Filomat 29:7 (2015), 1671–1680
DOI 10.2298/FIL1507671M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Resolvent Operator and Spectrum of New Type Boundary Value Problems

Oktay Sh. Mukhtarov[a,b], Hayati Olğar[a], Kadriye Aydemir[a]

[a] *Department of Mathematics, Faculty of Science-Art, Gaziosmanpaşa University, 60250 Tokat, Turkey*
[b] *Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan*

Abstract. The aim of this study is to investigate a new type boundary value problems which consist of the equation $-y''(x) + (\mathcal{B}y)(x) = \lambda y(x)$ on two disjoint intervals $(-1, 0)$ and $(0, 1)$ together with transmission conditions at the point of interaction $x = 0$ and with eigenparameter dependent boundary conditions, where $\mathcal{B}$ is an abstract linear operator, unbounded in general, in the direct sum of Lebesgue spaces $L_2(-1, 0) \oplus L_2(0, 1)$. By suggesting an own approaches we introduce modified Hilbert space and linear operator in it such a way that the considered problem can be interpreted as an eigenvalue problem of this operator. We establish such properties as isomorphism and coerciveness with respect to spectral parameter, maximal decreasing of the resolvent operator and discreteness of the spectrum. Further we examine asymptotic behaviour of the eigenvalues.

1. Introduction

In this study we shall investigate a new type boundary value problems, which consist of the equation

$$Ly := -y''(x) + (\mathcal{B}y)(x) = \lambda y(x) \tag{1}$$

on $(-1, 0) \cup (0, 1)$, together with boundary condition at $x = -1$

$$\Gamma_1(y) := \cos\alpha y(-1) + \sin\alpha y^{'}(-1) = 0, \quad \alpha \in [0, \pi) \tag{2}$$

transmission conditions at the point of interaction $x = 0$

$$\Gamma_2(y) := y(+0) - \beta_1 y(-0) - \gamma_1 y'(-0) = 0, \tag{3}$$

$$\Gamma_3(y) := y'(+0) - \beta_2 y(-0) - \gamma_2 y'(-0) = 0, \tag{4}$$

and eigenparameter dependent boundary condition at $x = 1$

$$\Gamma_4(y) := \lambda y(1) - y'(1) = 0, \tag{5}$$

2010 *Mathematics Subject Classification*. Primary 34L10 ; Secondary 34L20
Keywords. Boundary-value problems, transmission conditions, eigenvalues, coerciveness, spectrum, resolvent.
Received: 25 November 2013; Accepted: 07 July 2015
Communicated by Dragan S. Djordjević
Email addresses: omukhtarov@yahoo.com (Oktay Sh. Mukhtarov), hayati.olgar@gop.edu.tr (Hayati Olğar), kadriye.aydemir@gop.edu.tr (Kadriye Aydemir)

where $\mathcal{B}$ is an abstract linear operator in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$, λ is a complex spectral parameter and the coefficients $\beta_i, \gamma_i, (i = 1, 2)$ are real numbers.

We want to emphasize that the boundary value problem studied here is new, since it contains an non-differential term, namely abstract linear operator $\mathcal{B}$ in the equation. Furthermore, eigenvalue parameter appears not only in the equation but also in one of boundary conditions and at the point of interaction $x = 0$ are given two transmission conditions.

Note that the results of this study can be applied to the wide variety of boundary-value-transmission problems of the form (1)-(5). To illustrate this fact let us give some examples of perturbed operators $\mathcal{B}$ which are unbounded in $L_2(-1,0) \oplus L_2(0,1)$ and satisfy the conditions Theorems 2.3, 3.1, 3.2, 3.3, 4.8 and 4.9 of this study

1.
$$\mathcal{B}y := \sum_{i=1}^{n} q_{i0}(x)y(a_{i0}) + \sum_{j=1}^{m} q_{j1}(x)y'(a_{j1}),$$
where the functions $q_{ij}(x)$ are continuous in each intervals $[-1,0)$ and $(0,1]$ and has a finite one-hand limits $q_{ij}(\pm 0) \neq 0$.

2.
$$\mathcal{B}y := \int_{-1}^{0}(K_{10}(x,s)y(s) + K_{11}(x,s)y'(s))ds + \int_{0}^{1}(K_{20}(x,s)y(s) + K_{21}(x,s)y'(s))ds,$$
where the functions $K_{1j}(x,s) \neq 0$ and $K_{2j}(x,s) \neq 0$ are continuous in $[-1,1] \times [-1,0]$ and $[-1,1] \times [0,1]$, respectively. Clearly, these operators are unbounded in $L_2(-1,0) \oplus L_2(0,1)$, but are compact from $W_2^2(-1,0) \oplus W_2^2(0,1)$ to $L_2(-1,0) \oplus L_2(0,1)$.

Usually, the eigenvalue parameter appears linearly only in the differential equation of the classic Sturm-Liouville problems. However, in solving of many significant physics problems the eigenvalue parameter appear also in the boundary conditions. There is quite substantial literature on such type problems. Here we mention the results of [6–8, 11, 13, 14, 17–19, 21] and references cited therein. Walter [14] gave an operator-theoretic formulation of such problems. Fulton [6, 7] employed the residue calculus in a manner similar to Titchmarsh [9] to give a direct proof of the convergence properties of the eigenfunction expansion. He also presented an asymptotic representation for the eigenvalues and eigenfunctions and considered the case of an infinite and a semi-infinite interval. Hinton [8] obtained a uniform convergence theory for a larger class of functions. He gave a precise description of the class where uniform convergence takes place.

In recent years there has been increasing interest of such type problems but under supplementary so-called transmission conditions(see, for example [17, 18, 21]).

