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PROEFSCHRIFT

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CONTENTS

SCOPE OF THIS STUDY

Nowadays the types of circular accelerators vary from small cyclotrons to very large synchrotrons and storage rings. When designing such an accelerator profound knowledge of the beam dynamics is required. The particle motion is liable to three oscillation modes: two transverse (the betatron oscillations) and one longitudinal mode (the synchrotron oscillations) and coupling effects between the various modes have been observed experimentally in many accelerators.

The present investigation was undertaken to develop a general theory for the description of coupling and resonance effects. Therefore, a simultaneous treatment of all three oscillation modes has been set up with the use of the Hamilton formalism. The theory - which takes into account the HF accelerating electric field and the time dependence of the magnetic field - provides insight into various sources that can excite synchro-betatron resonances. In the examination of pure betatron resonances, the influence of the acceleration of the particles is disregarded.

The study was started as part of the design study for an electron storage ring in the Netherlands, used for synchrotron radiation, called PAMPUS (FOM76) and has been financed by the foundation "Fundamental Research on Matter" FOM. The PAMPUS project has been dismissed meanwhile by the Dutch government.
Afterwards, we got involved in the design study of a proton accumulator ring - called IKOR - which is part of the "Spallations-Neutronenquelle" (SNQ) project in West Germany (SNQ81I).
The theory developed in this thesis will be applied mainly on these two machines. Although PAMPUS will not be built, its design features are characteristic of common electron storage rings which are in progress now.

In chapter 1 we expand the general Hamilton function for the description of the relativistic particle motion in a time-dependent magnetic field and a HF accelerating electric field (in order to study transverse-longitudinal coupling effects) as well as for the motion in a time-independent magnetic field without acceleration (to study transverse coupling effects).

The results of this chapter initiate the further study. Moreover, the different lattices belonging to the machines considered are presented.

The linear transverse motion is discussed in chapter 2. Analytical formulae for the so-called Twiss parameters are derived from the linear Hamilton theory. We discuss some transformations that are useful in linear betatron theory and first and second order resonances are briefly considered using phase plane representations.

The simultaneous treatment of the betatron and synchrotron motion is developed in chapter 3. We start the theory for a cylindrical-symmetric magnetic field and the knowledge obtained is used to extend the theory for machines with an alternating gradient magnetic field structure. Various sources which lead to transverse-longitudinal coupling or synchro-betatron resonances become apparent and some of these are briefly discussed.

A theory for the description of the one-dimensional non-linear betatron motion is elaborated in chapter 4. Non-linear magnetic fields are often installed to correct the dynamical behaviour of the particles and special attention is paid to the use of sextupole magnets in IKOR. Resonances are generally studied by only retaining the slowly varying terms in the Hamilton function, whereas the fast oscillating terms are ignored. In this chapter, transformations are performed to remove these latter terms and result in a representation of "first" as well as "second order" non-linear effects. These second order effects may become important in "futuristic" accelerators.
Application of the theory leads to the required distance to a resonance or stipulates tolerances of the magnetic fields.

The two-dimensional non-linear betatron resonances are treated in chapter 5. The description of these resonances can be reduced rather simply to a one-dimensional problem and are treated by examination of trajectories in a phase plane. The study again leads to required distances to the resonance lines or to allowed magnetic field tolerances.

CHAPTER 1 GENERAL INTRODUCTION

1.1 Historical developments in accelerators and orbit theories

During the years after 1930 various types of circular accelerators have been developed. After the classical cyclotron (Law30) for proton energies up to about 20 MeV, the synchro-cyclotron was designed to avoid the energy limitation of the classical cyclotron by varying the frequency of the accelerating voltage (Ric46, Liv52). In this latter machine proton energies up to about 800 MeV are attained. Besides the proton, also heavier charged particles can be accelerated in (synchro-) cyclotrons.

Still higher energies have been attained later (since 1945) in synchrotrons in which a relatively modest amount of iron is used in a ring-shaped magnet. The frequency of the accelerating voltage rises to keep in step with that of the rotating particles and in the meantime the increasing magnetic field maintains the orbits at constant radius. There are two types of synchrotrons in which electrons, protons and heavier charged particles can be accelerated. On the one hand we have the constant gradient (C.G.) or weak-focusing synchrotron (McM45, Vek45, Boh46) in which proton energies up to 1 - 10 GeV are attained and on the other hand the alternating gradient (A.G.) or strong-focusing synchrotron (Chr50, Cou52) for proton energies up to 100 GeV and even higher. The terms "constant" and "alternating" imply that the radial gradient of the magnetic field either maintains a steady value or alternates in magnitude and sign when the azimuth changes. In A.G. machines the radial and vertical focusing can be greatly increased compared to C.G. machines and thus the magnet apertures can be reduced.

A special device for the acceleration of electrons - for energies to about 300 MeV - is the betatron in which the acceleration is achieved by the electric field induced by the change in magnetic flux going through the circular electron orbit (Ker41).

More recently (since about 1960) the general interest in the use of interacting beams has led to colliding beam facilities with intersecting storage rings. Furthermore electron storage rings, up to 5 GeV, for the production of synchrotron radiation have been in progress since 1970 (see e.g. proceedings of the "International Conference on High Energy Accelerators" 1971, 1974, 1977, 1980 and of the "Particle Accelerator Conference", ever since 1960, published in IEEE Transactions on Nuclear Science).

Nowadays the larger accelerators are frequently designed with so-called separated function guide fields, in which the focusing functions and bending functions are assigned to different magnetic elements. Such a guide field exists of a sequence of bending magnets (no radial gradient of the magnetic induction), quadrupoles (no magnetic induction on their axis) and field free sections in between. All these accelerators have a plane of symmetry, called the median plane, in the vicinity of which the particle trajectories lie.

The main task of the work presented in this thesis will be to study particle stability in circular accelerators. In case of stability the particles oscillate around an equilibrium orbit with limited amplitudes. The oscillations are in three dimensions: two transverse (betatron oscillations) and one longitudinal dimension (synchrotron oscillations). Coupling effects between the various oscillation modes can give rise to unstable motion. The frequencies of the associated oscillations can satisfy a resonance relation which might result in amplitude growth and beam loss.
This thesis deals with the study of coupling and resonance effects due to e.g. the accelerating electric field, perturbations in the magnetic field and non-linear magnetic fields which are often applied in the accelerator in order to cancel unwanted effects.

Important works on the subject of non-linear oscillations are those of Moser (Mos56) and Sturrock (Stu58). Up to now the synchrotron oscillations have usually been studied separately from the betatron oscillations. The Hamilton formalism has proved to be especially well-suited to study coupled betatron oscillations and non-linear phenomena (Scho57, Hag57, Hag62, Kol66, Lys73, Gui76, Ohn81).
In studying the properties of the magnetic field, the synchrotron oscillations can usually be ignored, i.e. the acceleration is neglected. The resulting pure betatron resonances are treated extensively and examinations of trajectories in a phase plane are applied to the uncoupled as well as the coupled betatron motion.

The aspects in which this work differs from the reports mentioned above will be discussed in more detail in this chapter.

Further, a general theory will be presented including the acceleration process, i.e. the longitudinal and transverse motions are treated simultaneously. For the description we use the Hamilton formalism and we start with the initial Hamilton function for the motion of a charged particle - with relativistic energy - in a time dependent magnetic and electric field. We will use curvi-linear coordinates and the time is the independent variable. This treatment enables us to study coupling effects between the longitudinal and transverse motions in (synchro-)cyclotrons, C.G. and A.G. synchrotrons and storage rings.
Recently Schulte and Hagedoorn were the first to develop a theory for the non-relativistic description of accelerated particles in cyclotrons, i.e. a simultaneous treatment of the radial and longitudinal motions (Schu78, Schu80). They used cartesian coordinates which turned out to be convenient for the description of the acceleration process and of the motion in the central region of the cyclotron, although the representation of the magnetic field is rather complex in this system. More differences between both treatments will be pointed out later in this thesis.

In the rest of this chapter we will bring the Hamilton function in a proper form to develop the theory in the subsequent chapters. Further some introductory notes on the application of the Hamilton theory are treated and at the end we discuss the contents of this thesis in more detail.

1.2 The general Hamilton function

The Hamilton formalism is appropriate to investigate particle orbits in circular accelerators. It gives a general point of view and the possibility of detailed descriptions. A Hamiltonian for the motion of a charged particle with relativistic energy in a time-dependent magnetic and electric field can be represented by

$$H(\vec{p},\vec{q},t) = \sqrt{E_r^2 + \left(\vec{p} - e\vec{A}(\vec{q},t)\right)^2 c^2} \tag{1.1}$$

where $\vec{p}$ and $\vec{q}$ are the vectors of the canonical momenta and coordinates, $\vec{A}$ is the magnetic vector potential, c is the velocity of light, E_r and e are respectively the rest energy and the charge of the particle. The time t is the independent variable and the value of H equals the total energy of the particle.

It is convenient to describe the motion of the particle in terms of coordinates related to the so-called reference orbit. This orbit is defined on a fixed time $t=t_o$ and the reference particle which has the nominal energy moves on this orbit. The reference orbit has the same symmetry as the unperturbed guide field and lies in the median plane.
In a cylindrical-symmetric magnetic field the reference orbit is a circle. More generally, in machines with a separated function guide field one can consider the orbit as being composed of arcs of a circle with radius ρ connected by straight lines. In case of a synchrotron or storage ring the reference orbit is the "design orbit".

