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Research Article

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Resolving resolution dimensions in triangulated categories

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Abstract: Let $\mathcal{T}$ be a triangulated category with a proper class ξ of triangles and $\mathcal{X}$ be a subcategory of $\mathcal{T}$. We first introduce the notion of $\mathcal{X}$-resolution dimensions for a resolving subcategory of $\mathcal{T}$ and then give some descriptions of objects having finite $\mathcal{X}$-resolution dimensions. In particular, we obtain Auslander-Buchweitz approximations for these objects. As applications, we construct adjoint pairs for two kinds of inclusion functors and characterize objects having finite $\mathcal{X}$-resolution dimensions in terms of a notion of ξ-cellular towers. We also construct a new resolving subcategory from a given resolving subcategory and reformulate some known results.

Keywords: triangulated categories, a proper class of triangles, resolving resolution dimensions, resolving subcategories, Auslander-Buchweitz approximations

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1 Introduction

Approximation theory is the main part of relative homological algebra and representation theory of algebras, and its starting point is to approximate arbitrary objects by a class of suitable subcategories. In particular, resolving subcategories play important roles in approximation theory (e.g., [1–3]). As an important example of resolving subcategories, Auslander and Buchweitz [4] studied the approximation theory of the subcategory consisting of maximal Cohen-Macaulay modules over an artin algebra, and Hernández et al. [5] developed an analogous theory for triangulated categories. Using the approximation triangles established by Hernández et al. [5, Theorem 5.4], Di and Wang [6] constructed additive functors (adjoint pairs) between additive quotient categories. On the other hand, Zhu [7] studied the resolution dimension with respect to a resolving subcategory in an abelian category, and Huang [8] introduced relative preresolving subcategories in an abelian category and defined homological dimensions relative to these subcategories, which generalized many known results (see [4,9,10]).

In analogy to relative homological algebra in abelian categories, Beligiannis [11] developed a relative version of homological algebra in a triangulated category $\mathcal{T}$, that is, a pair $(\mathcal{T}, \xi)$, in which ξ is a proper class of triangles (see Definition 2.4). Under this notion, a triangulated category is just equipped with a proper class consisting of all triangles. However, there are lots of non-trivial cases, for example, let $\mathcal{T}$ be a compactly generated triangulated category, then the class ξ consisting of pure triangles is a proper class ([12]), and the pair $(\mathcal{T}, \xi)$ is no longer triangulated in general. Later on, this theory has been paid more attentions and developed (e.g., [13–17]). It is natural to ask how the approximation theory acts on this relative setting of triangulated categories. In [18], Ma et al., introduced the notions of (pre)resolving

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subcategories and homological dimensions relative to these subcategories in this relative setting, which gives a parallel theory analogy to that of abelian categories [8]. In this paper, we devote to further studying relative homological dimensions in triangulated categories with respect to a resolving subcategory. The paper is organized as follows:

In Section 2, we give some terminology and some preliminary results.

In Section 3, some homological properties of resolving subcategories are obtained. In particular, we obtain Auslander-Buchweitz approximation triangles (see Proposition 3.10) for objects having finite resolving resolution dimensions. Our main result is the following:

Theorem. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$, a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. For any $M \in \mathcal{T}$, if $M \in \widehat{\mathcal{X}}$, then the following statements are equivalent*:

(1) $\mathcal{X}$-res.dim $M \leq m$.
(2) $\Omega^n(M) \in \mathcal{X}$ *for all* $n \geq m$.
(3) $\Omega^n_{\mathcal{X}}(M) \in \mathcal{X}$ *for all* $n \geq m$.
(4) $\xi xt^n_\xi(M, H) = 0$ *for all* $n > m$ *and all* $H \in \mathcal{H}$.
(5) $\xi xt^n_\xi(M, L) = 0$ *for all* $n > m$ *and all* $L \in \widehat{\mathcal{H}}$.
(6) *M admits a right $\mathcal{X}$-approximation $\varphi : X \to M$, where φ is ξ-proper epic, such that $K = \mathrm{Hoker}\ \varphi$ satisfying* $\mathcal{H}$-res.dim $K \leq m - 1$.
(7) *There are two triangles*

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in ξ such that X_M and X^M are in $\mathcal{X}$ and $\mathcal{H}$-res.dim $W_M \leq m - 1$, $\mathcal{H}$-res.dim $W^M = \mathcal{X}$-res.dim $W^M \leq m$.

In Section 4, we will further study objects having finite resolution dimensions with respect to a resolving subcategory $\mathcal{X}$. We first construct adjoint pairs for two kinds of inclusion functors. Then we characterize objects having finite resolution dimensions in terms of a notion of ξ-cellular towers.

As an application, in Section 5, given a resolving subcategory $\mathcal{X}$ of $\mathcal{T}$, we construct a new resolving subcategory $\mathcal{GP}_{\mathcal{X}}(\xi)$ with a ξxt-injective ξ-cogenerator $\mathcal{X} \cap {}^{\perp}\mathcal{X}$, which generalizes the Gorenstein projective subcategory $\mathcal{GP}(\xi)$ given by Asadollahi and Salarian [13]. Applying the obtained results to $\mathcal{GP}_{\mathcal{X}}(\xi)$, we generalize some known results in [13–15].

Throughout this paper, all subcategories are full, additive, and closed under isomorphisms.

2 Preliminaries

Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an additive functor. One defines the category $\mathrm{Diag}(\mathcal{T}, \Sigma)$ as follows:

- An object of $\mathrm{Diag}(\mathcal{T}, \Sigma)$ is a diagram in $\mathcal{T}$ of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$.
- A morphism in $\mathrm{Diag}(\mathcal{T}, \Sigma)$ between $X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} \Sigma X_i$, $i = 1, 2$, is a triple (α, β, γ) of morphisms in $\mathcal{T}$ such that the following diagram:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & \Sigma X_1 \\ \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\Sigma\alpha} \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \end{array}$$

commutes.

A triangulated category is a triple $(\mathcal{T}, \Sigma, \Delta)$, where $\mathcal{T}$ is an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ is an auto-equivalence of $\mathcal{T}$ (called suspension functor), and Δ is a full subcategory of $\mathrm{Diag}(\mathcal{T}, \Sigma)$ which is closed under isomorphisms and satisfies the axioms (T_1)–(T_4) in [11, Section 2.1] (also see [19]), where the axiom (T_4) is called the octahedral axiom. The elements in Δ are called *triangles*.

The following result is well known, which is an efficient tool in studying triangulated categories.

Remark 2.1. [11, Proposition 2.1] Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ an autoequivalence of $\mathcal{T}$, and Δ a full subcategory of $\mathrm{Diag}(\mathcal{T}, \Sigma)$ which is closed under isomorphisms. Suppose that the triple $(\mathcal{T}, \Sigma, \Delta)$ satisfies all the axioms of a triangulated category except possibly of the octahedral axiom. Then, the following statements are equivalent:

(1) **Octahedral axiom**. For any two morphisms $u : X \longrightarrow Y$ and $v : Y \longrightarrow Z$, there exists a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{u'} & Z' & \xrightarrow{u''} & \Sigma X \\
\| \downarrow & & \downarrow v & & \downarrow \alpha & & \downarrow \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{w} & Y' & \xrightarrow{w'} & \Sigma X \\
\downarrow u & & \| \downarrow & & \downarrow \beta & & \downarrow \Sigma u \\
Y & \xrightarrow{v} & Z & \xrightarrow{v'} & X' & \xrightarrow{v''} & \Sigma Y \\
\downarrow & & \downarrow 0 & & \downarrow (\Sigma u')v'' & & \downarrow \\
0 & \longrightarrow & \Sigma Z' & \xrightarrow{=} & \Sigma Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and the third column are triangles in Δ.

(2) **Base change**. For any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in Δ and any morphism $\alpha : Z' \longrightarrow Z$, there exists the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & X' & \xrightarrow{=} & X' & \longrightarrow & 0 \\
\downarrow & & \downarrow \beta' & & \downarrow \beta & & \downarrow \\
X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\
\| \downarrow & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow & & \downarrow \gamma' & & \downarrow \gamma & & \downarrow \\
0 & \longrightarrow & \Sigma X' & \xrightarrow{=} & \Sigma X' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

(3) **Cobase change**. For any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in Δ and any morphism $\beta : X \longrightarrow X'$, there exists the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1} Z' & \xrightarrow{=} & \Sigma^{-1} Z' & \longrightarrow & 0 \\
\downarrow & & \downarrow -\Sigma^{-1}\gamma & & \downarrow -\Sigma^{-1}\gamma' & & \downarrow \\
\Sigma^{-1} Z & \xrightarrow{-\Sigma^{-1} w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\| \downarrow & & \downarrow \beta & & \downarrow \beta' & & \downarrow \| \\
\Sigma^{-1} Z & \xrightarrow{-\Sigma^{-1} w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z \\
\downarrow & & \downarrow \alpha & & \downarrow \alpha' & & \downarrow \\
0 & \longrightarrow & Z' & \xrightarrow{=} & Z' & \longrightarrow & 0,
\end{array}$$

in which all rows and columns are triangles in Δ.

Throughout this paper, $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ is a triangulated category.

Definition 2.2. [11] A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called *split* if it is isomorphic to the triangle

$$X \xrightarrow{\binom{1}{0}} X \oplus Z \xrightarrow{(0,1)} Z \xrightarrow{0} \Sigma X.$$

We use Δ_0 to denote the full subcategory of Δ consisting of all split triangles.

Definition 2.3. [11] Let ξ be a class of triangles in $\mathcal{T}$.

(1) ξ is said to be *closed under base change* (resp. *cobase change*) if for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in ξ and any morphism $\alpha : Z' \longrightarrow Z$ (resp. $\beta : X \longrightarrow X'$) as in Remark 2.1(2) (resp. Remark 2.1(3)), the triangle

$$X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X \quad (\text{resp. } X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X')$$

is in ξ.

(2) ξ is said to be *closed under suspension* if for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in ξ and any $i \in \mathbb{Z}$ (the set of all integers), the triangle

$$\Sigma^i X \xrightarrow{(-1)^i \Sigma^i u} \Sigma^i Y \xrightarrow{(-1)^i \Sigma^i v} \Sigma^i Z \xrightarrow{(-1)^i \Sigma^i w} \Sigma^{i+1} X$$

is in ξ.

(3) ξ is called *saturated* if in the situation of base change as in Remark 2.1(2), whenever the third vertical and the second horizontal triangles are in ξ, then the triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is in ξ.

Definition 2.4. [11] A class ξ of triangles in $\mathcal{T}$ is called *proper* if the following conditions are satisfied.

(1) ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi$.

(2) ξ is closed under suspensions and is saturated.

(3) ξ is closed under base and cobase change.

