sigma^{(e)}$ is invertible in R. For any $V \in \mathrm{Mod}_R(G)$, $e_\sigma V = V^{U_\sigma^{(e)}}$ is the R-submodule of $U_\sigma^{(e)}$-invariant elements. For any polysimplicial subcomplex $\Sigma \subset \mathcal{B}(G)$ let $\Sigma^{(n)}$ be the collection of n-dimensional polysimplices of Σ. We put

$$C_n(\Sigma; V) := \bigoplus_{\sigma \in \Sigma^{(n)}} R\sigma \otimes_R e_\sigma V.$$

We fix an orientation of the polysimplices of $\mathcal{B}(G)$ and we identify $-\sigma$ with σ oriented in the opposite way. This allows us to write the boundary of σ in the polysimplicial sense [OpSo1, Section 2.1] as

$$\partial \sigma = \sum\nolimits_\tau [\sigma : \tau] \tau \quad \text{with} \quad [\sigma : \tau] \in \{1, 0, -1\}.$$

We have $[\sigma : \tau] = 0$ unless $\tau \subset \sigma$, and in that case Proposition 1.1.f tells us that $U_\tau^{(e)} \subset U_\sigma^{(e)}$ and $e_\tau V \supset e_\sigma V$. Thus we can define a differential

$$\begin{aligned} &\partial_n : C_n(\Sigma; V) \to C_{n-1}(\Sigma; V), \\ &\partial_n(\sigma \otimes v) = \partial\sigma \otimes v = \sum\nolimits_\tau [\sigma : \tau] \tau \otimes v \end{aligned}$$

and an augmentation

$$\begin{aligned} &\partial_0 : C_0(\Sigma; V) \to V = C_{-1}(\Sigma; V), \\ &\partial_0(x \otimes v) = v. \end{aligned}$$

Since $\partial^2 = 0$, $\big(C_*(\Sigma; V), \partial_*\big)$ is a differential complex. The group G acts on $C_n(\mathcal{B}(G); V)$ by

$$g(\sigma \otimes v) = g\sigma \otimes gv, \tag{5}$$

where $g\sigma$ is endowed with the orientation that makes $g : \sigma \to g\sigma$ orientation preserving. Clearly ∂_* is G-equivariant, so $\big(C_*(\mathcal{B}(G); V), \partial_*\big)$ is a complex of $\mathcal{H}(G; R)$-modules.

Theorem 1.2. *Let $\Sigma \subset \mathcal{B}(G)$ be convex.*
(a) The differential complex $\big(C_*(\Sigma; V), \partial_*\big)$ is acyclic and ∂_0 induces a bijection

$$H_0\big(C_*(\Sigma; V), \partial_*\big) \to \sum_{x \in \Sigma^{(0)}} V^{U_x^{(e)}}.$$

(b) If $V = \sum_{x \in \mathcal{B}(G)^{(0)}} V^{U_x^{(e)}}$, then $\big(C_*(\mathcal{B}(G); V), \partial_*\big)$ is a resolution of V in $\mathrm{Mod}_R(G)$. This resolution is projective if R is a field in which the order of $G_\sigma / U_\sigma^{(e)}$ is invertible, for every polysimplex σ.

Proof. (a) is [MeSo1, Theorem 2.4]. Although in [MeSo1] the affine building of $\mathcal{G}/Z(\mathcal{G})$ is used, this does not make any difference for the proof. In particular the crucial [MeSo1, Theorem 2.12] is also valid in our setup.
(b) The special case where $R = \mathbb{C}$ and $Z(G)$ is compact was proven in [ScSt1, Theorem II.3.1]. It remains to show that $C_n(\mathcal{B}(G); V)$ is projective under the indicated conditions. Let $\sigma_1, \ldots, \sigma_d$ be representatives for the G-orbits of n-dimensional polysimplices in $\mathcal{B}(G)$ and let $\epsilon_{\sigma_i} : G_{\sigma_i} \to \{1, -1\}$ be the orientation character of σ_i. By construction

$$C_n(\mathcal{B}(G); V) = \bigoplus_{i=1}^d \mathrm{ind}_{G_{\sigma_i}}^G (\epsilon_{\sigma_i} \otimes e_{\sigma_i} V) = \bigoplus_{i=1}^d \mathrm{ind}_{R[G_{\sigma_i}/U_{\sigma_i}^{(e)}]}^{\mathcal{H}(G;R)} \big(\epsilon_{\sigma_i} \otimes V^{U_{\sigma_i}^{(e)}}\big). \tag{6}$$

By assumption $|G_{\sigma_i}/U_{\sigma_i}^{(e)}| \in R^\times$, so the category of $R[G_{\sigma_i}/U_{\sigma_i}^{(e)}]$-modules is semisimple. In particular $\epsilon_{\sigma_i} \otimes V^{U_{\sigma_i}^{(e)}}$ is projective in this category, which by Frobenius reciprocity implies that $\mathrm{ind}_{R[G_{\sigma_i}/U_{\sigma_i}^{(e)}]}^{\mathcal{H}(G;R)} \big(\epsilon_{\sigma_i} \otimes V^{U_{\sigma_i}^{(e)}}\big)$ is projective in $\mathrm{Mod}(\mathcal{H}(G;R))$. □

2. A functorial contraction

From now on we fix $e \in \mathbb{N}$ and a special vertex $x_0 \in \mathcal{B}(G)$. By Proposition 1.1.d $K = U_{x_0}^{(e)}$ is a normal subgroup of the good maximal compact subgroup G_{x_0} of G. Let $\mathcal{H}(G,K;R)$ be the subalgebra of $\mathcal{H}(G;R)$ consisting of all K-biinvariant elements and let $\mathrm{Mod}_R(G,K)$ be the full subcategory of $\mathrm{Mod}_R(G)$ made of all objects V for which $V = \mathcal{H}(G;R)V^K$.

Theorem 2.1. *Suppose that R is an algebraically closed field whose characteristic is banal for G, that is, does not divide the pro-order of any compact subgroup of G.*

(a) *The exact functor*

$$\begin{array}{ccc} \mathrm{Mod}_R(G) & \to & \mathrm{Mod}(\mathcal{H}(G,K;R)), \\ V & \mapsto & e_K V = V^K \end{array}$$

provides an equivalence of categories $\mathrm{Mod}_R(G,K) \to \mathrm{Mod}(\mathcal{H}(G,K;R))$, with quasi-inverse $W \mapsto \mathcal{H}(G;R)e_K \otimes_{\mathcal{H}(G,K;R)} W$.

(b) *$\mathrm{Mod}_R(G,K)$ is a direct factor of the category $\mathrm{Mod}_R(G)$, so the above functors preserve projectivity.*

Proof. For $R = \mathbb{C}$ this is due to Bernstein, see [BeDe]. Vignéras [Vig2] observed that Bernstein's proof remains valid for R as indicated. □

We remark that it is likely that Theorem 2.1 is valid for much more general rings R. By [MeSo1, Section 3] this is the case for somewhat different idempotents of $\mathcal{H}(G;R)$.

Theorem 2.2. (a) *The augmented complex*

$$V^K \leftarrow C_*(\mathcal{B}(G);V)^K$$

admits a contraction which is natural in $V \in \mathrm{Mod}_R(G,K)$.

(b) *Suppose that R is an algebraically closed field whose characteristic is banal for G. Then*

$$\mathcal{H}(G,K;R) \leftarrow C_*(\mathcal{B}(G);\mathcal{H}(G;R)e_K)^K$$

is a projective $\mathcal{H}(G,K;R)$-bimodule resolution which admits a right $\mathcal{H}(G,K;R)$-linear contraction.

Remark. $C_*(\mathcal{B}(G);V)$ need not be generated its K-invariant vectors. The point is that there can be polysimplices σ that do not contain any element of Gx_0, and then $U_\sigma^{(e)}$ does not contain any conjugate of K.

Proof. (a) In the lowest degree it is easy, for $v \in V^K = C_{-1}(\mathcal{B}(G);V)^K$ we put

$$\gamma_{-1}(v) = x_0 \otimes v.$$

In higher degrees, consider an apartment A of $\mathcal{B}(G)$ containing x_0 and let T be the associated maximal split torus of G. In [OpSo1, Section 2.1] a contraction γ of the augmented differential complex $C_*(A;\mathbb{Z})$ is constructed, with the properties:

(1) γ is equivariant for the action of the Weyl group $N_{G_{x_0}}(T)/(T \cap G_{x_0})$ on A.
(2) For any polysimplex $\sigma \subset A$, the support of $\gamma(\sigma)$ is contained in the hull of $\sigma \cup \{x_0\}$.

Recall that the hull of a subset X of a thick affine building (as in property 1) is the intersection of all apartments that contain X. It is the best polysimplicial approximation of the convex closure of X. An alternative, more explicit, description is given in [OpSo1, Section 1.1]. This works well for the Bruhat–Tits building of $G/Z(G)$, which is the setting of [OpSo1].

For $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$ an intersection of apartments is not suitable because there is only one apartment. But by means of a basis of $X_*(Z(G))$ we have already identified this lattice with $\mathbb{Z}^d$. We define the hull of any subset $X \subset A$ as the hull of the projection of X on $\mathcal{B}(G/Z(G))$ times the smallest box

$$\prod\nolimits_{i=1}^d [n_i, n_i'] \subset X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{with} \quad n_i, n_i' \in \mathbb{Z}$$

that contains the projection of X on $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$. To reconcile this with [OpSo1], let $\{\beta_1, \ldots, \beta_d\}$ be the dual basis of $X^*(Z(G))$ and regard $\{\beta_1, -\beta_1, \ldots, \beta_d, -\beta_d\}$ as a root system of type A_1^d in $X^*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$. Then our hull agrees with the description given in [OpSo1, Section 1.1], so the construction of γ applies to A.