To define the deviation of the motion from the reference orbit, we define a curvi-linear orthogonal coordinate system x,z,s with x and z as the horizontal and vertical deviations from the reference orbit and s as the coordinate along the reference orbit. In this coordinate system a positively charged particle rotates in the s-direction in a magnetic field pointing in the positive z-direction and the length of the infinitesimal vector $d\vec{\sigma}$ is given by

$$d\sigma^2 = dx^2 + dz^2 + \{1 + \frac{x}{\rho(s)}\}^2 ds^2 \tag{1.2}$$

in which $\rho(s)$ is the local radius of curvature of the reference orbit. The Hamiltonian (1.1) can be written as

$$H = \sqrt{E_r^2 + (p_x - eA_x)^2c^2 + (p_z - eA_z)^2c^2 + \left(\frac{p_s}{1 + \frac{x}{\rho(s)}} - eA_s\right)^2c^2} \tag{1.3}$$

A logical next step would be a series expansion of $\vec{A}$ in the coordinates using div grad $\vec{A} = 0$. However, in accelerator physics it is common use to express $\vec{A}$ in the components of the magnetic field via $\vec{B} = \text{curl } \vec{A}$, where $\vec{B}$ is fixed by the relations div $\vec{B} = 0$ and curl $\vec{B} = \vec{0}$. The time variation of the magnetic field - which is very slow - can be represented by a simple multiplying factor and

therefore it is sufficient to give the constant representation.
Expressed as function of the coordinates the magnetic field is :

$$
\begin{aligned}
B_x &= b_1 z + b_2 xz - \tfrac{1}{6} b_3 (z^3 - 3x^2 z) + \\
&\quad + \frac{1}{12}\left[\left(\frac{B_o'}{\rho}\right)' + \frac{B_o''}{\rho} - \frac{b_2}{\rho} + \frac{b_1}{\rho^2} - b_1''\right](z^3 + 3x^2 z) \\
B_z &= B_o + b_1 x + \tfrac{1}{2} b_2 (x^2 - z^2) - \tfrac{1}{2}\left(\frac{b_1}{\rho} + B_o''\right) z^2 + \tfrac{1}{6} b_3 (x^3 - 3xz^2) + \\
&\quad + \frac{1}{12}\left[\left(\frac{B_o'}{\rho}\right)' + \frac{B_o''}{\rho} - \frac{b_2}{\rho} + \frac{b_1}{\rho^2} - b_1''\right](x^3 + 3xz^2) \\
B_s &= B_o' z + \left(b_1' - \frac{B_o'}{\rho}\right) xz + \ldots
\end{aligned}
\tag{1.4}
$$

with $'$ = d/ds and $''$ = d^2/ds^2.
The coefficients are periodic functions of s and for pure multipoles it holds:

$$
\begin{aligned}
B_o &= B_z(x=z=0) \\
b_1 &= \left(\frac{\partial B_z}{\partial x}\right)_o , \quad b_2 = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_o , \quad b_3 = \left(\frac{\partial^3 B_z}{\partial x^3}\right)_o
\end{aligned}
\tag{1.5}
$$

where the subscript "o" means that all quantities are evaluated on the reference orbit (x=z=0).
The degree of the polynomial in (1.4) is determined by the number of terms of the multipole expansion. The reader can see that the third degree takes into account octupoles and a further expansion of lower order poles.
Non-linear fields can arise from errors in the guide field, from fringing fields and from extra elements intentionally put in the accelerator.
Since $\vec{A}$ is defined in terms of $\vec{B}$ by $\vec{B}$ = curl $\vec{A}$, the vector potential is arbitrary to the extent that the gradient of some scalar function can be added as long as the magnetic field does not change in time. However, in case of a time-dependent magnetic field the simple multiplying factor for $\vec{B}$ is extended to the vector potential. When we combine this with the betatron accelerating fields along the reference orbit by $\vec{E} = -\partial\vec{A}/\partial t$, we see that $\vec{A}$ is fixed by the relation $\oint A_s ds = \Phi$ is the enclosed magnetic flux.

In case of a synchrotron or storage ring a related vector potential is [†]

$$A_x = -\frac{1}{2}B_o'z^2 - \frac{1}{2}\left(b_1' - \frac{B_o'}{\rho}\right)xz^2$$

$$A_z = 0$$

$$\begin{aligned} A_s = & - B_o x + \frac{1}{2}\left(\frac{B_o}{\rho} - b_1\right)x^2 + \frac{1}{2}b_1 z^2 + \\ & + \frac{1}{6}\left(\frac{b_1}{\rho} - 3\frac{B_o}{\rho^2}\right)x^3 - \frac{1}{6}b_2\left(x^3 - 3xz^2\right) \\ & - \frac{1}{24}b_3\left(x^4 - 6x^2z^2 + z^4\right) + \frac{1}{24}\left(12\frac{B_o}{\rho^3} - 4\frac{b_1}{\rho^2} + \frac{b_2}{\rho}\right)x^4 \\ & - \frac{1}{48}\left[\left(\frac{B_o'}{\rho}\right)' + \frac{B_o''}{\rho} - \frac{b_2}{\rho} + \frac{b_1}{\rho^2} - b_1''\right]\left(x^4 - 6x^2z^2 - z^4\right) \quad . \end{aligned} \tag{1.6}$$

For the (synchro-)cyclotron with a cylindrical-symmetric magnetic field (which is constant in time) the derivatives to s disappear.

Substituting the vector potential into (1.3), the Hamiltonian can be expressed in a power series of the canonical variables. The coordinates x and z are considered to be small quantities: they are assumed to be much smaller than the local radius of curvature of the trajectory. The equations of motion are given by the Hamilton equations. In general we will not solve these equations directly, but canonical transformations obtained from a generating function $G_1(\bar{p},q,t)$ are applied to simplify the Hamiltonian:

$$\bar{q} = \frac{\partial G_1}{\partial \bar{p}} \ , \ p = \frac{\partial G_1}{\partial q} \ ; \ \bar{H} = H + \frac{\partial G_1}{\partial t} \quad . \tag{1.7}$$

Three other forms of the generating function, viz $G_2(p,\bar{q},t)$, $G_3(q,\bar{q},t)$ and $G_4(p,\bar{p},t)$ can be used and give expressions which are similar to (1.7) but with a minus sign if $\bar{p}$ or q are calculated.

In the following paragraphs we will develop the Hamilton theory so that it becomes more suited to study coupled orbit motion. First we

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[†] Taking a cylindrical-symmetric magnetic field, we find for the betatron:

$$A_x = A_z = 0$$

$$A_s = - B_o\rho - \frac{1}{2}\left(\frac{B_o}{\rho} + b_1\right)x^2 + \frac{1}{2}b_1 z^2 + \left(\frac{1}{2}\frac{B_o}{\rho^2} + \frac{1}{6}\frac{b_1}{\rho}\right)x^3 + \dots \tag{1.6†}$$

will discuss the case of a time-dependent magnetic and electric field (section 1.3) and afterwards the motion without acceleration in a time-independent magnetic field will be considered (section 1.4).

1.3 Time-dependent magnetic and electric field

The acceleration of charged particles to higher energies is done by external electric fields (except in the case of the betatron). In cyclotrons and synchro-cyclotrons the accelerating structures involve one or more Dees, whereas synchrotrons and storage rings have at least one cavity around the ring. In electron storage rings used as synchrotron radiation sources, the energy must be added in order to compensate for radiation losses.

In general the electric field will depend upon the coordinate s and upon the time t. It has a "fast" oscillating time dependence and its period is comparable with the period of revolution and the periods of the betatron oscillations.

As a rule the change of the magnetic field occurs extremely slowly with respect to the motion along the reference orbit. The characteristic times are e.g. 10^6 larger than the revolution period.

In general we can write

$$\vec{E} = - \text{grad } V - \frac{\partial \vec{A}}{\partial t} \quad . \tag{1.8}$$

In machines with a Dee structure a HF potential function $V(\vec{r},t)$ is possible inside the acceleration region and the effect of the accelerating voltage is represented by adding this $V(\vec{r},t)$ to the Hamiltonian (1.3)(Schu78). This potential function has a non-zero value in the Dee and equals zero in the dummy Dee.
However this procedure is less evident when dealing with, for instance, one cavity in a ring-shaped accelerator. In case of cavities, a fast oscillating vector potential $\vec{A}(t)$ can be introduced, corresponding to the HF magnetic flux of the cavities. Note that in case of an odd number of cavities there is a net flux through the surface enclosed by the reference orbit i.e. $\oint Eds \neq 0$ (at a fixed time).

To deduce a Hamiltonian in which the time-dependent electric and magnetic fields are visible separately, we split the chosen vector potential in a fast and a slowly varying part.

For convenience we omit the fringing fields of the electric field and consequently there are no transverse components of the electric field. For the longitudinal component of the vector potential we write

$$A_s(t) = A_{s,f}(t) + A_{s,s}(t) \tag{1.9}$$

where $A_{s,f}(t)$ is the fast oscillating part due to the HF electric field field E_s :

$$E_s(t) = -\frac{\partial}{\partial t} A_{s,f}(t) \tag{1.10}$$

and $A_{s,s}(t)$ is the slowly varying component originating from the magnetic field which is discussed in the previous section.

As the Hamiltonian should be expanded in a power series of the canonical variables, these should all be small quantities. We define a longitudinal reference momentum $p_{so}(t)$ by the relation †

$$p_{so}(t) = P(x=z=0;t) + eA_{s,s}(x=z=0;t) \tag{1.11}$$

in which the kinetic momentum in the time-dependent magnetic field is $P(x=z=0;t) = eB(x=z=0;t)\rho$.

In case of an A.G. synchrotron or storage ring the cavity is placed in a straight section and studying the Hamiltonian (1.3) it is convenient to apply a transformation generated by the function

$$G = \bar{p}_x x + \bar{p}_z z + \bar{p}_s s + p_{so}(t)s + e\int^s A_{s,f}(s',t)ds'$$

with

$$\begin{aligned} &\bar{x} = x \qquad \bar{z} = z \qquad \bar{s} = s \\ &\bar{p}_x = p_x \qquad \bar{p}_z = p_z \qquad \bar{p}_s = p_s - p_{so}(t) - eA_{s,f}(s,t) \ . \end{aligned} \tag{1.12}$$

All variables remain unchanged except the longitudinal canonical momentum and the Hamiltonian (1.3) becomes (with $A_x = A_z = 0$)

$$\bar{H} = \sqrt{E_r^2 + \bar{p}_x^2 c^2 + \bar{p}_z^2 c^2 + \left(\frac{\bar{p}_s + p_{so}}{1 + \frac{\bar{x}}{\rho}} - eA_{s,s}\right)^2 c^2} + \bar{s}\frac{d}{dt}p_{so}(t) - e\int^s E_s(s',t)ds' \ . \tag{1.13}$$

---

† We recall the note on page 8 concerning the choice of the vector potential and the relation between $\oint A_s ds$ and the enclosed magnetic flux.

The influence of the time-dependent magnetic field is incorporated via $A_{s,s}(t)$ and $p_{so}(t)$ and the acceleration by the HF electric field via the "potential-like" function $- e\int^{s} E_s(s',t)ds'$.
In general the variation of the HF electric field with time has the form of a cosine function and the "potential-like" function can be written as

$$- e\int^{s} E_s(s')ds' \cdot \cos \int \omega_{HF}(t)dt \tag{1.14}$$

in which $\omega_{HF}(t)$ is the angular frequency of the HF accelerating system. We introduce a term in $\bar{H}$,

$$eV(s)\cdot\cos \int\omega_{HF}(t)dt \qquad \text{with} \qquad V(s) = - \int^{s} E_s(s')ds' \tag{1.15}$$

where V(s) is a unique function of s but, in contrast with $V(\vec{r})$ in case of a cyclotron, not necessarily of the position (note: s has increased with the circumference after one turn, whereas the position has not changed). In case of a cyclotron $V(s) = \hat{V}$ is the voltage in the Dee and $V(s) = 0$ in the dummy Dee.