Throughout this paper, we always assume that ξ is a proper class of triangles in $\mathcal{T}$.

Definition 2.5. [11] Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle in ξ. Then, the morphism u (resp. v) is called *ξ-proper monic* (resp. *ξ-proper epic*), and u (resp. v) is called the *hokernel* of v (resp. the *hocokernel* of u).

We use Hoker v to denote the hokernel of $v : Y \longrightarrow Z$. Dually, we use Hocok u to denote the hocokernel of $u : X \to Y$. For any triangle,

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ. We say that $\mathcal{X}$ is *closed under ξ-extensions* if $X, Z \in \mathcal{X}$, it holds that $Y \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under hokernels of ξ-proper epimorphisms* (resp. *hocokernels of ξ-proper monomorphisms*) if Y, $Z \in \mathcal{X}$ (resp. X, $Y \in \mathcal{X}$), it holds that $X \in \mathcal{X}$ (resp. $Z \in \mathcal{X}$).

Definition 2.6. (see [11, 4.1]) An object P (resp. I) in $\mathcal{T}$ is called *ξ-projective* (resp. *ξ-injective*) if for any triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ in ξ, the induced complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P, X) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P, Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(P, Z) \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z, I) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Y, I) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, I) \longrightarrow 0)$$

is exact. We use $\mathcal{P}(\xi)$ (resp. $\mathcal{I}(\xi)$) to denote the full subcategory of $\mathcal{T}$ consisting of ξ-projective (resp. ξ-injective) objects.

We say that $\mathcal{T}$ has *enough ξ-projective objects* if for any object $M \in \mathcal{T}$, there exists a triangle $K \longrightarrow P \longrightarrow M \longrightarrow \Sigma K$ in ξ with $P \in \mathcal{P}(\xi)$. Dually, we say that $\mathcal{T}$ has *enough ξ-injective objects* if for any object $M \in \mathcal{T}$, there exists a triangle $M \longrightarrow I \longrightarrow K \longrightarrow \Sigma M$ in ξ with $I \in \mathcal{I}(\xi)$.

Remark 2.7. $\mathcal{P}(\xi)$ is closed under direct summands, hokernels of ξ-proper epimorphisms, and ξ-extensions. Dually, $\mathcal{I}(\xi)$ is closed under direct summands, hocokernels of ξ-proper monomorphisms, and ξ-extensions.

Definition 2.8. Let $\mathcal{E}$ be a subcategory of $\mathcal{T}$.
(1) A triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in ξ is called *$\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, -)$-exact* (resp. *$\mathrm{Hom}_{\mathcal{T}}(-,\mathcal{E})$-exact*) if for any object E in $\mathcal{E}$, the induced complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, X) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, Z) \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z, E) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Y, E) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, E) \longrightarrow 0)$$

is exact.
(2) [13] A *ξ-exact* complex is a complex

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \tag{2.1}$$

in $\mathcal{T}$ such that for any $n \in \mathbb{Z}$, there exists a triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1} \tag{2.2}$$

in ξ and the differential d_n is defined as $d_n = g_{n-1} f_n$. A ξ-exact complex as (2.1) is called *$\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, -)$-exact* (resp. *$\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{E})$-exact*) if the triangle (2.2) is $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, -)$-exact (resp. $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{E})$-exact) for any $n \in \mathbb{Z}$.

Asadollahi and Salarian [13] introduced the notion of ξ-Gorenstein projective objects.

Definition 2.9. [13, Definition 3.6] Let $\mathcal{T}$ be a triangulated category with enough ξ-projective objects and X an object in $\mathcal{T}$. A *complete ξ-projective resolution* is a $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P}(\xi))$-exact ξ-exact complex

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow \cdots$$

in $\mathcal{T}$ with all P_i ξ-projective objects. The objects K_n as in (2.2) are called *ξ-Gorenstein projective* objects. We use $\mathcal{GP}(\xi)$ to denote the full subcategory of $\mathcal{T}$ consisting of all ξ-Gorenstein projective objects.

Throughout this paper, we always assume that $\mathcal{T}$ is a triangulated category with enough ξ-projective objects and ξ-injective objects.

Let M be an object in $\mathcal{T}$. Beligiannis [11] defined the ξ-extension groups $\xi xt^n_{\xi}(-, M)$ to be the nth right ξ-derived functor of the functor $\mathrm{Hom}_{\mathcal{T}}(-, M)$, that is,

$$\xi xt^n_{\xi}(-, M) := \mathcal{R}^n_{\xi}\mathrm{Hom}_{\mathcal{T}}(-, M).$$

Remark 2.10. Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a triangle in ξ. By [11, Corollary 4.12], there exists a long exact sequence

$$0 \longrightarrow \xi xt_\xi^0(Z, M) \longrightarrow \xi xt_\xi^0(Y, M) \longrightarrow \xi xt_\xi^0(X, M) \longrightarrow$$
$$\xi xt_\xi^1(Z, M) \longrightarrow \xi xt_\xi^1(Y, M) \longrightarrow \xi xt_\xi^1(X, M) \longrightarrow \cdots$$

of "ξxt" functor. If $\mathcal{T}$ has enough ξ-injective objects and N is an object in $\mathcal{T}$, then there exists a long exact sequence

$$0 \longrightarrow \xi xt_\xi^0(N, X) \longrightarrow \xi xt_\xi^0(N, Y) \longrightarrow \xi xt_\xi^0(N, Z) \longrightarrow$$
$$\xi xt_\xi^1(N, X) \longrightarrow \xi xt_\xi^1(N, Y) \longrightarrow \xi xt_\xi^1(N, Z) \longrightarrow \cdots$$

of "ξxt" functor.

Following Remark 2.10, we usually use the strategy of "dimension shifting," which is an important tool in relative homological theory of triangulated categories.

Now, we set

$$\mathcal{X}^\perp = \{M \in \mathcal{T} | \xi xt_\xi^{n \geq 1}(X, M) = 0 \text{ for all } X \in \mathcal{X}\},$$
$${}^\perp\mathcal{X} = \{M \in \mathcal{T} | \xi xt_\xi^{n \geq 1}(M, X) = 0 \text{ for all } X \in \mathcal{X}\}.$$

For two subcategories $\mathcal{H}$ and $\mathcal{X}$ of $\mathcal{T}$, we say $\mathcal{H} \perp \mathcal{X}$ if $\mathcal{H} \subseteq {}^\perp\mathcal{X}$ (equivalently, $\mathcal{X} \subseteq \mathcal{H}^\perp$).

Taking $C = \mathcal{E} = \mathcal{P}(\xi)$ in [18, Definitions 3.1 and 3.2], we have the following definitions.

Definition 2.11. (cf. [18, Definition 3.1]) Let $\mathcal{H}$ and $\mathcal{X}$ be two subcategories of $\mathcal{T}$ with $\mathcal{H} \subseteq \mathcal{X}$. Then, $\mathcal{H}$ is called a ξ-cogenerator of $\mathcal{X}$ if for any object X in $\mathcal{X}$, there exists a triangle

$$X \longrightarrow H \longrightarrow Z \longrightarrow \Sigma X$$

in ξ with H an object in $\mathcal{H}$ and Z an object in $\mathcal{X}$. In particular, a ξ-cogenerator $\mathcal{H}$ is called *ξxt-injective* if $\mathcal{X} \perp \mathcal{H}$.

Definition 2.12. (cf. [18, Definition 3.2]) Let $\mathcal{T}$ be a triangulated category with enough ξ-projective objects and $\mathcal{X}$ a subcategory of $\mathcal{T}$. Then, $\mathcal{X}$ is called a *resolving* subcategory of $\mathcal{T}$ if the following conditions are satisfied.

(1) $\mathcal{P}(\xi) \subseteq \mathcal{X}$.
(2) $\mathcal{X}$ is *closed under ξ-extensions*.
(3) $\mathcal{X}$ is *closed under hokernels of ξ-proper epimorphisms*.

3 Resolution dimensions with respect to a resolving subcategory

Taking $\mathcal{E} = \mathcal{P}(\xi)$ in [18, Definition 3.5], we first have the following definition.

Definition 3.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and M an object in $\mathcal{T}$. The $\mathcal{X}$-resolution dimension of M, written $\mathcal{X}$-res.dim M, is defined by

$$\mathcal{X}\text{-res.dim}\, M = \inf\{n \geq 0 \,|\, \text{there exists a } \xi\text{-exact complex}$$
$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \longrightarrow X_0 \longrightarrow M \longrightarrow 0 \text{ in } \mathcal{T} \text{ with all } X_i \text{ objects in } \mathcal{X}\}.$$

For a ξ-exact complex

$$\cdots \xrightarrow{f_{n+1}} X_n \longrightarrow \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \longrightarrow 0$$

with all $X_i \in \mathcal{X}$. The Hoker f_{n-1} is called an nth ξ-$\mathcal{X}$-syzygy of M, denoted by $\Omega^n_{\mathcal{X}}(M)$. In case for $\mathcal{X} = \mathcal{P}(\xi)$, we write ξ-pd $M := \mathcal{X}$-res.dim M and $\Omega^n(M) := \Omega^n_{\mathcal{P}(\xi)}(M)$. In case for $\mathcal{X} = \mathcal{GP}(\xi)$, $\mathcal{X}$-res.dim M coincides with ξ-$\mathcal{G}$pd M defined in [13] as ξ-Gorenstein projective dimension. We use $\widehat{\mathcal{X}}$ to denote the full subcategory of $\mathcal{T}$ whose objects have finite $\mathcal{X}$-resolution dimension.

Lemma 3.2. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a resolving subcategory of $\mathcal{T}$. For any object $M \in \mathcal{T}$, if*

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M \longrightarrow 0$$

are ξ-exact complexes with all X_i and Y_i in $\mathcal{X}$ for $0 \le i \le n-1$, then $X_n \in \mathcal{X}$ if and only if $Y_n \in \mathcal{X}$.

Proof. For $M \in \mathcal{T}$, there exists a ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_i \in \mathcal{P}(\xi)$ for $0 \le i \le n-1$.

