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Resolvent estimates and local energy decay for hyperbolic equations

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Abstract We examine the cut-off resolvent $R_\chi(\lambda) = \chi(-\Delta_D - \lambda^2)^{-1}\chi$, where Δ_D is the Laplacian with Dirichlet boundary condition and $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighborhood of the obstacle K. We show that if $R_\chi(\lambda)$ has no poles for $\operatorname{Im}\lambda \geq -\delta$, $\delta > 0$, then $\|R_\chi(\lambda)\|_{L^2 \to L^2} \leq C|\lambda|^{n-2}$, $\lambda \in \mathbb{R}$, $|\lambda| \geq C_0$. This estimate implies a local energy decay. We study the spectrum of the Lax-Phillips semigroup $Z(t)$ for trapping obstacles having at least one trapped ray.

Keywords Trapping obstacles · Resonances · Local energy decay · Cut-off resolvent

Mathematics Subject Classification (2000) 35P25 · 35L05 · 47A40

1 Introduction

Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega = \mathbb{R}^n \setminus \overline{K}$. Such K is called an *obstacle* in $\mathbb{R}^n$. We consider the

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Dirichlet problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R} \times \Omega, \\ u = 0 \text{ on } \mathbb{R} \times \partial K, \\ u(0,x) = f_0(x),\, \partial_t u(0,x) = f_1(x). \end{cases} \tag{1.1}$$

Let $K \subset B_a = \{x \in \mathbb{R}^n : \, |x| \leq a\}$ and for $m \geq 0$ set

$$p_m(t) = \sup \Big[\frac{\|\nabla_x u\|_{L^2(B_a \cap \Omega)} + \|\partial_t u\|_{L^2(B_a \cap \Omega)}}{\|\nabla_x f_0\|_{H^m(B_a \cap \Omega)} + \|f_1\|_{H^m(B_1 \cap \Omega)}},$$

$$(0,0) \neq (f_0, f_1) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega),\, \mathrm{supp}\, f_i \subset B_a,\, i = 1,2\} \Big].$$

For $\mathrm{Im}\,\lambda > 0$ consider the cut-off resolvent $R_\chi(\lambda) = \chi R(\lambda) \chi : L^2(\Omega) \longrightarrow L^2(\Omega)$, where $R(\lambda) = (-\Delta_D - \lambda^2)^{-1}$, $\chi \in C_0^\infty(B_{a+1})$, $\chi = 1$ on B_a and Δ_D is the Dirichlet Laplacian with domain $D(\Delta_D) = H_0^2(\Omega)$.

The following result of Vodev generalized the classical one of Morawetz for $n \geq 3$ odd.

Theorem 1.1 **([20])** *The following conditions are equivalents:*
(a) $\lim_{t \to +\infty} p_0(t) = 0$,
(b) *There exist* $C_0 > 0$, $C_1 > 0$ *so that*

$$\|\lambda R_\chi(\lambda)\| \leq C_1, \lambda \in \mathbb{R},\, |\lambda| \geq C_0,$$

(c) *There exist constants* $C > 0$, $\gamma > 0$ *so that*

$$p_0(t) \leq \begin{cases} Ce^{-\gamma t},\, n \,\text{odd}, \\ Ct^{-n}, n \,\text{even}. \end{cases}$$

It is known that (b) holds if the obstacle K is non-trapping, that is the singularities of the solution of the Dirichlet problem with initial data with compact support leave any compact $\omega \subset \mathbb{R}^n$ for $t \geq t(\omega)$ (see for instance [4] for more details). For trapping obstacles without any condition on the geometry of K we have the following

Theorem 1.2 **([3])** *We have the estimate*

$$\|R_\chi(\lambda)\| \leq Ce^{C|\lambda|},\, \lambda \in \mathbb{R},\, |\lambda| \geq C_0$$

and for every integer $m > 1$ *we have*

$$p_m(t) \leq \frac{C_m}{(\log t)^m},\, t > 1. \tag{1.2}$$

The cut-off resolvent $R_\chi(\lambda)$ has a meromorphic continuation in $\mathbb{C}$ for n odd and in $\mathbb{C}' = \{z \in \mathbb{C} : z \neq -i\mu,\, \mu \in \mathbb{R}^+\}$ for n even ([10], [19]). There are many examples when we have a domain

$$\{z \in \mathbb{C} :\, -\delta \leq \operatorname{Im} z \leq 0\},\, \delta > 0$$

without poles (resonances) of $R_\chi(\lambda)$ (cf. for example [7]). In this talk we obtain some results showing that in this case we have a polynomial bound of the cut-off resolvent $R_\chi(\lambda)$ on $\mathbb{R}$ and a better local energy decay than (1.2). Our main result is the following

Theorem 1.3 *Assume that the cut-off resolvent $R_\chi(\lambda)$ has no poles for* $\operatorname{Im}\lambda \geq -\delta,\, \delta > 0$. *Then*

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C|\lambda|^{n-2},\, \lambda \in \mathbb{R},\, |\lambda| \geq C_0. \tag{1.3}$$

Remark 1.1 Notice that if for some $M \geq 0$ we have the estimate

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C_1|\lambda|^M,\, \operatorname{Im}\lambda \geq -\delta,\, |\operatorname{Re}\lambda| \geq C_0,$$

then a result of N. Burq [5] says that

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C_2 \frac{\log(2+|\lambda|^2)}{1+|\lambda|},\, \lambda \in \mathbb{R},\, |\lambda| \geq C_0.$$

In particular, a such estimate holds for two strictly convex disjoint obstacles and under some conditions for several strictly convex disjoint obstacles ([7]).