For an elementary tensor $\sigma \otimes v_\sigma$ we define

$$\gamma_n(\sigma \otimes v_\sigma) = \gamma(\sigma) \otimes v_\sigma \in C_{n+1}(A; \mathbb{Z}) \otimes_{\mathbb{Z}} V.$$

By property 2 this does not depend on the choice of the apartment A and it clearly is functorial in V. Recall from [BrTi, Proposition 7.4.8] that, whenever $B \subset A, g \in G$ and $g(B) \subset A$, there exists $n \in N_G(T)$ such that $n(b) = g(b)$ for all $b \in B$. With property 1 it follows that γ_n extends to a G_{x_0}-equivariant map

$$\gamma_n : C_n(\mathcal{B}(G); V) \to C_{n+1}(\mathcal{B}(G); \mathbb{Z}) \otimes_{\mathbb{Z}} V.$$

Consider a typical K-invariant element $e_K(\sigma \otimes v_\sigma) \in C_n(\mathcal{B}(G); V)^K$. By the K-equivariance of γ_n

$$\gamma_n(e_K(\sigma \otimes v_\sigma)) = e_K(\gamma(\sigma) \otimes v_\sigma) = e_K\big(\sum\nolimits_\tau \gamma_{\sigma\tau} \tau \otimes v_\sigma\big) \in C_{n+1}(\mathcal{B}(G); V)^K \tag{7}$$

By property 2 the support of (7) lies in the K-orbit of the hull of $\{x_0\} \cup \sigma$. We fix a polysimplex τ_0 in this support and we have a closer look at the value of (7) at τ_0. Let us write G_τ^+ for the subgroup of G_τ consisting of the elements that preserve the orientation of the polysimplex τ. Then the value of (7) at τ_0 is

$$\sum_{\tau \subset A \cap K\tau_0} \; \sum_{k \in K/(K \cap G_\tau^+), k\tau = \pm\tau_0} \gamma_{\sigma\tau} \epsilon(k,\tau) k \frac{e_{K \cap G_\tau^+}}{[K : K \cap G_\tau^+]} v_\sigma, \tag{8}$$

where $\epsilon(k,\tau) \in \{\pm 1\}$ is defined by $k\tau = \epsilon(k,\tau)\tau_0$. By Proposition 1.1.g we have $U_\tau^{(e)} \subset U_{x_0}^{(e)} U_\sigma^{(e)}$ for all τ occurring in the above sum. However, we need a more precise version. Let Φ be the root system of (G,T) and let Φ^+ be a system of positive roots such that σ lies in the positive Weyl chamber A^+. Let U^+ and U^- be the unipotent subgroups of G associated to Φ^+ and $-\Phi^+$. The constructions in [ScSt1, Section 1.2] entail that

$$\begin{array}{lllll} U_{x_0}^{(e)} \cap U^+ & \subset & U_\tau^{(e)} \cap U^+ & \subset & U_\sigma^{(e)} \cap U^+, \\ U_{x_0}^{(e)} \cap Z_G(T) & = & U_\tau^{(e)} \cap Z_G(T) & = & U_\sigma^{(e)} \cap Z_G(T), \\ U_{x_0}^{(e)} \cap U^- & \supset & U_\tau^{(e)} \cap U^- & \supset & U_\sigma^{(e)} \cap U^-. \end{array} \tag{9}$$

By [ScSt1, Corollary I.2.8]

$$U_\tau^{(e)} = (U_\tau^{(e)} \cap U^-)(U_\tau^{(e)} \cap Z_G(T))(U_\tau^{(e)} \cap U^+), \tag{10}$$

which is contained in

$$\begin{aligned}&(U_\tau^{(e)} \cap U^-)(U_\sigma^{(e)} \cap Z_G(T))(U_\sigma^{(e)} \cap U^+) \ \subset \\ &(U^- \cap G_\tau^+ \cap U_{x_0}^{(e)})(U_\sigma^{(e)} \cap Z_G(T)U^+) \ \subset \ (G_\tau^+ \cap K)U_\sigma^{(e)}.\end{aligned}$$

By assumption $v_\sigma \in V^{U_\sigma^{(e)}}$, so the above means that

$$k e_{K \cap G_\tau^+} v_\sigma = k e_{K \cap G_\tau^+} e_\tau v_\sigma = k e_\tau e_{K \cap G_\tau^+} v_\sigma = e_{\tau_0} k e_{K \cap G_\tau^+} v_\sigma \in V^{U_{\tau_0}^{(e)}},$$

where we used that G_τ normalizes $U_\tau^{(e)}$ for the second equality. Consequently (8) lies in $V^{U_{\tau_0}^{(e)}}$ and

$$\gamma_n(e_K(\sigma \otimes v_\sigma)) \in C_{n+1}(\mathcal{B}(G); V)^K. \tag{11}$$

Because $\gamma\partial + \partial\gamma = \text{id}$ on $C_*(A;\mathbb{Z})$, γ_* is a contraction of the augmented complex $\big(C_*(\mathcal{B}(G), V), \partial_*\big)$.

(b) It only remains to show that $C_n(\mathcal{B}(G); \mathcal{H}(G;R)e_K)^K$ is projective as a $\mathcal{H}(G,K;R)$-bimodule. By (6) it equals

$$\bigoplus_{i=1}^d \big(\text{ind}_{G_{\sigma_i}}^G (\epsilon_{\sigma_i} \otimes e_{\sigma_i}\mathcal{H}(G;R)e_K)\big)^K = \bigoplus_{i=1}^d e_K \mathcal{H}(G;R) \underset{\mathcal{H}(G_{\sigma_i};R)}{\otimes} \epsilon_{\sigma_i} \otimes e_{\sigma_i}\mathcal{H}(G;R)e_K. \tag{12}$$

Consider the $\mathcal{H}(G;R)$-bimodule $\mathcal{H}(G;R) \otimes_{\mathcal{H}(G_\sigma;R)} \epsilon_\sigma \otimes e_\sigma \mathcal{H}(G;R)$, for any polysimplex σ. Since $U_\sigma^{(e)}$ is normalized by G_σ and fixes σ pointwise, $e_\sigma \in \mathcal{H}(G_\sigma;R)$ is a central idempotent and $e_\sigma \epsilon_\sigma = \epsilon_\sigma$. As R has banal characteristic, the group algebra

$$R[G_\sigma / U_\sigma^{(e)}] = \mathcal{H}(G_\sigma, U_\sigma^{(e)}; R) = e_\sigma \mathcal{H}(G_\sigma;R) \tag{13}$$

is a semisimple direct summand of $\mathcal{H}(G_\sigma;R)$. Given an irreducible representation ρ of (13) let e_ρ be the corresponding central idempotent and $p_\rho \in e_\rho R[G_\sigma/U_\sigma^{(e)}]$ an idempotent of rank 1. Then $\sum_\rho (p_{\epsilon_\sigma \otimes \rho} \otimes p_\rho)$ is an idempotent in $\mathcal{H}(G;R) \otimes_R \mathcal{H}(G;R)^{op}$ and as $\mathcal{H}(G;R)$-bimodules

$$\begin{aligned}&\mathcal{H}(G;R) \underset{\mathcal{H}(G_\sigma;R)}{\otimes} \epsilon_\sigma \otimes e_\sigma \mathcal{H}(G;R) = \\ &\bigoplus\nolimits_\rho \mathcal{H}(G;R) e_{\epsilon_\sigma \otimes \rho} \underset{\mathcal{H}(G_\sigma;R)}{\otimes} e_\rho \mathcal{H}(G;R) = \\ &\bigoplus\nolimits_\rho \mathcal{H}(G;R) p_{\epsilon_\sigma \otimes \rho} \otimes_R p_\rho \mathcal{H}(G;R) = \\ &\big(\mathcal{H}(G;R) \otimes_R \mathcal{H}(G;R)^{op}\big) \sum\nolimits_\rho (p_{\epsilon_\sigma \otimes \rho} \otimes p_\rho).\end{aligned} \tag{14}$$

In particular $\mathcal{H}(G;R) \otimes_{\mathcal{H}(G_\sigma;R)} \epsilon_\sigma \otimes e_\sigma \mathcal{H}(G;R)$ is projective as a $\mathcal{H}(G;R)$-bimodule. By Theorem 2.1, applied to the category of smooth $G \times G^{op}$-representations on R-modules,

$$e_K \mathcal{H}(G;R) \underset{\mathcal{H}(G_\sigma;R)}{\otimes} \epsilon_\sigma \otimes e_\sigma \mathcal{H}(G;R) e_K$$

is projective in $\text{Mod}(\mathcal{H}(G,K;R) \otimes_R \mathcal{H}(G,K;R)^{op})$. Together with (12) this completes the proof. □

We record some useful properties of the contraction constructed above.

Corollary 2.3. *The contraction γ from Theorem 2.2.a satisfies:*

(*a*) *γ is G_{x_0}-equivariant.*
(*b*) *For any polysimplex σ, the support of $\gamma(\sigma)$ is contained in the hull of $\sigma \cup \{x_0\}$. In particular it intersects every G_{x_0}-orbit in at most one polysimplex.*
(*c*) *The exists $M_\gamma \in \mathbb{N}$ such that $\gamma(\sigma) = \sum_\tau \gamma_{\sigma\tau}\tau$ with $|\gamma_{\sigma\tau}| \leq M_\gamma$ for all polysimplices σ and τ of $\mathcal{B}(G)$.*

Proof. (a) and the first half of (b) follow from the properties of the contraction of $C_*(A;\mathbb{Z})$ that we used in the proof of Theorem 2.2. Let A' be an apartment containing σ and x_0, corresponding to a maximal split torus T'. Then the hull of $\sigma \cup \{x_0\}$ is contained in a Weyl chamber for the Weyl group $N_{G_{x_0}}(T')/(T' \cap G_{x_0})$. Two points of A' are in the same G_{x_0}-orbit if and only if they are in one $N_{G_{x_0}}(T')/(T' \cap G_{x_0})$-orbit. Hence this hull, and in particular the support of $\gamma(\sigma)$, intersects every G_{x_0}-orbit in at most one polysimplex.
(c) is a direct consequence of the definition of γ and of the corresponding property of the contraction from [OpSo1, Section 2.1]. $\Box$

Let d be a G-invariant metric on $\mathcal{B}(G)$. Then the restriction of d to an apartment A comes from an inner product on A. We may and will assume that:

- in any apartment A, $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$ is orthogonal to $A \cap \mathcal{B}(G/Z(G))$;
- the chosen basis of $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$ is orthogonal.

The following elementary property of the hull of a polysimplex of $\mathcal{B}(G)$ will be used in the proof of Theorem 4.2.