Consider the following triangle:

$$K_1^M \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K_1^M$$

in ξ. As a similar argument to that of [11, Proposition 4.11], we get the following ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow X_{n-1} \oplus P_{n-2} \longrightarrow \cdots \longrightarrow X_2 \oplus P_1 \longrightarrow X_1 \oplus P_0 \longrightarrow X_0 \longrightarrow 0.$$

Similarly, we have the following ξ-exact complex

$$0 \longrightarrow K_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow Y_{n-1} \oplus P_{n-2} \longrightarrow \cdots \longrightarrow Y_2 \oplus P_1 \longrightarrow Y_1 \oplus P_0 \longrightarrow Y_0 \longrightarrow 0.$$

Set

$$X := \operatorname{Hoker}(X_{n-1} \oplus P_{n-2} \longrightarrow X_{n-2} \oplus P_{n-3})$$

and

$$Y := \operatorname{Hoker}(Y_{n-1} \oplus P_{n-2} \longrightarrow Y_{n-2} \oplus P_{n-3}).$$

Since $\mathcal{X}$ is resolving, we have that X and Y are objects in $\mathcal{X}$. Consider the following triangles:

$$K_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow X \longrightarrow \Sigma K_n$$

and

$$K_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow Y \longrightarrow \Sigma K_n$$

in ξ, we have that $X_n \oplus P_{n-1} \in \mathcal{X}$ if and only if $K_n \in \mathcal{X}$ if and only if $Y_n \oplus P_{n-1} \in \mathcal{X}$.

But from the following triangles in ξ

$$X_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow P_{n-1} \stackrel{0}{\longrightarrow} \Sigma X_n \quad \text{and} \quad Y_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow P_{n-1} \stackrel{0}{\longrightarrow} \Sigma Y_n,$$

we have that $X_n \in \mathcal{X}$ if and only if $X_n \oplus P_{n-1} \in \mathcal{X}$, and $Y_n \in \mathcal{X}$ if and only if $Y_n \oplus P_{n-1} \in \mathcal{X}$. Thus, $X_n \in \mathcal{X}$ if and only if $Y_n \in \mathcal{X}$. □

Using the above, we can get:

Proposition 3.3. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. Then, the following statements are equivalent*:

(1) $\mathcal{X}$-res.dim $M \le m$.
(2) $\Omega^n(M) \in \mathcal{X}$ *for* $n \ge m$.
(3) $\Omega^n_{\mathcal{X}}(M) \in \mathcal{X}$ *for* $n \ge m$.

Proof. Apply Lemma 3.2. □

Now we can compare resolution dimensions in a given triangle in ξ as follows.

Proposition 3.4. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$, and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

be a triangle in ξ. Then, we have the following statements:
(1) $\mathcal{X}$-res.dim $B \leq \max\{\mathcal{X}\text{-res.dim}\, A, \mathcal{X}\text{-res.dim}\, C\}$.
(2) $\mathcal{X}$-res.dim $A \leq \max\{\mathcal{X}\text{-res.dim}\, B, \mathcal{X}\text{-res.dim}\, C - 1\}$.
(3) $\mathcal{X}$-res.dim $C \leq \max\{\mathcal{X}\text{-res.dim}\, A + 1, \mathcal{X}\text{-res.dim}\, B\}$.

Proof. For any $A \in \mathcal{T}$, if $\mathcal{X}$-res.dim $A = m$, by Proposition 3.3, we have the following ξ-exact complex

$$0 \longrightarrow P_m^A \longrightarrow P_{m-1}^A \longrightarrow \cdots \longrightarrow P_1^A \longrightarrow P_0^A \longrightarrow A \longrightarrow 0$$

in $\mathcal{T}$ with $P_i^A \in \mathcal{P}(\xi)$ for $0 \leq i \leq m-1$ and $P_m^A \in \mathcal{X}$.

(1) Assume $\mathcal{X}$-res.dim $A = m$ and $\mathcal{X}$-res.dim $C = n$. We proceed it by induction on m and n. The case $m = n = 0$ is trivial. Without loss of generality, we assume $m \leq n$, then we can let $P_i^A = 0$ for $i > m$. As a similar argument to that of [11, Proposition 4.11], we get the following ξ-exact complex:

$$0 \longrightarrow P_n^A \oplus P_n^C \longrightarrow P_{n-1}^A \oplus P_{n-1}^C \longrightarrow \cdots \longrightarrow P_0^A \oplus P_0^C \longrightarrow B \longrightarrow 0$$

in $\mathcal{T}$. Thus, $\mathcal{X}$-res.dim $B \leq n$ and the desired assertion are obtained.

(2) Assume $\mathcal{X}$-res.dim $B = m$ and $\mathcal{X}$-res.dim $C = n$. We proceed it by induction on m and n. The case $m = n = 0$ is trivial. Without loss of generality, we assume $m \leq n-1$, then we can let $P_i^B = 0$ for $i > m$. By [18, Theorem 3.7], there exist a ξ-exact complex

$$0 \longrightarrow P_n^C \oplus P_{n-1}^B \longrightarrow P_{n-1}^C \oplus P_{n-2}^B \longrightarrow \cdots \longrightarrow P_2^C \oplus P_1^B \longrightarrow K \longrightarrow A \longrightarrow 0$$

and a triangle

$$K \longrightarrow P_1^C \oplus P_0^B \longrightarrow P_0^C \longrightarrow K[1]$$

in ξ, it follows that $K \in \mathcal{P}(\xi)$ by Remark 2.7. Thus, $\mathcal{X}$-res.dim $A \leq n-1$ and the desired assertion is obtained.

(3) Assume $\mathcal{X}$-res.dim $A = m$ and $\mathcal{X}$-res.dim $B = n$. We proceed it by induction on m and n. The case $m = n = 0$ is trivial. Without loss of generality, we assume $m + 1 \leq n$, then we can let $P_i^A = 0$ for $i > m$. By [18, Theorem 3.8], we have the following ξ-exact complex

$$0 \longrightarrow P_n^B \oplus P_{n-1}^A \longrightarrow \cdots \longrightarrow P_2^B \oplus P_1^A \longrightarrow P_1^B \oplus P_0^A \longrightarrow P_0^B \longrightarrow C \longrightarrow 0$$

in $\mathcal{T}$, thus $\mathcal{X}$-res.dim $A \leq n$ and the desired assertion is obtained. □

As direct results, we have the following closure properties for the subcategory $\widehat{\mathcal{X}}$.

Remark 3.5. If $\mathcal{X}$ is a resolving subcategory of $\mathcal{T}$, then $\widehat{\mathcal{X}}$ is closed under hokernels of ξ-proper epimorphisms, hocokernels of ξ-proper monomorphisms, and ξ-extensions.

Corollary 3.6. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$, and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

be a triangle in ξ. Then, we have the following statements:
(1) (cf. [18, Proposition 3.11]) *Assume that C is an object in $\mathcal{X}$. Then*, $\mathcal{X}$-res.dim $A = \mathcal{X}$-res.dim B.
(2) *Assume that B is an object in $\mathcal{X}$. Then, either $A \in \mathcal{X}$ or else* $\mathcal{X}$-res.dim $A = \mathcal{X}$-res.dim $C - 1$.
(3) (cf. [18, Proposition 3.13]) *Assume that A is an object in $\mathcal{X}$ and neither B nor C in $\mathcal{X}$. Then*, $\mathcal{X}$-res.dim $B =$ $\mathcal{X}$-res.dim C.

Proposition 3.7. *Let $\mathcal{H}$ and $\mathcal{X}$ be two subcategories of $\mathcal{T}$ with $\mathcal{H} \subseteq \mathcal{X}$.*

(1) $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{X}}$.

(2) *If $\mathcal{X}$ is resolving, then for any $M \in \widehat{\mathcal{H}}$, $\mathcal{H}$-res.dim $M = \mathcal{X}$-res.dim M if and only if $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.*

In particular, if $\mathcal{X} \perp \mathcal{H}$, and $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands, then $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.

Proof.

(1) It is clear.

(2) ($\Rightarrow$) Clearly, $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap \mathcal{X}$. Let $M \in \widehat{\mathcal{H}} \cap \mathcal{X}$. By the assumption, we have $\mathcal{H}$-res.dim $M = \mathcal{X}$-res.dim $M = 0$, then $M \in \mathcal{H}$, so $\widehat{\mathcal{H}} \cap \mathcal{X} \subseteq \mathcal{H}$. Thus, $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.

($\Leftarrow$) Let $M \in \widehat{\mathcal{H}}$. Suppose $\mathcal{H}$-res.dim $M = n$ and $\mathcal{X}$-res.dim $M = m$. Clearly, $m \leq n$. Consider the following ξ-exact complexes:

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $H_i \in \mathcal{H}$ and $X_j \in \mathcal{X}$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Since $\mathcal{H} \subseteq \mathcal{X}$, we have $\Omega^m_{\mathcal{H}}(M) \in \mathcal{X}$ by Lemma 3.2. Then, $\Omega^m_{\mathcal{H}}(M) \in \widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$, and thus, $\mathcal{H}$-res.dim $M \leq m$ and the desired equality is obtained.

Now, we assume that $\mathcal{X} \perp \mathcal{H}$ and $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. Clearly, $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap \mathcal{X}$. Conversely, let $M \in \widehat{\mathcal{H}} \cap \mathcal{X}$. There exists a ξ-exact complex

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0.$$

Set $K_i = \operatorname{Hoker}(H_i \to H_{i-1})$ for $0 \leq i \leq n-2$, where $H_{-1} = M$. Since $\mathcal{X}$ is resolving, we have $K_i \in \mathcal{X}$, and hence, $K_i \in \widehat{\mathcal{H}} \cap \mathcal{X}$. Consider the following triangle:

$$H_n \longrightarrow H_{n-1} \longrightarrow K_{n-2} \longrightarrow \Sigma H_n \tag{1}$$

in ξ. Since $\xi xt^1_{\xi}(K_{n-2}, H_n) = 0$ by the assumption that $\mathcal{X} \perp \mathcal{H}$, we have that the triangle (1) is split. It follows that $H_{n-1} \cong H_n \oplus K_{n-2}$ and there exists a triangle

$$K_{n-2} \longrightarrow H_{n-1} \longrightarrow H_n \xrightarrow{0} \Sigma K_{n-2}$$

in ξ. Since $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands by assumption, we have $K_{n-2} \in \mathcal{H}$. Repeating this process, we can obtain each $K_i \in \mathcal{H}$, hence, $M \in \mathcal{H}$ and $\widehat{\mathcal{H}} \cap \mathcal{X} \subseteq \mathcal{H}$. Thus, $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$. □

Now we give the following definition.

Definition 3.8. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and M an object in $\mathcal{T}$. A ξ-proper epimorphism $X \longrightarrow M$ is said to be a right $\mathcal{X}$-approximation of M if $\operatorname{Hom}_{\mathcal{T}}(\widetilde{X}, X) \longrightarrow \operatorname{Hom}_{\mathcal{T}}(\widetilde{X}, M) \longrightarrow 0$ is exact for any $\widetilde{X} \in \mathcal{X}$. In this case, there is a triangle $K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$ in ξ.

We need the following easy and useful observation.