Remark 1.2 For the semiclassical Schrödinger operators $-h^2\Delta + V(x)$ in the case of dimension 1 a polynomial bound $\mathscr{O}(h^{-M})$ of the cut-off resolvent in

$$W = \{z \in \mathbb{C} : 0 < a_0 \leq \operatorname{Re} z \leq a_1,\, \operatorname{Im} z \geq -a_2 h, a_i > 0, i = 0,1,2\}$$

has been obtained in [2], provided that we have no resonances in a neighborhood of W. It is natural to conjecture that under the condition of Theorem 1.3, the cut-off resolvent $R_\chi(\lambda)$ is bounded uniformly on $\mathbb{R}$ for any dimension $n \geq 3$.

2 Estimates of $R_\chi(\lambda)$

It is useful to transform the problem to a semi-classical one. Setting $\lambda = \frac{\sqrt{z}}{h},\, 0 < h \leq 1$, we have

$$(-\Delta_D - \lambda^2)^{-1} = h^2(-h^2\Delta_D - z)^{-1}$$

and we will study the operator $\chi(P(h) - z)^{-1}\chi$ with $P(h) = -h^2\Delta_D,\, h > 0$, in the domain

$$\mathscr{D}_{c_1} = \{z \in \mathbb{C} :\, 0 < a_0 \leq |\operatorname{Re} z| \leq a_1,\, -c_1 h \leq \operatorname{Im} z \leq c_0,\, a_i > 0,\, c_i > 0,\, i = 0,1\}.$$

We will work in the "black box" setup ([15], [17]). For this purpose define $\mathscr{H}_a = L^2(\Omega \cap B_a)$ and set

$$\mathscr{L} = \mathscr{H}_a \oplus L^2(\mathbb{R}^n \setminus B_a).$$

We consider $P(h)$ as an operator $P(h) : \mathscr{L} \longrightarrow \mathscr{L}$ with domain $\mathscr{D}(P) \subset \mathscr{L}$ and the hypothesis in [15], [17] for a "black box" framework are satisfied. In particular, setting

$$\mathscr{H}^\sharp = \mathscr{H}_a \oplus L^2(\mathbb{T}_a^n \setminus B_1),\ \mathbb{T}_a^n = \mathbb{R}^n/(a\mathbb{Z}^n),$$

we introduce $P^\sharp(h)$ by replacing $-h^2\Delta_D$ by $-h^2\Delta_{\mathbb{T}_a^n}$. The operator $P^\sharp(h)$ has a discrete spectrum and we denote by $N(P^\sharp(h),\lambda)$ the number of eigenvalues of $P^\sharp(h)$ in $[-\lambda,\lambda]$. Then we we have

$$N(P^\sharp(h),\lambda) = \mathscr{O}\Big(\Big(\frac{\lambda}{h^2}\Big)^{n/2}\Big), \text{ for } \lambda \geq 1.$$

This follows from the Weyl asymptotic for the counting function for eigenvalues of $P^\sharp(h)$. In the following for simplicity we will write P instead of $P(h)$.

We will examine the resolvent of the complex dilated operator $P_\theta(h)$ defined as follows. Introduce a function $f_\theta(t) : \mathbb{R}^+ \longrightarrow \mathbb{C}$ having the properties:

$$f_\theta(t) = t \text{ for } t \leq a+1,$$

$$f_\theta(t) = e^{i\theta}t,\ t \gg 1,$$
$$0 \leq \arg f_\theta(t) \leq \theta,\ \partial_t f_\theta(t) \neq 0,$$
$$\arg f_\theta(t) \leq \arg \partial_t f_\theta(t) \leq \arg f_\theta(t) + \varepsilon$$

with small $\varepsilon > 0$. Let $\mu_\theta(t\omega) = f_\theta(t)\omega$, $t = |x| \in \mathbb{R}^+$, $\omega \in S^{n-1}$ and set $\Gamma_\theta = \mu_\theta(\mathbb{R}^n)$. Let $\Psi \in C_0^\infty(B_{a+1})$ be equal to 1 near B_a. As in [15], [16], we introduce the dilated operator P_θ

$$P_\theta u = P(\Psi u) - \Delta_{\Gamma_\theta}(1-\Psi)u$$

with domain

$$D_\theta = \{u \in L^2(\Gamma_\theta) : \Psi u \in D(P),\ (1-\Psi)u \in H^2(\Gamma_\theta)\},$$