Lemma 2.4. *There exists $\delta \in \mathcal{B}(G)$ such that, for every polysimplex σ, the diameter of the hull of $\sigma \cup \{x_0\}$ is at most $d(x_0,\sigma) + d(x_0,\delta)$.*

Proof. Denote the hull of $\sigma \cup \{x_0\}$ by $\mathcal{H}$. Let the apartment A and the root system $R = R(G,T) \cup \pm\{\beta_1,\ldots,\beta_d\}$ be as in the proof of Theorem 2.2. The above assumptions on d imply that there exist $c_\alpha > 0$ such that

$$d(x,x_0)^2 = \sum\nolimits_{\alpha \in R} c_\alpha \alpha(x)^2 \quad \text{for all } x \in A. \tag{15}$$

Choose a system of positive roots R^+ such that σ (and hence $\mathcal{H}$) lies in the positive Weyl chamber, with x_0 as origin. Then

$$0 \leq \alpha(x) \leq \min_{y \in \sigma} \alpha(y) + 1 \quad \text{for all } x \in \mathcal{H}, \alpha \in R^+,$$

see [OpSo1, §1.1]. Let $\alpha^\vee \in A$ be the coroot of α and put $\delta = \sum_{\alpha \in R^+} \alpha^\vee/2$. It is well-known that $\beta(\delta)$ equals the height of $\beta \in R$, which by definition is at least 1 for all $\beta \in R^+$. With (15) it follows that, for all $x_1, x_2 \in \mathcal{H}$:

$$0 \leq |\alpha(x_1) - \alpha(x_2)| \leq \min_{y \in \sigma} \alpha(y) + \alpha(\delta) \quad \text{for all } \alpha \in R^+,$$
$$d(x_1,x_2) \leq d(x_0,\sigma+\delta) \leq d(x_0,\sigma) + d(x_0,\delta).$$

Thus δ works for this particular σ. For other polysimplices σ' the above argument would produce a possibly different δ'. But since all Weyl chambers in all apartments containing x_0 are conjugate under G_{x_0}, $d(x_0,\delta') = d(x_0,\delta)$. $\Box$

3. Fréchet $\mathcal{S}(G,K)$-modules

In this section we consider only complex G-representations, so we fix $R = \mathbb{C}$ and we suppress it from the notation. Recall from [Vig1] that the Harish-Chandra–Schwartz algebra $\mathcal{S}(G)$ is a convolution algebra of functions $G \to \mathbb{C}$ that are rapidly decreasing in comparison with a suitable length function ℓ. It is the union, over all compact open subgroups $U \subset G$, of the subalgebras

$$\mathcal{S}(G,U) = e_U \mathcal{S}(G) e_U.$$

Each such subalgebra $\mathcal{S}(G,U)$ is nuclear and Fréchet, and it is the completion of $\mathcal{H}(G,U)$ with respect to the family of norms $f \mapsto \|\ell^m f\|_2$ $(m \in \mathbb{N})$. Nevertheless $\mathcal{S}(G)$ is not a Fréchet algebra, it is an inductive limit of Fréchet spaces and its multiplication is only separately continuous.

Before we move on to Fréchet modules, we recall some facts about algebraic $\mathcal{S}(G)$-modules. Let $\mathrm{Mod}(\mathcal{S}(G))$ denote the category of smooth $\mathcal{S}(G)$-modules. Let $\mathrm{Mod}(\mathcal{S}(G),U)$ be the subcategory of all V with $V = \mathcal{S}(G)V^U$. We write $K = U_{x_0}^{(e)}$ as before. The analogue of Theorem 2.1 reads:

Proposition 3.1. [ScZi1, Lemmas 2.2 and 2.3]

(a) *The exact functor*

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{S}(G)) & \to & \mathrm{Mod}(\mathcal{S}(G,K)), \\ V & \mapsto & e_K V = V^K \end{array}$$

provides an equivalence of categories $\mathrm{Mod}(\mathcal{S}(G),K) \to \mathrm{Mod}(\mathcal{S}(G,K))$*, with quasi-inverse* $W \mapsto \mathcal{S}(G)e_K \otimes_{\mathcal{S}(G,K)} W$*.*

(b) *Mod*$(\mathcal{S}(G),K)$ *is a direct factor of the category Mod*$(\mathcal{S}(G))$ *and all it objects* V *satisfy* $V = \mathcal{H}(G)V^K$*.*

By part (b) the module $\mathcal{S}(G)e_K \in \mathrm{Mod}(\mathcal{S}(G),K)$ lies also in $\mathrm{Mod}(G,K)$. Hence

$$\mathcal{S}(G)e_K = \mathcal{H}(G)(\mathcal{S}(G)e_K)^K = \mathcal{H}(G)e_K\mathcal{S}(G)e_K = \mathcal{H}(G)e_K \underset{\mathcal{H}(G,K)}{\otimes} \mathcal{S}(G,K). \tag{16}$$

Let $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$ be the category of Fréchet $\mathcal{S}(G,K)$-modules, with continuous module maps as morphisms. We decree that an exact sequence in this category is admissible if it is split exact as a sequence of Fréchet spaces.

Let F be any Fréchet space and write the completed projective tensor product as $\widehat{\otimes}$. With respect to the indicated exact structure, modules of the form $\mathcal{S}(G,K)\widehat{\otimes}F$ are projective in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$. Hence this exact category has enough projective objects and all derived functors are well-defined.

Now we specify a category of $\mathcal{S}(G)$-modules that is equivalent to $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$. As objects we take all modules $V \in \mathrm{Mod}(\mathcal{S}(G),K)$ such that $W := V^K$ belongs to $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$. Because of Proposition 3.1 we have

$$V \cong \mathcal{S}(G)e_K \underset{\mathcal{S}(G,K)}{\otimes} W \quad \text{with} \quad W \in \mathrm{Mod}_{Fr}(\mathcal{S}(G,K)). \tag{17}$$

Since $\mathcal{S}(G)e_K$ is the union of the subalgebras $e_\sigma^{(r)}\mathcal{S}(G)e_K$, we can write

$$V = \bigcup_{r\in\mathbb{N}} V^{U_\sigma^{(r)}} \cong \bigcup_{r\in\mathbb{N}} e_\sigma^{(r)}\mathcal{S}(G)e_K \underset{\mathcal{S}(G,K)}{\otimes} W. \tag{18}$$

Moreover $e_\sigma^{(r)} \mathcal{S}(G) e_K$ is of finite rank as a right $\mathcal{S}(G,K)$-module, so every term $e_\sigma^{(r)} \mathcal{S}(G) e_K \underset{\mathcal{S}(G,K)}{\otimes} W$ becomes in a natural way a Fréchet space. We endow V with the inductive limit topology coming from these subspaces, thus making it into an LF-space.

Clearly every $\mathcal{S}(G)$-module map $\phi : V \to \tilde{V}$ sends V^U to $\tilde{V}^U$, and by definition ϕ is continuous if and only if $\phi|_{V^U}$ is continuous for every compact open subgroup $U \subset G$. On the other hand, as soon as $\phi|_{V^K} : V^K \to \tilde{V}^K$ is continuous, the aforementioned finite rank property assures that $\phi|_{V^U}$ is continuous for all U. We define $\mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$ to be the category of all $\mathcal{S}(G)$-modules of the form (17), with continuous $\mathcal{S}(G)$-module maps as morphisms. Exact sequences in this category are required to be split as sequences of LF-spaces.

Our arguments will also apply to certain modules that are not generated by their U-invariants for any compact open subgroup $U \subset G$. Let $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ be the category of all topological $\mathcal{S}(G)$-modules V such that:

- V^U is a Fréchet $\mathcal{S}(G,U)$-module for every compact open subgroup $U \subset G$;
- V has the inductive limit topology from the subspaces V^U.

Of course the morphisms are continuous module maps. As before, we require that exact sequences in this category are split as sequences of topological vector spaces. In view of [ScSt2, Proposition 1], $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ naturally contains all admissible $\mathcal{S}(G)$-modules. On the other hand, not every LF-space which is a topological $\mathcal{S}(G)$-module belongs to $\mathrm{Mod}_{LF}(\mathcal{S}(G))$, as we require its objects to be LF-spaces in a specific way. For example, the regular representation on $\mathcal{S}(G)$ itself is not in $\mathrm{Mod}_{LF}(\mathcal{S}(G))$.

For $V \in \mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$ the vector space

$$V^U = e_U \mathcal{S}(G) e_K \underset{\mathcal{S}(G,K)}{\otimes} V^K$$

has a natural Fréchet topology, because $e_U \mathcal{S}(G) e_K$ is of finite rank over $\mathcal{S}(G,K)$. Hence $\mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$ is contained in $\mathrm{Mod}_{LF}(\mathcal{S}(G))$. In view of Proposition 3.1 and the above considerations we have:

Corollary 3.2. *The functors*

$$\begin{array}{ccc} \mathrm{Mod}_{Fr}(\mathcal{S}(G), K) & \longleftrightarrow & \mathrm{Mod}_{Fr}(\mathcal{S}(G,K)) \\ V & \mapsto & V^K \\ \mathcal{S}(G) e_K \underset{\mathcal{S}(G,K)}{\otimes} W & \leftarrow & W \end{array}$$

are equivalences of exact categories. Moreover $\mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$ *is a direct factor of* $\mathrm{Mod}_{LF}(\mathcal{S}(G))$.

The category $\mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$ has enough projectives because it is equivalent with $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$. Hence $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ also has enough projectives, and derived functors are well-defined in that exact category.

For $V \in \mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$ (6) and (16) show that

$$\mathcal{S}(G,K) \underset{\mathcal{H}(G,K)}{\otimes} C_n(\mathcal{B}(G); V)^K = \bigoplus_{i=1}^d \mathcal{S}(G,K) \underset{\mathcal{H}(G,K)}{\otimes} e_K \mathcal{H}(G) \underset{\mathcal{H}(G_{\sigma_i})}{\otimes} \epsilon_{\sigma_i} \otimes e_{\sigma_i} V$$

$$= \bigoplus_{i=1}^d e_K \mathcal{S}(G) \underset{\mathcal{H}(G_{\sigma_i})}{\otimes} \epsilon_{\sigma_i} \otimes e_{\sigma_i} V = \bigoplus_{i=1}^d e_K \mathcal{S}(G) e_{\sigma_i} \underset{\mathcal{H}(G_{\sigma_i}; U_{\sigma_i}^{(e)})}{\otimes} \epsilon_{\sigma_i} \otimes e_{\sigma_i} V. \tag{19}$$

By Frobenius reciprocity and Theorem 1.2 this is a projective object of $\mathrm{Mod}(\mathcal{S}(G,K))$. As a vector space it is complete with respect to the projective tensor product topology if and only if $e_{\sigma_i} V$ has finite dimension for all i. Since $V = \mathcal{H}(G)V^K$, this happens if and only if V is admissible.

Unfortunately, for inadmissible V the modules (19) need not form a resolution of V. The simplest counterexample occurs when $G = \mathbb{F}^\times$, a onedimensional torus. Then $\mathcal{B}(G) \cong \mathbb{R}$ with $\mathbb{F}^\times$ acting by translations. Furthermore $K = 1+\omega^{e+1}\mathcal{O}$, where ω is a uniformizer of the ring of integers $\mathcal{O}$ of $\mathbb{F}$. For $V = \mathcal{S}(G)e_K \cong \mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K)$ the modules (19) form an augmented differential complex

$$\mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K) \leftarrow \mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K) \underset{\mathcal{H}(\mathcal{O}^\times)}{\otimes} \mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K) \\ \leftarrow \mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K) \underset{\mathcal{H}(\mathcal{O}^\times)}{\otimes} \mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K).$$

With the appropriate identifications

$$\partial_0(f\otimes f') = ff' \quad \text{and} \quad \partial_1(f\otimes f') = f\otimes f' - \omega f\otimes \omega^{-1} f'.$$

For any $s \in \mathcal{S}(\mathbb{Z}\times\mathcal{O}^\times/K)\setminus C_c(\mathbb{Z}\times\mathcal{O}^\times/K)$ we have $s\otimes e_K - e_K\otimes s \in \ker\partial_0 \setminus \mathrm{im}\,\partial_1$. The problem is that, in order to move $s\otimes e_K$ to $e_K\otimes s$, one would need infinitely long shifts in the direction of (ω,ω^{-1}), whereas the image of ∂_1 can only take care of finitely many such shifts.