High energy electron accelerators differ from other machines by the radiation losses. The stochastic quantum emission results in a damping (or anti-damping) of the three oscillation modes by the accelerating system. The effects of this damping play a role on a time scale which is very large compared with the period of the betatron and synchrotron oscillations (San71). The effect of the radiation losses will not be discussed in this thesis.

Expanding the Hamiltonian (1.13) into a power series of the variables, it can be used to treat the synchrotron and betatron motions simultaneously. It is convenient to eliminate the constants in (1.13) by a scale transformation.

1.3.1 The scale transformation

In order to eliminate the constants e,c, E_r in the Hamiltonian (1.13) we define new relative variables and a new dimensionless time unit. The variables are normalized on quantities belonging to the reference orbit and the reference particle.

We emphasize that the reference orbit is a solution of the Hamiltonian H with $\vec{A}(\vec{q},t)$ replaced by $\vec{A}(\vec{q},t=t_o)$.
The new variables are defined by:

$$\bar{\bar{x}} = \frac{\bar{x}}{R} \qquad \bar{\bar{z}} = \frac{\bar{z}}{R} \qquad \bar{\bar{s}} = \frac{\bar{s}}{R} \tag{1.16a}$$

$$\bar{\bar{p}}_x = \frac{\bar{p}_x}{P_o} \qquad \bar{\bar{p}}_z = \frac{\bar{p}_z}{P_o} \qquad \bar{\bar{p}}_s = \frac{\bar{p}_s}{P_o} \quad .$$

The quantity R is the length of the reference orbit divided by 2π. It is common to speak of R as the mean radius. The momentum P_o is the kinetic momentum of the reference particle.
The new time unit is based on the revolution period of the reference particle:

$$\tau = \omega_o t \qquad \text{with} \qquad \omega_o = \frac{c}{R}\sqrt{1 - \frac{1}{\gamma_o^2}} \tag{1.16b}$$

$$\text{and} \quad P_o^2 c^2 = \left(1 - \frac{1}{\gamma_o^2}\right) W_o^2 = W_o^2 - E_r^2$$

and where the subscript "o" refers to the reference particle, ω_o is its angular revolution frequency and W_o is its total energy.

In order to maintain Hamilton's equations, the Hamiltonian must be adjusted accordingly

$$\bar{\bar{H}} = \frac{\bar{H}}{\left(1 - \frac{1}{\gamma_o^2}\right) W_o} = \frac{\bar{H}}{\omega_o P_o R} \quad . \tag{1.16c}$$

In chapter 3 this Hamiltonian will serve as the starting-point to describe the betatron and synchrotron motion simultaneously and to study coupling effects between them.

We notice that an equivalent angular variable θ along the reference orbit is defined by $\theta = s/R$, which is exactly the variable $\bar{\bar{s}}$ of (1.16a). For that reason $\bar{\bar{s}}$ and its canonical conjugate $\bar{\bar{p}}_s$ are written as $\bar{\bar{\theta}}$ and $\bar{\bar{p}}_\theta$ in future, i.e. chapter 3.

Finally a remark on the method used to study coupling effects: since we consider x/ρ as a small quantity - in order to be able to expand the term $(1 + x/\rho)^{-1}$ in the Hamiltonian (1.13) - this theory is not generally suitable for studies at very small radii i.e. for central region studies in (synchro-)cyclotrons.

Schulte developed a theory for the non-relativistic description of accelerated particles in central regions of cyclotrons by using cartesian coordinates and splitting the horizontal motion into a circle motion and a centre motion (Schu78, Schu80).

1.4 Time-independent magnetic field, no acceleration

Considering the motion in a time-independent magnetic field and having no acceleration, A_x, A_z and A_s do not depend on the time. Then $dH/dt = \partial H/\partial t = 0$ expressing the fact that the energy is constant for a particle moving in a magnetic field.

We change to the longitudinal coordinate s being the new independent variable. The equations of motion are still of Hamiltonian form and $-p_s$ acts as the new Hamiltonian (see e.g. Kol66):

$$- p_s = \bar{H} = -\left(1 + \frac{x}{\rho}\right)\sqrt{P^2 - \left(p_x - eA_x\right)^2 - \left(p_z - eA_z\right)^2} - e\left(1 + \frac{x}{\rho}\right)A_s \tag{1.17}$$

with $P^2 = m^2c^2 = H^2 - E_r^2$.

The number of degrees of freedom is now reduced from three (in (1.3)) to two (in (1.17)). The new Hamiltonian $\bar{H}$ is a periodic function of the independent variable s.

Normalizing the variables on the mean radius R and the kinetic momentum P (analogue to (1.16a)) and introducing the azimuthal angle θ as the independent variable ($ds = Rd\theta$) the Hamiltonian becomes

$$\bar{\bar{H}} = \frac{\bar{H}}{P} \quad . \tag{1.18}$$

Since $\bar{\bar{x}},\bar{\bar{z}},\bar{\bar{p}}_x$ and $\bar{\bar{p}}_z$ are all small quantities, the Hamiltonian can be expanded in powers of these variables.

Studying the particle motion in an accelerator with a separated function guide field and considering a particle with nominal energy, i.e. the energy of the reference particle, the final Hamiltonian up to the fourth degree in the variables - which is consistent with the neglect of multipoles higher than the octupole (see (1.4)) - is (substitute (1.6) into (1.17)/(1.18)):

$$
\begin{aligned}
\bar{\bar{H}} = {} & \frac{1}{2}\,\bar{\bar{p}}_x^2 + \frac{1}{2}(\varepsilon^2 - n)\bar{\bar{x}}^2 + \frac{1}{2}\,\bar{\bar{p}}_z^2 + \frac{1}{2}\,n\bar{\bar{z}}^2 + \\
& \frac{1}{2}\,\varepsilon\bar{\bar{x}}(\bar{\bar{p}}_x^2 + \bar{\bar{p}}_z^2) - \frac{1}{6}\,S(\bar{\bar{x}}^3 - 3\bar{\bar{x}}\bar{\bar{z}}^2) + \frac{1}{2}\,\dot{\varepsilon}\bar{\bar{p}}_x\bar{\bar{z}}^2 + \\
& \frac{1}{8}(\bar{\bar{p}}_x^2 + \bar{\bar{p}}_z^2)^2 - \frac{1}{2}\,\dot{n}\bar{\bar{x}}\bar{\bar{p}}_x\bar{\bar{z}}^2 - \frac{1}{24}\,O(\bar{\bar{x}}^4 - 6\bar{\bar{x}}^2\bar{\bar{z}}^2 + \bar{\bar{z}}^4) + \\
& \frac{1}{48}\left((\dot{\varepsilon})^2 + 2(\ddot{\varepsilon^2}) + \ddot{n}\right)(\bar{\bar{x}}^4 - 6\bar{\bar{x}}^2\bar{\bar{z}}^2 + \bar{\bar{z}}^4) + \\
& \frac{1}{24}\left(2(\dot{\varepsilon})^2 - 2(\ddot{\varepsilon^2}) - \ddot{n}\right)\bar{\bar{z}}^4
\end{aligned}
\tag{1.19}
$$

in which θ is the independent variable and $\dot{} = d/d\theta$, $\ddot{} = d\ /d\theta^2$ and for the pure multipoles holds:

$\varepsilon = \dfrac{R}{\rho}$ is the "normalized" dipole component

$n = -\dfrac{R^2}{B_o\rho}\left(\dfrac{\partial B_z}{\partial x}\right)_o$ is the "normalized" quadrupole component

$S = -\dfrac{R^3}{B_o\rho}\left(\dfrac{\partial^2 B_z}{\partial x^2}\right)_o$ is the "normalized" sextupole component

$O = -\dfrac{R^4}{B_o\rho}\left(\dfrac{\partial^3 B_z}{\partial x^3}\right)_o$ is the "normalized" octupole component.

This Hamiltonian (1.19) is suited to study linear and non-linear transverse coupled orbit motion. The linearized equations of motions are obtained by considering the terms of second degree only.

Field errors and magnet alignment errors affect the vector potential and can therefore be incorporated in the Hamiltonian. The effect of a momentum deviation ΔP from the momentum P_o can be studied by substituting $P = P_o(1 + \Delta P/P_o)$ with $P_o = eB_o\rho$ in (1.18) (see chapter 4).
Errors can give rise to first degree terms in $\bar{\bar{x}}$ and/or $\bar{\bar{z}}$ in the Hamiltonian that indicate the presence of a new equilibrium orbit (having the symmetry of the unperturbed linear guide field) or a disturbed closed orbit (not having this symmetry). A general trajectory will execute betatron oscillations around this new equilibrium orbit or disturbed closed orbit, which will be indicated by $(\bar{\bar{x}}_e, \bar{\bar{p}}_{xe})$ and/or $(\bar{\bar{z}}, \bar{\bar{p}}_{ze})$. These betatron motions are studied by applying the transformation

$$G(\overline{\overline{x}},\tilde{p}_x,\theta) = \tilde{p}_x\overline{\overline{x}} - \tilde{p}_x\overline{\overline{x}}_e(\theta) + \overline{\overline{x}}\overline{\overline{p}}_{xe}(\theta)$$

$$\tilde{x} = \overline{\overline{x}} - \overline{\overline{x}}_e(\theta) \quad \text{and} \quad \tilde{p}_x = \overline{\overline{p}}_x - \overline{\overline{p}}_{xe}(\theta) \tag{1.20}$$

and the same for the vertical motion.
The new Hamiltonian $\tilde{H} = \overline{\overline{H}} + \partial G/\partial\theta$ does not contain first degree terms in the variables.

Before starting the study of the linear and especially the non-linear orbit motion (i.e. the study of resonances) in following chapters, we will first discuss some general aspects how to deal with these problems.

1.5 Mathematical treatment of resonances

Essential quantities of the orbits are the betatron numbers Q_x and Q_z (often called "tunes"). These numbers represent the number of betatron oscillations in one revolution and are representative of the focusing effect; the lower these numbers are, the weaker the focusing.
Furthermore we have the synchrotron oscillation number Q_s, representing the number of synchrotron oscillations in one turn.

Generally resonance effects occur when the condition

$$m_1Q_x + m_2Q_z + m_3Q_s = p_r \qquad m_1,m_2,m_3,p_r \text{ integers} \tag{1.21}$$

is fulfilled; $|m_1|+|m_2|+|m_3|$ is called the order of the resonance.
In this section we will mainly restrict ourselves to the problem how to deal with betatron resonances, although the synchro-betatron resonances ($m_3 \neq 0$) can be treated in a similar way.
The betatron resonances are divided in the uncoupled or one-dimensional and the coupled or two-dimensional resonances.