$D(P)$ being the domain of P. Here $-\Delta_{\Gamma_\theta}$ is the dilated Laplacian corresponding to the change $\mathbb{R}^n \ni x \longrightarrow f_\theta(t)\omega \in \mathbb{C}^n$ and we refer to [15], [16], [17] for more details. Next set $\theta = c_1 h$ so that in the domain

$$\Omega_\theta = \{z \in \mathbb{C} : |z-\omega| \leq \theta,\ -\theta \leq \operatorname{Im} z \leq a_2\theta\} \subset \mathscr{D}_{c_1},\ a_2 \gg 1$$

there are no eigenvalues of P_θ. Note that the eigenvalues of P_θ coincide with their multiplicities with the resonances of P ([15], [16], [17]). From [8], [11] the counting function of the eigenvalues of $P^\#(h)$ satisfies

$$N(P^\#, [\lambda - h, \lambda + h]) = \mathscr{O}(h^{1-n}),$$

for $\lambda \in [a_0, a_1]$. Then, following a construction, given by one of the authors [1], we may find a finite rank operator L so that

$$(P_\theta + \theta L - z)^{-1} = \mathscr{O}(\theta^{-1}),\ \forall z \in \Omega_\theta, \tag{2.1}$$

$$(P_\theta - z)^{-1} = \mathscr{O}(\theta^{-1}),\ z \in \Omega_\theta^+ = \Omega_\theta \cap \{\operatorname{Im} z \geq \varepsilon\theta\},\ 0 < \varepsilon < a_2, \tag{2.2}$$

$$\|L\|_{D(P_\theta)\to L^2(\Gamma_\theta)} = \mathscr{O}(1),\ \operatorname{rank} L = \kappa = C_0 h^{-n+1}$$

with a constant $C_0 > 0$ independent on h. This construction generalises that of Sjöstrand [16] with a finite rank operator K_0, rank $K_0 = Ch^{-n}$. Now consider the Grushin problem

$$\begin{cases} (P_\theta - z)u + R_-(z)u_- = v, \\ R_+(z)u = v_+, \end{cases} \tag{2.3}$$

where $u \in D(P_\theta), v \in L^2(\Gamma_\theta)$, while $u_-, v_+ \in \mathbb{C}^\kappa$. Given an orthonormal basis $(e_1, ..., e_\kappa)$ in Image L^*, the operators $R_\pm$ have the form

$$R_+u = (u, e_j)_{j=1,...,\kappa},$$

$$R_-u_- = \sum_{j=1}^{\kappa} u_{-,j}(P_\theta + \theta L - z)e_j,\ u_- = (u_{-,1}, ..., u_{-,\kappa}),$$

where $(,)$ is the scalar product in $L^2(\Gamma_\theta)$.

Following the results in Section 6, [16], the problem (2.3) is invertible and the inverse operator is given by

$$\begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-,+}(z) \end{pmatrix}.$$

To estimate the operators $E(z)$, $E_-(z)$, $E_+(z)$, $E_{-,+}(z)$, consider an orthonormal basis

$$(e_{\kappa+1}, ..., e_m, ...)$$

in $\left(\text{Image}\, L^*\right)^\perp = \operatorname{Ker} L$. Let

$$u = \sum_{j=1}^{\kappa} u_j e_j + \sum_{j=\kappa+1}^{\infty} u_j e_j = u' + u''.$$

From (2.3) we get

$$(P_\theta - z)(u' + u'') + (P_\theta + \theta L - z)(\sum_{j=1}^{\kappa} u_{-,j} e_j) = v,$$

$$(u, e_j) = u_j = v_{+,j},\ j = 1, ..., \kappa.$$

This implies

$$(P_\theta + \theta L - z)(u'' + \sum_{j=1}^{\kappa} u_{-,j} e_j) = v - (P_\theta - z)u'$$

$$= v - (P_\theta - z) \sum_{j=1}^{\kappa} u_j e_j,$$

hence

$$\Big(u'' + \sum_{j=1}^{\kappa} u_{-,j} e_j\Big) = \Big(P_\theta + \theta L - z\Big)^{-1} v$$

$$-\Big(P_\theta + \theta L - z\Big)^{-1} (P_\theta - z) \sum_{j=1}^{\kappa} v_{+,j} e_j = A + B.$$

The estimate (2.1) leads to $A = \mathscr{O}(\theta^{-1})\|v\|$. Using once more (2.1), we get

$$B = \Big(-I + (P_\theta + \theta L - z)^{-1} \theta L\Big) \sum_{j=1}^{\kappa} v_{+,j} e_j = \mathscr{O}(1)\|v_+\|_{\mathbb{C}^\kappa}.$$