Recently, Ekin Uğurlu and Elgiz Bairamov [10] have investigated a singular dissipative boundary value problem with transmission conditions. A. Boumenir [1] use sampling techniques to reconstruct the characteristic function associated with the eigenvalues of two linked Sturm-Liouville operators by a transmission condition. J. Ao et al. [13] have considered the finite spectrum of Sturm-Liouville problems with transmission conditions and eigenparameter-dependent boundary conditions. E.Şen and A. Bayramov [11] studied a discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the boundary condition and with transmission conditions at the point of discontinuity. B. Chanane [5] computed the eigenvalues of Sturm-Liouville problems with discontinuity conditions inside a finite interval by using the regularized sampling method.

Such properties as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel bases of a system of root functions, distributions of eigenvalues of some discontinuous boundary value problems with transmission conditions and its applications to the corresponding initial boundary value problems for parabolic equations have been investigated in [16–18, 21]. The various physics applications of this kind of problems arise in heat and mass transfer problems [3], in vibrating string problems when the string loaded additionally with point masses [2], in diffraction problems [15], etc.

2. Isomorphism and Coerciveness of the Problem in Modified Hilbert Spaces

For operator-theoretic formulation we shall assume that $\theta := \beta_1\gamma_2 - \beta_2\gamma_1 > 0$ and in the Hilbert space $(L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}$ introduce a new inner product by

$$< F, G >_{\Xi^0_{2,\theta}} = \int_{-1}^{0} y(x)\overline{z(x)}dx + \frac{1}{\theta}\left\{\int_0^1 y(x)\overline{z(x)}dx + y_1\overline{z_1}\right\} \tag{6}$$

for $F := \begin{pmatrix} y(x)\,, & y_1 \end{pmatrix}, G =: \begin{pmatrix} z(x)\,, & z_1 \end{pmatrix} \in (L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}$. It easy to verify that $\Xi^0_{2,\theta} := ((L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}, <,>_{\Xi^0_{2,\theta}})$ is a Hilbert space.

Suppose that $D(\mathcal{B}) = W_2^2(-1,0) \oplus W_2^2(0,1)$ and define a linear operator $\widetilde{\mathcal{A}} : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ by

$$\widetilde{\mathcal{A}}\begin{pmatrix} y(x)\,, & y(1) \end{pmatrix} = \begin{pmatrix} -y''(x) + (\mathcal{B}y)(x)\,, & y'(1) \end{pmatrix}, \quad F \in D(\widetilde{\mathcal{A}}) \tag{7}$$

on the domain $D(\widetilde{\mathcal{A}})$ which is defined as the set of all $F := \begin{pmatrix} y(x)\,, & y_1 \end{pmatrix} \in \Xi^0_{2,\theta}$ satisfying the following conditions:

(i) $\quad y(x) \in W_2^2(-1,0) \oplus W_2^2(0,1)$

(ii) $\quad \Gamma_1(y) = 0, \Gamma_2(y) = 0, \Gamma_3(y) = 0$ (8)

(iii) $\quad y_1 = y(1).$ (9)

Then, the considered problem (1) – (5) can be written in the operator form as

$$\widetilde{\mathcal{A}}F = \lambda F, \quad F \in D(\widetilde{\mathcal{A}}). \tag{10}$$

Now we introduce a new inner product space $\Xi^2_{2,\Gamma} = (D(\widetilde{\mathcal{A}})), < .,. >_{2,\Gamma})$ with the inner-product

$$< F, G >_{\Xi^2_{2,\Gamma}} =< y, z >_{W_2^2(-1,0)} + < y, z >_{W_2^2(0,1)} \tag{11}$$

for $F = \begin{pmatrix} y(x) \\ y(1) \end{pmatrix}$, $G = \begin{pmatrix} z(x) \\ z(1) \end{pmatrix} \in D(\widetilde{\mathcal{A}})$.

Remark 2.1. *It can be shown that all axioms of inner product are satisfied. Indeed, let* $< F, F >_{\Xi^2_{2,\Gamma}} = 0$. *From this, by (11) it follows immediately that the first component* $y(x)$ *of this element is zero vector of* $W_2^2(-1,0) \cup W_2^2(0,1)$ *and consequently* $y(1) = 0$. *So,* $F = (y(x), y(1)) = (0,0)$ *is zero vector of* $\Xi^2_{2,\Gamma}$. *The other axioms are satisfied, obviously.*

Lemma 2.2. $\Xi^2_{2,\Gamma}$ *is a Hilbert space.*

Proof. Let $F_n = \begin{pmatrix} y_n(x) \\ y_n(1) \end{pmatrix}$; $n = 1, 2, ...$ be any Cauchy sequence in $\Xi^2_{2,\Gamma}$. Then the sequence $\{y_n(x)\}$ consisting of the first components of sequence $\{F_n\}$, $n = 1, 2, ...$ forms a Cauchy sequence in the Hilbert space $W_2^2(-1,0) \oplus W_2^2(0,1)$. Thus, there exist $z(x) \in W_2^2(-1,0) \oplus W_2^2(0,1)$ such that

$$\left\| y_n(x) - z(x) \right\|_{W_2^2(-1,0) \oplus W_2^2(0,1)} \to 0 \ (n \to \infty).$$

Since the embeddings $W_2^2(-1,0) \subset C^1[-1,0]$ and $W_2^2(0,1) \subset C^1[0,1]$ are continuous we have,

$$| \Gamma_i y_n - \Gamma_i z | \leq c \parallel y_n - z \parallel_{W_2^2(-1,0) \oplus W_2^2(0,1)} \ (i = 1, 2, 3)$$

for some $c > 0$. Hence

$$\Gamma_i z = \lim_{n \longrightarrow \infty} \Gamma_i y_n = 0 \ \ (i = 1, 2, 3). \tag{12}$$