To avoid this problem we have to complete (19), and the best way is to introduce, for $V \in \mathrm{Mod}_{LF}(\mathcal{S}(G))$:

$$C_n^t(\mathcal{B}(G);V) := \bigcup_{r\in\mathbb{N}} \bigoplus_{i=1}^{d} \big(e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_{\sigma_i} \underset{\mathcal{H}(G_{\sigma_i};U_{\sigma_i}^{(e)})}{\widehat{\otimes}} \epsilon_{\sigma_i} \otimes e_{\sigma_i} V \big). \tag{20}$$

This lies in $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ and is a topological completion of $C_n(\mathcal{B}(G);V)$. Notice the specific order of the operations, which is necessary because (completed) projective tensor products do not always commute with inductive limits. We have

$$C_n^t(\mathcal{B}(G);V) = \mathcal{S}(G)\otimes_{\mathcal{H}(G)} C_n(\mathcal{B}(G);V)$$

if and only if V is admissible.

For the augmentation we simply put $C_{-1}^t(V) = V$.

Lemma 3.3. *Let $V \in \mathrm{Mod}_{LF}(\mathcal{S}(G))$. The boundary maps ∂_n of $C_*(\mathcal{B}(G);V)$ extend to continuous $\mathcal{S}(G)$-linear boundary maps ∂_n^t of $C_*^t(\mathcal{B}(G);V)$.*

Remark. An analogous result was used implicitly in [OpSo1, Section 2.3]. The proof given here also applies in the setting of [OpSo1].

Proof. Since the map of Fréchet spaces

$$\begin{array}{ccc} e_{U_{x_0}^{(r)}}\mathcal{S}(G)e_{\sigma_i}\times e_{\sigma_i}V & \to & e_{U_{x_0}^{(r)}}V, \\ (f,v) & \mapsto & fv \end{array}$$

is continuous and $\mathcal{H}(G_{\sigma_i}/U_{\sigma_i}^{(e)})$-balanced, ∂_0 extends to a continuous map

$$\partial_0^t : C_0^t(\mathcal{B}(G);V)^{U_{x_0}^{(r)}} \to V^{U_{x_0}^{(r)}}. \tag{21}$$

To see that the higher boundary maps are also continuous, we fix an $n+1$-polysimplex τ with faces τ_j. The bilinear map of Fréchet spaces

$$\begin{array}{ccc} e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_\tau \times \epsilon_\tau \otimes e_\tau V & \to & \bigoplus_j e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_{\tau_j} \widehat{\otimes} \epsilon_{\tau_j} \otimes e_{\tau_j} V, \\ (f, \tau \otimes v) & \mapsto & f \widehat{\otimes} \partial(\tau) \otimes v = \sum_j f \widehat{\otimes} [\tau : \tau_j] \tau_j \otimes v \end{array}$$

is continuous, for it is made from the identity map of $e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_\tau \widehat{\otimes} e_\tau V$, a finite linear combination and the embeddings

$$e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_\tau \widehat{\otimes} e_\tau V \to e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_{\tau_j} \widehat{\otimes} e_{\tau_j} V.$$

For σ_i in the G-orbit of τ_j

$$e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_{\tau_j} \widehat{\otimes} e_{\tau_j} V \to e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_{\sigma_i} \underset{\mathcal{H}(G_{\sigma_i}; U_{\sigma_i}^{(e)})}{\widehat{\otimes}} \epsilon_{\sigma_i} \otimes e_{\sigma_i} V$$

is a quotient map, so in particular continuous. Hence the composition

$$e_{U_{x_0}^{(r)}} \mathcal{S}(G) e_\tau \times \epsilon_\tau \otimes e_\tau V \to C_n^t(\mathcal{B}(G); V)^{U_{x_0}^{(r)}}$$

is also continuous. By construction the latter map is $\mathcal{H}(G_\tau, U_\tau^{(e)})$-balanced and it extends ∂_{n+1} on $e_{U_{x_0}^{(r)}} \mathcal{H}(G) e_\tau \times \epsilon_\tau \otimes e_\tau V$. Now the universal property of $\underset{\mathcal{H}(G_\tau; U_\tau^{(e)})}{\widehat{\otimes}}$ says that there exists a unique continuous map

$$\partial_{n+1}^t : C_{n+1}^t(\mathcal{B}(G); V)^{U_{x_0}^{(r)}} \to C_n^t(\mathcal{B}(G); V)^{U_{x_0}^{(r)}} \tag{22}$$

which extends ∂_{n+1}.

By definition (21) and (22) are $\mathcal{S}(G, U_{x_0}^{(e)})$-linear. As $C_n^t(\mathcal{B}(G); V)$ is endowed with the inductive limit topology, we can take the union over $r \in \mathbb{N}$ in (21) and (22) to find the required continuous maps ∂_n^t. These are homomorphisms of modules over $\bigcup_{r \in \mathbb{N}} \mathcal{S}(G, U_{x_0}^{(e)}) = \mathcal{S}(G)$. $\square$

Whereas Lemma 3.3 says that $\big(C_*^t(\mathcal{B}(G); V)^K, \partial_*^t \big)$ is a differential complex in $\mathrm{Mod}_{Fr}(\mathcal{S}(G, K))$, the next lemma implies that it consists of projective objects.

Lemma 3.4. *Let* $V \in \mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$.

(a) $C_n^t(\mathcal{B}(G); \mathcal{S}(G) e_K)^K$ *is projective in* $\mathrm{Mod}_{Fr}(\mathcal{S}(G, K) \widehat{\otimes} \mathcal{S}(G, K)^{op})$.
(b) $C_n^t(\mathcal{B}(G); V)^K$ *is projective in* $\mathrm{Mod}_{Fr}(\mathcal{S}(G, K))$.

Proof. There are some technical complications, caused by the fact that $C_n^t(\mathcal{B}(G); V)$ is not necessarily generated by its K-invariant vectors.

(a) Since G acts transitively on the chambers of $\mathcal{B}(G)$, we may assume that x_0 and all the σ_i lie in the closure of a fixed chamber c. Then

$$U_{x_0}^{(e+1)} \subset U_c^{(e+1)} \subset U_{\sigma_i}^{(e)} \text{ for all } i = 1, \ldots, d,$$

so abbreviating $U = U_{x_0}^{(e+1)}$ and $\sigma = \sigma_i$ we have

$$e_U e_\sigma = e_\sigma = e_\sigma e_U.$$

It follows that a typical direct summand of $C^t_n(\mathcal{B}(G);\mathcal{S}(G)e_U)^U$ looks like

$$\big(\mathcal{S}(G,U)\widehat{\otimes}\mathcal{S}(G,U)^{op}\big)\sum_\rho (p_{\epsilon_\sigma\otimes\rho}\otimes p_\rho) = S(G,U)e_\sigma \underset{\mathcal{H}(G_\sigma;U_\sigma^{(e)})}{\widehat{\otimes}} \epsilon_\sigma\otimes e_\sigma\mathcal{S}(G,U) = \\ e_U S(G)e_\sigma \underset{\mathcal{H}(G_\sigma;U_\sigma^{(e)})}{\widehat{\otimes}} \epsilon_\sigma\otimes e_\sigma\mathcal{S}(G)e_U .$$

These modules are projective in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,U)\widehat{\otimes}\mathcal{S}(G,U)^{op})$, because the left hand side is a direct summand of a free module. It follows readily from the definition of nuclearity that

$$\mathcal{S}(G,U)\widehat{\otimes}\mathcal{S}(G,U)^{op}\cong\mathcal{S}(G\times G^{op},U\times U^{op}) \tag{23}$$

as Fréchet algebras, see the proof of [Mey2, Lemma 1]. By Proposition 3.1, applied to $G\times G^{op}$, the functors

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{S}(G\times G^{op},U\times U^{op})) & \longleftrightarrow & \mathrm{Mod}(\mathcal{S}(G\times G^{op},K\times K^{op})) \\ Y & \mapsto & Y^{K\times K^{op}} \\ e_{U\times U^{op}}\mathcal{S}(G\times G^{op})e_{K\times K^{op}}Z & \leftarrow & Z \end{array} \tag{24}$$

provide an equivalence between $\mathrm{Mod}(\mathcal{S}(G\times G^{op},K\times K^{op}))$ and a direct factor of $\mathrm{Mod}(\mathcal{S}(G\times G^{op},U\times U^{op}))$. Because $\mathcal{S}(G\times G^{op},U\times U^{op})$ is of finite rank as a module over $\mathcal{S}(G\times G^{op},K\times K^{op})$,

$$e_{U\times U^{op}}\mathcal{S}(G\times G^{op})e_{K\times K^{op}}Z = \mathcal{S}(G\times G^{op},U\times U^{op}) \underset{\mathcal{S}(G\times G^{op},K\times K^{op})}{\widehat{\otimes}} Z.$$

It follows that the functors (24) preserve the property Fréchet and preserve continuity of morphisms, so they remain equivalences for the appropriate categories of Fréchet modules. In particular $Y\mapsto Y^{K\times K^{op}}$ preserves projectivity, so

$$\Big(e_U S(G)e_\sigma \underset{\mathcal{H}(G_\sigma;U_\sigma^{(e)})}{\widehat{\otimes}} \epsilon_\sigma\otimes e_\sigma\mathcal{S}(G)e_U\Big)^{K\times K^{op}} = e_K S(G)e_\sigma \underset{\mathcal{H}(G_\sigma;U_\sigma^{(e)})}{\widehat{\otimes}} \epsilon_\sigma\otimes e_\sigma\mathcal{S}(G)e_K$$

is projective in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K)\widehat{\otimes}\mathcal{S}(G,K)^{op})$. By (20) $C^t_n(\mathcal{B}(G);\mathcal{S}(G)e_K)^K$ is a finite direct sum of such modules.

(b) Apply the functor $\underset{\mathcal{S}(G,K)}{\widehat{\otimes}} V^K$ to part (a) and use $V=\mathcal{S}(G)V^K$. □

4. Continuity of the contraction

We want to show that the contraction from Theorem 2.2 extends to $C^t_n(\mathcal{B}(G);V)^K$ by continuity. To that end we need a more concrete description of $C^t_n(\mathcal{B}(G);V)^K$, at least in the universal case $V=\mathcal{S}(G)e_K$.