Lemma 3.9. *Let $\mathcal{H}$ and $\mathcal{X}$ be two subcategories of $\mathcal{T}$.*

(1) *If $\mathcal{X} \perp \mathcal{H}$, then $\mathcal{X} \perp \widehat{\mathcal{H}}$. In particular, if $\mathcal{H} \perp \mathcal{H}$, then $\mathcal{H} \perp \widehat{\mathcal{H}}$.*

(2) *If $M \in {}^{\perp}\mathcal{H}$, then $M \in {}^{\perp}\widehat{\mathcal{H}}$.*

Proof. Apply Remark 2.10. □

The following is an analogous theory of Auslander-Buchweitz approximations (see [4,5]).

Proposition 3.10. *Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ closed under ξ-extensions, and let $\mathcal{H}$ be a subcategory of $\mathcal{T}$ such that $\mathcal{H}$ is a ξ-cogenerator of $\mathcal{X}$. Then, for each $M \in \mathcal{T}$ with $\mathcal{X}\text{-res.dim}\, M = n < \infty$, there exist two triangles*

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{2}$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M \tag{3}$$

in ξ, where $X, X' \in \mathcal{X}$, $\mathcal{H}\text{-res.dim}\, K \leq n-1$ and $\mathcal{H}\text{-res.dim}\, W \leq n$ (if $n = 0$, this should be interpreted as $K = 0$).

In particular, if $\mathcal{X} \perp \mathcal{H}$, then the ξ-proper epimorphism $X \longrightarrow M$ is a right $\mathcal{X}$-approximation of M.

Proof. We proceed by induction on n. The case for $n = 0$ is trivial. If $n = 1$, there exists a triangle

$$X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma X_1 \tag{4}$$

in ξ with $X_0, X_1 \in \mathcal{X}$. Since $\mathcal{H}$ is a ξ-cogenerator of $\mathcal{X}$, there is a triangle

$$X_1 \longrightarrow H \longrightarrow X_1' \longrightarrow \Sigma X_1$$

in ξ with $H \in \mathcal{H}$ and $X_1' \in \mathcal{X}$. Applying cobase change for the triangle (4) along the morphism $X_1 \longrightarrow H$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X_1' & \overset{=}{\longrightarrow} & \Sigma^{-1}X_1' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
\Sigma^{-1}M & \longrightarrow & H & \overset{u}{\longrightarrow} & X_0' & \longrightarrow & M \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow \\
0 & \longrightarrow & X_1' & \overset{=}{\longrightarrow} & X_1' & \longrightarrow & 0.
\end{array}$$

Since ξ is closed under cobase changes, we obtain that the triangle

$$H \longrightarrow X_0' \longrightarrow M \longrightarrow \Sigma H \tag{5}$$

is in ξ with $\mathcal{H}\text{-res.dim}\, H = 0$. Note that $\alpha' u = \alpha$ is ξ-proper epic, so we have that α' is ξ-proper epic by [16, Proposition 2.7]; hence, the triangle

$$X_0 \longrightarrow X_0' \longrightarrow X_1' \longrightarrow \Sigma X_0$$

is in ξ. Since $\mathcal{X}$ is closed under ξ-extensions by assumption, we have $X_0' \in \mathcal{X}$. So, (5) is the first desired triangle.

For X_0', there is a triangle

$$X_0' \longrightarrow H_0 \longrightarrow X_0'' \longrightarrow \Sigma X_0'$$

in ξ with $H_0 \in \mathcal{H}$ and $X_0'' \in \mathcal{X}$. Applying cobase change for the triangle (5) along the morphism $X_0' \longrightarrow H_0$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X_0'' & \overset{=}{\longrightarrow} & \Sigma^{-1}X_0'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H & \overset{u}{\longrightarrow} & X_0' & \longrightarrow & M & \longrightarrow & \Sigma H \\
\| \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow \| \\
H & \overset{u'}{\longrightarrow} & H_0 & \overset{v'}{\longrightarrow} & U & \longrightarrow & \Sigma H \\
\downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma'} & & \downarrow \\
0 & \longrightarrow & X_0'' & \overset{=}{\longrightarrow} & X_0'' & \longrightarrow & 0.
\end{array} \tag{6}$$

Note that $u' = \beta u$ is ξ-proper monic by [16, Proposition 2.6], so the third horizontal triangle is in ξ. Since $y'v' = y$ is ξ-proper epic, y' is ξ-proper epic by [16, Proposition 2.7]. So the triangle

$$M \longrightarrow U \longrightarrow X_0'' \longrightarrow \Sigma M$$

is in ξ with $\mathcal{H}$-res.dim $U \leq 1$ and $X_0'' \in \mathcal{X}$, which is the second desired triangle.

Now suppose $n \geq 2$. Then, there is a triangle

$$K' \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K' \tag{7}$$

in ξ with $\mathcal{X}$-res.dim $K' \leq n-1$ and $X_0 \in \mathcal{X}$. For K', by the induction hypothesis, we get a triangle

$$K' \longrightarrow K \longrightarrow X_2 \longrightarrow \Sigma K'$$

in ξ with $\mathcal{H}$-res.dim $K \leq n-1$ and $X_2 \in \mathcal{X}$. Applying cobase change for the triangle (7) along the morphism $K' \longrightarrow K$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X_2 & \overset{=}{\longrightarrow} & \Sigma^{-1}X_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M & \longrightarrow & K' & \longrightarrow & X_0 & \longrightarrow & M \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}M & \longrightarrow & K & \overset{\kappa}{\longrightarrow} & X & \longrightarrow & M \\
\downarrow & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda'} & & \downarrow \\
0 & \longrightarrow & X_2 & \overset{=}{\longrightarrow} & X_2 & \longrightarrow & 0.
\end{array}$$

Note that $\lambda'\kappa = \lambda$ is ξ-proper epic, then λ' is ξ-proper epic by [16, Proposition 2.7], so the triangle

$$X_0 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_0$$

is in ξ. It follows that $X \in \mathcal{X}$ from the assumption that $\mathcal{X}$ is closed under ξ-extensions. Since ξ is closed under cobase changes, we obtain the first desired triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{8}$$

in ξ with $\mathcal{H}$-res.dim $K \leq n-1$ and $X \in \mathcal{X}$.

For X, since $\mathcal{H}$ is a ξ-cogenerator of $\mathcal{X}$, we get the following triangle

$$X \longrightarrow H_1 \longrightarrow X' \longrightarrow \Sigma X$$

in ξ with $H_1 \in \mathcal{H}$ and $X' \in \mathcal{X}$.

Applying cobase change for the triangle (8) along the morphism $X \longrightarrow H_1$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X' & \overset{=}{\longrightarrow} & \Sigma^{-1}X' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & X & \longrightarrow & M & \longrightarrow & \Sigma K \\
\| & & \downarrow & & \downarrow & & \| \\
K & \longrightarrow & H_1 & \longrightarrow & W & \longrightarrow & \Sigma K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X' & \overset{=}{\longrightarrow} & X' & \longrightarrow & 0.
\end{array}$$

As a similar argument to that of the diagram (6), we obtain that the triangles

$$K \longrightarrow H_1 \longrightarrow W \longrightarrow \Sigma K$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M \tag{9}$$

are in ξ. Thus, (9) is the second desired triangle in ξ with $\mathcal{H}$-res.dim $W \leq n$ and $X' \in \mathcal{X}$.

In particular, suppose $\mathcal{X} \perp \mathcal{H}$, by Lemma 3.9, we have $\mathcal{X} \perp \widehat{\mathcal{H}}$. Then, $\xi xt^1_\xi(\widetilde{X}, K) = 0$ for any $\widetilde{X} \in \mathcal{X}$, it follows that $\mathrm{Hom}_\mathcal{T}(\widetilde{X}, X) \longrightarrow \mathrm{Hom}_\mathcal{T}(\widetilde{X}, M) \longrightarrow 0$ is exact. Thus, the ξ-proper epimorphism $X \longrightarrow M$ is a right $\mathcal{X}$-approximation of M. □

Proposition 3.11. *Keep the notion as* Proposition 3.10. *Assume $M \in \widehat{\mathcal{X}}$ with $\mathcal{X}$-res.dim $M = n < \infty$.*

(1) *If $\mathcal{X}$ is resolving, then in the triangles* (2) *and* (3), *we have $\mathcal{H}$-res.dim $K = n - 1$ and $\mathcal{H}$-res.dim $W = \mathcal{X}$-res.dim $W = n$.*

In particular, if $\mathcal{X} \perp \mathcal{H}$, then the ξ-proper epimorphism $X \to M$ in the triangle (2) *is a right $\mathcal{X}$-approximation of M, such that $\mathcal{H}$-res.dim $K = n - 1$ (if $n = 0$, it should be interpreted $K = 0$).*

(2) *If $\mathcal{X} \perp \mathcal{H}$ and $\mathcal{X}$ is resolving, then there is a triangle*

$$M \longrightarrow M' \longrightarrow X \longrightarrow \Sigma M$$

in ξ with $M' \in \mathcal{X}^\perp$, $X \in \mathcal{X}$ and $\mathcal{X}$-res.dim $M = \mathcal{X}$-res.dim M'.

(3) (*a*) *Let $\omega_\mathcal{H} = \mathcal{H}^\perp \cap \mathcal{H}$. If $\omega_\mathcal{H}$ is a ξ-cogenerator of $\mathcal{H}$ and $\mathcal{H}$ is closed under ξ-extensions, then $\mathcal{X} \perp \omega_\mathcal{H}$ if and only if $\mathcal{X} \perp (\mathcal{H}^\perp \cap \widehat{\mathcal{H}})$.*

(*b*) *If $\mathcal{X}$ is a resolving and $\omega_\mathcal{X} = \mathcal{X} \cap \mathcal{X}^\perp$ is a ξ-cogenerator of $\mathcal{X}$ and $M \in \mathcal{X}^\perp$, then $\mathcal{X}$-res.dim $M = \omega_\mathcal{X}$-res.dim M.*

(4) *Suppose that $\mathcal{H}$ and $\mathcal{X}$ are resolving. If $\omega_\mathcal{H} = \mathcal{H} \cap \mathcal{H}^\perp$ is a ξ-cogenerator of $\mathcal{H}$ and $\mathcal{X} \perp \omega_\mathcal{H}$, then M admits a right $\mathcal{X}$-approximation $X' \longrightarrow M$ such that $K'' \longrightarrow X' \longrightarrow M \longrightarrow \Sigma K''$ is a triangle in ξ, where $\mathcal{H}$-res.dim $K'' = n - 1$. In fact, we have $\omega_\mathcal{H}$-res.dim $K'' = n - 1$.*

Proof.

(1) Suppose $\mathcal{X}$ is resolving. Applying Corollary 3.6(2) to the triangle (2) yields that $\mathcal{X}$-res.dim $K = n - 1$. On the other hand, since $\mathcal{H} \subseteq \mathcal{X}$, we have $n - 1 = \mathcal{X}$-res.dim $K \le \mathcal{H}$-res.dim $K \le n - 1$. Thus, $\mathcal{H}$-res.dim $K = n - 1$.