Thus

$$\|u''\| + \|u_-\|_{\mathbb{C}^\kappa} = \mathscr{O}(\theta^{-1})\|v\| + \mathscr{O}(1)\|v_+\|_{\mathbb{C}^\kappa}$$

and $\|u'\|_{\mathbb{C}^\kappa} = \mathscr{O}(1)\|v_+\|_{\mathbb{C}^\kappa}$. Consequently,

$$\|u\| + \|u\|_{\mathbb{C}^\kappa} = \mathscr{O}(\theta^{-1})\|v\| + \mathscr{O}(1)\|v_+\|_{\mathbb{C}^\kappa}$$

and we get the estimates

$$\|E(z)\| = \mathscr{O}(\theta^{-1}),\ \|E_-(z)\| = \mathscr{O}(\theta^{-1}),$$

$$\|E_+(z)\| = \mathscr{O}(1),\ \|E_{-,+}(z)\| = \mathscr{O}(1),$$

where $E_{-,+}(z) : \mathbb{C}^\kappa \longrightarrow \mathbb{C}^\kappa$. Moreover, the resolvent $(P_\theta - z)^{-1}$ and $(E_{-,+}(z))^{-1}$ are related by the equality (see for instance, [16])

$$(P_\theta - z)^{-1} = E(z) - E_+(z)((E_{-,+}(z))^{-1} E_-(z)$$

and the above estimates yield

$$\begin{aligned}\|(P_\theta - z)^{-1}\| \leq & \|E(z)\| + \|E_+(z)\| \|(E_{-,+}(z))^{-1}\| \|E_-(z)\| \\ = & \mathscr{O}(\theta^{-1})(1 + \|(E_{-,+}(z))^{-1}\|).\end{aligned}$$

Obviously,

$$(E_{-,+})^{-1} = \frac{{}^t\text{comatrix}(E_{-,+})}{\det(E_{-,+})} = \frac{\mathscr{O}(e^{C\kappa})}{\det(E_{-,+})}$$

and the problem is reduced to obtain a lower bound of $D(z) = \det(E_{-,+})$.

Since $\|E_{-,+}(z)\|_{\mathbb{C}^\kappa \to \mathbb{C}^\kappa} = \mathscr{O}(1)$, we have $|D(z)| \leq e^{C\kappa}$, $z \in \Omega_\theta$. On the other hand, in Ω_θ^+ we get

$$(E_{-,+}(z))^{-1} u_+ = -R_+(z)(P_\theta - z)^{-1} R_-(z) u_+$$

$$= -R_+(z)\Big(I + (P_\theta - z)^{-1} \theta L\Big) \sum_{j=1}^{\kappa} u_{-,j} e_j = \mathscr{O}(1)$$

and the estimate (2.2) for $z \in \Omega_\theta^+$ yields

$$|D(z)| = |\det(E_{-,+}(z))| = \Big|\frac{1}{\det((E_{-,+}(z))^{-1})}\Big| \geq e^{-C_1\kappa},\ z \in \Omega_\theta^+.$$

Recall that P_θ has no eigenvalues in Ω_θ, hence $D(z)$ has no zeros in Ω_θ. This makes possible to introduce the positive harmonic function $G(z) = C\kappa - \log|D(z)| \geq 0$, $z \in \Omega_\theta$. We have in Ω_θ^+ the estimate $\log|D(z)| \geq -C_1\kappa$, so we can apply the Harnack inequality for positive harmonic functions. In fact, for every $M \subset\subset \Omega_\theta$ we have

$$\sup_{z\in M} G(z) \leq C_M \inf_{z\in M} G(z) \leq C_M \inf_{z\in M\cap\Omega_\theta^+} G(z).$$

Making a small decrease of Ω_θ, which means to replace c_1 by a constant $0 < c_2 < c_1$, we deduce

$$G(z) \leq C_2\kappa,\ \log|D(z)| \geq -C_3\kappa,\ z \in \Omega_\theta,\ \theta = c_2 h.$$

Next suppose that Ω_θ is defined by c_2 instead of c_1. Combining the above estimates with the fact that $\kappa = C_0 h^{-n+1}$, we conclude that

$$\|(P_\theta - z)^{-1}\| \leq C_5 e^{C_4 h^{-n+1}},\ z \in \Omega_\theta.$$

Moreover, the same estimate is uniform with respect to choice of ω in Ω_θ, provided ω runs over a compact interval in $\mathbb{R}^+$ so that P_θ has no eigenvalues in Ω_θ. Thus we obtain

$$\|(P_\theta - z)^{-1}\| \leq C_5 e^{C_5 h^{-n+1}},\ z \in \mathscr{D}_{c_2}.$$

The complex scaling was chosen so that $f_\theta(t) = 1$ for $t \leq a+1$. Since $\operatorname{supp}\chi \subset B_{a+1}$, it is easy to see that

$$\chi(P-z)^{-1}\chi = \chi(P_\theta - z)^{-1}\chi,$$

hence

$$\|\chi(-h^2\Delta - z)^{-1}\chi\|_{L^2(\Omega)\longrightarrow L^2(\Omega)} \leq C_6 e^{C_6 h^{-n+1}},\ z \in \mathscr{D}_{c_2}.$$