Let $\{\sigma_i\}$ a set of representatives for the G-orbits of polysimplices of $\mathcal{B}(G)$. We may assume that x_0 is among them and that all the σ_i lie in a single chamber. We normalize the G-invariant metric d on $\mathcal{B}(G)$ so that the diameter of a chamber is 1. Let $\ell : G\to\mathbb{R}_{\geq 0}$ be the length function

$$\ell(g)=d(gx_0,x_0)+1.$$

As was shown in [Vig1, Section 9], the topology on $\mathcal{S}(G)$ is defined by the norms

$$p_m(f)=\|\ell^m f\|_2 \qquad m\in\mathbb{N}.$$

More precisely, $e_\tau\mathcal{S}(G)e_\sigma$ is the completion of $e_\tau\mathcal{H}(G)e_\sigma$ with respect to this collection of norms.

Note that the identification (6) of the two appearances of $C_n(\mathcal{B}(G); \mathcal{S}(G)e_K)$ goes via the map

$$
\begin{array}{cccc}
(25) & \alpha : \bigoplus_{\sigma \in \mathcal{B}(G)^{(n)}} \mathbb{C}\sigma \otimes_{\mathbb{C}} e_\sigma \mathcal{S}(G) e_K & \to & \bigoplus_{i=1}^{d} \mathcal{H}(G) e_{\sigma_i} \underset{\mathcal{H}(G_{\sigma_i})}{\otimes} \epsilon_{\sigma_i} \otimes e_{\sigma_i} \mathcal{S}(G) e_K \\
& \sum_\sigma \sigma \otimes f_\sigma & \mapsto & \sum_\sigma e_\sigma g_\sigma^{-1} \otimes g_\sigma f_\sigma .
\end{array}
$$

Here we have chosen for each σ an element $g_\sigma \in G$ such that $g_\sigma \sigma = \pm \sigma_i$, where σ_i is the chosen representative of the G-orbit of σ. We fix such a choice of such g_σ once and for all. Hence $g_\sigma f_\sigma \in e_{\sigma_i} \mathcal{S}(G) e_K$ for all σ. The fact that we tensor over $\mathcal{H}(G_{\sigma_i})$ makes the map α independent of the choices of the g_σ. The argument of (14) shows that

$$
e_K \mathcal{S}(G) e_{\sigma_i} \underset{\mathcal{H}(G_{\sigma_i})}{\widehat{\otimes}} \epsilon_{\sigma_i} \otimes e_{\sigma_i} \mathcal{S}(G) e_K \cong \big(e_K \mathcal{S}(G) e_\sigma \widehat{\otimes} (e_K \mathcal{S}(G) e_\sigma)^{op} \big) \sum_\rho (e_\rho \otimes e_{\epsilon_{\sigma_i} \rho}).
$$

As a Fréchet space this is a direct summand of

$$
e_K \mathcal{S}(G) e_{\sigma_i} \widehat{\otimes} e_{\sigma_i} \mathcal{S}(G) e_K, \tag{26}
$$

so the topology on $C_n^t(\mathcal{B}(G); \mathcal{S}(G)e_K)^K$ can be described with any defining family of seminorms on (26).

A general element of $C_n(\mathcal{B}(G); \mathcal{S}(G)e_K)$ can be written as $x = \sum_\sigma \sigma \otimes f_\sigma$ where $f_\sigma \in e_\sigma \mathcal{S}(G) e_K$. We define $f_{-\sigma}$ by $f_{-\sigma} = -f_\sigma$ for all σ (recall that we have fixed an orientation for all simplexes of $\mathcal{B}(G)$). We will use this notational convention from now on. Then x is K-invariant if and only if $f_{k\sigma} = k f_\sigma$. We define a norm q_m ($m \in \mathbb{N}$) on $C_n(\mathcal{B}(G); \mathcal{S}(G)e_K)^K$ by

$$
q_m \big(\sum\nolimits_\sigma \sigma \otimes f_\sigma \big) = \big(\sum\nolimits_\sigma \| (1 + d(\sigma, x_0) + \ell)^m g_\sigma f_\sigma \|_2^2 \big)^{1/2}. \tag{27}
$$

We remark at this point that this family of seminorms does depend on the choices of the elements g_σ, but not up to equivalence.

Lemma 4.1. *The Fréchet space $C_n^t(\mathcal{B}(G); \mathcal{S}(G)e_K)^K$ is the completion of $C_n(\mathcal{B}(G); \mathcal{H}(G)e_K)^K$ with respect to the family of norms $\{q_m \mid m \in \mathbb{N}\}$.*

Proof. By [Vig1, Section 9], applied to $G \times G$, one defining family of norms on (26) is

$$
q'_m(f) = \| (\ell_1 + \ell_2)^m f \|_2 = \Big(\int_{G \times G} (\ell(g_1) + \ell(g_2))^m |f(g_1, g_2)|^2 dg_1 dg_2 \Big)^{1/2}.
$$

We retract q'_m to $C_n(\mathcal{B}(G); \mathcal{H}(G)e_K)^K$ via (25). Notice that for $g \in U_\sigma^{(e)} g_\sigma^{-1}$ the difference between $\ell(g)$ and $d(\sigma, x_0) + 1$ is at most $d(g x_0, g \sigma_i) \leq 1$, and hence inessential when it comes to these norms. Consider $x = \sum_\sigma \sigma \otimes f_\sigma \in C_n(\mathcal{B}(G); \mathcal{H}(G)e_K)^K$ and

$$
\alpha(x) = \sum\nolimits_\sigma e_\sigma g_\sigma^{-1} \otimes g_\sigma f_\sigma . \tag{28}
$$

Since the right hand side of (25) is a finite direct sum over the polysimplices σ_i, it suffices to consider the case that x is supported on the G-orbit of one such σ_i. Then $e_\sigma g_\sigma^{-1}$ has support in $\{g \in G : g \sigma_i = \sigma\}$, so the different $e_\sigma g_\sigma^{-1}$ have disjoint supports. Hence the sum (28) is orthogonal for the L_2-norm, and this remains true if we multiply it with the function $(\ell_1 + \ell_2)^m$. The L_2-norm of $e_\sigma g_\sigma^{-1}$ is $\mathrm{vol}\big(U_{\sigma_i}^{(e)}\big)^{-1/2}$, which is independent of σ.

It follows that $q'_m(\alpha(x))$ equals the right hand side of (27), up to a constant factor. Consequently the norms q_m with $m \in \mathbb{N}$ define the topology of $C^t_n(\mathcal{B}(G); \mathcal{S}(G)e_K)^K$. Now the result follows from the obvious density of $C_n(\mathcal{B}(G); \mathcal{H}(G)e_K)^K$. $\Box$

Theorem 4.2. (a) *The differential complex*

$$\mathcal{S}(G,K) \leftarrow C^t_*(\mathcal{B}(G); \mathcal{S}(G)e_K)^K$$

is a projective resolution in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K)\widehat{\otimes}\mathcal{S}(G,K)^{op})$. *It admits a continuous contraction which is right* $\mathcal{S}(G,K)$*-linear.*

(b) *Let* $V \in \mathrm{Mod}_{Fr}(\mathcal{S}(G),K)$. *Then*

$$V^K \leftarrow C^t_*(\mathcal{B}(G); V)^K \quad \text{and} \quad V \leftarrow \mathcal{S}(G)e_K C^t_*(\mathcal{B}(G); V)$$

are projective resolutions in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$ *and in* $\mathrm{Mod}_{Fr}(\mathcal{S}(G),K)$.

Proof. (a) The projectivity was already established in Lemma 3.4.
Like in the proof of Lemma 4.1 it suffices to consider an element $x = \sum_\sigma \sigma \otimes f_\sigma \in C_n(\mathcal{B}(G); \mathcal{H}(G)e_K)^K$. Recall from the text just above (27) that the K-invariance is equivalent with $f_{k\sigma} = kf_\sigma$. Then

$$\gamma_n(x) = \sum_{\tau \in \mathcal{B}(G)^{(n+1)}} \tau \otimes \sum_{\sigma \in \mathcal{B}(G)^{(n)}} \gamma_{\sigma,\tau} f_\sigma . \tag{29}$$

By (11)

$$F_\tau := \sum\nolimits_{\sigma \in \mathcal{B}(G)^{(n)}} \gamma_{\sigma,\tau} f_\sigma$$

is invariant under $U_\tau^{(e)}$. By the K-equivariance of γ the element $\gamma_n(x)$ is K-invariant, and by our convention $F_{-\tau} = -F_\tau$. In view of Lemma 4.1 we have

$$q_m(\gamma_n(x))^2 = \sum\nolimits_{\tau \in \mathcal{B}(G)^{(n+1)}} \|(1 + d(\tau, x_0) + \ell)^m g_\tau F_\tau\|^2.$$

The K-invariance implies that $F_{k\tau} = kF_\tau$ for all τ and $k \in K$. From the definition of g_τ it is clear that $g_{k\tau}k \in G_{\tau_i} g_\tau$. For all $h \in G_{\tau_i}$ and $g \in G$ we have $\ell(hg) \leq \ell(g) + 2$. Since $d(k\tau, x_0) = d(\tau, x_0)$ for all $k \in K$, we obtain

$$\|(1 + d(k\tau, x_0) + \ell)^m g_{k\tau} F_{k\tau}\|^2 \leq \|(3 + d(\tau, x_0) + \ell)^m g_\tau F_\tau\|^2.$$

Hence $q_m(\gamma_n(x))^2$ is bounded by

$$\sum_{\tau \in K\backslash\mathcal{B}(G)^{(n+1)}} [K : K \cap G_\tau] \, \|(3 + d(\tau, x_0) + \ell)^m g_\tau F_\tau\|^2 =$$

$$\sum_{\tau \in K\backslash\mathcal{B}(G)^{(n+1)}} [K : K \cap G_\tau] \, \Big\|(3 + d(\tau, x_0) + \ell)^m \sum_{\sigma \in \mathcal{B}(G)^{(n)}} \gamma_{\sigma,\tau} g_\tau f_\sigma \Big\|_2^2 .$$

By Corollary 2.3.c

$$\leq M_\gamma^2 \sum_{\tau \in K\backslash\mathcal{B}(G)^{(n+1)}} [K : K \cap G_\tau] \sum_{\sigma \in \mathcal{B}(G)^{(n)} : \gamma_{\sigma,\tau} \neq 0} \Big\|(3 + d(\tau, x_0) + \ell)^m g_\tau g_\sigma^{-1} (g_\sigma f_\sigma) \Big\|_2^2 . \tag{30}$$