Moreover, applying Corollary 3.6(1) to the triangle (3) implies $\mathcal{X}$-res.dim $W = \mathcal{X}$-res.dim $M = n$. So, $n = \mathcal{X}$-res.dim $W \le \mathcal{H}$-res.dim $W \le n$. Hence, $\mathcal{H}$-res.dim $W = \mathcal{X}$-res.dim $W = n$.

The last assertion follows from the above argument and Proposition 3.10.

(2) Since $\mathcal{X} \perp \mathcal{H}$, we have $\mathcal{X} \perp \widehat{\mathcal{H}}$ by Lemma 3.9, and so the result immediately follows from (1).

(3) (a) (⇐) Suppose $\mathcal{X} \perp (\mathcal{H}^\perp \cap \widehat{\mathcal{H}})$. Clearly, $\omega_\mathcal{H} = \mathcal{H}^\perp \cap \mathcal{H} \subseteq \mathcal{H}^\perp \cap \widehat{\mathcal{H}} \subseteq \mathcal{X}^\perp$, that is, $\mathcal{X} \perp \omega_\mathcal{H}$.

(⇒) Suppose $\mathcal{X} \perp \omega_\mathcal{H}$. Let $L \in \mathcal{H}^\perp \cap \widehat{\mathcal{H}}$. By Proposition 3.10, there exists a triangle

$$K' \longrightarrow H_0 \longrightarrow L \longrightarrow \Sigma K'$$

in ξ with $H_0 \in \mathcal{H}$ and $\omega_\mathcal{H}$-res.dim $K' \le \mathcal{H}$-res.dim $L - 1 < \infty$. Note that $K' \in \mathcal{H}^\perp$ by Lemma 3.9, so $L \in \mathcal{H}^\perp$ implies $H_0 \in \mathcal{H}^\perp$. Then, $H_0 \in \omega_\mathcal{H}$, and so, $L \in \widehat{\omega_\mathcal{H}}$. Since $\mathcal{X} \perp \omega_\mathcal{H}$, we have $L \in \mathcal{X}^\perp$ by Lemma 3.9. Thus, $\mathcal{X} \perp (\mathcal{H}^\perp \cap \widehat{\mathcal{H}})$.

(b) Suppose $\mathcal{X}$-res.dim $M = n$, by (1), there exists a triangle

$$K \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K$$

in ξ with $X_0 \in \mathcal{X}$ and $\omega_\mathcal{X}$-res.dim $K = n - 1$. Note that $M \in \mathcal{X}^\perp$ and $K \in \mathcal{X}^\perp$, so $X_0 \in \mathcal{X}^\perp$, and hence, $X_0 \in \omega_\mathcal{X}$. It follows that $\omega_\mathcal{X}$-res.dim $M \le n$. But $n = \mathcal{X}$-res.dim $M \le \omega_\mathcal{X}$-res.dim $M \le n$, thus $\mathcal{X}$-res.dim $M = \omega_\mathcal{X}$-res.dim M.

(4) Suppose $\mathcal{X}$-res.dim $M = n$, by (1), there exists a triangle

$$K \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K \tag{10}$$

in ξ with $X_0 \in \mathcal{X}$ and $\mathcal{H}$-res.dim $K = n - 1$. By (2), there is a triangle

$$K \longrightarrow K'' \longrightarrow H \longrightarrow \Sigma K$$

in ξ with $H \in \mathcal{H}$, $K'' \in \mathcal{H}^{\perp}$ and $\mathcal{H}$-res.dim $K'' = \mathcal{H}$-res.dim K. Then, $K'' \in \mathcal{H}^{\perp} \cap \widehat{\mathcal{H}}$. Applying cobase change for the triangle (10) along the morphism $K \longrightarrow K''$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}H & \xrightarrow{=} & \Sigma^{-1}H & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M & \longrightarrow & K & \longrightarrow & X_0 & \longrightarrow & M \\
\| \downarrow & & \downarrow & & \downarrow & & \downarrow \| \\
\Sigma^{-1}M & \longrightarrow & K'' & \longrightarrow & X' & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H & \xrightarrow{=} & H & \longrightarrow & 0.
\end{array}$$

One can see that the triangle

$$K'' \longrightarrow X' \longrightarrow M \longrightarrow \Sigma K'' \tag{11}$$

is in ξ and $X' \in \mathcal{X}$. Note that $\mathcal{X} \perp \omega_{\mathcal{H}}$, so $\mathcal{X} \perp \mathcal{H}^{\perp} \cap \widehat{\mathcal{H}}$ by (3)(a). Then, $\xi xt^1_{\xi}(\widetilde{X}, K'') = 0$ for any $\widetilde{X} \in \mathcal{X}$, and so, $\mathrm{Hom}_{\mathcal{T}}(\widetilde{X}, X') \longrightarrow \mathrm{Hom}_{\mathcal{T}}(\widetilde{X}, M) \longrightarrow 0$ is exact. Thus, the ξ-proper epimorphism $X' \longrightarrow M$ is a right $\mathcal{X}$-approximation of M and $\mathcal{H}$-res.dim $K'' = n-1$ in the triangle (11). Note that $K'' \in \mathcal{H}^{\perp}$, so we have $\omega_{\mathcal{H}}$-res.dim $K'' = \mathcal{H}$-res.dim $K'' = n-1$ by (3)(b). ☐

Lemma 3.12. *Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$ with $\mathcal{H} \perp \mathcal{H}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. Then, $\mathcal{H} = \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H}$.*

Proof. Clearly, $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H}$.

Conversely, let $M \in \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H}$. Consider the following ξ-exact complex:

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0.$$

Set $K_i = \mathrm{Hoker}\,(H_i \to H_{i-1})$ for $0 \le i \le n-2$, where $H_{-1} = M$. Then, $M \in {}^{\perp}\mathcal{H}$ yields $K_i \in {}^{\perp}\mathcal{H}$, and so the triangle

$$H_n \longrightarrow H_{n-1} \longrightarrow K_{n-2} \longrightarrow \Sigma H_n$$

is split. It follows that $H_{n-1} \cong H_n \oplus K_{n-2}$ and there exists a triangle

$$K_{n-2} \longrightarrow H_{n-1} \longrightarrow H_n \xrightarrow{0} \Sigma K_{n-2}$$

in ξ. Since $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands by assumption, we have $K_{n-2} \in \mathcal{H}$. Repeating this process, we can obtain $K_i \in \mathcal{H}$, hence $M \in \mathcal{H}$ and $\widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H} \subseteq \mathcal{H}$. Thus, $\widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H} = \mathcal{H}$. ☐

Proposition 3.13. *Let $\mathcal{X}$ be a resolving subcategory and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. Then, $\mathcal{X} = \widehat{\mathcal{X}} \cap {}^{\perp}\widehat{\mathcal{H}} = \widehat{\mathcal{X}} \cap {}^{\perp}\mathcal{H}$.*

Proof. Clearly, $\mathcal{X} \subseteq \widehat{\mathcal{X}} \cap {}^{\perp}\mathcal{H}$ and $\widehat{\mathcal{X}} \cap {}^{\perp}\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{X}} \cap {}^{\perp}\mathcal{H}$.

Now, let $M \in \widehat{\mathcal{X}} \cap {}^{\perp}\mathcal{H}$. Then, by Lemma 3.9, we have $M \in \widehat{\mathcal{X}} \cap {}^{\perp}\widehat{\mathcal{H}}$, and hence, $\widehat{\mathcal{X}} \cap {}^{\perp}\mathcal{H} \subseteq \widehat{\mathcal{X}} \cap {}^{\perp}\widehat{\mathcal{H}}$.

On the other hand, by Proposition 3.10, there is a triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{12}$$

in ξ with $X \in \mathcal{X}$ and $\mathcal{H}$-res.dim $K < \infty$. Note that $M \in {}^{\perp}\mathcal{H}$ implies $K \in {}^{\perp}\mathcal{H}$, and hence, $K \in \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H} = \mathcal{H}$ by Lemma 3.12. Note that $\xi xt^1_{\xi}(M, K) = 0$, so the triangle (12) is split; hence, $X \cong K \oplus M$. Consider the following triangle

$$M \longrightarrow X \longrightarrow K \xrightarrow{0} \Sigma M$$

in ξ. It follows that $M \in \mathcal{X}$ from the assumption that $\mathcal{X}$ is resolving. Thus, $\widehat{\mathcal{X}} \cap {}^{\perp}\mathcal{H} \subseteq \mathcal{X}$. □

Our main result is the following.

Theorem 3.14. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of ξ-proper epimorphisms or closed under direct summands. For any $M \in \mathcal{T}$, if $M \in \widehat{\mathcal{X}}$, then the following statements are equivalent*:

(1) $\mathcal{X}$-res.dim $M \leq m$.
(2) $\Omega^n(M) \in \mathcal{X}$ *for all* $n \geq m$.
(3) $\Omega^n_{\mathcal{X}}(M) \in \mathcal{X}$ *for all* $n \geq m$.
(4) $\xi xt^n_\xi(M, H) = 0$ *for all* $n > m$ *and all* $H \in \mathcal{H}$.
(5) $\xi xt^n_\xi(M, L) = 0$ *for all* $n > m$ *and all* $L \in \widehat{\mathcal{H}}$.
(6) *M admits a right $\mathcal{X}$-approximation $\varphi : X \to M$, where φ is ξ-proper epic, such that $K =$ Hoker φ satisfying* $\mathcal{H}$-res.dim $K \leq m - 1$.
(7) *There are two triangles*

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in ξ such that $X_M, X^M \in \mathcal{X}$ and $\mathcal{H}$-res.dim $W_M \leq m - 1$, $\mathcal{H}$-res.dim $W^M = \mathcal{X}$-res.dim $W^M \leq m$.