Taking into account the scaling $\lambda = \frac{\sqrt{z}}{h}$, for $z \in \mathscr{D}_{c_2}$ we get

$$\operatorname{Re} z = h^2(\operatorname{Re}^2\lambda - \operatorname{Im}^2\lambda) \geq a_0,\ \operatorname{Im} z = 2h^2 \operatorname{Re}\lambda \operatorname{Im}\lambda \geq -c_2 h$$

which imply

$$\operatorname{Re}\lambda \geq \frac{a_0}{h} \geq a_0 > 0,\ \operatorname{Im}\lambda \geq -\frac{c_2}{2\sqrt{a_0}} = -a_2.$$

Consequently, we obtain

$$\|R_\chi(\lambda)\| \leq C_7 e^{C_7|\lambda|^{n-1}},\ \operatorname{Re}\lambda \geq a_0,\ \operatorname{Im}\lambda \geq -a_2. \tag{2.4}$$

In the same way we treat the domain $\operatorname{Re}\lambda \leq -a_0$, $\operatorname{Im}\lambda \geq -a_2$ and we get (2.4) for $|\operatorname{Re}\lambda| \geq a_0 > 0$.

3 Estimates on the real axis and decay of local energy

Proposition 3.1 *Let $f(z)$ be a holomorphic function in*

$$U_\alpha = \{z \in \mathbb{C} :\ \mathrm{Im}\, z \geq -\alpha\},\ \alpha > 0,$$

such that

$$|f(z)| \leq C_0 e^{C|z|^m},\ z \in U_\alpha,\ m \geq 1,$$

$$|f(z)| \leq \frac{C_1}{|z|\,\mathrm{Im}\, z},\ \mathrm{Im}\, z > 0.$$

Then we have $|f(z)| \leq C_2(1+|z|)^{m-1}$, $z \in \mathbb{R}$.

Proof Introduce the function $g(z) = e^{-iAz^{m+1}} f(z)$, where $A > 0$ is sufficiently large. Consider the domain bounded by the curves:

$$\gamma_+ = \{z \in \mathbb{C} :\ \mathrm{Im}\, z = \frac{1}{|z|^m},\ \mathrm{Re}\, z \geq 1\},$$

$$\gamma_- = \{z \in \mathbb{C} :\ \mathrm{Im}\, z = -\alpha,\ \mathrm{Re}\, z \geq 1\},$$

$$\gamma_0 = \{z \in \mathbb{C} :\ -\alpha \leq \mathrm{Im}\, z \leq \frac{1}{|z|^m},\ \mathrm{Re}\, z = 1\}.$$

For $z \in \gamma_-$ and $\mathrm{Re}\, z \gg 1$ we have

$$|g(z)| \leq C_0 e^{C'(\mathrm{Re}\, z)^m} \exp\Big(A\frac{(m+1)}{2}(\mathrm{Re}\, z)^m\, \mathrm{Im}\, z\Big) \leq C_3$$

taking $2C' - A(m+1)\alpha < 0$. On the curve γ_+ we obtain

$$|g(z)| \leq C_4 |z|^{m-1} \exp\Big((m+1)A(\mathrm{Re}\, z)^m\, \mathrm{Im}\, z\Big[1 + \mathscr{O}\Big(\frac{1}{|\mathrm{Re}\, z|}\Big)\Big]\Big)$$

$$\leq C_4 |z|^{m-1} \exp\Big(\frac{B(\mathrm{Re}\, z)^m}{|z|^m}(m+1)\Big) \leq C_5 |z|^{m-1}.$$

To obtain the estimate, we apply the Pragmen-Lindelöf theorem for the function $g(z)$ and deduce

$$|g(z)| \leq C_6 |z|^{m-1}$$

for

$$\mathrm{Re}\, z \geq 1,\ -\alpha \leq \mathrm{Im}\, z \leq \frac{1}{|z|^m}.$$

In particular, for $z \in \mathbb{R}$, $z \geq 1$ we get

$$|f(z)| \leq C_6 |z|^{m-1}.$$

In a similar way we treat the case $z \leq -1$. □

To apply Proposition 3.1, notice that the operator $-\Delta_D$ with Dirichlet boundary condition on $\partial\Omega$ is a self-adjoint positive operator and it is easy to see that

$$\|R(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{C}{|z|\,\mathrm{Im}\,z},\ \mathrm{Im}\,z > 0.$$

Combining the estimate (2.4) and Proposition 3.1 with $m = n-1$, we obtain (1.3) and the proof of Theorem 1.3 is complete.