The length function ℓ satisfies

$$\begin{aligned}\ell(g_\tau g_\sigma^{-1} g) &\leq \ell(g_\tau g_\sigma^{-1}) + \ell(g) = d(g_\tau g_\sigma^{-1} x_0, x_0) + 1 + \ell(g)\\ &= d(g_\sigma^{-1} x_0, g_\tau^{-1} x_0) + 1 + \ell(g)\\ &\leq d(\sigma,\tau) + d(g_\sigma^{-1}\sigma_i, g_\sigma^{-1} x_0) + d(g_\tau^{-1}\tau_i, g_\tau^{-1} x_0) + 1 + \ell(g)\\ &\leq d(\sigma,\tau) + 3 + \ell(g).\end{aligned}$$

Corollary 2.3.b says that τ lies in the hull of $\sigma \cup \{x_0\}$ when $\gamma_{\sigma,\tau} \neq 0$, in which case

$$d(\sigma,\tau) + d(\tau, x_0) \leq 2d(\sigma, x_0) + 2d(\delta, x_0)$$

by Lemma 2.4. We combine these length estimates to

$$\begin{aligned}3 + d(\tau, x_0) + \ell(g_\tau g_\sigma^{-1} g) &\leq 6 + 2d(\sigma, x_0) + 2d(\delta, x_0) + \ell(g)\\ &\leq \frac{6 + 2d(x_0,\delta) + 1}{1+1}(1 + d(\sigma, x_0) + \ell(g)) := c\,(1 + d(\sigma, x_0) + \ell(g)).\end{aligned} \tag{31}$$

Therefore we may continue the estimate (30) with

$$\leq M_\gamma^2 \sum_{\tau \in K \backslash \mathcal{B}(G)^{(n+1)}} [K : K \cap G_\tau] \sum_{\sigma \in \mathcal{B}(G)^{(n)} : \gamma_{\sigma,\tau} \neq 0} \big\| c^m (1 + d(\sigma, x_0) + \ell)^m g_\sigma f_\sigma \big\|_2^2. \tag{32}$$

By Corollary 2.3.b the sets $S_\tau := \{\sigma \in \mathcal{B}(G)^{(n)} : \gamma_{\sigma,\tau} \neq 0\}$ and $kS_\tau = \{\sigma \in \mathcal{B}(G)^{(n)} : \gamma_{\sigma,k\tau} \neq 0\}$ are disjoint if $k\tau \neq \pm\tau$. By Lemma 2.4 their union is contained in

$$\mathcal{B}(G)_\tau^{(n)} := \{\sigma \in \mathcal{B}(G)^{(n)} : d(\sigma, x_0) \geq d(\tau, x_0) - d(\delta, x_0)\},$$

so (32) is bounded by (using the invariance $f_{k\sigma} = k f_\sigma$)

$$\leq M_\gamma^2 c^{2m} \sum_{\tau \in K \backslash \mathcal{B}(G)^{(n+1)}} \sum_{\sigma \in \mathcal{B}(G)_\tau^{(n)}} \big\| (1 + d(\sigma, x_0) + \ell)^m g_\sigma f_\sigma \big\|_2^2. \tag{33}$$

By the Cartan decomposition $G_{x_0} \backslash \mathcal{B}(G)$ is in bijection with a Weyl chamber in an apartment A. As K is of finite index in G_{x_0}, this shows that $K \backslash \mathcal{B}(G)^{(d)}$ is of polynomial growth. Choose $N \in 2\mathbb{N}$ such that

$$b_N := \sum\nolimits_{K \backslash \mathcal{B}(G)^{(n+1)}} (1 + d(\tau, x_0))^{-N} \text{ is finite.}$$

This enables us to estimate (33) by

$$\begin{aligned}&\leq M_\gamma^2 c^{2m} \sum_{\tau \in K \backslash \mathcal{B}(G)^{(n+1)}} (1 + d(\tau, x_0))^{-N} \sum_{\sigma \in \mathcal{B}(G)_\tau^{(n)}} \big\| (1 + d(\sigma, x_0) + \ell)^{m + N/2} g_\sigma f_\sigma \big\|_2^2\\ &\leq M_\gamma^2 c^{2m} b_N \sum_{\sigma \in \mathcal{B}(G)^{(n)}} \big\| (1 + d(\sigma, x_0) + \ell)^{m + N/2} g_\sigma f_\sigma \big\|_2^2 \;=\; M_\gamma^2 c^{2m} b_N q_{m+N/2}(x)^2.\end{aligned}$$

Altogether we obtained

$$q_m(\gamma_n(x)) \leq M_\gamma c^m \sqrt{b_N}\, q_{m+N/2}(x), \tag{34}$$

from which we conclude that on $C_n(\mathcal{B}(G); \mathcal{S}(G) e_K)^K$ the map γ_n is continuous with respect to the family of norms $\{q_m \mid m \in \mathbb{N}\}$. From Theorem 2.2 we know that γ_n is right $\mathcal{H}(G,K)$-linear, so by Lemma 4.1 it extends continuously to a right-$\mathcal{S}(G,K)$-linear map

$$\gamma_n^t : C_n^t(\mathcal{B}(G); \mathcal{S}(G) e_K)^K \to C_{n+1}^t(\mathcal{B}(G); \mathcal{S}(G) e_K)^K.$$

The relation $\partial_{n+1}\gamma_n + \gamma_{n-1}\partial_n = \mathrm{id}$ extends by continuity to $\partial^t_{n+1}\gamma^t_n + \gamma^t_{n-1}\partial^t_n = \mathrm{id}$.
(b) The first statement follows from (a) upon applying the functor $\widehat{\otimes}_{\mathcal{S}(G,K)} V^K$. The second is a consequence of the first and Corollary 3.2. $\square$

We remark that Theorem 4.2 does not imply that $V \leftarrow C^t_*(\mathcal{B}(G); V)$ is a resolution. Although this is true, one needs more sophisticated techniques to prove it – see the next section. The main use of Theorem 4.2 is to compute and compare Ext-groups:

Proposition 4.3. *Let $V, W \in \mathrm{Mod}(\mathcal{S}(G))$ with V admissible.*
(*a*) *There is a natural isomorphism*

$$\mathrm{Ext}^n_{\mathcal{H}(G)}(V, W) \cong \mathrm{Ext}^n_{\mathcal{S}(G)}(V, W).$$

If $W \in \mathrm{Mod}_{LF}(\mathcal{S}(G))$, then these are also isomorphic to $\mathrm{Ext}^n_{\mathrm{Mod}_{LF}(\mathcal{S}(G))}(V, W)$.
(*b*) *Suppose that moreover $V, W \in \mathrm{Mod}(\mathcal{S}(G), K)$. There are natural isomorphisms*

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{H}(G)}(V, W) &\cong \mathrm{Ext}^n_{\mathcal{H}(G,K)}(V^K, W^K) \\ &\cong \mathrm{Ext}^n_{\mathcal{S}(G,K)}(V^K, W^K) \cong \mathrm{Ext}^n_{\mathrm{Mod}(\mathcal{S}(G),K)}(V, W). \end{aligned}$$

(*c*) *If furthermore $V, W \in \mathrm{Mod}_{Fr}(\mathcal{S}(G), K)$, then the groups from (b) are also naturally isomorphic to*

$$\mathrm{Ext}^n_{\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))}(V^K, W^K) \quad \textit{and} \quad \mathrm{Ext}^n_{\mathrm{Mod}_{Fr}(\mathcal{S}(G),K)}(V, W).$$

Proof. (b) The outer isomorphisms follow from Theorem 2.1 and Corollary 3.2. For the middle one, we observe that by Theorem 2.2.a

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{H}(G,K)}(V^K, W^K) &= H^n\Big(\mathrm{Hom}_{\mathcal{H}(G,K)}\big(C_*(\mathcal{B}(G); V)^K, W^K\big), \mathrm{Hom}(\partial_*, W^K)\Big) \\ &= H^n\Big(\mathrm{Hom}_{\mathcal{S}(G,K)}\big(\mathcal{S}(G,K) \underset{\mathcal{H}(G,K)}{\otimes} C_*(\mathcal{B}(G); V)^K, W^K\big), \mathrm{Hom}(\partial_*, W^K)\Big). \end{aligned} \tag{35}$$

As V is admissible,

$$\mathcal{S}(G,K) \underset{\mathcal{H}(G,K)}{\otimes} C_*(\mathcal{B}(G); V)^K = C^t_*(\mathcal{B}(G); V)^K.$$

By Theorem 1.2, Frobenius reciprocity and Lemma 3.4.b this module is projective in $\mathrm{Mod}(\mathcal{S}(G,K)$ and in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$. Moreover it is finitely generated, so every module map to a Fréchet $\mathcal{S}(G,K)$-module is automatically continuous. Therefore (35) equals

$$H^n\Big(\mathrm{Hom}_{\mathcal{S}(G,K)}\big(C^t_*(\mathcal{B}(G); V)^K, W^K\big), \mathrm{Hom}(\partial_*, W^K)\Big),$$

which by Theorem 4.2 is $\mathrm{Ext}^n_{\mathcal{S}(G,K)}(V^K, W^K)$.
(c) In case $W^K \in \mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$, the above argument also shows that we obtain the same answer if we work in $\mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$. By Corollary 3.2, these Ext-groups are naturally isomorphic to $\mathrm{Ext}^n_{\mathrm{Mod}_{Fr}(\mathcal{S}(G),K)}(V, W)$.
(a) The first statement was proven in [ScZi2, Section 9], using the results of Meyer [Mey2]. Here we provide an alternative proof. Recall that the Bernstein decomposition of $\mathrm{Mod}(\mathcal{H}(G))$ is given by idempotents in the centre of the category $\mathrm{Mod}(\mathcal{H}(G))$ [BeDe]. Hence $\mathrm{Mod}(\mathcal{S}(G))$ and $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ admit an analogous decomposition.

This persists to Ext-groups, so we may and will assume that V and W live in a single Bernstein component Ω. Choose $e \in \mathbb{N}$ such that all representations in Ω are generated by their $U_{x_0}^{(e)}$-invariant vectors. Then $V, W \in \mathrm{Mod}(\mathcal{S}(G), U_{x_0}^{(e)})$ and moreover $W^K \in \mathrm{Mod}_{Fr}(\mathcal{S}(G,K))$ if $W \in \mathrm{Mod}_{LF}(\mathcal{S}(G))$. Now we can apply parts (b) and (c). $\Box$

The admissibility of V is necessary in Proposition 4.3. The difference can already be observed in degree $n = 0$: $\mathrm{Ext}^0_{\mathcal{S}(G)}(V,W) = \mathrm{Hom}_{\mathcal{S}(G)}(V,W)$ can be smaller than $\mathrm{Ext}^0_{\mathcal{H}(G)}(V,W) = \mathrm{Hom}_{\mathcal{H}(G)}(V,W)$ for general $V, W \in \mathrm{Mod}(\mathcal{S}(G))$. In $\mathrm{Mod}_{LF}(\mathcal{S}(G))$ we usually get an even smaller space of morphisms, because they are required to be continuous.