1.5.1 One-dimensional resonances

The one-dimensional betatron resonances are of the type

$$mQ = p_r \qquad Q = Q_x,\, Q_z \;. \tag{1.22}$$

Resonances of order 1 and 2 belong to the linear theory and resonances with $m \geq 3$ are the non-linear resonances. In fact - as we will see later - a resonance like (1.22) will be excited by the p_r-th azimuthal Fourier component of the magnetic field that drives the resonance.

For the moment we consider an unperturbed quadratic Hamiltonian H_o with constant coefficients and an extra term H_1 is added to H_o:

$$\begin{aligned} H &= H_o + H_1 \\ &= \frac{1}{2} p_x^2 + \frac{1}{2} Q^2 x^2 + f(\theta) x^m \end{aligned} \tag{1.23}$$

with $f(\theta) = \sum_p f_p \, e^{ip\theta}$.

One may study the resonances by the use of action and angle variables J, ϕ. These variables lead to simplified equations in canonical form. The transition to J, ϕ is defined by a canonical transformation (Cor60)

$$\begin{aligned} &G(x,\phi) = \frac{1}{2} Qx^2 \tan \phi \\ &x = \sqrt{2J/Q} \cos \phi \qquad p_x = \sqrt{2QJ} \sin \phi \end{aligned} \tag{1.24}$$

and the Hamiltonian becomes

$$K = K_o + K_1 = QJ + K_1 \quad . \tag{1.25}$$

The solutions of K_o are known : $\phi = - Q\theta$ and J = constant. Subsequently this solution for ϕ is substituted in K_1 which then consists of fast and slowly oscillating terms. The nature of the resonance can be understood quite well by keeping only the low frequency or resonant terms which may occur for $\ell Q = p_r$ with $\ell = \pm m, \pm (m-2), \ldots$ In general a specific term of m-th degree in the Hamiltonian will have its main influence on a resonance of the order m and the effect of resonances with $\ell = \pm (m-2), \pm (m-4), \ldots$ are of less importance in this respect.
A simple transformation to a coordinate system rotating with the resonance frequency, generated by

$$\begin{aligned} &G = \left(\bar{\phi} - \frac{p_r \theta}{m}\right) J \\ &\text{with } \bar{J} = J \quad \text{and} \quad \bar{\phi} = \phi + \frac{p_r \theta}{m} \end{aligned} \tag{1.26}$$

converts the Hamiltonian into

$$\bar{K} = \delta Q\bar{J} + 2|\tilde{f}_{p_r}|\bar{J}^{\frac{m}{2}} \cos m\bar{\phi} \qquad (1.27)$$

with $\delta Q = Q - p_r/m$ and $\tilde{f}_{p_r}$ is the Fourier component f_{p_r} multiplied with some constant which is irrelevant for the moment.
This Hamiltonian does not depend on the independent variable θ so that the phase plane trajectories follow from $\bar{K}$ = constant. A survey of these trajectories in the phase plane in which $\sqrt{2\bar{J}}$ and $\bar{\phi}$ are the polar coordinates, gives information about the amplitude behaviour near resonance. The function $\bar{K}(\sqrt{2\bar{J}},\bar{\phi})$ can be visualized.
The case $m = 1$, i.e. an imperfection in the dipole field, leads to a shifted closed orbit with respect to the reference orbit. This shift may tend towards infinity when the tune Q is an integer.
For $m = 2$, i.e. an imperfection in the field gradient, the flowlines are ellipses or hyperbolas depending on the excitation term $\tilde{f}_{p_r}$ and δQ. Both cases, $m = 1$ and $m = 2$, are briefly discussed in chapter 2.
More generally, for non-linear resonances the phase plane is divided into a stable (limited amplitudes) and an unstable region (unlimited amplitudes) separated by the separatrix. Interesting points on this separatrix are the so-called unstable fixed points (saddle points of $\bar{K}$), which satisfy the relations (as also do stable fixed points, which are extrema of $\bar{K}$):

$$\dot{\bar{J}} = 0 \quad \text{and} \quad \dot{\bar{\phi}} = 0 \qquad \text{with} \quad \dot{} = d/d\theta \;. \qquad (1.28)$$

The distance from the origin to the unstable fixed point is related to the maximum oscillation amplitude and depends on the distance from the resonance δQ and the strength of the excitation term $\tilde{f}_{p_r}$.
To excite a resonance the Q value does not have to lie exactly on a resonance but within a band about the resonance. This is the origin of the term stopband width.
Given an amplitude and working point Q, the allowed strength $\tilde{f}_{p_r}$ can be determined in order to maintain stable motion (computations of tolerances). On the other hand, given the excitation term, the required distance to the resonance can be fixed (choice of working point Q).

The analysis given here, taking constant coefficients in H_o (see (1.23)), is valid in case of cylindrical-symmetric magnetic fields in which most perturbations are usually negligible except those with $m = 1$. The situation is different in case of A.G. field structures in which the excitation field for non-linear resonances ($m \geq 3$) might be "large". Then the periodic terms in H_o have to be eliminated first - this procedure is sketched in chapter 2 - and afterwards the effect of the non-linearities can be examined.

In new, large machines in which the non-linear correction elements might reach a new order of magnitude (Don77), their effects have to be examined by transforming the rapidly oscillating terms in K_1 to higher degree. This procedure - carried out in chapter 4 - shows that a given non-linearity of m-th degree in the Hamiltonian can also contribute to resonances of higher order than m.

1.5.2 Two-dimensional resonances

The scheme outlined in the preceding sub-section is now generalized for the case of coupled resonances:

$$m_1 Q_x + m_2 Q_z = p_r \ . \tag{1.29}$$

To illustrate the effects of these resonances we examine the Hamiltonian for the linear betatron oscillations, to which a coupling term $f(\theta)x^j z^l$ with $j = |m_1|$ and $l = |m_2|$ is added. When we transform to action and angle variables and take into account the resonant terms only, the Hamiltonian is

$$K = Q_x J_x + Q_z J_z + J_x^{\left|\frac{m_1}{2}\right|} J_z^{\left|\frac{m_2}{2}\right|} \left(\tilde{f}_{p_r} e^{i(m_1\phi_x + m_2\phi_z + p_r\theta)} + \text{c.c.} \right) \tag{1.30}$$

in which c.c. means the complex conjugate.

In this two-dimensional case, the phase space is four-dimensional and therefore Guignard developed a treatment with the so-called "resonance curves" (Gui76). However, a simplification of the problem is obtained by finding the invariants of the motion and reducing the number of degrees of freedom. Using (1.30), the equations of motion give

$$m_2 J_x - m_1 J_z = \text{constant} \ . \tag{1.31}$$

Applying the transformation, generated by the function G (Hag64):

$$G = - J_1\phi_x - \frac{m_1}{m_2} J_2\phi_x - J_2\phi_z - Q_x J_1\theta - \frac{p_r}{m_2} J_2\theta \tag{1.32}$$

with $J_x = J_1 + \frac{m_1}{m_2} J_2 \qquad \phi_1 = \phi_x + Q_x\theta$

$$J_z = J_2 \qquad \phi_2 = \phi_z + \frac{m_1}{m_2}\phi_x + \frac{p_r\theta}{m_2}$$

the Hamiltonian is simplified to

$$\bar{K} = \delta Q J_2 + 2|\tilde{f}_{p_r}|\left(J_1 + \frac{m_1}{m_2}J_2\right)^{\left|\frac{m_1}{2}\right|} J_2^{\left|\frac{m_2}{2}\right|} \cos m_2\phi_2 \tag{1.33}$$

with $\delta Q = Q_z + \frac{m_1}{m_2} Q_x - \frac{p_r}{m_2}$.

ϕ_1 does not appear in this Hamiltonian and hence J_1 is an invariant of the motion in accordance with (1.31). Furthermore $\bar{K}$ itself is the second invariant. The two-dimensional resonance is thus reduced to a problem with only one degree of freedom. Examinations of trajectories in a phase plane may lead to allowed tolerances or a required distance to the resonance, similar to the one-dimensional resonance. This will be illustrated in chapter 5.
Two types of resonances can be distinguished. If m_1 and m_2 have different signs (difference resonance) there is an exchange of energy between the two oscillation modes and the amplitudes remain limited (see (1.31)). If m_1 and m_2 have the same sign (sum resonance) both amplitudes may have an unlimited growth leading to instability.
In case of a difference resonance an arbitrary parameter is obviously required to define a stopband width. Such a parameter might be the maximum allowed energy transfer from one direction to the other.

Considering transverse coupling, the existence of resonances appears as forbidden bands in the Q_x, Q_z diagram around the lines $m_1Q_x + m_2Q_z = p_r$. The diagram will be divided into regions within which the operating point (Q_x,Q_z) must be chosen.
When the non-linear fields have a well-marked periodicity, harmonics of this periodicity are most relevant. But on the other hand these so-called systematic resonance lines are much rarer in the Q_x, Q_z diagram than the lines corresponding to perturbations. An example of a working diagram is plotted in figure 1.4.

1.5.3 Synchro-betatron resonances

Synchro-betatron resonances are resonances of the type (1.21) with $m_3 \neq 0$. Up to now these resonances have not been considered with a general Hamiltonian in which the betatron and synchrotron motion are treated simultaneously. In chapter 3 we will develop such a theory, using the results of section 1.3. The Hamiltonian (1.16c) is expanded in powers of the canonical variables and action and angle variables can be used again for all three oscillation modes. The six-dimensional phase space can be reduced to a two-dimensional one by two successive transformations of the type (1.32). Strictly speaking the treatment is similar to the one of the preceding sub-section dealing with pure betatron oscillations. The theory will be illustrated briefly with some examples.

1.6 Parameters of "FODO", PAMPUS and IKOR

The theory which will be developed in the subsequent chapters is applied on several lattice configurations. They are all made up of sequences of N identical cells, each cell containing a prescribed set of bending magnets, quadrupoles and extra non-linear magnetic field elements such as sextupoles and octupoles. The guide field is isomagnetic, i.e. all bending magnets have the same radius of curvature. The magnetic fields are periodic functions in θ and can be expanded in Fourier series. For convenience we shall assume these functions to be stepped functions (hard-edge approximation), but this is not a restriction of the theory, see e.g. the Hamiltonian (1.19).