Theorem 1.3 makes possible to apply a result of G.Popov and G. Vodev (see Proposition 1.4 in [13]) in order to obtain the following

Theorem 3.1 *Under the hypothesis of Theorem* 1.3 *for every* $m > 0$ *and* $t > 1$ *we have for n odd the estimate*

$$p_m(t) \leq C(t^{-1}\log t)^{m/(n-1)},$$

while for n even and $t > 1$ *we have*

$$p_m(t) \leq \begin{cases} C(t^{-1}\log t)^{m/(n-1)}, \text{ for } 0 < m \leq n(n-1), \\ Ct^{-n} \text{ for } m > n(n-1). \end{cases}$$

The factor $m/(n-1)$ comes from the estimate of the resolvent of the generator G of the unitary group $U(t) = e^{itG}$ related to the problem (1.1). More precisely, we have

$$G = -i\begin{pmatrix} 0 & Id \\ \Delta_D & 0 \end{pmatrix}$$

with domain

$$D(G) = \{(u,v) : u \in H_0^2(\Omega),\ v \in H_D(\Omega)\} \subset \mathscr{H},$$

where $\mathscr{H} = \{(u,v) : u \in H_D(\Omega),\ v \in L^2(\Omega)\}$ and $H_D(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\|_D^2 = \int_\Omega |\nabla\varphi|^2 dx.$$

For the resolvent $(G-\lambda)^{-1}$ we have the representation

$$(G-\lambda)^{-1} = \begin{pmatrix} \lambda R(\lambda) & -iR(\lambda) \\ -i\Delta_D R(\lambda) & \lambda R(\lambda) \end{pmatrix}. \tag{3.1}$$

Therefore (1.3) implies the estimates (see [20], [5])

$$\|\lambda R_\chi(\lambda)\|_{H_D\to H_D} \leq C|\lambda|^{n-1},\ \|\chi\Delta_D R(\lambda)\chi\|_{H_D\to L^2} \leq C|\lambda|^{n-1},$$

$$\|R_\chi(\lambda)\|_{L^2\to H_D} \leq C|\lambda|^{n-1},\ \lambda \in \mathbb{R}, |\lambda| \geq C_0$$

and we obtain

$$\|\chi(G-\lambda)^{-1}\chi\|_{\mathscr{H}\to\mathscr{H}} \leq C|\lambda|^{n-1},\ \lambda \in \mathbb{R},\ |\lambda| \geq C_0.$$

4 Spectre of the Lax-Phillips semigroup $Z(t)$

In this section we assume $n \geq 3$, n, odd and we examine the spectrum of the Lax-Phillips semigroup $Z^b(t) = P^b_+ U(t) P^b_-$, $t \geq 0$, where $U(t)$ is the unitary group introduced in Section 3 and $P^b_\pm$ are the orthogonal projections on the orthogonal complements of the spaces

$$D^b_\pm = \{f \in \mathscr{H} : U_0(t) f = 0, |x| < \pm t + b\}, b > a.$$

Here $U_0(t)$ is the unitary group related to the Cauchy problem for the wave equation in $\mathbb{R}_t \times \mathbb{R}^n$ (see [10]). We choose $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi = 1$ for $|x| \leq a$, $\chi = 0$ for $|x| \geq b$. We fix $b > a$ with this property and note that $P^b_\pm \chi = \chi = \chi P^b_\pm$ and for simplicity we will write $Z(t)$ instead of $Z^b(t)$. Let B be the generator of $Z(t)$. Therefore,

$$\sigma(B) \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

and the eigenvalues z_j of iB coincide with their multiplicities with the poles of $R_\chi(\lambda)$ (see [10]). The condition

$$\sup_{\lambda \in \mathbb{R}} \|\lambda R_\chi(\lambda)\|_{L^2(\Omega) \to L^2(\Omega)} = +\infty \tag{4.1}$$

implies

$$\sup_{\lambda \in \mathbb{R}} \|(B + i\lambda)^{-1}\|_{\mathscr{H} \to \mathscr{H}} = +\infty. \tag{4.2}$$

In fact, for $\operatorname{Re} \lambda > 0$ we have

$$\chi(iG - \lambda)^{-1} \chi = -\int_0^\infty e^{-\lambda t} \chi e^{itG} \chi dt$$

$$= -\int_0^\infty e^{-\lambda t} \chi Z(t) \chi dt = \chi (B - \lambda)^{-1} \chi$$

and by analytic continuation for $\operatorname{Re} \lambda \geq 0$ we obtain

$$\chi(iG + i\lambda)^{-1} \chi, = \chi (B + i\lambda)^{-1} \chi, \forall \lambda \in \mathbb{R}$$

and we may exploit the representation (3.1). On the other hand, (4.1) means that the condition (b) of Theorem 1.1 is not satisfied, so we have not an uniform decay of the local energy. This holds for obstacles having at least one generalized non-degenerate trapping ray (see [14] and [12] for more details).