Proof. (1) ⇔ (2) ⇔ (3) It follows from Proposition 3.3.
(1) ⇔ (6) It follows from Proposition 3.11(1).
(1) ⇔ (7) It follows from Proposition 3.11(1).
(1) ⇒ (4) Suppose $\mathcal{X}$-res.dim $M \leq m$. There is a ξ-exact complex

$$0 \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with all X_i in $\mathcal{X}$. Since $\mathcal{H}$ is a ξxt-injective ξ-cogenerator of $\mathcal{X}$, we have $\xi xt^{k\geq 1}_\xi(X_i, H) = 0$ for all $H \in \mathcal{H}$. So, $\xi xt^n_\xi(M, H) \cong \xi xt^{n-m}_\xi(X_m, H) = 0$ for $n > m$.
(4) ⇒ (5) It follows from Lemma 3.9.
(5) ⇒ (4) It is clear.
(4) ⇒ (1) Since $M \in \widehat{\mathcal{X}}$, by Proposition 3.11(1), there is a triangle $K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$ in ξ with $\mathcal{H}$-res.dim $K < \infty$ and $X \in \mathcal{X}$. Then, $\xi xt^i_\xi(K, H) \cong \xi xt^{i+1}_\xi(M, H)$ for $H \in \mathcal{H}$ and $i \geq 1$ since $\xi xt^{i\geq 1}_\xi(X, H) = 0$. So, $\xi xt^{i\geq m}_\xi(K, H) = 0$. Note that $\mathcal{H}$-res.dim $K < \infty$, so we have the following ξ-exact complex

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_0 \longrightarrow K \longrightarrow 0$$

with all $H_i \in \mathcal{H}$. Then,

$$\xi xt^i_\xi(\Omega^{m-1}_{\mathcal{H}}(K), H) \cong \xi xt^{i+m-1}_\xi(K, H) = 0$$

for $i \geq 1$ and all $H \in \mathcal{H}$, which means $\Omega^{m-1}_{\mathcal{H}}(K) \in {}^{\perp}\mathcal{H}$. Note that $\mathcal{H}$-res.dim $\Omega^{m-1}_{\mathcal{H}}(K) < \infty$, hence, $\Omega^{m-1}_{\mathcal{H}}(K) \in \widehat{\mathcal{H}} \cap {}^{\perp}\mathcal{H}$. It follows that $\Omega^{m-1}_{\mathcal{H}}(K) \in \mathcal{H}$ from Lemma 3.12, so $\mathcal{H}$-res.dim $K \leq m - 1$. Thus, $\mathcal{X}$-res.dim $M \leq m$. □

4 Additive quotient categories and ξ-cellular towers with respect to a resolving subcategory

In this section, we will further study objects having finite resolution dimension with respect to a resolving subcategory $\mathcal{X}$. We first construct adjoint pairs for two kinds of inclusion functors. Then, we characterize objects having finite resolution dimension in terms of a notion of ξ-cellular towers.

4.1 Adjoint pairs

Suppose that $\mathcal{D}$ and $\mathcal{X}$ are two subcategories of $\mathcal{T}$. Denote by $[\mathcal{D}]$ the ideal of $\mathcal{X}$ consisting of morphisms factoring through some object in $\mathcal{D}$. Thus, we have a quotient category $\mathcal{X}/[\mathcal{D}]$, which is also an additive category.

Lemma 4.1. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Assume that $f : X \longrightarrow M$ is a morphism in $\mathcal{T}$ with $X \in \mathcal{X}$ and $M \in \widehat{\mathcal{X}}$, then the following statements are equivalent:*
(1) *f factors through an object in $\mathcal{H}$.*
(2) *f factors through an object in $\widehat{\mathcal{H}}$.*

Proof. It suffices to show that (2) $\Rightarrow$ (1). Suppose that f factors through an object $L \in \widehat{\mathcal{H}}$. Then, $f = gh$, where $h : X \to L$ and $g : L \to M$. Consider the following triangle

$$L' \longrightarrow H \longrightarrow L \longrightarrow \Sigma L'$$

in ξ with $H \in \mathcal{H}$ and $L' \in \widehat{\mathcal{H}}$. Note that $\mathcal{H}$ is a ξxt-injective ξ-cogenerator of $\mathcal{X}$, by Lemma 3.9, we have $\xi xt^1_\xi(X, L') = 0$. So, h factors through H, it follows that f factors through H. □

Lemma 4.2. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$, and let $M, N \in \widehat{\mathcal{X}}$. Assume that $f : M \to N$ is a morphism in $\mathcal{T}$, consider two triangles*

$$W_M \xrightarrow{\alpha} X_M \xrightarrow{p} M \longrightarrow \Sigma W_M \quad \text{and} \quad W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \longrightarrow \Sigma W_N$$

in ξ with $X_M, X_N \in \mathcal{X}$ and $W_M, W_N \in \widehat{\mathcal{H}}$ (see Proposition 3.10*), then we have the following statements:*
(1) *There exists a morphism $g : X_M \to X_N$ such that $qg = fp$.*
(2) *If $g, g' : X_M \to X_N$ are two morphisms such that $qg = fp$ and $qg' = fp$, then $[g] = [g']$ in* $\mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N)$.

Proof.
(1) Apply Proposition 3.10.
(2) Suppose $g, g' : X_M \to X_N$ are two morphisms such that $qg = fp$ and $qg' = fp$, then $q(g' - g) = qg' - qg = 0$, and so there exists a morphism $h : X_M \to W_N$ such that $g' - g = \beta h$, that is, there is a commutative diagram as follows:

$$\begin{array}{ccccccc} & & X_M & & & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle g'-g} & & & & \\ W_N & \xrightarrow{\beta} & X_N & \xrightarrow{q} & N & \longrightarrow & \Sigma W_N. \end{array}$$

Note that $W_N \in \widehat{\mathcal{H}}$, so $g' - g : X_M \to X_N$ factors through an object in $\mathcal{H}$ by Lemma 4.1. Thus, $[g] = [g']$ in $\mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N)$. □

By Lemma 4.2, there exists a well-defined additive functor

$$F : \widehat{\mathcal{X}} \to \mathcal{X}/[\mathcal{H}],$$

which maps an object $M \in \widehat{\mathcal{X}}$ to X_M and a morphism $f : M \to N \in \mathrm{Hom}_{\widehat{\mathcal{X}}}(M, N)$ to $[g] \in \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N)$ as described in Lemma 4.2.

Clearly, we have $F(H) = 0$ for any object $H \in \mathcal{H}$. Hence, F factors through $\widehat{\mathcal{X}}/[\mathcal{H}]$. That is, there exists an additive functor $\mu : \widehat{\mathcal{X}}/[\mathcal{H}] \to \mathcal{X}/[\mathcal{H}]$ making the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathcal{X}} & \xrightarrow{F} & \mathcal{X}/[\mathcal{H}] \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\mu} & \\ & \widehat{\mathcal{X}}/[\mathcal{H}], & \end{array}$$

where π is the canonical quotient functor.

Now we show that the additive functor μ defined above and the inclusion functor between additive quotients $\mathcal{X}/[\mathcal{H}]$ and $\widehat{\mathcal{X}}/[\mathcal{H}]$ are adjoint.

Theorem 4.3. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Then, the additive functor $\mu : \widehat{\mathcal{X}}/[\mathcal{H}] \longrightarrow \mathcal{X}/[\mathcal{H}]$ defined above is right adjoint to the inclusion functor $\mathcal{X}/[\mathcal{H}] \longrightarrow \widehat{\mathcal{X}}/[\mathcal{H}]$.*

Proof. Let $X \in \mathcal{X}$ and $N \in \widehat{\mathcal{X}}$. By Proposition 3.10, there is a triangle

$$W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \longrightarrow \Sigma W_N$$

in ξ with $W_N \in \widehat{\mathcal{H}}$ and $X_N \in \mathcal{X}$. Note that the additive map

$$[q]_* : \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, \mu(N)) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X, N)$$

is natural in both X and N by Lemma 4.2. We claim that $[q]_*$ is an isomorphism.

Indeed, since $\mathcal{H}$ is a ξxt-injective ξ-cogenerator of $\mathcal{X}$, by Lemma 3.9, we have $\xi xt^1_\xi(X, W_N) = 0$, and hence, $\mathrm{Hom}_\mathcal{T}(X, X_N) \to \mathrm{Hom}_\mathcal{T}(X, N)$ is an epimorphism, so $[q]_*$ is still an epimorphism.

Now, assume that $g : X \to X_N$ is a morphism such that $[qg] = [q][g] = [q]_*[g] = [0] \in \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$. Then, there exists an object $H \in \mathcal{H}$ such that $qg = ts$ as the following commutative diagram:

$$\begin{array}{ccccccc} & & X & \overset{s}{\dashrightarrow} & H & & \\ & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle\theta} & \downarrow{\scriptstyle t} & & \\ W_N & \xrightarrow{\beta} & X_N & \xrightarrow{q} & N & \longrightarrow & \Sigma W_N. \end{array}$$

Note that $\xi xt^1_\xi(H, W_N) = 0$ by assumption, so there exists a morphism $\theta : H \to X_N$ such that $t = q\theta$. Since $q(g - \theta s) = qg - q\theta s = ts - ts = 0$, so $g - \theta s$ factors through W_N. By Lemma 4.1, $g - \theta s$ factors through an object in $\mathcal{H}$. It follows that $[g - \theta s] = 0 \in \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$. Since $\theta s = 0 \in \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$, we have $0 = [g] \in \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$. So $[q]_*$ is a monomorphism, and thus, $[q]_*$ is an isomorphism. □

Corollary 4.4. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under direct summands. For any $N \in \widehat{\mathcal{X}}$, the following statements are equivalent:*

(1) $N \in \widehat{\mathcal{H}}$.

(2) *There is a triangle*

$$W_N \longrightarrow X_N \xrightarrow{q} N \longrightarrow \Sigma W_N$$

in ξ with $W_N \in \widehat{\mathcal{H}}$ and $X_N \in \mathcal{X}$ such that $[q] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X, N)$.

Proof. The assertion (1) ⇒ (2) follows from Lemma 4.1. It suffices to show (2) ⇒ (1). Note that the adjunction isomorphism established in Theorem 4.3 implies that the additive map

$$[q]_* : \mathrm{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_N, X_N) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X_N, N)$$

is isomorphic. Since $[q]_*[\mathrm{id}_{X_N}] = [q\mathrm{id}_{X_N}] = [q] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X_N, N) = 0$, so $[\mathrm{id}_{X_N}] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X_N, X_N)$, and thus, id_{X_N} factors through an object $H \in \mathcal{H}$. It follows that X_N is a direct summand of W_N. Since $\mathcal{H}$ is closed under direct summands, we have $X_N \in \mathcal{H}$. Thus, $N \in \widehat{\mathcal{H}}$. □

Next, we compare additive quotients $\widehat{\mathcal{H}}/[\mathcal{X}]$ and $\widehat{\mathcal{X}}/[\mathcal{X}]$.