5. Bornological modules

The content of Sections 3 and 4 can be formulated nicely with bornologies. In this section we work in the category $\mathrm{Mod}_{bor}(A)$ of complete bornological modules over a bornological algebra A, as in [Mey1]. The corresponding tensor product is the completed bornological tensor product over A, which we denote by $\widetilde{\otimes}_A$. In case $A = \mathbb{C}$, we suppress it from the notation.

We endow $\mathcal{H}(G)$ with the fine bornology, so a subset of $\mathcal{H}(G)$ is considered to be bounded if it is contained in a finite dimensional linear subspace of $\mathcal{H}(G)$ and over there is bounded in the usual sense. On $\mathcal{S}(G)$ we use the precompact bornology, which means that a subset is bounded if and only if it is contained in some compact subset. Since we use the inductive limit topology on $\mathcal{S}(G)$, every bounded set is contained in a compact subset of $\mathcal{S}(G,U)$ for some compact open subgroup $U \subset G$.

By [Mey2, Lemma 2] we have

$$\mathcal{H}(G) \widetilde{\otimes} \mathcal{H}(G) \cong \mathcal{H}(G \times G) \quad \text{and} \quad \mathcal{S}(G) \widetilde{\otimes} \mathcal{S}(G) \cong \mathcal{S}(G \times G).$$

These isomorphisms, the second of which does not hold for the algebraic or the completed projective tensor product, to some extent explain why bornology is a convenient technique in our situation.

Since $\mathcal{H}(G)$ has the fine bornology, the projectivity properties from Theorems 1.2 and 2.2 carry over. Hence for any $V \in \mathrm{Mod}_{bor}(\mathcal{H}(G))$ the modules

$$\begin{array}{lll} C_n(\mathcal{B}(G); V) & \in & \mathrm{Mod}_{bor}(\mathcal{H}(G)), \\ C_n(\mathcal{B}(G); \mathcal{H}(G)) & \in & \mathrm{Mod}_{bor}(\mathcal{H}(G) \widetilde{\otimes} \mathcal{H}(G)^{op}), \\ C_n(\mathcal{B}(G); V)^K & \in & \mathrm{Mod}_{bor}(\mathcal{H}(G,K)), \\ C_n(\mathcal{B}(G); \mathcal{H}(G) e_K)^K & \in & \mathrm{Mod}_{bor}(\mathcal{H}(G,K) \widetilde{\otimes} \mathcal{H}(G,K)^{op}) \end{array} \tag{36}$$

are projective in the respective categories.

The categories of topological $\mathcal{S}(G)$-modules that we used in the previous sections are full subcategories of $\mathrm{Mod}_{bor}(\mathcal{S}(G))$. To see this, we endow all $V, W \in \mathrm{Mod}_{LF}(\mathcal{S}(G))$ with the precompact bornology. Any $\mathcal{S}(G)$-module map $\phi : V \to W$ sends V^U to W^U, for any compact open subgroup U. By the definition of the inductive limit topology, ϕ is continuous if and only if $\phi\big|_{V^U}$ is continuous for all U. Since V^U and W^U are Fréchet spaces, the latter condition is equivalent to boundedness of $\phi\big|_{V^U}$. As the bornology on V is the inductive limit of the bornologies on the V^U, this in turn is equivalent to boundedness of ϕ. Hence

$$\mathrm{Hom}_{\mathrm{Mod}_{bor}(\mathcal{S}(G))}(V,W) = \mathrm{Hom}_{\mathrm{Mod}_{LF}(\mathcal{S}(G))}(V,W).$$

Since bornological tensor products commute with inductive limits, the definition (20) can be simplified to

$$C_n^t(\mathcal{B}(G);V) = \bigoplus_{i=1}^d \mathcal{S}(G) \underset{\mathcal{H}(G_{\sigma_i};U_{\sigma_i}^{(e)})}{\widetilde{\otimes}} \epsilon_{\sigma_i} \otimes V^{U_{\sigma_i}^{(e)}} \quad \text{for} \quad V \in \mathrm{Mod}_{bor}(\mathcal{S}(G)).$$

The same argument as in the proof of Theorem 1.2 shows that this is a projective object of $\mathrm{Mod}_{bor}(\mathcal{S}(G))$. By (14)

$$C_n^t(\mathcal{B}(G);\mathcal{S}(G)) = \bigoplus_{i=1}^d \big(\mathcal{S}(G)\widetilde{\otimes}\mathcal{S}(G)^{op}\big) \sum_\rho (p_\rho \otimes p_{\epsilon_{\sigma_i}\rho})$$

is projective in $\mathrm{Mod}_{bor}(\mathcal{S}(G)\widetilde{\otimes}\mathcal{S}(G)^{op}) = \mathrm{Mod}_{bor}(\mathcal{S}(G \times G^{op}))$. Just like for Fréchet modules, Proposition 3.1 remains valid for bornological modules. It follows that

$$\begin{aligned} (37) \qquad & C_n^t(\mathcal{B}(G);V)^K = \bigoplus_{i=1}^d e_K \mathcal{S}(G) \underset{\mathcal{H}(G_{\sigma_i};U_{\sigma_i}^{(e)})}{\widetilde{\otimes}} \epsilon_{\sigma_i} \otimes e_{U_{\sigma_i}^{(e)}} V \in \mathrm{Mod}_{bor}(\mathcal{S}(G,K)), \\ & C_n^t(\mathcal{B}(G);\mathcal{S}(G)e_K)^K \in \mathrm{Mod}_{bor}(\mathcal{S}(G,K)\widetilde{\otimes}\mathcal{S}(G,K)^{op}) \end{aligned}$$

are projective. Furthermore we note that, by the associativity of bornological tensor products and by (16),

$$(38) \qquad C_n^t(\mathcal{B}(G);V) \;=\; \mathcal{S}(G) \underset{\mathcal{H}(G)}{\widetilde{\otimes}} C_n(\mathcal{B}(G);V),$$

$$C_n^t(\mathcal{B}(G);V)^K \;=\; e_K\mathcal{S}(G) \underset{\mathcal{H}(G)}{\widetilde{\otimes}} C_n(\mathcal{B}(G);V)^K \;=\; \mathcal{S}(G,K) \underset{\mathcal{H}(G,K)}{\widetilde{\otimes}} C_n(\mathcal{B}(G);V)^K.$$

According to [Mey2, Theorem 22] the embedding of bornological algebras $\mathcal{H}(G) \to \mathcal{S}(G)$ is isocohomological. Together with Theorem 1.2.b and [Mey1, Theorem 35] this implies that, for any $V \in \mathrm{Mod}_{bor}(\mathcal{S}(G))$ which is generated by its $U_{x_0}^{(e)}$-invariant vectors for some $e \in \mathbb{N}$,

$$(39) \qquad \big(\mathcal{S}(G) \underset{\mathcal{H}(G)}{\widetilde{\otimes}} C_*(\mathcal{B}(G);V), \partial_*^t\big) = \big(C_*^t(\mathcal{B}(G);V), \partial_*^t\big)$$

is a resolution of V in $\mathrm{Mod}_{bor}(\mathcal{S}(G))$.

Theorem 5.1. *Let* $V, W \in \mathrm{Mod}_{bor}(\mathcal{S}(G))$.

(a) *There is a natural isomorphism*

$$\mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{H}(G))}(V,W) \cong \mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{S}(G))}(V,W).$$

(b) *Suppose that moreover* $V, W \in \mathrm{Mod}(G,K)$. *There are natural isomorphisms*

$$\begin{aligned} \mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{H}(G))}(V,W) &\cong \mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{H}(G,K))}(V^K,W^K) \\ &\cong \mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{S}(G,K))}(V^K,W^K) \cong \mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{S}(G))}(V,W). \end{aligned}$$

Proof. Of course this is a straightforward consequence of Meyer's result (39). Even Theorem 2.2 is not really needed, only the existence of some projective resolution. Here we show how the theorem can be derived from Theorem 4.2.

(b) The outer isomorphisms follow from the bornological versions of Theorem 2.1 and Proposition 3.1. As concerns the middle one, by Theorem 2.2.a

$$\mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{H}(G,K))}(V^K, W^K) = \\ H^n\big(\mathrm{Hom}_{\mathrm{Mod}_{bor}(\mathcal{H}(G,K))}\big(C_*(\mathcal{B}(G);V)^K, W^K\big), \mathrm{Hom}_{bor}(\partial_*, W^K)\big).$$

By (38) and by Frobenius reciprocity this is isomorphic to

$$H^n\big(\mathrm{Hom}_{\mathrm{Mod}_{bor}(\mathcal{S}(G,K))}\big(C^t_*(\mathcal{B}(G);V)^K, W^K\big), \mathrm{Hom}_{bor}(\partial^t_*, W^K)\big),$$

which by Theorem 4.2 and (37) is $\mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{S}(G,K))}(V^K, W^K)$.
(a) By the same argument as for Proposition 4.3, this follows from part (b). □

We remark that in general

$$\mathrm{Ext}^n_{\mathrm{Mod}_{bor}(\mathcal{H}(G))}(V, W) \ncong \mathrm{Ext}^n_{\mathcal{H}(G)}(V, W).$$

The reason is that morphisms in $\mathrm{Mod}_{bor}(\mathcal{H}(G))$ have to be bounded, which is a nontrivial condition if V is not admissible.

6. Generalization to disconnected reductive groups

In the final section we take a more general point of view, we let $G = \mathcal{G}(\mathbb{F})$ be an algebraic group whose identity component $G^\circ = \mathcal{G}^\circ(\mathbb{F})$ is linear and reductive. We will show how the results of the previous sections can be generalized to such groups.

First we discuss the categorical issues. Since G acts on G° by conjugation, it acts on $\mathrm{Mod}(G^\circ) = \mathrm{Mod}(\mathcal{H}(G^\circ;\mathbb{C}))$ and on the centre of this category. If e°_Ω is a central idempotent of $\mathrm{Mod}(G^\circ)$, then

$$e_\Omega := \sum_{g \in G/\mathrm{Stab}_G(\Omega^\circ)} g e^\circ_\Omega g^{-1}$$

is a central idempotent of $\mathrm{Mod}(G)$. It follows that the category of smooth G-representations on complex vector spaces admits a factorization, parametrized by the G-orbits of Bernstein components of G°:

$$\mathrm{Mod}(G) = \prod_{\Omega = G\Omega^\circ/G} \mathrm{Mod}_\Omega(G) = \prod_{\Omega = G\Omega^\circ/G} e_\Omega \,\mathrm{Mod}(G). \tag{40}$$

However, in contrast with the Bernstein decomposition for connected reductive p-adic groups, it is possible that $\mathrm{Mod}_\Omega(G)$ is decomposable.