In the following we assume the condition (4.1) satisfied. Suppose that there are only finite number of resonances in the domain

$$\{z \in \mathbb{C} : \operatorname{Im} z \geq -\delta\},\ \delta > 0.$$

Choose $0 \leq \alpha \leq \delta$ so that we have no resonances on the line $\{z \in \mathbb{C} : \operatorname{Im} z = -\alpha\}$, hence the resolvent $(B + \alpha + i\lambda)^{-1}$ exists for every $\lambda \in \mathbb{R}$. It is easy to see that

$$\sup_{\lambda \in \mathbb{R}} \|(B + \alpha + i\lambda)^{-1}\|_{\mathscr{H} \to \mathscr{H}} = +\infty. \tag{4.3}$$

Indeed, if the resolvent $(B+\alpha+i\lambda)^{-1}$ is uniformly bounded with respect to $\lambda \in \mathbb{R}$, the cut-off resolvent $\|\lambda R_\chi(-i\alpha+\lambda)\|_{L^2\to L^2}$ will be also bounded uniformly with respect to $\lambda \in \mathbb{R}$. Consider the domain

$$\{z\in\mathbb{C} : -\alpha \leq \operatorname{Im} z \leq c_0,\, |\operatorname{Re} z| \geq c_1,\, c_i > 0,\, i=0,1\}$$

with sufficiently large c_1. For all z in this domain we have an estimate (see for example [18])

$$\|zR_\chi(z)\|_{L^2\to L^2} \leq Ce^{C|z|^n}$$

and an application of the Pragmen-Lindelöf theorem leads to a contradiction with (4.1). Next, assume that

$$e^{-\alpha-i\beta} \notin \sigma(e^B),\, \forall\beta\in\mathbb{R}.$$

Then $\|(e^{-\alpha-i\beta}-e^B)^{-1}\| \leq C_\alpha,\, \forall\beta\in\mathbb{R}$ and from the equality

$$I-e^{B+\alpha+i\beta} = -(B+\alpha+i\beta)\int_0^1 e^{t(B+\alpha+i\beta)}dt$$

we deduce

$$(B+\alpha+i\beta)^{-1} = -\int_0^1 e^{t(B+\alpha+i\beta)}dt\,(I-e^{B+\alpha+i\beta})^{-1}.$$

Consequently, the resolvent $(B+\alpha+i\beta)^{-1}$ is uniformly bounded with respect to $\beta\in\mathbb{R}$ and we obtain a contradiction with (4.3). This shows that there exists $\beta_0\in\mathbb{R}$ so that

$$e^{-\alpha-i\beta_0} \in \sigma(e^B)\setminus e^{\sigma(B)}.$$

Now we are in position to apply the result in [9] saying that there exists a set $\mathscr{M}_\alpha \subset \mathbb{R}^+$ with Lebesgue measure zero so that for all $t\in]0,\infty[\setminus\mathscr{M}_\alpha$ we have

$$e^{t(-\alpha-i\beta_0)}e^{i\omega} \in \sigma(Z(t)) : \forall\omega\in\mathbb{R},$$

hence

$$e^{-\alpha t+i\omega} \in \sigma(Z(t)),\, \forall\omega\in\mathbb{R}.$$

Assume that for $\frac{p_n}{q_n}\in\mathbb{Q},\, 0<\frac{p_n}{q_n}\leq\delta$ we have no resonances on the line

$$\{z\in\mathbb{C} :\, \operatorname{Im} z = -\frac{p_n}{q_n}\}.$$

The above argument implies the existence of a set $\mathscr{M}_n\subset\mathbb{R}^+$ with Lebesgue measure zero such that for $t\in]0,\infty[\setminus\mathscr{M}_n$ we have

$$e^{-t\frac{p_n}{q_n}+i\omega} \in \sigma(Z(t)).$$

The rationals are dense in $]0,\delta[$ and the spectrum $\sigma(Z(t))$ is closed. Thus for $t\in]0,\infty[\setminus\Big(\cup_{n\in\mathbb{N}}\mathscr{M}_n\Big)$ we get the relation

$$\{z=e^{-ty+i\omega}\in\sigma(Z(t)) :\, 0\leq y\leq\delta,\, \omega\in\mathbb{R}\}.$$

Finally, we have the following

Theorem 4.1 *Suppose that we have a finite number of resonances z with* $\operatorname{Im} z \geq -\delta$, $\delta > 0$. *If the condition* (4.1) *holds, there exists a set $\mathscr{R} \subset \mathbb{R}^+$ with Lebesgue measure zero so that for all $t \in]0, \infty[\setminus \mathscr{R}$ we have*

$$\{z \in \mathbb{C} : e^{-t\delta} \leq |z| \leq 1\} \subset \sigma(Z(t)).$$

Next we will examine the singularities of the cut-off resolvent $\chi(U(t)-z)^{-1}\chi$ for $z \to z_0 \in \mathbb{S}^1$, $|z| > 1$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\psi(x) = 1$ for $|x| \leq a+1$, $\psi(x) = 0$ for $|x| \geq a+2$. Introduce the operator

$$L_\psi(g,h) = \Big(0, \langle \nabla_x \psi, \nabla_x g\rangle + (\Delta \psi) g\Big).$$