Separated function lattices with a so-called FODO structure are common in accelerator and storage ring designs. An example is given in figure 1.1a. The corresponding linear guiding and focusing functions - $\varepsilon^2(\theta)$ and $n(\theta)$ (see 1.19) - are plotted in figure 1.1b and 1.1c. Magnets of the sector type are used quite often, i.e. orbits enter and leave perpendicularly to the magnet boundary. Sometimes it is appropriate to design bending magnets whose pole faces are rectangular. With such a magnet the reference orbit must enter and leave the magnet at a non-rectangular angle. There will be radial gradients at the edges, leading to edge-focusing effects. In principle a guide field constructed of rectangular magnets together with quadrupoles does not strictly satisfy the definition of "separated function", although it is often referred to as such.

The rectangular magnets were applied in a proposal for a Dutch electron storage ring, called PAMPUS (Bac79a). A unit cell of this lattice is shown in figure 1.2. While writing this thesis, authorities decided not to build the proposed Dutch synchrotron radiation facility PAMPUS. However, the lattice and its properties are characteristic of recent electron storage rings used for synchrotron radiation.

Finally we will discuss the lattice of IKOR. This is a proposed proton accumulator ring in the "Spallations-Neutronenquelle" (SNQ) project, a cooperation between the institutes KFA Jülich and KfK Karlsruhe in West Germany (SNQ81 I,II,III; Jül81).
A unit cell of IKOR - "Isochrone KOmpressor Ring" - is given in figure 1.3. From 1979 to 1982 we contributed to the work of the IKOR Study Group (Jül81). We were especially concerned with the consequences of the choice of the working point on resonance widths and tolerances. These problems are strongly related to the specification of correction elements inasmuch as the presence of correction elements affects the properties of the resonances.
A list of parameters of the machines is given in table 1.1.

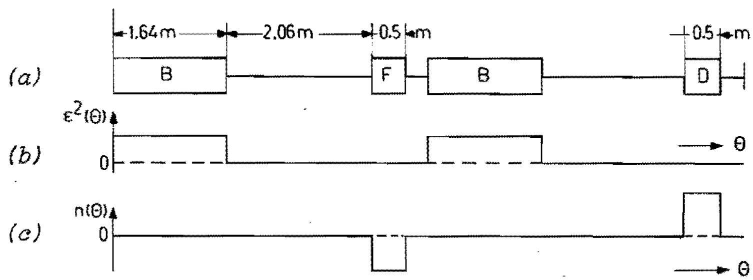

Figure 1.1 Unit cell of a simple FODO lattice (a) (Bac79a) and its guiding (b) and focusing functions (c).

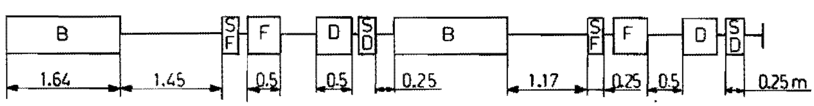

Figure 1.2 Unit cell of the - meanwhile dismissed - PAMPUS electron storage ring (FODOBOOFODOBOO type; Bac79a), SF and SD are sextupole magnets.

Table 1.1 List of parameters of PAMPUS † *and IKOR.*

	PAMPUS †	IKOR
particles	electrons	protons
lattice type	isomagnetic/separated function	
period structure	FODOBOOFODOBOO	FODOODOOBO
number of cells	8	11
kinetic energy (GeV)	1.5	1.1
γ_o	2936	2.173
mean radius R (m)	13.74	32.18
betatron number Q_x	2.10 - 6.25	3.25
Q_z	$Q_x \approx Q_z$	4.40
emittance $\pi\varepsilon_x$ (mm.mrad)	100π - 5π	150π
$\pi\varepsilon_z$ (mm.mrad)	$\varepsilon_z \leq 0.1\ \varepsilon_x$	50π
bending magnets	rectangular	sector type
field B_o (Tesla)	1.2	1.3
radius of curvature ρ (m)	4.17	4.64
magnetic length (m)	1.64	2.65
quadrupole lenses		
gradient (T/m)	up to 10	up to 3.5
magnetic length (m)	0.5	0.4
momentum compaction factor α	0.2 - 0.014	0.202
$\gamma_{tr} = \alpha^{-\frac{1}{2}}$	2.24 - 8.45	2.226
peak voltage $\hat{V}$ of HF system (kV)	up to 800	-
radiation loss U_o per turn (keV)	107.5	-
frequency HF system (MHz)	500	-
harmonic number h	144	-
synchrotron oscillation number Q_s	0.03 - 0.005	-

for more data see e.g. Bac79a and Jül81.

† Authorities decided in the meantime not to build the proposed Dutch synchrotron radiation facility PAMPUS.

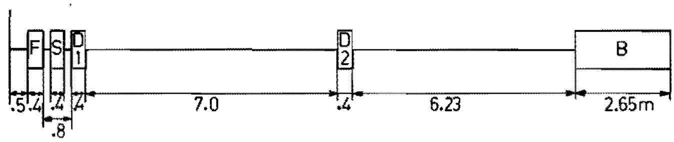

Figure 1.3 Unit cell of the proposed proton accumulator ring IKOR (Jül81) (FODOODOOBO type), S is a sextupole magnet.

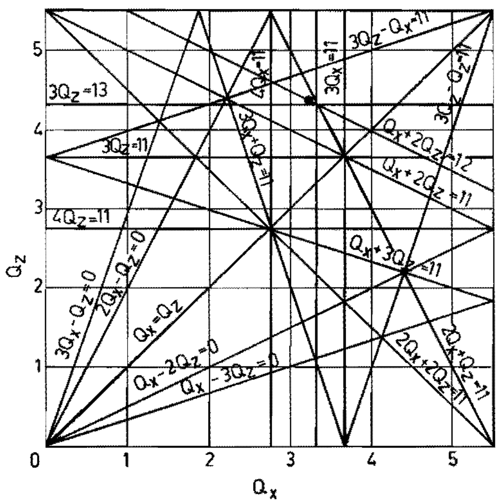

Figure 1.4 Resonance lines for IKOR, due to systematic machine harmonics (N=11) and imperfection harmonics. The working point is $Q_x \approx 3.25$, $Q_z \approx 4.4$.

1.7 The purpose and contents of this thesis

The influence of resonances on the orbit motion is of particular interest in designing new accelerators and storage rings. The purpose of this study is to develop a universal theory for the investigation of three-dimensional resonances, i.e. not only coupling effects between the transverse motions, but investigation of coupling between the transverse and longitudinal motion as well. To achieve this, we use the Hamilton formalism.
Resonance problems are related to the linear guide field, to the specification of non-linear correction elements and to the accelerating system in the accelerator.

In this chapter we have already considered some introductory and general features how to study resonance effects.

Expressions given in this chapter are part of the beginning of the elaborate study.

In chapter 2 we will consider problems related to the linear betatron theory. Generally the coefficients of the quadratic terms in the Hamiltonian depend upon the azimuthal angle θ (see (1.19)). In order to study a resonance in the way as mentioned before, the Hamiltonian must be transformed into a form with constant coefficients. This is carried out and subsequently analytical expressions for quantities which describe the linear betatron motion (i.e. the Twiss parameters) have been derived. These expressions contain Fourier components of the unperturbed linear guide field. Perturbations in this guide field can excite linear resonances. A brief analysis of these resonances is presented, using phase plane representations instead of solving the equations of motion.

The simultaneous treatment of the betatron and synchrotron motions - i.e. a theory which includes the acceleration process - is developed in chapter 3 starting from the advance knowledge of section 1.3. We start with the description of the particle motion in a cylindrical-symmetric magnetic field and afterwards the theory is extended to magnetic fields with an alternating gradient (A.G.) structure. The theory enables us to consider coupling effects in circular accelerators. As an illustration we will briefly discuss radial-longitudinal coupling in cyclotrons. Further, different sources which can excite synchro-betatron resonances in A.G. synchrotrons and storage rings become obvious and some of them will be treated in more detail.

It is a general procedure in studying resonances to neglect the rapidly oscillating terms (zero average) which are present in the Hamiltonian, besides the resonant terms. However, in principle these terms must be transformed away and they reappear in higher degrees, resulting in "higher order" effects. This is illustrated in chapter 4 for the one-dimensional non-linear betatron motion. A separate presentation of the one-dimensional case is justified by its simplicity, which facilitates the demonstration of characteristic phenomena and their numerical evaluation. We will restrict ourselves chiefly to a detailed examination of non-linear effects due to sextupole and octupole fields. Special attention will be paid to the use of sextupoles in IKOR.

The two-dimensional betatron resonances are discussed in chapter 5. A comprehensive treatment - studying trajectories in a phase plane - gives a good insight into the resonance behaviour and enables us to calculate tolerances and stopband widths. Results will be compared with results of Guignard's theory (Gui76, Gui78).

Finally we have to note that the interaction between the particles in the beam or between the particles and their surroundings (i.e. vacuum chamber, beam pipe etc.) will not be discussed in this thesis.

CHAPTER 2 LINEAR BETATRON THEORY

2.1 Introduction: the betatron oscillations

As pointed out in chapter 1, the quadratic Hamiltonian generally has periodic coefficients (see (1.19)). To explore the Hamilton theory in order to study resonances in the way as outlined before, these periodic coeffiecients have to be eliminated by canonical transformations. In order to find these transformations we first consider the solution of the equation of motion in the non-accelerated, periodic case: i.e. the betatron oscillations.

The betatron oscillations are described by the quadratic part of (1.19). This Hamiltonian yields a set of uncoupled equations of motion with periodic coefficients, known as Hill's equations:

$$\frac{d^2\bar{\bar{y}}}{d\theta^2} + K_y(\theta)\bar{\bar{y}} = 0 \qquad \bar{\bar{y}} = \bar{\bar{x}}\ ,\ \bar{\bar{z}}$$

$$K_y(\theta + 2\pi) = K_y(\theta) \tag{2.1}$$

$$K_x = \varepsilon^2 - n\ , \quad K_z = n$$

in which $\bar{\bar{y}}$ is the reduced transverse coordinate and K_y is the normalized guide field. The alternating gradient (A.G.) synchrotron ideally consists of N identical sections or "unit cells", so that K_y also satisfies $K_y(\theta + 2\pi/N) = K_y(\theta)$.