Consequently, defining $G = \begin{pmatrix} z(x) \\ z(1) \end{pmatrix}$ we see that $G \in \Xi^2_{2,\Gamma}$ and $\|F_n - G\|_{\Xi^2_{2,\Gamma}} \longrightarrow 0$ $(n \to \infty)$. The proof is complete. □

Theorem 2.3. *If the operator $\mathcal{B}$ acted compactly from the Hilbert space $W_2^2(-1,0)\oplus W_2^2(0,1)$ into the Hilbert space $L_2(-1,0)\oplus L_2(0,1)$, then for any $\varepsilon>0$ there exists $R_\varepsilon>0$ and $C_\varepsilon>0$, such that for all $\lambda\in\left\{\lambda\in\mathbb{C}:\left|arg\lambda-\pi\right|<\pi-\varepsilon\right\}$ for which $|\lambda|>R_\varepsilon$ the operator $\lambda I-\widetilde{\mathcal{A}}$ from $\Xi^2_{2,\Gamma}$ onto $\Xi^0_{2,\theta}$ is an isomorphism and the coercive inequality*

$$\|F\|_{\Xi^2_{2,\Gamma}}+|\lambda|\,\|F\|_{\Xi^0_{2,\theta}}\le C_\varepsilon\,\|G\|_{\Xi^0_{2,\theta}} \tag{13}$$

holds for the solution $F=F(\lambda)$ of the equation $(\lambda I-\widetilde{\mathcal{A}})F=G,\ G\in\Xi^0_{2,\theta}$.

Proof. Let $\sin\alpha\neq0$, $\gamma_1\neq0$ (other cases are similar). Denote

$L_0y:=-y''(x)-\lambda y(x),\Gamma_{10}(y):=\sin\alpha y'(-1),\Gamma_{20}(y):=-\gamma_1y'(-0),\Gamma_{30}(y):=y'(+0)-\gamma_2y'(-0),\Gamma_{40}(y):=-y'(1).$

We shall define the operator $\widetilde{L}_0(\lambda)$ by $\widetilde{L}_0(\lambda)y:=\left(L_0(\lambda)y,\Gamma_{10}(y),\Gamma_{20}(y),\Gamma_{30}(y),\Gamma_{40}(y)\right)$.

Then, by virtue of [16, Lemma 3.1] for any $\varepsilon>0$ there exists $R_\varepsilon>0$ and $C_\varepsilon>0$ such that for all $\lambda\in G_\varepsilon:=\left\{\lambda\in\mathbb{C}:\left|arg\lambda\right|<\pi-\varepsilon,\ |\lambda|>R_\varepsilon\right\}$ this operator is an isomorphism from $W_2^2(-1,0)\oplus W_2^2(0,1)$ onto $\left(L_2(-1,0)\oplus L_2(0,1)\right)\oplus\mathbb{C}^4$ and for these λ for the solution of the problem $\widetilde{L}_0(\lambda)y=f(x)$, $\Gamma_{i0}(y)=g_i$, $i=1,2,3,4$, the coercive estimate

$$|\lambda|\,\left\|y\right\|_{W_2^2}+\left\|y\right\|_{L_2}\le C(\varepsilon)\left\|\left(f(.)\,,g_1\,,g_2\,,g_3\,,g_4\right)\right\|_{L_2\,\oplus\,\mathbb{C}^4} \tag{14}$$

holds; where $\left(f(.)\,,g_1\,,g_2\,,g_3\,,g_4\right)\in\left(L_2(-1,0)\oplus L_2(0,1)\right)\oplus\mathbb{C}^4$ and $\left\|.\right\|_{W_2^2}$, $\left\|.\right\|_{L_2}$ and $\left\|.\right\|_{L_2\,\oplus\,\mathbb{C}^4}$ denotes the norms of the spaces $W_2^2(-1,0)\oplus W_2^2(0,1)$, $L_2(-1,0)\oplus L_2(0,1)$ and $\left(L_2(-1,0)\oplus L_2(0,1)\right)\oplus\mathbb{C}^4$, respectively.

Since the embedding $W_2^2(-1,0)\oplus W_2^2(0,1)\subset L_2(-1,0)\oplus L_2(0,1)$ is continuous (see, for example [12]) the linear functionals $\Gamma_{i1}(y):=\Gamma_1(y)-\Gamma_{i0}(y)$ $(i=1,2,3,4)$ are continuous in the space $W_2^k(-1,0)\oplus W_2^k(0,1)$ for any integer $k\ge1$.

Let us define the operator $\widetilde{L}(\lambda)$ by $\widetilde{L}(\lambda)y=\left((\lambda I-\widetilde{\mathcal{A}})y,\Gamma_4(\lambda)y\right)$ and consider the following problem

$$\widetilde{L}(\lambda)y=F,\quad F\in\Xi^0_{2,\theta}\oplus\mathbb{C}. \tag{15}$$

Since the operator $\ell(y):=-y''$ is continuous from $W_2^2(-1,0)\oplus W_2^2(0,1)$ into $L_2(-1,0)\oplus L_2(0,1)$ and the operator $\mathcal{B}$ is compact from $W_2^2(-1,0)\oplus W_2^2(0,1)$ into $L_2(-1,0)\oplus L_2(0,1)$, the operator $\lambda I-\widetilde{\mathcal{A}}$ is continuous from $\Xi^2_{2,\Gamma}$ into $\Xi^0_{2,\theta}$. By [20, Lemma 1.2.8/3] for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\left\|\mathcal{B}u\right\|_{L_2}\le\varepsilon\left\|u\right\|_{W_2^2}+C(\varepsilon)\left\|u\right\|_{L_2},\quad u\in W_2^2(-1,0)\oplus W_2^2(0,1). \tag{16}$$

Taking in view (14)-(16) and applying [17, Theorem 4] to the equation (16) we have that for any $\varepsilon>0$ (small enough) there exists $R_\varepsilon>0$ and $C_\varepsilon>0$ such that for all $\lambda\in G_\varepsilon$ for which $|\lambda|>R_\varepsilon$ the operator $\widetilde{L}(\lambda)$ is an isomorphism between the spaces $\left\{y:y\in W_2^2(-1,0)\oplus W_2^2(0,1)\,,\ \Gamma_i(y)=0\ (i=1,2,3),\ \|y\|:=\|y\|_{W_2^2}\right\}$ and $\left(L_2(-1,0)\oplus L_2(0,1)\right)\oplus\mathbb{C}$ and for these λ the coercive estimate (13) holds. The proof is complete. □

3. The Resolvent Operator and Discreteness of the Spectrum

We can now prove the following important results for the resolvent operator $R(\lambda,\widetilde{\mathcal{A}})=(\lambda I-\widetilde{\mathcal{A}})^{-1}$.