Lemma 4.5. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$, and let $M, N \in \widehat{\mathcal{X}}$. Assume that $f : M \to N$ is a morphism in $\mathcal{T}$, consider two triangles*

$$M \xrightarrow{s} W^M \xrightarrow{l} X^M \longrightarrow \Sigma M \quad \textit{and} \quad N \xrightarrow{t} W^N \xrightarrow{r} X^N \longrightarrow \Sigma N$$

in ξ with $X^M, X^N \in \mathcal{X}$ and $W^M, W^N \in \widehat{\mathcal{H}}$ (see Proposition 3.10*), then, we have the following statements*:
1. *There exists a morphism $g : W^M \to W^N$ such that $gs = tf$.*
2. *If $g, g' : W^M \longrightarrow W^N$ are two morphisms such that $gs = tf$ and $g's = tf$, then $[g] = [g']$ in $\mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(X_M, X_N)$.*

Proof.
(1) Since $\mathcal{X} \perp \mathcal{H}$ by assumption, we have $\xi xt^1_{\xi}(X^M, W^N) = 0$ by Lemma 3.9. So, there exists a morphism $g : W^M \longrightarrow W^N$ such that $gs = tf$.
(2) Suppose $g, g' : W^M \longrightarrow W^N$ are two morphisms such that $gs = tf$ and $g's = tf$, then $(g' - g)s = g's - gs = 0$, and so there exists a morphism $h' : X^M \to W^N$ such that $g' - g = h'l$, that is, there is a commutative diagram as follows:

$$\begin{array}{ccccccc} M & \xrightarrow{s} & W^M & \xrightarrow{l} & X^M & \longrightarrow & \Sigma M \\ & & {\scriptstyle g'-g}\downarrow & \swarrow{\scriptstyle h} & & & \\ & & W^N & & & & \end{array}$$

Note that $X^M \in \mathcal{X}$, so $g' - g : W^M \to W^N$ factors through an object in $\mathcal{X}$. Thus, $[g] = [g']$ in $\mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, W^N)$. □

By Lemma 4.5, there exists a well-defined additive functor

$$G : \widehat{\mathcal{X}} \to \widehat{\mathcal{H}}/[\mathcal{X}],$$

which maps an object $M \in \widehat{\mathcal{X}}$ to W^M and a morphism $f : M \to N \in \mathrm{Hom}_{\widehat{\mathcal{X}}}(M, N)$ to $[g] \in \mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, W^N)$ as described in Lemma 4.5.

Clearly, we have $G(X) = 0$ for any object $X \in \mathcal{X}$. Hence, G factors through $\widehat{\mathcal{X}}/[\mathcal{X}]$. That is, there exists an additive functor $\eta : \widehat{\mathcal{X}}/[\mathcal{X}] \to \widehat{\mathcal{H}}/[\mathcal{X}]$ making the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathcal{X}} & \xrightarrow{G} & \widehat{\mathcal{H}}/[\mathcal{X}] \\ {\scriptstyle \pi}\searrow & & \nearrow{\scriptstyle \eta} \\ & \widehat{\mathcal{X}}/[\mathcal{X}], & \end{array}$$

where η is the canonical quotient functor.

Now we show that the additive functor η defined above and the inclusion functor between additive quotients $\widehat{\mathcal{H}}/[\mathcal{X}]$ and $\widehat{\mathcal{X}}/[\mathcal{X}]$ are adjoint.

Theorem 4.6. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Then, the additive functor $\eta : \widehat{\mathcal{X}}/[\mathcal{X}] \to \widehat{\mathcal{H}}/[\mathcal{X}]$ defined above is left adjoint to the inclusion functor $\widehat{\mathcal{H}}/[\mathcal{X}] \to \widehat{\mathcal{X}}/[\mathcal{X}]$.*

Proof. Let K be an object in $\widehat{\mathcal{H}}$ and M an object in $\widehat{\mathcal{X}}$. By Proposition 3.10, there is a triangle

$$M \xrightarrow{s} W^M \xrightarrow{l} X^M \longrightarrow \Sigma M$$

in ξ with $W^M \in \widehat{\mathcal{H}}$ and $X^M \in \mathcal{X}$. Note that the additive map

$$[s]^* : \mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(\eta(M), K) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{X}}/\mathcal{X}}(M, K)$$

is natural in both M and K by Lemma 4.5. We claim that $[s]^*$ is an isomorphism.

Indeed, since $\mathcal{H}$ is a ξxt-injective cogenerator of $\mathcal{X}$, by Lemma 3.9, we have $\xi xt^1_\xi(X^M, K) = 0$, and hence, $\mathrm{Hom}_{\mathcal{T}}(W^M, K) \to \mathrm{Hom}_{\mathcal{T}}(M, K)$ is an epimorphism, so $[s]^*$ is still an epimorphism.

Now, assume that $g : W^M \to K$ is a morphism such that $[gs] = [g][s] = [s]^*[g] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(M, K)$. Then, there exists an object $X \in \mathcal{X}$ such that $gs = kv$. Since $\mathcal{H}$ is a ξxt-injective ξ-cogenerator of $\mathcal{X}$, there exists a triangle

$$X \longrightarrow H \longrightarrow X' \longrightarrow \Sigma X$$

in ξ with $H \in \mathcal{H}$ and $X' \in \mathcal{X}$. Note that $\xi xt^1_\xi(X^M, H) = 0$ and $\xi xt^1_\xi(X', K) = 0$, so we get the following commutative diagram:

$$\begin{array}{ccccccc}
M & \xrightarrow{s} & W^M & \xrightarrow{l} & X^M & \longrightarrow & \Sigma M \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} & & & & \\
X & \longrightarrow & H & \longrightarrow & X' & \longrightarrow & \Sigma X \\
{\scriptstyle k}\downarrow & \swarrow{\scriptstyle v''} & & & & & \\
K & & & & & &
\end{array}$$

It follows that $[v''v'] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{H}}/\mathcal{X}}(W^M, K)$ as $H \in \mathcal{X}$. Since $v''v's = kv = gs \in \mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(M, K)$, by Lemma 4.5(2), we have $[g] = [v''v'] \in \mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, K)$, and hence, $[g] = 0$. So $[s]^*$ is a monomorphism, and thus, $[s]^*$ is an isomorphism. ☐

Corollary 4.7. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a ξxt-injective ξ-cogenerator of $\mathcal{X}$. Assume that $\mathcal{X}$ is closed under direct summands. For any $N \in \widehat{\mathcal{X}}$, the following statements are equivalent*:
(1) $N \in \mathcal{X}$.
(2) *There is a triangle*

$$N \xrightarrow{s} W^N \longrightarrow X^N \longrightarrow \Sigma N$$

in ξ with $W^N \in \widehat{\mathcal{H}}$ and $X^N \in \mathcal{X}$ such that $[s] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(N, W^N)$.

Proof. The assertion (1) $\Rightarrow$ (2) is obvious. It suffices to show (2) $\Rightarrow$ (1). Note that the adjunction isomorphism established in Theorem 4.6 implies that the additive map

$$[s]^* : \mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^N, W^N) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{X}}/\mathcal{X}}(N, W^N)$$

is isomorphic. Since $[s]^*[\mathrm{id}_{W^N}] = [\mathrm{id}_{W^N} s] = [s] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(N, W^N) = 0$, so $[\mathrm{id}_{W^N}] = [0] \in \mathrm{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^N, W^N)$, and thus, id_{W^N} factors through an object $X' \in \mathcal{X}$. It follows that W^N is a direct summand of X'. Since $\mathcal{X}$ is closed under direct summands, we have $W^N \in \mathcal{X}$. Thus, $N \in \mathcal{X}$. ☐

4.2 A characterization of finite resolution dimension via ξ-cellular towers

For $M \in \widehat{\mathcal{X}}$, there exists a triangle

$$K_1 \xrightarrow{f_0} X_0 \xrightarrow{g_0} M \xrightarrow{h_0} \Sigma K_1 \tag{13}$$

in ξ with $X_0 \in \mathcal{X}$ and $K_1 \in \widehat{\mathcal{X}}$. Similarly, there exists a triangle

$$K_2 \xrightarrow{f_1} X_1 \xrightarrow{g_1} K_1 \xrightarrow{h_1} \Sigma K_2$$

in ξ with $X_1 \in \mathcal{X}$ and $K_2 \in \widehat{\mathcal{X}}$. Continuing the above procedure for K_n, there exists a triangle

$$K_{n+1} \xrightarrow{f_n} X_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in ξ with $X_n \in \mathcal{X}$ and $K_{n+1} \in \widehat{\mathcal{X}}$.

Applying cobase change for the triangle (13) along the morphism $h_1 : K_1 \longrightarrow \Sigma K_2$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
\Sigma^{-1}M & \longrightarrow & K_1 & \xrightarrow{f_0} & X_0 & \xrightarrow{g_0} & M \\
\| & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \gamma_1} & & \| \\
\Sigma^{-1}M & \longrightarrow & \Sigma K_2 & \xrightarrow{u_2} & C_2 & \xrightarrow{v_2} & M,
\end{array}$$

where the triangle

$$\Sigma K_2 \xrightarrow{u_2} C_2 \xrightarrow{v_2} M \longrightarrow \Sigma^2 K_2 \tag{14}$$

is in ξ. Next consider the triangle (14) along the morphism $-\Sigma h_2 : \Sigma K_2 \longrightarrow \Sigma^2 K_3$, we get the following commutative diagram:

$$\begin{array}{ccccccc}
\Sigma^{-1}M & \longrightarrow & \Sigma K_2 & \xrightarrow{u_2} & C_2 & \xrightarrow{v_2} & M \\
\| & & \downarrow{\scriptstyle -\Sigma h_2} & & \downarrow{\scriptstyle \gamma_2} & & \| \\
\Sigma^{-1}M & \longrightarrow & \Sigma^2 K_3 & \xrightarrow{u_3} & C_3 & \xrightarrow{v_3} & M,
\end{array}$$

where the triangle $\Sigma^2 K_3 \xrightarrow{u_3} C_3 \xrightarrow{v_3} M \longrightarrow \Sigma^3 K_3$ is in ξ.

Continuing in this manner, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
K_1 & \xrightarrow{f_0} & X_0 & \xrightarrow{g_0} & M & \xrightarrow{h_0} & \Sigma K_1 \\
\downarrow & & \downarrow{\scriptstyle \gamma_1} & & \| & & \downarrow \\
\Sigma K_2 & \xrightarrow{u_2} & C_2 & \xrightarrow{v_2} & M & \longrightarrow & \Sigma^2 K_2 \\
\downarrow & & \downarrow{\scriptstyle \gamma_2} & & \| & & \downarrow \\
\Sigma^2 K_3 & \xrightarrow{u_3} & C_3 & \xrightarrow{v_3} & M & \longrightarrow & \Sigma^3 K_3 \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\Sigma^{n-1} K_n & \xrightarrow{u_n} & C_n & \xrightarrow{v_n} & M & \longrightarrow & \Sigma^n K_n \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where all the horizontal triangles are in ξ.

Set $C_0 = 0$ and $C_1 = X_0$. The above construction produces a tower

$$0 \longrightarrow C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} \cdots \longrightarrow C_{n-1} \xrightarrow{\gamma_{n-1}} C_n \cdots,$$

which we call the ξ-cellular tower of M with respect to $\mathcal{X}$.