The solution of (2.1) can be written in the well-known Floquet form. Returning to the coordinate $y = R\bar{\bar{y}}$ expressed in a length-unit, we can write

$$\underline{y}(\theta) = \begin{pmatrix} y(\theta) \\ y'(\theta) \end{pmatrix} = \frac{1}{2}\, C\left(\underline{w}(\theta)e^{i(Q_y\theta + \chi_y)} + \text{c.c.}\right) \qquad ' = \frac{1}{R}\frac{d}{d\theta} \tag{2.2}$$

where C and χ_y are constants of the motion and Q_y - the betatron number - follows from the characteristic equation for the transfer matrix $\underline{\underline{M}}$ over one period 2π of $K_y(\theta)$:

$$\underline{\underline{M}}(2\pi/0)\underline{w}(0) = e^{i2\pi Q_y}\,\underline{w}(0) \quad . \tag{2.3}$$

The complex Floquet factors $\begin{pmatrix} w_1(\theta) \\ w_2(\theta) \end{pmatrix} = \underline{w}(\theta)$ are periodic in θ with the same period as $K_y(\theta)$ and are related to the eigenvector $\underline{w}(0)$ by

$$\underline{w}(\theta) = e^{-iQ_y\theta}\, \underline{\underline{M}}(\theta/0)\underline{w}(0) \quad . \tag{2.4}$$

Eq.(2.2) is for any χ_y the representation of an ellipse in the y,y' plane. Its shape depends on the observation point and the ellipse - often called eigenellipse - is periodic with 2π.

An important quantity of the oscillation is its amplitude. To obtain a real amplitude - the amplitude in (2.2) is complex - we define a new vector $\underline{u}(\theta)$ by

$$\underline{u}(\theta) = \begin{pmatrix} u_1(\theta) \\ u_2(\theta) \end{pmatrix} = \underline{w}(\theta) e^{-i \arg w_1(\theta)} \tag{2.5}$$

so that $u_1(\theta)$ is real and the solution $\underline{y}(\theta)$ of (2.2) can be written as (using the real representation)

$$\underline{y}(\theta) = C\underline{u}(\theta) \cos\, (Q_y\theta + \arg w_1(\theta) + \chi_y) \quad . \tag{2.6}$$

This result is similar to the notation introduced by Courant and Snyder (Cou58), writing the amplitude as

$$y(\theta) = \sqrt{\varepsilon_y}\sqrt{\beta_y(\theta)}\, \cos\, \left(\int_o^\theta \frac{R\, d\Theta}{\beta_y(\Theta)} + \chi_y \right) \tag{2.7}$$

ε_y is constant and $\beta_y(\theta)$ is the so-called amplitude or betatron function. Furthermore the quantity

$$\mu_y(\theta) = \int_o^\theta \frac{R\, d\Theta}{\beta_y(\Theta)} \tag{2.8}$$

is the betatron phase which is strongly related to Q_y:

$$Q_y = \frac{1}{2\pi}\, \mu_y(2\pi) \quad . \tag{2.9}$$

The relation between Floquet's theorem and the notation of Courant and Snyder can be written as

$$\begin{aligned} u_1(\theta) &= \sqrt{\beta_y}(\theta) \\ \arg w_1(\theta) &= \mu_y(\theta) - Q_y\theta \\ C &= \sqrt{\varepsilon_y} \quad . \end{aligned} \tag{2.10}$$

The betatron function $\beta_y(\theta)$ is uniquely determined by the function $K_y(\theta)$ and therefore it can serve as an alternate "representation" of the focusing characteristics of the magnetic field.
An important feature of the betatron motion is clear from (2.7): at each azimuth the displacement y of the particle is at most $\sqrt{\varepsilon_y}\sqrt{\beta_y(\theta)}$. The complete trajectory of a particle falls within an envelope defined by $\pm \sqrt{\varepsilon_y}\sqrt{\beta_y(\theta)}$. Of course the "aperture" A_y of the machine must satisfy the condition $A_y^2 > \varepsilon_y \beta_{y,max}$.

The eigenellipse in the phase plane (y,y') has a constant area $\pi\varepsilon_y$. A particle with oscillation amplitude y will lie on such an ellipse at the successive turns. The area of the ellipse which belongs to the maximum amplitude - this ellipse surrounds all particles in a beam - is often called the emittance of the beam. †
Analogously to the concept "emittance" - which is a property of the beam - the concept "acceptance" has been introduced being a property of the machine. Only particles whose trajectories (y,y') lie inside that acceptance will be accelerated.

From the form of (2.7) it can be shown that the eigenellipse is described by (Cou58)

$$\gamma_y y^2 + 2\alpha_y yy' + \beta_y (y')^2 = \varepsilon_y = \frac{\text{area}}{\pi} \qquad \text{with } ' = \frac{1}{R}\frac{d}{d\theta} \tag{2.11}$$

in which the quantities $\gamma_y(\theta)$, $\alpha_y(\theta)$ and $\beta_y(\theta)$ - called the Twiss parameters - are periodic functions of θ related by

$$\begin{aligned} \beta_y\gamma_y - 1 &= \alpha_y^2 \\ \beta_y' &= -2\alpha_y \\ \alpha_y' &= R^{-2}K_y\beta_y - \gamma_y \ . \end{aligned} \tag{2.12}$$

The description of the eigenellipse with the Twiss parameters is illustrated in figure 2.1.

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† Sometimes the quantity ε_y - i.e. the area of the ellipse divided by π - is called the emittance.

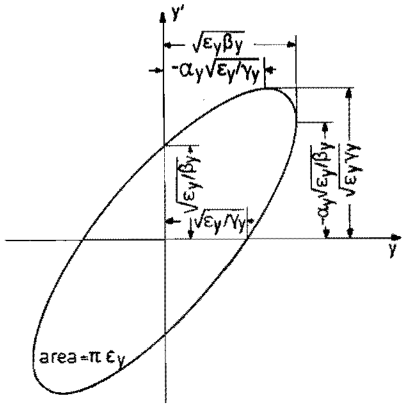


Figure 2.1
The phase space ellipse.

The Twiss parameters are very useful in matrix calculations † for the passage of the vector $\underline{y}(\theta)$ through the accelerator (see e.g. Cou58, Brü66, Ste71) and will be frequently used in this thesis.

Generally the Twiss parameters are calculated by a matrix code. When the parameters are known, they can directly be incorporated in canonical transformations in order to remove the periodic coefficients in the quadratic Hamiltonian. Before illustrating this procedure in section 2.3, we will derive analytical formulae for the Twiss parameters expressed in Fourier components of the linear guide field. In principle such formulae can be obtained from (2.12) by substituting a Fourier series for $K_y(\theta)$. But to fit the derivation into the general Hamilton formalism we will calculate these formulae in a different way, starting from the initial quadratic Hamiltonian and using the theory of canonical transformations.

---

† The propagation of beams of light through a media is also described by Hill's equation and matrix calculations are often used. In e.g. laser physics it is common use to define a complex beam parameter p by (see Ver79; or q = 1/p see Kog65):

$$p = u_2/u_1$$

where u_1 and u_2 are defined in (2.5).
This $p(\theta)$ is strongly related with the Twiss parameters from accelerator physics. Calculations show that (see Ver79)

$$p(\theta) = \beta^{-1}(\theta)\{i - \alpha(\theta)\}$$

and the equations (2.12) simplify to

$$p' = - R^{-2}K(\theta) - p^2 \; .$$

2.2 Analytical expressions for the Twiss parameters

In this section we will link the Hamilton theory and its canonical transformations with the well-known Twiss parameters (see also Cor81c). We note that the entire discussion of this section applies equally to the vertical as well as the horizontal motion.

The Hamiltonian leading to eq.(2.1) is (omit the bars above the variables)

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} K_x x^2 \quad . \tag{2.13}$$

The linear guide field is expanded in a Fourier series

$$K_x(\theta) = \varepsilon^2(\theta) - n(\theta) = \sum_{p\geq 0}^{\infty} A_p \cos pN\theta + B_p \sin pN\theta \tag{2.14}$$

with N the number of unit cells, i.e. the periodicity of the linear guide field.

In dealing with a problem represented by (2.13) we transform to action and angle variables J,ϕ (see (1.24)) and the Hamiltonian $K(J,\phi)$ then consists of a constant and an oscillating part. Subsequently, the elimination of the oscillating part is achieved by a canonical transformation of the form (Hag62)

$$\begin{aligned} G(\bar{J},\phi,\theta) &= -\bar{J}\phi - \bar{J}U_2(\phi,\theta) \\ J &= \bar{J}\left(1 + \frac{\partial U_2}{\partial \phi}\right) \\ \phi &= \bar{\phi} - U_2(\phi,\theta) \quad . \end{aligned} \tag{2.15}$$

The function $U_2(\phi,\theta)$ is determined by the requirement that all oscillating parts in the Hamiltonian vanish, resulting in a Hamiltonian of the form $\bar{K} = Q_x\bar{J}$.

Thus eq.(2.15) transforms the phase plane ellipse x,x' into a circle with radius $\sqrt{2\bar{J}}$ and the relation between $\bar{J}$ and the emittance of the beam is (we recall the use of reduced variables in (2.13)):

$$\bar{J} = \varepsilon_x/2R \quad . \tag{2.16}$$

The function U_2 is periodic in θ and ϕ and can be written as (see also Bac79b)

$$U_2(\phi,\theta) = \sum_{k=1}^{\infty} a_{2k}(\theta)\cos 2k\phi + b_{2k}(\theta)\sin 2k\phi \quad . \tag{2.17}$$

The coefficients $a_{2k}(\theta)$ and $b_{2k}(\theta)$ have the same periodicity as the linear field $K_x(\theta)$ and contain its Fourier components A_p, B_p.
The relation between U_2 (and thus A_p, B_p) and the Twiss parameters is obvious when we recall that the initial variables x, p_x lie on an eigenellipse so that we can write (see also fig. 2.1)

$$
\begin{aligned}
x(p_x = 0) &= \frac{1}{R}\sqrt{\varepsilon_x/\gamma_x} = \sqrt{\frac{2\bar{J}}{Q_x}\left(1 + \frac{\partial U_2}{\partial\phi}\right)_{\phi=0}} \\
p_x(x = 0) &= \sqrt{\varepsilon_x/\beta_x} = \sqrt{2Q_x\bar{J}\left(1 + \frac{\partial U_2}{\partial\phi}\right)_{\phi=\pi/2}} \quad .
\end{aligned}
\qquad (2.18)
$$

Substitution of (2.16) and (2.17) results in the analytical expressions for the Twiss parameters β_x and γ_x:

$$
\begin{aligned}
\beta_x(\theta) &= \frac{R}{Q_x}\,\frac{1}{1 + \sum_{k=1}^{\infty}(-1)^k 2kb_{2k}(\theta)} \\
\gamma_x(\theta) &= \frac{Q_x}{R}\,\frac{1}{1 + \sum_{k=1}^{\infty} 2kb_{2k}(\theta)} \quad .
\end{aligned}
\qquad (2.19)
$$

In case of cylindrical-symmetric magnetic fields these equations are reduced to

$$
\beta_x = \frac{R}{Q_x} = \frac{R}{\sqrt{1-n}} \qquad \text{and} \qquad \gamma_x = \beta_x^{-1} \quad . \qquad (2.20)
$$

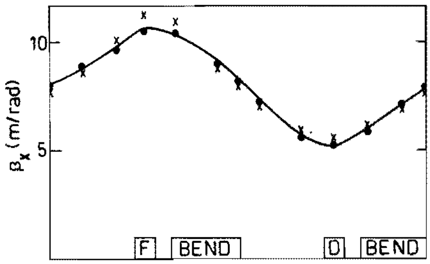
Up to the first degree in the Fourier components A_p, B_p of the linear guide field, we get for k = 1 (see Cor80c)

$$
b_2^{(1)} = \sum_{p\geq 1} \frac{A_p \cos pN\theta + B_p \sin pN\theta}{(pN)^2 - 4Q_x^2} \quad . \qquad (2.21)
$$

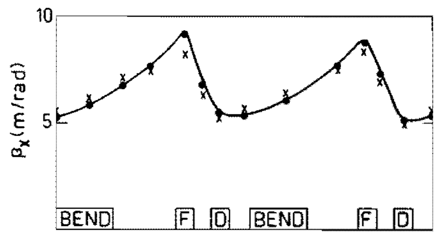
The derivation of more coefficients b_{2k} is a time-consuming procedure and the analytical expressions become increasingly complicated (Cor80c).