Theorem 3.1. *Let the operator $\mathcal{B}:W_2^2(-1,0)\oplus W_2^2(0,1)\longrightarrow L_2(-1,0)\oplus L_2(0,1)$ be compact. Then, for any $\varepsilon>0$ there exists $R_\varepsilon>0$ such that all $\lambda\in\left\{\lambda\in\mathbb{C}:\left|arg\lambda-\pi\right|<\pi-\varepsilon\right\}$ for which $|\lambda|>R_\varepsilon$ is regular value of the operator $\widetilde{\mathcal{A}}$ and for the resolvent operator $R(\lambda,\widetilde{\mathcal{A}})=(\lambda I-\widetilde{\mathcal{A}})^{-1}$ the following inequality holds*

$$\|R(\lambda,\widetilde{\mathcal{A}})\|\le C_\varepsilon\,|\lambda|^{-1}. \tag{17}$$

Proof. By taking $F = R(\lambda, \widetilde{\mathcal{A}})G$ in (13) we get

$$\|R(\lambda, \widetilde{\mathcal{A}})G\|_{\Xi^2_{2,\Gamma}} + |\lambda| \, \|R(\lambda, \widetilde{\mathcal{A}})G\|_{\Xi^0_{2,\theta}} \leq C_\varepsilon \, \|G\|_{\Xi^0_{2,\theta}} \tag{18}$$

from which it follows immediately that $\|R(\lambda, \widetilde{\mathcal{A}})G\|_{\Xi^0_{2,\theta}} \leq C_\varepsilon |\lambda|^{-1} \|G\|_{\Xi^0_{2,\theta}}$ that is the estimate (17) is hold. □

Theorem 3.2. *The Resolvent operator $R(\lambda, \widetilde{\mathcal{A}})$ acted boundedly from the Hilbert space $\Xi^0_{2,\theta}$ to the Hilbert space $\Xi^2_{2,\Gamma}$.*

Proof. Again, as in the proof of the previous theorem, by taking $F = R(\lambda, \widetilde{\mathcal{A}})G$ in (13) we have immediately that $\|R(\lambda, \widetilde{\mathcal{A}})G\|_{\Xi^2_{2,\Gamma}} \leq C_\varepsilon \, \|G\|_{\Xi^0_{2,\theta}}$. □

Theorem 3.3. *If the operator $\mathcal{B}$ from $W^2_2(-1,0) \oplus W^2_2(0,1)$ to $L_2(-1,0) \oplus L_2(0,1)$ is compact, then for any $\varepsilon > 0$ there exist $R_\varepsilon > 0$ such that for all $\lambda \in \left\{\lambda \in \mathbb{C} : \left|arg\lambda - \pi\right| < \pi - \varepsilon\right\}$ for which $|\lambda| > R_\varepsilon$ the resolvent operator $R(\lambda, \widetilde{\mathcal{A}}) : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ is compact.*

Proof. Firstly, let us show that the embedding $\Xi^2_{2,\Gamma} \subset \Xi^0_{2,\theta}$ is compact.

Let $F_n = \begin{pmatrix} y_n(x) \\ y_n(1) \end{pmatrix} \in \Xi^2_{2,\Gamma}$, $n = 1, 2, 3, ...$ be any bounded sequence. Then, $\{y_n(x)\}$ must be bounded in the Hilbert space $W^2_2(-1,0) \oplus W^2_2(0,1)$.

Since the embeddings $W^2_2(-1,0) \subset L_2(-1,0)$ and $W^2_2(0,1) \subset L_2(0,1)$ are compact (see, [12]), there exist the subsequence $\{y_{n_k}(x)\}$ and the function $y_0(x) \in (L_2(-1,0) \oplus L_2(0,1))$ such that

$$\|y_{n_k} - y_0\|_{L_2(-1,0)} \longrightarrow 0 \text{ and } \|y_{n_k} - y_0\|_{L_2(0,1)} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{19}$$

Moreover, since the embeddings $W^2_2(-1,0) \subset C[-1,0]$ and $W^2_2(0,1) \subset C[0,1]$ are continuous, the numerical sequence $\{y_{n_k(1)}\}$ is bounded. Thus, there exist a convergent subsequence $\{y_{n_{k_s}}(1)\}$ of sequence $\{y_{n_k}(1)\}$. Denoting $y_1 := \lim_{s \longrightarrow \infty} y_{n_{k_s}}(1)$ consider $F_0 = \begin{pmatrix} y_0(x) \\ y_1 \end{pmatrix} \in \Xi^0_{2,\theta}$. Since

$$\|F_{n_{k_s}} - F_0\|^2_{\Xi^0_{2,\theta}} = \|y_{n_{k_s}} - y_0\|^2_{L_2(-1,0)} + \frac{1}{\theta}(\|y_{n_{k_s}} - y_0\|^2_{L_2(0,1)} + |y_{n_{k_s}}(1) - y_1|^2_{\mathbb{C}}) \tag{20}$$

we have

$$\|F_{n_{k_s}} - F_0\|^2_{\Xi^0_{2,\theta}} \longrightarrow 0 \ (n \longrightarrow \infty). \tag{21}$$