In particular, we define $L_\psi(U(t)f)$ and $L_\psi(U_0(t)f)$ and will write simply $L_\psi U(t)$ and $L_\psi U_0(t)$. It is easy to see that we have the following equalities:

$$(1-\psi)U(t) = U_0(t)(1-\psi) + \int_0^t U_0(t) L_\psi U(t-s) ds,$$

$$U(t)(1-\psi) = (1-\psi)U_0(t) + \int_0^t U(t-s) L_\psi U_0(s) ds.$$

Applying these equalities, we get

$$U(t) = U(t)\psi + (1-\psi)U_0(t) + \int_0^t \psi U(t-s) L_\psi U_0(s) ds$$

$$+ \int_0^t U_0(t-s)(1-\psi) L_\psi U_0(s) ds + \int_0^t \int_0^{t-s} U_0(\tau) L_\psi U(t-s-\tau) L_\psi U_0(s) ds d\tau$$

$$= \psi U(t)\psi + U_0(t)\psi(1-\psi) + (1-\psi)U_0(t) + \int_0^t \psi U(t-s) L_\psi U_0(s) ds$$

$$+ \int_0^t U_0(s) L_\psi U(t-s)\psi ds + \int_0^t U_0(t-s)(1-\psi) L_\psi U_0(s) ds$$

$$+ \int_0^t \int_0^{t-s} U_0(\tau) L_\psi U(t-s-\tau) L_\psi U_0(s) ds d\tau.$$

Now let $z \in \mathbb{C}$ be such that $|z| > 1$. Let $g \in C_0^\infty(B_{a+2})$ be a cut-off function equal to 1 on B_{a+1}. We choose the projectors $P_\pm^b = P_\pm$ so that

$$P_\pm \psi = \psi = \psi P_\pm, \; P_\pm g = g = g P_\pm.$$

Next we fix $b > 0$ and the projectors $P_\pm$ with these properties and note that $g L_\psi = L_\psi = L_\psi g$. Let $T_0 > 0$ be chosen so that $P_+ U_0(t) P_- = 0$ for $t \geq T_0$. Given a $t > 0$, we have

$$(Z(t)-z)^{-1} = -\sum_{j=0}^\infty z^{-j-1} P_+ U(jt) P_-$$

$$= P_+ \psi (U(t)-z)^{-1} \psi P_- - \sum_{jt \leq T_0} z^{-j-1} P_+ U_0(jt) \psi (1-\psi) P_-$$

$$- \sum_{jt \leq T_0} z^{-j-1} P_+ (1-\psi) U_0(jt) P_-$$

$$-\int_0^{T_0} P_+U_0(s)L_\psi(U(t)-z)^{-1}\Phi U(-s)\psi P_- ds$$

$$-\int_0^{T_0} P_+\psi(U(t)-z)^{-1}\Phi U(-s)L_\psi U_0(s)P_- ds$$

$$-\sum_{jt\leq T_1}\int_0^{T_0} z^{-j-1}P_+U_0(jt)\Phi U_0(-s)(1-\psi)L_\psi U_0(s)P_- ds$$

$$-\int_0^{T_0}\int_0^{T_0} P_+U_0(\tau)L_\psi(U(t)-z)^{-1}\Phi_1 U(-s-\tau)L_\psi U_0(s)P_- dsd\tau+G(z)$$

with a function $G(z)$ holomorphic for $z\neq 0$. Here Φ and Φ_1 are cut-off functions with compact support determined by the finite speed of propagation so that

$$(1-\Phi)U_0(-s)g=0 \text{ for } 0\leq s\leq T_0,$$

$$(1-\Phi_1)U(-t)g=0 \text{ for } 0\leq t\leq 2T_0.$$

Finally, $T_1>0$ is chosen so that $P_+U(t)\Phi=0$ for $t\geq T_1$. The terms in the above presentation of $(Z(t)-z)^{-1}$ given by finite sums are holomorphic functions with respect to z. Consequently, if

$$\lim_{z\to z_0,|z|>1}\|\Psi(U(t)-z)^{-1}\Psi\|<\infty$$

for $\Psi\in C_0^\infty(x\in\mathbb{R}^n:\ |x|\leq c+1)$ and equal to 1 for $|x|\leq c$ for some suitably large and fixed constant $c>0$, we conclude that $(Z(t)-z)^{-1}$ is not singular at $z_0\in\mathbb{S}^1$. Combining this argument with the fact under the condition (4.1) we have $\mathbb{S}^1\subset\sigma(Z(t))$ for almost $t>0$, we obtain the following

Theorem 4.2 *Assume the condition* (4.1) *fulfilled. Then for almost all* $t\in]0,\infty[$ *and all* $z_0\in\mathbb{S}^1$ *we have*

$$\lim_{z\to z_0,|z|>1}\|\Psi(U(t)-z)^{-1}\Psi\|=+\infty.$$

This result is important for the analysis of the analytic continuation of the cut-off resolvent $U_\chi(z)=\chi(U(T,0)-z)^{-1}\chi$ of the monodromy operator $U(T,0)$ related to the propagator $U(t,s)$ for time-periodic perturbations of the wave equation. In particular,we conclude that for trapping periodically moving obstacles we have not a meromorphic continuation of $U_\chi(z)$ from $\{z\in\mathbb{C}:|z|\geq A\gg 1\}$ across the unit circle $\mathbb{S}$.

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