According to the above construction, one can obtain the following result by Proposition 3.3.

Theorem 4.8. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$. For any $M \in \mathcal{T}$, if $M \in \widehat{\mathcal{X}}$, then the following statements are equivalent*:

(1) $\mathcal{X}$-res.dim $M \leq n$.

(2) *For each $i > 0$, the morphisms $v_{n+i} : C_{n+i} \to M$ of the ξ-cellular tower of M with respect to $\mathcal{X}$ constructed above are isomorphisms.*

5 Applications

In this section, we will construct a new resolving subcategory from a given resolving subcategory, which generalizes the notion of ξ-Gorenstein projective objects given by Asadollahi and Salarian [13]. By applying the previous results to this subcategory, we obtain some known results in [13–15].

Definition 5.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and M an object in $\mathcal{T}$. A complete $\mathcal{P}(\xi)\mathcal{X}$-resolution of M is a $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{X})$-exact ξ-exact complex

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

in $\mathcal{T}$ with all $P_i \in \mathcal{P}(\xi)$, $X^i \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$ such that both

$$K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow \Sigma K_1 \quad \text{and} \quad M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M$$

are corresponding triangles in ξ. The $\mathcal{GP}_{\mathcal{X}}(\xi)$-*Gorenstein category* is defined as

$$\mathcal{GP}_{\mathcal{X}}(\xi) = \{M \in \mathcal{T} | M \text{ admits a complete } \mathcal{P}(\xi)\mathcal{X}\text{-resolution}\}.$$

Remark 5.2.

(1) Since $\mathcal{X}$ is a resolving subcategory of $\mathcal{T}$, we have $\mathcal{P}(\xi) \subseteq \mathcal{X}$, so $\mathcal{P}(\xi) \subseteq \mathcal{X} \cap {}^{\perp}\mathcal{X}$. Then, we have $K_1 \in \mathcal{GP}_{\mathcal{X}}(\xi)$.

(2) If $M \in \mathcal{GP}_{\mathcal{X}}(\xi)$, then $\xi xt^0_{\xi}(M, X) \cong \mathrm{Hom}_{\mathcal{T}}(M, X)$ and $\xi xt^1_{\xi}(M, X) = 0$ for any $X \in \mathcal{X}$. In fact, the following ξ-exact complex:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a ξ-projective resolution of M (see [11]), which is $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{X})$-exact.

Evidently, $M \in \mathcal{GP}_{\mathcal{X}}(\xi)$ if and only if $\xi xt^0_{\xi}(M, X) \cong \mathrm{Hom}_{\mathcal{T}}(M, X)$ and $\xi xt^1_{\xi}(M, X) = 0$ for any $X \in \mathcal{X}$, and M admits a $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{X})$-exact ξ-exact complex

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

with $X^i \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$.

(3) If $\mathcal{X} = \mathcal{P}(\xi)$, then we have $\mathcal{X} \cap {}^{\perp}\mathcal{X} = \mathcal{P}(\xi)$ by Lemma 3.12, and thus, $\mathcal{GP}_{\mathcal{X}}(\xi)$ coincides with $\mathcal{GP}(\xi)$ defined in [13].

We have the following result.

Theorem 5.3. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$. Then, $\mathcal{GP}_{\mathcal{X}}(\xi)$ is a resolving subcategory of $\mathcal{T}$.*

Proof. Let P be a ξ-projective object. Consider the following ξ-exact complex:

$$\cdots \longrightarrow 0 \xrightarrow{0} P \xrightarrow{\mathrm{id}_P} P \xrightarrow{0} 0 \longrightarrow \cdots$$

in $\mathcal{T}$. Clearly, it is $\mathrm{Hom}_{\mathcal{T}}(-,\mathcal{X})$-exact. In particular,

$$0 \xrightarrow{0} P \xrightarrow{\mathrm{id}_P} P \xrightarrow{0} 0 \quad \text{and} \quad P \xrightarrow{\mathrm{id}_P} P \xrightarrow{0} 0 \xrightarrow{0} \Sigma P$$

are corresponding triangles in ξ. Since $P \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$ by Remark 5.2(1). we have $\mathcal{P}(\xi) \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$.

As a similar argument to the proof of [18, Theorem 4.3(1)], we obtain that $\mathcal{GP}_{\mathcal{X}}(\xi)$ is closed under ξ-extensions and hokernels of ξ-proper epimorphisms. Thus, $\mathcal{GP}_{\mathcal{X}}(\xi)$ is a resolving subcategory of $\mathcal{T}$. □

Lemma 5.4. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ satisfying $\mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$. Then, $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is a ξxt-injective ξ-cogenerator of $\mathcal{GP}_{\mathcal{X}}(\xi)$ and is closed under hokernels of ξ-proper epimorphisms.*

Proof. Let $M \in \mathcal{GP}_{\mathcal{X}}(\xi)$. There is a $\mathrm{Hom}_{\mathcal{T}}(-,\mathcal{X})$-exact triangle

$$M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M \tag{15}$$

in ξ with $X^0 \in \mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$. For any $\widetilde{X} \in \mathcal{X}$, applying the functor $\mathrm{Hom}_{\mathcal{T}}(-,\widetilde{X})$ to the triangle (15) yields the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(K^1,\widetilde{X}) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(X^0,\widetilde{X}) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(M,\widetilde{X}) & \longrightarrow & 0 & \\
& \downarrow & & \downarrow \cong & & \downarrow \cong & & & \\
0 \longrightarrow & \xi xt^0_{\xi}(K^1,\widetilde{X}) & \longrightarrow & \xi xt^0_{\xi}(X^0,\widetilde{X}) & \longrightarrow & \xi xt^0_{\xi}(M,\widetilde{X}) & \longrightarrow & \xi xt^1_{\xi}(K^1,\widetilde{X}) & \longrightarrow \xi xt^1_{\xi}(X^0,\widetilde{X})(=0),
\end{array}$$

where the two isomorphisms follow from the assumption that $X^0, M \in \mathcal{GP}_{\mathcal{X}}(\xi)$ and Remark 5.2(2). It follows that $\xi xt^1_{\xi}(K^1,\widetilde{X}) = 0$ and $\xi xt^0_{\xi}(K^1,\widetilde{X}) \cong \mathrm{Hom}_{\mathcal{T}}(K^1,\widetilde{X})$, so $K^1 \in \mathcal{GP}_{\mathcal{X}}(\xi)$ by Remark 5.2(2), then $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is a ξ-cogenerator of $\mathcal{GP}_{\mathcal{X}}(\xi)$. Obviously, $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is a ξxt-injective ξ-cogenerator of $\mathcal{GP}_{\mathcal{X}}(\xi)$.

It is obvious that $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ is closed under hokernels of ξ-proper epimorphisms. □

As an application of Theorem 3.14, we have:

Proposition 5.5. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ satisfying $\mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$ and $M \in \mathcal{T}$. If $M \in \widehat{\mathcal{GP}_{\mathcal{X}}(\xi)}$, then the following statements are equivalent:*

(1) $\mathcal{GP}_{\mathcal{X}}(\xi)$-res.dim $M \leq m$.
(2) $\Omega^n(M) \in \mathcal{GP}_{\mathcal{X}}(\xi)$ *for all* $n \geq m$.
(3) $\Omega^n_{\mathcal{GP}_{\mathcal{X}}(\xi)}(M) \in \mathcal{GP}_{\mathcal{X}}(\xi)$ *for all* $n \geq m$.
(4) $\xi xt^n_{\xi}(M,H) = 0$ *for all* $n > m$ *and all* $H \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$.
(5) $\xi xt^n_{\xi}(M,L) = 0$ *for all* $n > m$ *and all* $L \in \widehat{\mathcal{X} \cap {}^{\perp}\mathcal{X}}$.
(6) *M admits a right $\mathcal{GP}_{\mathcal{X}}(\xi)$-approximation $\varphi : X \to M$, where φ is ξ-proper epic, such that $K =$ Hoker φ satisfying $\mathcal{H}$-*res.dim $K \leq m-1$.
(7) *There are two triangles*

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

*in ξ such that X_M, $X^M \in \mathcal{GP}_{\mathcal{X}}(\xi)$ and $\mathcal{X} \cap {}^{\perp}\mathcal{X}$-*res.dim $W_M \leq m-1$, $\mathcal{X} \cap {}^{\perp}\mathcal{X}$-res.dim $W^M = \mathcal{GP}_{\mathcal{X}}(\xi)$-res.dim $W^M \leq m$.

Immediately, we have:

Corollary 5.6. *Let $\mathcal{T}$ be a triangulated category and $M \in \mathcal{T}$. If $M \in \widehat{\mathcal{GP}(\xi)}$, then the following statements are equivalent*:

(1) $\mathcal{GP}(\xi)$-res.dim $M \leq m$.

(2) $\Omega^n(M) \in \mathcal{GP}(\xi)$ *for all* $n \geq m$.

(3) $\Omega^n_{\mathcal{GP}(\xi)}(M) \in \mathcal{GP}(\xi)$ *for all* $n \geq m$.

(4) $\xi xt^n_\xi(M, H) = 0$ *for all* $n > m$ *and all* $P \in \mathcal{P}(\xi)$.

(5) $\xi xt^n_\xi(M, L) = 0$ *for all* $n > m$ *and all* $L \in \widehat{\mathcal{P}(\xi)}$.

(6) *M admits a $\mathcal{GP}(\xi)$-approximation $\varphi : X \to M$, where φ is ξ-proper epic, such that $K = \mathrm{Hoker}\,\varphi$ satisfying ξ-pd $K \leq m - 1$.*

(7) *There are two triangles*

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in ξ such that X_M and X^M are in $\mathcal{X}$ and ξ-pd $W_M \leq m - 1$, ξ-pd $W^M = \mathcal{GP}(\xi)$-res.dim $W^M \leq m$.

Remark 5.7. As in Corollary 5.6, (1) ⇔ (2) ⇔ (6) is [13, Theorem 4.6 (ii) ⇔ (iii) ⇔ (iv)], (1) ⇔ (5) is [13, Proposition 3.19]. (1) ⇔ (4) is [14, Remark 2.14].

Following Theorems 4.8 and 5.3, we have the following result, which is a generalization of [15, Proposition 5.1].

Proposition 5.8. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$. For any $M \in \mathcal{T}$, if $M \in \widehat{\mathcal{GP}_\mathcal{X}(\xi)}$, then the following statements are equivalent*:

(1) $\mathcal{GP}_\mathcal{X}(\xi)$-res.dim $M \leq n$.

(2) *For each $i > 0$, the morphisms $v_{n+i} : C_{n+i} \to M$ of the ξ-cellular tower of M with respect to $\mathcal{GP}_\mathcal{X}(\xi)$ constructed above are isomorphisms.*

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