Consequently, the embedding $\Xi^2_{2,\Gamma} \subset \Xi^0_{2,\theta}$ is compact.Moreover by Theorem 3.2 that for any $\varepsilon > 0$ there exist $R_\varepsilon > 0$ such that for all $\lambda \in \left\{\lambda \in \mathbb{C} : \left|arg\lambda - \pi\right| < \pi - \varepsilon\right\}$ for which $|\lambda| > R_\varepsilon$ the resolvent operator $R(\lambda, \widetilde{\mathcal{A}})$ acted boundedly from $\Xi^0_{2,\theta}$ to $\Xi^2_{2,\Gamma}$. Thus, the resolvent operator $R(\lambda, \widetilde{\mathcal{A}}) : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ is compact. □

Corollary 3.4. *The spectrum of the operator $\widetilde{\mathcal{A}}$ is discrete.*

4. Asymptotic Behaviour of the Eigenvalues

Define the operator $\mathcal{A} : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ by action law

$$\mathcal{A}\begin{pmatrix} y(x) \\ y(1) \end{pmatrix} = \begin{pmatrix} -y''(x) \\ y'(1) \end{pmatrix} \tag{22}$$

with the same domain as $\widetilde{\mathcal{A}}$ and the operator $\mathcal{B}_1 : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ with domain $D(\mathcal{B}_1) \supseteq D(\mathcal{A})$ and action law

$$\mathcal{B}_1 F = \begin{pmatrix} (\mathcal{B}y)(x) \\ 0 \end{pmatrix}. \tag{23}$$

It can be seen easily that the eigenvalues of boundary value transmission problem (1) – (5) and the operator $\widetilde{\mathcal{A}} = \mathcal{A} + \mathcal{B}_1$ are the same. Because of this reason, we will examine the eigenvalues of the operator $\widetilde{\mathcal{A}}$.

Lemma 4.1. *The operator $\mathcal{A} : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ is symmetric.*

Proof. Suppose that y and z are satisfied the boundary condition (2) and transmission conditions (4) – (5). Then the direct calculations gives

$$W(y,z;-1) = 0 \ \text{ and } \ W(y,z;-0) = -\frac{1}{\theta}\, W(y,z;+0) \tag{24}$$

where as usual by $W(f,g;x)$ we denote the wronskians $W(f,g;x) := f(x)g'(x) - f'(x)g(x)$. Taking into account the equalities (24) we can derive by two partial integration that $< \mathcal{A}Y, Z >_{\Xi^0_{2,\theta}} = < Y, \mathcal{A}Z >_{\Xi^0_{2,\theta}}$ for any $Y = \begin{pmatrix} y(x) \\ y_1 \end{pmatrix}$, $Z = \begin{pmatrix} z(x) \\ z_1 \end{pmatrix} \in D(\mathcal{A})$, so $\mathcal{A}$ is symmetric. □

Corollary 4.2. *All eigenvalues of the operator $\mathcal{A}$ are real.*

Now we shall define one-hand eigensolutions $\Phi_1(x,\lambda)$ and $\Phi_2(x,\lambda)$.
Let the function $\Phi_1(x,\lambda)$ is defined as

$$\Phi_1(x,\lambda) = \sin\alpha \cos\sqrt{\lambda}(x+1) - \frac{1}{\sqrt{\lambda}}\cos\alpha \sin\sqrt{\lambda}(x+1). \tag{25}$$

It can be checked directly that this function satisfies the first boundary condition (2) and the equation (1) in the case $\mathcal{B} = 0$.

In terms of this solution we shall construct the following initial-value problem:

$$-y''(x) = \lambda y(x), \quad x \in [0,1], \tag{26}$$

$$y(0) = \beta_1\Phi_1(0,\lambda) + \gamma_1\Phi_1'(0,\lambda), \ \ y'(0) = \beta_2\Phi_1(0,\lambda) + \gamma_2\Phi_1'(0,\lambda). \tag{27}$$

Obviously this problem has an unique solution $y := \Phi_2(x,\lambda)$ given by

$$\begin{aligned}\Phi_2(x,\lambda) \ &= \ (\beta_1\sin\alpha - \gamma_1\cos\alpha)\cos\sqrt{\lambda}\cos\sqrt{\lambda}x - (\gamma_2\sin\alpha + \frac{\beta_2}{\lambda}\cos\alpha)\sin\sqrt{\lambda}\sin\sqrt{\lambda}x \\ &+ \ (\frac{\beta_2}{\sqrt{\lambda}}\sin\alpha - \frac{\gamma_2}{\sqrt{\lambda}}\cos\alpha)\cos\sqrt{\lambda}\sin\sqrt{\lambda}x - (\sqrt{\lambda}\gamma_1\sin\alpha + \frac{\beta_1}{\sqrt{\lambda}}\cos\alpha)\sin\sqrt{\lambda}\cos\sqrt{\lambda}x\end{aligned} \tag{28}$$

Let us define an eigensolution $\Phi(x,\lambda)$ by

$$\Phi(x,\lambda) = \begin{cases} \Phi_1(x,\lambda), & x \in [-1,0) \\ \Phi_2(x,\lambda), & x \in (0,1] \end{cases}. \tag{29}$$

Lemma 4.3. *The eigenvalues of the operator $\mathcal{A}$ coincide with the zeros of the function $w(\lambda) = \Gamma_4(\Phi)$.*