As an illustration we compare the analytically calculated β_x with the one obtained with a matrix code. The results for the different lattice configurations are given in figure 2.2. The first order result for β_x - obtained by only taking into account $b_2^{(1)}$ of (2.21) - turns out to give a good approximation on condition that the modulation of β_x is not too large. For higher field modulations it is essential to involve higher order terms in (2.19). Taking into account the two next relevant terms in b_{2k}, eq.(2.19) leads to rather good results as shown in figure 2.2.

"FODO" , $Q_x = 1.50$
$B_o = 1.2$ T
gradient F : 1.10 T/m
gradient D :-2.00 T/m



PAMPUS , $Q_x = 2.10$
$B_o = 1.2$ T
gradient F : 3.56 T/m
gradient D : -3.06 T/m



IKOR , $Q_x = 3.25$
$B_o = 1.3$ T
gradient F : 1.34 T/m
gradient D1: -0.20 T/m
gradient D2: -3.09 T/m

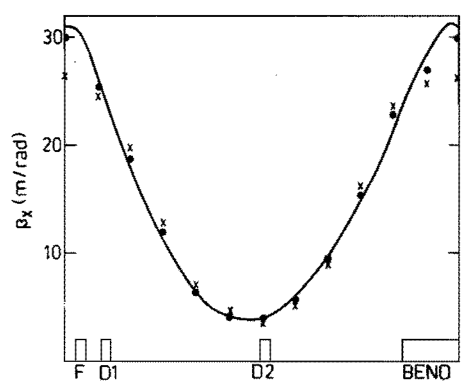


Figure 2.2

The horizontal betatron function β_x for the various lattices:
× *analytically calculated β_x (2.19) taking into account only $b_2^{(1)}$,*
• *analytically calculated β_x taking into account $b_2^{(1)}$, $b_2^{(2)}$, $b_4^{(2)}$*
— *result from a matrix code.*

Substitution of the complete function U_2 in the Hamiltonian will generally lead to tedious calculations and comprehensive formulae. It turns out to be more advantageous to apply transformations which directly contain the Twiss parameters - being known now - instead of the series of (2.17) and (2.19). These kind of transformations will be discussed in the next section.

2.3 Canonical transformations useful in linear Hamilton theory

In the transformations - containing the Twiss parameters - some advance knowledge obtained from the treatment of Courant and Snyder (Cou58, see section 2.1) is included.
The first transformation is generated by the function

$$G(x,\bar{p}_x,z,\bar{p}_z,\theta) = \sqrt{\frac{R}{\beta_x}}\, x\bar{p}_x + \frac{1}{4}\frac{R}{\beta_x}\beta'_x\, x^2 + \sqrt{\frac{R}{\beta_z}}\, z\bar{p}_z + \frac{1}{4}\frac{R}{\beta_z}\beta'_z\, z^2 \qquad ' = \frac{1}{R}\frac{d}{d\theta}$$

so that (2.22)

$$x = \sqrt{\frac{\beta_x}{R}}\,\bar{x}$$

$$p_x = \sqrt{\frac{R}{\beta_x}}\left(\bar{p}_x + \frac{1}{2}\beta'_x\,\bar{x}\right)$$

and similar for z and p_z.
This is a transformation to new axes Re $\underline{u}(\theta)$ and Im $\underline{u}(\theta)$ with $\underline{u}(\theta)$ defined in eq.(2.5).
Substitution of these new variables $\bar{x},\bar{p}_x,\bar{z}$ and $\bar{p}_z$ into (2.13) or into the quadratic part of (1.19) results in a new Hamiltonian of the form

$$\bar{H} = \frac{R}{2\beta_x(\theta)}\left(\bar{p}_x^2 + \bar{x}^2\right) + \frac{R}{2\beta_z(\theta)}\left(\bar{p}_z^2 + \bar{z}^2\right) \tag{2.23}$$

where the amplitude function β_x satisfies the relation

$$\frac{1}{2}\beta_x\beta''_x - \frac{1}{4}(\beta'_x)^2 + R^{-2}K_x\beta_x^2 = 1 \qquad ' = \frac{1}{R}\frac{d}{d\theta} \tag{2.24}$$

which is in agreement with (2.12)(and of course a similar relation holds for the vertical betatron function).
This Hamiltonian (2.23) corresponds with the solution (2.7).
We shall treat now the uncoupled and coupled motion separately.

2.3.1 Uncoupled betatron motion

To investigate only one degree of freedom we consider e.g. the horizontal motion.
The θ-dependence of the coefficient in (2.23) is easily removed by the transition to a new independent variable ψ, defined by (compare with (2.8))

$$d\psi = \frac{Rd\theta}{\beta_x Q_x} \quad . \tag{2.25}$$

We notice that $d\psi \approx d\theta$ when the modulation of β_x is small.
Subsequently the Hamiltonian has the well-known form of a harmonic oscillator

$$\bar{\bar{H}}(\bar{x},\bar{p}_x;\psi) = \frac{\beta_x Q_x}{R}\bar{H} = \frac{1}{2}Q_x(\bar{p}_x^2 + \bar{x}^2) \tag{2.26}$$

and substitution of action and angle variables leads to a simple Hamiltonian:

$$\bar{x} = \sqrt{2J}\cos\phi \ , \ \bar{p}_x = \sqrt{2J}\sin\phi \ ; \quad K = Q_x J \quad . \tag{2.27}$$

The phase space ellipse is again transformed to a circle and the relation between J and the emittance is again $J = \varepsilon_x/2R$ (see (2.16)).

Thus the transformations (2.22) and (2.25) simplify the initial quadratic Hamiltonian with its periodic coefficients and are the starting-point for the investigation of linear (section 2.4) and non-linear one-dimensional betatron resonances (chapter 4).

2.3.2 Coupled betatron motion

Since the variable ψ of (2.25) contains β_x (or β_z in case of the vertical motion), it seems unfavourable to use this variable in the two-dimensional case. In that case coefficients depending on the independent variable still remain.
Therefore we maintain the azimuth θ as the independent variable.
The θ-dependence of the coefficients in the Hamiltonian (2.23) is now removed by a canonical transformation generated by the function

$$G(\bar{x},\phi_x,\bar{z},\phi_z) = \frac{1}{2}\bar{x}^2\tan(\phi_x + \Psi_x(\theta)) + \frac{1}{2}\bar{z}^2\tan(\phi_z + \Psi_z(\theta))$$
$$\text{with } \Psi_{x,z}(\theta) = Q_{x,z}\theta - \int_o^\theta \frac{Rd\Theta}{\beta_{x,z}(\Theta)} = Q_{x,z}\theta - \mu_{x,z}(\theta) \tag{2.28}$$

so that

$$\bar{x},\bar{z} = \sqrt{2J}_{x,z}\cos(\phi_{x,z} + \Psi_{x,z}(\theta))$$

$$\bar{p}_{x,z} = \sqrt{2J}_{x,z}\sin(\phi_{x,z} + \Psi_{x,z}(\theta)) \quad .$$

This is a transformation with Re $\underline{w}(\theta)$ and Im $\underline{w}(\theta)$ as new axes (see (2.2)). The new Hamiltonian becomes

$$K = \bar{H} + \partial G/\partial\theta = Q_x J_x + Q_z J_z \quad . \tag{2.29}$$

The solutions $J_{x,z}$ = constant and $\phi_x = -Q_x\theta$, $\phi_z = -Q_z\theta$ correspond to the solution (2.2).

Of course also the one-dimensional motion can be treated with this procedure.

Summarizing we conclude that the Hamiltonian - representing linear betatron motion - has been transformed to a constant one (2.29) and the complete transformation of the coordinates and momenta is ((2.22) plus (2.28))

$$\begin{aligned} x &= \sqrt{2J_x\beta_x/R}\cos(\phi_x + \Psi_x(\theta)) \\ p_x &= \sqrt{2J_x R/\beta_x}\{\sin(\phi_x + \Psi_x(\theta)) + \tfrac{1}{2}\beta_x'\cos(\phi_x + \Psi_x(\theta))\} \end{aligned} \tag{2.30}$$

and the same vertically for z and p_z.

A similar transformation has already been mentioned in e.g. Sno69.

The transformation (2.30) will be used in following chapters, particularly in chapter 5 to study non-linear coupled betatron resonances.

To conclude this chapter we give a short review of some resonances due to linear machine imperfections.

2.4 Linear machine imperfections

The effect of imperfections in the linear guide field in A.G. synchrotrons or storage rings is usually studied by solving the equations of motion (San71, Kei77). In this section the effect is examined by considerations of the phase plane, as an introduction to the treatment of resonances later in this thesis.

A few results for IKOR will be given. The lattice functions in this accumulator ring are plotted in figure 2.3.

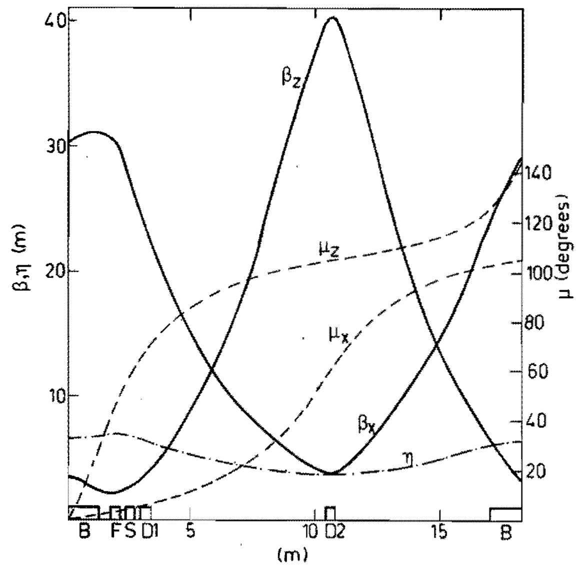


Figure 2.3
Lattice functions in IKOR.