Following [BuKu] we call a compact open subgroup $U \subset G$ (or more precisely the idempotent $e_U \in \mathcal{H}(G)$) a type if the category $\mathrm{Mod}(G, U)$ is closed under the formation of subquotients in $\mathrm{Mod}(G)$. In case $\mathcal{G}$ is connected, [BuKu, Proposition 3.6] shows that these are precisely the compact open subgroups for which Theorem 2.1 holds.

Lemma 6.1. *Let $U \subset G^\circ$ be a type for G°. Then Theorem 2.1 holds for (G, U). In particular $V \mapsto V^U$ defines an equivalence of categories $\mathrm{Mod}(G, U) \to \mathrm{Mod}(\mathcal{H}(G, U))$ and $\mathrm{Mod}(G, U)$ is a direct factor of $\mathrm{Mod}(G)$.*

Proof. Part (a) of Theorem 2.1 is [BuKu, Proposition 3.3]. By [BuKu, Proposition 3.6] there exists a finite collection Λ of Bernstein components for G° such that

$$\mathrm{Mod}(G^\circ, U) = \prod_{\Omega^\circ \in \Lambda} \mathrm{Mod}_{\Omega^\circ}(G^\circ).$$

We claim that

$$\mathrm{Mod}(G, U) = \prod_{\Omega \in G\Lambda/G} \mathrm{Mod}_\Omega(G). \tag{41}$$

First we consider $V \in \mathrm{Mod}(G,U)$ as a $\mathcal{H}(G^\circ)$-module. As such

$$V = \mathcal{H}(G)V^U = \sum_{g \in G/G^\circ} \mathcal{H}(G^\circ) g V^U = \sum_{g \in G/G^\circ} \mathcal{H}(G^\circ) V^{gUg^{-1}},$$

so V is generated by $\sum_{g \in G/G^\circ} V^{gUg^{-1}}$. Since

$$\mathcal{H}(G^\circ) V^{gUg^{-1}} \in \prod_{\Omega^\circ \in g\Lambda} \mathrm{Mod}_{\Omega^\circ}(G^\circ) = \prod_{\Omega^\circ \in g\Lambda} e_{\Omega^\circ} \mathrm{Mod}(G^\circ),$$

V lies in

$$\prod_{\Omega^\circ \in G\Lambda} e_{\Omega^\circ} \mathrm{Mod}(G^\circ) = \prod_{\Omega^\circ \in G\Lambda/G} \sum_{g \in G/\mathrm{Stab}_G(\Omega^\circ)} g e_{\Omega^\circ} g^{-1} \mathrm{Mod}(G^\circ)$$
$$= \prod_{\Omega = G\Omega^\circ/G \in G\Lambda/G} e_\Omega \mathrm{Mod}(G^\circ).$$

In particular $V \in \prod_{\Omega \in G\Lambda/G} \mathrm{Mod}_\Omega(G)$. Conversely, let $W \in \mathrm{Mod}_\Omega(G)$ with $\Omega = G\Omega^\circ/G \in G\Lambda/G$. Then

$$W = e_\Omega W = \sum_{g \in G/\mathrm{Stab}_G(\Omega^\circ)} g e^\circ_\Omega g^{-1} W \in \prod_{G/\mathrm{Stab}_G(\Omega^\circ)} \mathrm{Mod}_{g\Omega^\circ}(G^\circ).$$

As $\Omega^\circ \in \Lambda$, the latter category is a direct factor of $\prod_{g \in G/N_G(U)} \mathrm{Mod}(G^\circ, gUg^{-1})$. Now we can write

$$W = \sum_{g \in G/N_G(U)} \mathcal{H}(G^\circ) W^{gUg^{-1}} = \sum_{g \in G/N_G(U)} \mathcal{H}(G^\circ) g W^U = \mathcal{H}(G) W^U,$$

which verifies our claim (41). Finally, by (40) $\prod_{\Omega \in G\Lambda/G} \mathrm{Mod}_\Omega(G)$ is a direct factor of $\mathrm{Mod}(G)$. □

Because the decomposition (40) is defined in terms of central idempotents of $\mathrm{Mod}(G)$, it can be handled in the same way as the usual Bernstein decomposition. In particular the proofs of [ScZi1, Lemmas 2.2 and 2.3] remain valid. Using these proofs in various categories leads to:

Corollary 6.2. *Let $U \subset G^\circ$ be a type for G°. Then Theorem 2.1 holds for (G,U) in* $\mathrm{Mod}(\mathcal{S}(G)), \mathrm{Mod}_{LF}(\mathcal{S}(G)), \mathrm{Mod}_{bor}(\mathcal{S}(G))$ *and* $\mathrm{Mod}_{bor}(\mathcal{H}(G))$.

Next we deal with the affine building of G. As a metric space, it is defined as

$$\mathcal{B}(G) = \mathcal{B}(\mathcal{G}, \mathbb{F}) := \mathcal{B}(\mathcal{G}^\circ, \mathbb{F}) = \mathcal{B}(G^\circ/Z(G^\circ)) \times X_*(Z(G^\circ)) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The action of G° on $\mathcal{B}(G^\circ/Z(G^\circ))$ is extended to G in the following way. There is a bijection between maximal compact subgroups of $G^\circ/Z(G^\circ)$ and vertices of

$\mathcal{B}(G^\circ/Z(G^\circ))$, which associates to a vertex x its stabilizer K_x. For any $g \in G$ the subgroup $gK_xg^{-1} \subset G^\circ/Z(G^\circ)$ is again maximal compact, so of the form K_y for a unique vertex y of $\mathcal{B}(G^\circ/Z(G^\circ))$. We define $g(x) = y$ and extend this by interpolation to an isometry of $\mathcal{B}(G^\circ/Z(G^\circ))$.

Since $Z(G^\circ)$ is a characteristic subgroup of G° and G° is normal in G, G acts on $Z(G^\circ)$ by conjugation. This induces an action of G on $X_*(Z(G^\circ))$ which extends the action of G°.

Because the polysimplicial structure on $\mathcal{B}(G^\circ/Z(G^\circ))$ is natural, it is preserved by G. Unfortunately, no such thing holds for $X_*(Z(G^\circ)) \otimes_{\mathbb{Z}} \mathbb{R}$, so our choice of a polysimplicial structure is in general not stable under the G-action. Even worse, if G acts in a complicated way on $X_*(Z(G^\circ))$, it can be very difficult to find a suitable polysimplicial structure on $X_*(Z(G^\circ)) \otimes_{\mathbb{Z}} \mathbb{R}$. A serious investigation of this problem would lead us quite far away from the theme of the paper, so we avoid it by means of the following assumption.

Condition 6.3. *There exists a G-stable root system of full rank in $X^*(Z(G^\circ))$.*

Under this condition the affine Coxeter complex of the root system is a suitable polysimplicial structure on $X_*(Z(G^\circ)) \otimes_{\mathbb{Z}} \mathbb{R}$. In most examples G/G° is small and the condition is easily seen to be fulfilled.

We also need a slightly improved version of the groups $U_\sigma^{(e)}$. To define it, we have to go through a part of the construction from [ScSt1]. Let $T = \mathcal{T}(\mathbb{F})$ be a maximal $\mathbb{F}$-split torus of G° and let Φ be the root system of (G°, T). Furthermore denote by U^+ and U^- the unipotent subgroups of G° corresponding to some choice of positive and negative roots. The new group $U_\sigma^{[e]}$ will admit a factorization like (10).

First we assume that $\mathcal{G}^\circ$ is quasi-split over $\mathbb{F}$, so that $Z_{G^\circ}(T)$ is a maximal torus of G°. We can keep $U_\sigma^{(e)} \cap U^+$ and $U_\sigma^{(e)} \cap U^-$, but we have to change $U_\sigma^{(e)} \cap Z_{G^\circ}(T)$. Let $Z_{G^\circ}(T)_r^{mc}$ ($r \in \mathbb{R}_{\geq 0}$) be the "minimal congruent filtration" of the torus $Z_{G^\circ}(T)$, as defined in [Yu, §5], and put

$$Z_{G^\circ}(T)_{e+}^{mc} = \bigcup\nolimits_{r>e} Z_{G^\circ}(T)_r^{mc}.$$

Following Yu we define, for $e \in \mathbb{N}$ and a polysimplex σ of $\mathcal{B}(G) = \mathcal{B}(\mathcal{G}^\circ, \mathbb{F})$:

$$U_\sigma^{[e]} := (U_\sigma^{(e)} \cap U^+)\, Z_{G^\circ}(T)_{e+}^{mc}\, (U_\sigma^{(e)} \cap U^-).$$

For general (not quasi-split) $\mathcal{G}^\circ$ the subgroups $U_\sigma^{[e]} \subset G^\circ$ are obtained from those in the quasi-split case via etale descent, as in [ScSt1, Section 1.2]. For this system of subgroups Proposition 1.1 and all the results of [ScSt1] hold.

According to [Yu, §9.4], there exists an affine group scheme $\mathcal{G}_\sigma^\circ$ such that $\mathcal{G}_\sigma^\circ(\mathcal{O}) = G_\sigma^\circ$ is the pointwise stabilizer of σ and

$$U_\sigma^{[e]} = \ker \big(\mathcal{G}_\sigma^\circ(\mathcal{O}) \to \mathcal{G}_\sigma^\circ(\mathcal{O}/\pi^{e+1}\mathcal{O}) \big).$$

Consequently $U_\sigma^{[e]}$ is stable under any automorphism of the affine group scheme $\mathcal{G}_\sigma^\circ$, which is not guaranteed in full generality in [ScSt1]. Clearly this applies to the action of $N_G(G_\sigma^\circ)$ on $\mathcal{G}_\sigma^\circ$, so $gU_\sigma^{[e]}g^{-1} = U_\sigma^{[e]}$ for all $g \in N_G(G_\sigma^\circ)$. Proposition 1.1.d and the definition of the action of G on $\mathcal{B}(G)$ then entail

$$gU_\sigma^{[e]}g^{-1} = U_{g\sigma}^{[e]} \quad \text{for all } g \in G.$$

Therefore Proposition 1.1 holds for G with the system of subgroups $U_\sigma^{[e]}$, which is enough to make everything from [MeSo1] work.

With the above adjustments the preceeding sections generalize almost to disconnected reductive groups, mostly in a trivial way. Only the proof of Theorem 2.2.a needs a little more care, it is there that we use the condition 6.3. The construction of the contraction γ of $C_*(A;\mathbb{Z})$ in [OpSo1, Section 2.1] applies to an apartment A spanned by an integral root system. Thus the assumed root system in $X^*(Z(G^\circ))$, together with the roots of (G°, T), functions as a book-keeping device to write down a contraction which has some nice properties. The remainder of the proof of Theorem 2.2.a needs no modification.

We conclude that:

Theorem 6.4. *Under Condition 6.3 all the results of Sections 1–5 are valid for G with the system of subgroups $U_\sigma^{[e]}$.*

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