Proof. It is easy to verify that for all $\lambda \in \mathbb{C}$, $-\Phi''(x,\lambda) = \lambda\Phi(x,\lambda)$, $x \in [-1,0) \cup (0,1]$ and $\Gamma_1(\Phi(.,\lambda)) = \Gamma_2(\Phi(.,\lambda)) = \Gamma_3(\Phi(.,\lambda)) = 0$. Let $w(\lambda_0) = 0$, i.e. $\Gamma_4(\Phi(.,\lambda_0)) = 0$. Then, we have that

$$\widetilde{\Phi}_0 := \begin{pmatrix} \Phi(.,\lambda_0) \\ \Phi(1,\lambda_0) \end{pmatrix} \in D(\mathcal{A}) \ \text{ and } \ \mathcal{A}\widetilde{\Phi}_0 = \lambda_0\widetilde{\Phi}_0.$$

Moreover, it is obvious that, if $\lambda = \lambda_0$ is an eigenvalue of $\mathcal{A}$, then $w(\lambda_0) = \Gamma_4(\Phi(.,\lambda_0))$. □

Remark 4.4. *If* $\lambda = \lambda_0$ *is an eigenvalue of the operator* $\mathcal{A}$, *then* $\begin{pmatrix} \Phi(x,\lambda_0) \\ \Phi(1,\lambda_0) \end{pmatrix}$ *would be eigenelement corresponding to this eigenvalue.*

Theorem 4.5. *The eigenvalues of the operator* $\mathcal{A}$ *are bounded below.*

Proof. Consider the case $\sin\alpha = 0$. Let $\lambda = \mu^2$. By substituting the above formulas for $\Phi_i(x,\lambda)$ $(i = 1,2)$ in the definition of $w(\lambda)$ we have

$$\begin{aligned} w(\lambda) &= -\gamma_1 \cos\alpha\,\mu^2\,\cos^2\mu - \left(\gamma_2 + \beta_1 + \gamma_1 - \frac{\beta_2}{\mu^2}\right)\cos\alpha\,\mu\,\sin\mu\cos\mu \\ &- (\beta_2 + \beta_1)\cos\alpha\sin^2\mu + \gamma_2\cos\alpha\cos^2\mu. \end{aligned} \tag{30}$$

Putting $s = it$ $(t > 0)$ in this formula we have $w(-t^2) \to \infty$ as $t \to \infty$. Consequently, $w(\lambda) \neq 0$ for λ negative and sufficiently large in module. □

We are now ready to find the asymptotic approximation formulas for the eigenvalues of the considered operator $\mathcal{A}$. Since the eigenvalues are coincide with the zeros of the entire function $\omega(\lambda)$, it follows that they have no finite limit. Moreover, we already know that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0 \le \lambda_1 \le \lambda_2 \le ....,$ which counted according to their multiplicity.

Theorem 4.6. *The operator* $\mathcal{A}$ *has an precisely numerable many real eigenvalues, whose behavior may be expressed by two sequence* $\{\lambda_{n,1}\}$ *and* $\{\lambda_{n,2}\}$ *with following asymptotic as* $n \to \infty$:

Case 1. *If* $\sin\alpha = 0$, *then*

$$\mu_{n,1} = \left(n - \frac{1}{2}\right)\pi + O\left(\frac{1}{\sqrt{n}}\right) \quad , \quad \mu_{n,2} = \left(n + \frac{1}{2}\right)\pi + O\left(\frac{1}{\sqrt{n}}\right) \tag{31}$$

Case 2. *If* $\sin\alpha \neq 0$, *then*

$$\mu_{n,1} = \left(n - 1\right)\pi + O\left(\frac{1}{n}\right) \quad , \quad \mu_{n,2} = \left(n + \frac{1}{2}\right)\pi + O\left(\frac{1}{n}\right) \tag{32}$$

where $\lambda_{n,1} = \mu_{n,1}^2$, $\lambda_{n,2} = \mu_{n,2}^2$.

Proof. Consider the *Case*1. Let $\lambda = \mu^2$ and $\mu = \sigma + it$. Denote $\omega_0(\lambda) = -\gamma_1\cos\alpha\,\mu^2\,\cos^2\mu$.

Let $\{\xi_{n,1}\}$ and $\{\xi_{n,2}\}$ be two sequence, such that $0 < \xi_{n,i} < \frac{1}{2}$ and let $J_{n,i}(i = 1,2)$ are the bounds of the domains $\{\mu \in \mathbb{C} : |\sigma| \le \pi(n + \xi_{n,1}),\ |t| \le \pi(n + \xi_{n,2})\}$. We can choose the sequences $\{\xi_{n,1}\}$ and $\{\xi_{n,2}\}$ so that $\{\pi(n + \xi_{n,i}) \neq \pi m\}$ for every integers n and m. Taking in view that $w(\lambda)$ and $w_0(\lambda)$ are analytic inside and on a closed contours $J_{n,i}$ respectively and the fact that $0 < \xi_{n,i} < \frac{1}{2}$, it easy to show that $|w_0(\lambda)| > |w(\lambda) - w_0(\lambda)|$ on both $J_{n,1}$ and $J_{n,2}$ for sufficiently large n. Then by applying Rouche's theorem on sufficiently large contours $J_{n,i}$ it follows that $w(\lambda)$ and $w_0(\lambda)$ have the same number zeros inside $J_{n,i}$ provided that all zeros are counted according to their multiplicity. Since inside the contour $J_{n,1}$ the function $w_0(\mu^2)$ has zeros at points $\mu_0 = 0$ and $\mu_n = \left(n - \frac{1}{2}\right)\pi$, $n = \mp 1, \mp 2, ...,$ (with multiplicity 2), the zeros of $w(\lambda)$ may be represented as two sequence $\lambda_{n,1} = \mu_{n,1}^2$ and $\lambda_{n,2} = \mu_{n,2}^2, n = 0,1,2,...,$ so that $\pi(n - 1 + \xi_{n,1}) < \mu_{n,1} < \pi(n + \xi_{n,1})$ and $\pi(n + \xi_{n,2}) < \mu_{n,2} < \pi(n + 1 + \xi_{n,2})$ for sufficiently large n. Consequently,

$$\mu_{n,1} = \left(n - \frac{1}{2}\right)\pi + \delta_{n,1} \text{ and } \mu_{n,2} = \left(n + \frac{1}{2}\right)\pi + \delta_{n,2}$$

where $|\delta_{n,1}| < \frac{\pi}{2}$ and $|\delta_{n,2}| < \frac{\pi}{2}$ for sufficiently large n. Putting back in (30) we can derive that $\delta_{n,1} = O(\frac{1}{\sqrt{n}})$ and $\delta_{n,2} = O(\frac{1}{\sqrt{n}})$, so the formula (31) is proved. The proof of the other case is similar. □

Below we shall write $f(r) \sim g(r)\ r \to \infty$ if $\lim_{r\to\infty}\frac{f(r)}{g(r)} = 1$.

Lemma 4.7. *Let the linear operator $\mathcal{B}_1(\mathcal{A}-\lambda_0 I)^{-1}$ be compact for some regular point λ_0 of $\mathcal{A}$. Then the spectrum of the operator $\widetilde{\mathcal{A}} = \mathcal{A} + \mathcal{B}_1$ is discrete and for the eigenvalues $(\lambda_{n,i\ ,\alpha})$ $(i = 1, 2)$ which are arranged in the form $|\lambda_{1,i,\alpha}| \leq |\lambda_{2,i,\alpha}| \leq ...$ in angle $\left\{\lambda \in \mathbb{C} : \left|arg\lambda - \pi\right| < \pi - \alpha\right\}$ where all eigenvalues are written as multiple as its, the following asymptotic formulas for modulus $|\lambda_{n,i,\alpha}|$ are hold as $n \to \infty$:*

Case 1. If $\sin\alpha = 0$, *then*

$$|\lambda_{n,1,\alpha}| = \pi^2 n^2 + o\left(n^2\right) \quad , \quad |\lambda_{n,2,\alpha}| = \pi^2 n^2 + o\left(n^2\right) \tag{33}$$

Case 2. If $\sin\alpha \neq 0$, *then*

$$|\lambda_{n,1,\alpha}| = (n-1)^2\pi^2 + o\left(n^2\right) \quad , \quad |\lambda_{n,2,\alpha}| = (n+\frac{1}{2})^2\pi^2 + o\left(n^2\right) \tag{34}$$

Proof. Let $\sin\alpha = 0$. Employing the same method as in proof of Theorem 5.1 in [18] we can show that the operator $\mathcal{A}$ is self-adjoint. Since

$$\lambda_{n,1} = \pi^2 n^2 + O(n), \tag{35}$$

there exists real numbers α and β such that

$$\pi^2 n^2 + \alpha n \leq \lambda_{n,1} \leq \pi^2 n^2 + \beta n. \tag{36}$$

Denoting $N_+(r,\mathcal{A}) = \sum\limits_{0 \leq \lambda_{n,1} \leq r} 1$ and applying the asymptotic equality

$$\left\{\sqrt{1+\frac{M}{r}} - \sqrt{\frac{M}{r}}\right\} = \left(1 + O(\frac{1}{\sqrt{r}})\right),\ r \to \infty$$

where $M > 0$ any real number, from (36) we deduce that

$$N_+(r,\mathcal{A}) = \frac{\sqrt{r}}{\pi} + O(1),\ r \to \infty. \tag{37}$$

From this asymptotic equality it follows that $\lim\limits_{\substack{r \to \infty \\ \varepsilon \to 0}} \frac{N_+(r(1+\varepsilon),\mathcal{A})}{N_+(r,\mathcal{A})} = 1$. Then denoting $N_+(r,\alpha,\widetilde{\mathcal{A}}) = \sum_{\lambda_{n,i,\alpha} \in \left\{\lambda \in \mathbb{C}: \left|arg\lambda - \pi\right| < \pi - \alpha\right\}} 1$, by virtue of [4] we have

$$\lim_{r \longrightarrow \infty} \frac{N_+\left(r,\alpha,\widetilde{\mathcal{A}}\right)}{N_+(r,\mathcal{A})} = 1. \tag{38}$$

Consequently

$$N_+\left(r,\alpha,\widetilde{\mathcal{A}}\right) = N_+(r,\mathcal{A}) + o(N_+(r,\mathcal{A})) = \frac{\sqrt{r}}{\pi} + o(\sqrt{r}),\ r \to \infty. \tag{39}$$

From (39) it follows that $n = \frac{\sqrt{|\lambda_{n,1,\ \alpha}|}}{\pi} + o(\sqrt{|\lambda_{n,1,\ \alpha}|})$ and consequently

$$|\lambda_{n,1,\ \alpha}| = \pi^2 n^2 + o(n^2)\ ,\ n \to \infty. \tag{40}$$

The proof of the other case is similar. □

Theorem 4.8. *Under conditions of the previous Lemma, the spectrum of boundary value transmission problem* $(1)-(5)$ *is discrete and for the eigenvalues* $\{\widetilde{\lambda}_{n,i}\}(i=1,2)$, *that arranged in decreasing modulus, the following asymptotic formulas are valid:*

Case 1. *If* $\sin\alpha = 0$, *then*

$$\widetilde{\lambda}_{n,1} = \pi^2 n^2 + o\left(n^2\right) \quad , \quad \widetilde{\lambda}_{n,2} = \pi^2 n^2 + o\left(n^2\right). \tag{41}$$

Case 2. *If* $\sin\alpha \neq 0$, *then*

$$\widetilde{\lambda}_{n,1} = (n-1)^2\pi^2 + o\left(n^2\right) \quad , \quad \widetilde{\lambda}_{n,2} = (n+\frac{1}{2})^2\pi^2 + o\left(n^2\right) \tag{42}$$

Proof. Let $\sin\alpha = 0$. Because of the Theorem 2.3, for every α $(0<\alpha<\frac{\pi}{2})$, the number of eigenvalues of the operator $\widetilde{\mathcal{A}}$ which lies outside of the angle $\left\{\lambda\in\mathbb{C} : \left|arg\lambda - \pi\right| < \pi-\alpha\right\}$ is finite. We denote the number of these eigenvalues by k_α. We can arrange the eigenvalues of the operator $\widetilde{\mathcal{A}}$ as following

$$\widetilde{\lambda}_{n+k_\alpha,i} = \lambda_{n,i,\,\alpha} \quad , \quad n=1,2,... \quad i=1,2 \tag{43}$$

Then from (40) and (43) we obtain

$$|\widetilde{\lambda}_{n,i}| = |\lambda_{n-k_\alpha,i,\alpha}| = \pi^2(n-k_\alpha)^2 + o((n-k_\alpha)^2) = \pi^2n^2 + o(n^2)\,,\ n\longrightarrow\infty,\ \ i=1,2. \tag{44}$$

Further, for every $\alpha(0<\alpha<\frac{\pi}{2})$ there exist a natural number n_α such that

$$|\widetilde{\lambda}_{n,i}|^{-1}Re\widetilde{\lambda}_{n,i} > \cos\alpha, \quad |\widetilde{\lambda}_{n,i}|^{-1}|Im\widetilde{\lambda}_{n,i}| < \sin\alpha$$

for all $n>n_\alpha$. From the last inequalities we have

$$\cos\alpha \le \liminf_{n\to\infty}\frac{Re\widetilde{\lambda}_{n,i}}{|\widetilde{\lambda}_{n,i}|} \le \limsup_{n\to\infty}\frac{Re\widetilde{\lambda}_{n,i}}{|\widetilde{\lambda}_{n,i}|} \le 1 \ \text{ and } \ 0 \le \liminf_{n\to\infty}\frac{|Im\widetilde{\lambda}_{n,i}|}{|\widetilde{\lambda}_{n,i}|} \le \limsup_{n\to\infty}\frac{|Im\widetilde{\lambda}_{n,i}|}{|\widetilde{\lambda}_{n,i}|} \le \sin\alpha\ .$$

Letting $\alpha\to 0$ in the last inequalities we have $\lim_{n\longrightarrow\infty}\frac{Re\widetilde{\lambda}_{n,i}}{|\widetilde{\lambda}_{n,i}|} = 1$ and $\lim_{n\longrightarrow\infty}\frac{|Im\widetilde{\lambda}_{n,i}|}{|\widetilde{\lambda}_{n,i}|} = 0$. Consequently $Re\widetilde{\lambda}_{n,i} = |\widetilde{\lambda}_n| + o(|\widetilde{\lambda}_{n,i}|) = \pi^2n^2 + o(n^2)$ and $|Im\widetilde{\lambda}_{n,i}| = o(|\widetilde{\lambda}_{n,i}|) = o(n^2)$, respectively. From these asymptotic equations we find that $\widetilde{\lambda}_{n,i} = \pi^2n^2 + o(n^2)\,,\ n\longrightarrow\infty$. The proof of the other case is similar. □

The main result of this section is the following Theorem.

Theorem 4.9. *If the operator* $\mathcal{B}$ *is compact from* $W_2^2(-1,0)\oplus W_2^2(0,1)$ *into* $L_2(-1,0)\oplus L_2(0,1)$, *then the spectrum of boundary value transmission problem* $(1)-(5)$ *is discrete and for the eigenvalues* $\{\widetilde{\lambda}_{n,i}\}$ $(\ i=1,2, n=1,2,...)$ *arranged as* $|\widetilde{\lambda}_{1,i}| \le |\widetilde{\lambda}_{2,i}| \le |\widetilde{\lambda}_{3,i}| \le ....,$ *the following asymptotic formulas are hold:*

Case 1. *If* $\sin\alpha = 0$, *then*

$$\widetilde{\lambda}_{n,1} = \pi^2 n^2 + o\left(n^2\right) \quad , \quad \widetilde{\lambda}_{n,2} = \pi^2 n^2 + o\left(n^2\right). \tag{45}$$

Case 2. *If* $\sin\alpha \neq 0$, *then*

$$\widetilde{\lambda}_{n,1} = (n-1)^2\pi^2 + o\left(n^2\right) \quad , \quad \widetilde{\lambda}_{n,2} = (n+\frac{1}{2})^2\pi^2 + o\left(n^2\right) \tag{46}$$

Proof. By virtue of Theorem 3.3 the resolvent operator $R(\lambda,\mathcal{A}) : \Xi^0_{2,\theta} \longrightarrow \Xi^2_{2,\Gamma}$ is bounded. On the other hand, since $\mathcal{B} : W_2^2(-1,0)\oplus W_2^2(0,1) \to L_2(-1,0)\oplus L_2(0,1)$ is compact, then the operator $\mathcal{B}_1 : \Xi^2_{2,\Gamma} \longrightarrow \Xi^0_{2,\theta}$ is also compact. Consequently, the operator $\mathcal{B}_1R(\lambda,\mathcal{A}) : \Xi^0_{2,\theta} \longrightarrow \Xi^0_{2,\theta}$ is compact. Finally, applying Theorem 2.3 we complete the proof. □

Remark 4.10. *It is well-known that for special case $\mathcal{B}u = q(x)u$ (i.e. for standard Sturm-Liouville problems) the eigenvalues are real and the first asymptotic term has the form $O(n)$. However, in our problem, the eigenvalues may be also nonreal complex numbers and the first asymptotic term appears in the weak form as $o(n^2)$ because of the abstract linear operator $\mathcal{B}$ in the equation.*

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