2.4.1 The effect of a dipole imperfection

An imperfection ΔB in the dipole field B_o appears in the Hamiltonian (1.19) as a first degree term in $\bar{\bar{x}}$ with coefficient $\varepsilon\Delta B(\theta)/B_o = \varepsilon b(\theta)$.

The final Hamiltonian to study the resonance effect for $Q_x \simeq p_r$ in the way outlined in the previous sections (see e.g. section 1.5.1 and 2.3), is

$$\bar{K}(\bar{J},\bar{\phi};\psi) = \delta Q\bar{J} + \frac{1}{2}\sqrt{2\bar{J}}\,\{A_{p_r}\cos\bar{\phi} + B_{p_r}\sin\bar{\phi}\} \tag{2.31}$$

with $\delta Q = Q_x - p_r$ and where A_{p_r} and B_{p_r} are the components of the p_r-th harmonic of the Fourier expansion of the imperfection:

$$\varepsilon(\psi)b(\psi)\beta_x^{3/2}(\psi)R^{-3/2}Q_x = \sum_{p\geq 0} A_p\cos p\psi + B_p\sin p\psi\ . \tag{2.32}$$

When returning to cartesian coordinates $\tilde{x},\tilde{y}$ ($\tilde{x} = \sqrt{2\bar{J}}\cos\bar{\phi}$ and $\tilde{y} = \sqrt{2\bar{J}}\sin\bar{\phi}$) it becomes obvious that the flowlines $\tilde{K}(\tilde{x},\tilde{y})$ = constant are circles:

$$\tilde{K} = \frac{1}{2}\,\delta Q\left\{\left(\tilde{x} + \frac{A_{p_r}}{2\delta Q}\right)^2 + \left(\tilde{y} + \frac{B_{p_r}}{2\delta Q}\right)^2\right\}\ . \tag{2.33}$$

When there is no field imperfection the circles are concentric around the stable fixed point $(\tilde{x},\tilde{y}) = (0,0)$. Due to the p_r-th harmonic of the dipole imperfection, the stable fixed point shifts across a relative distance $\Delta r/R$ with

$$\Delta r = \frac{R}{2\delta Q} \sqrt{A_{p_r}^2 + B_{p_r}^2} \quad . \tag{2.34}$$

This expression clearly shows its resonance properties at $Q_x = p_r$.
The resonance displaces an area in phase plane as schematically illustrated in figure 2.4 - see also the effect of a first harmonic perturbation in cyclotrons (Nie72) - and it is obvious that the shifted beam area must lie within the machine acceptance.
The shift Δr is proportional to the strength of the imperfection and depends on the value of β_x at the position of the perturbation, thus β_x is a measure of the "sensitivity" to perturbations.

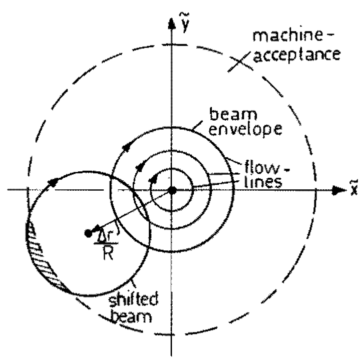


Figure 2.4
Displacement of an area in $(\tilde{x},\tilde{y})$ phase plane, due to a p_r-th harmonic of a dipole imperfection. The shaded area is lost in practice.

As an illustration we consider the effect of a dipole imperfection in IKOR. As the machine has been proposed to operate at $Q_x \approx 3.25$, the relevant harmonic is $p_r = 3$. The amplitude of this third harmonic is $6.5\ 10^{-2}|\Delta B/B_o|$ where $\Delta B/B_o$ is the relative imperfection in one dipole. Due to a $\Delta B/B_o = 5.10^{-4}$, i.e. $\Delta B = 6.5$ Gauss, the shift $\Delta r = 2.1$ mm.
In practice all dipoles may have disturbances. In the design stage a detailed behaviour $\Delta B(\theta)$ is not known and the field deviations are true errors with a statistical distribution. A statistical analysis must be made to obtain a statistical estimate of the maximum amplitude of the disturbed closed orbit (Bov70). We will not go into that subject here, but roughly speaking Δr must then be multiplied with the square root of the number of dipoles that cause these errors.
In IKOR, with 11 dipoles, $\Delta r \approx 7$ mm for $\Delta B = 6.5$ Gauss.

For horizontal correction of the closed orbit (i.e. correction of Δr) auxiliary windings on the bending magnets are provided (Jül81).

2.4.2 The effect of an imperfection in the field gradient

The influence of a gradient imperfection $\Delta n(\theta)$ on e.g. the vertical motion, is represented by a second degree term in $\bar{\bar{z}}$ with coefficient $\frac{1}{2}\Delta n(\theta)$ in the Hamiltonian (1.19) (the horizontal motion can be treated similarly). After application of the various transformations (see section 1.5.1 and 2.3), the resulting Hamiltonian shows a resonant term for $Q_z \approx p_r/2$:

$$\bar{K}(\bar{J},\bar{\phi};\psi) = (Q_z - \frac{p_r}{2})\bar{J} + C_o\bar{J} + \frac{1}{2}\bar{J}\{C_{p_r}\cos 2\bar{\phi} + D_{p_r}\sin 2\bar{\phi}\} \tag{2.35}$$

with $\frac{1}{2}\Delta n(\psi)\beta_z^2(\psi)R^{-2}Q_z = \sum_{p\geq 0} C_p\cos p\psi + D_p\sin p\psi$.

The flowlines $\tilde{K}(\tilde{x},\tilde{y})$ = constant - where $\tilde{x}$ and $\tilde{y}$ are again cartesian coordinates - are ellipses (bounded amplitudes, stable motion) or hyperbolae (unlimited amplitudes, unstable motion). The unstable motion occurs when the following condition is fulfilled:

$$\frac{p_r}{2} - \Delta Q_z - \frac{1}{2}\sqrt{C_{p_r}^2 + D_{p_r}^2} < Q_z < \frac{p_r}{2} - \Delta Q_z + \frac{1}{2}\sqrt{C_{p_r}^2 + D_{p_r}^2} \tag{2.36}$$

where ΔQ_z is the tune shift caused by the zeroeth harmonic of the gradient imperfection (see (2.35))

$$\Delta Q_z = C_o = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\Delta n\beta_z^2R^{-2}Q_z d\psi = \frac{1}{4\pi}\int_0^{L_o} \Delta n\beta_z R^{-2}ds \ . \tag{2.37}$$

In other words, the stopband width only exists when there is a p_r-th harmonic of the gradient imperfection. The width is given by the amplitude of this p_r-th harmonic.

As an illustration, we consider the influence of a gradient disturbance in IKOR which has been proposed to operate not too far from a half integer value for the vertical tune ($Q_z \approx 4.4$, see also table 1.1). Since the Fourier components and consequently the stopband width depend on β_z at the position of the imperfection, we consider the effect where β_z has its maximum value, i.e. position D2 (see fig. 2.3: $\beta_z \approx 40$ m/rad). The amplitude of the 9th harmonic is 7.10^{-3} for a relative imperfection $\Delta n/n=5.10^{-3}$ in one quadrupole.

The tune shift $\Delta Q_z = 3.3\ 10^{-3}$ and the forbidden region for Q_z becomes

$$4.493 < Q_z < 4.500 \qquad \text{(IKOR)} .$$

This narrow stopband may grow in case of extra imperfections in other quadrupoles.

Additionally we notice that a half integer resonance can also be excited when a disturbed closed orbit is combined with sextupole fields since effective gradients appear.

CHAPTER 3 SIMULTANEOUS TREATMENT OF BETATRON AND SYNCHROTRON MOTION

3.1 Introduction

In general the transverse and the longitudinal motions are coupled in a circular accelerator.
The present development was undertaken to describe the betatron and synchrotron oscillations simultaneously, resulting in a theory for the description of coupling effects.

The synchrotron oscillations are the result of the existing longitudinal electric fields. These fields, used for acceleration or compensation of radiation losses, are generated by a Dee - dummy Dee structure (in cyclotrons) or by so-called cavities (in synchrotrons). All accelerating structures have HF longitudinal electric fields, i.e. the characteristic time is comparable with or even smaller than the revolution period of the particles.

For simplicity we will assume an accelerating gap with an infinitesimal small width, equivalent to stepwise acceleration.
The representation of the electric fields in the general Hamiltonian has already partly been discussed in chapter 1, section 1.3.

First we will explore the theory for the description of coupling effects in accelerators with a cylindrical-symmetric magnetic field, which might be time-dependent. †
We start with the expansion of the Hamiltonian (1.16c) in its variables neglecting the electric fields, to obtain a deliberate preamble. A coupling term between the radial and longitudinal motion appears in second degree in this Hamiltonian. A canonical transformation is applied to separate the two modes of oscillation in the linearized

† In principle the C.G. synchrotron does not have an exact cylindrical-symmetric magnetic field. But the total path length in the straight sections is usually much smaller than the path length in all magnets and as far as the derivation of the theory is concerned, we might represent the magnetic field - as being cylindrical-symmetric - by its average value on the reference orbit and its average gradient.

case, and the meaning of the new canonical variables will be explained (section 3.2).

Afterwards, the acceleration by the electric fields is taken into account. Elaboration of the total Hamiltonian will result in a theory which is principally suited to study coupling effects between the transverse and longitudinal motions (section 3.3).

In section 3.4 we will briefly discuss radial-longitudinal coupling in a cylindrical-symmetric magnetic field.

The extension of the theory to accelerators with an alternating gradient (A.G.) magnetic field structure is carried out in section 3.5. This treatment is simplified by the advance knowledge of the preceding sections.

Finally we will discuss some features of two specific synchro-betatron resonances in A.G. synchrotrons or storage rings (section 3.6).

3.2 Time-dependent cylindrical-symmetric magnetic field, no HF accelerating structures

The reference orbit in a cylindrical-symmetric magnetic field is a circle with radius R equal to the radius of curvature ρ.

In general the magnetic field B varies slowly in time and we write

$$B(t) = B_o\left[1 + b(t)\right] \tag{3.1}$$

with B_o the initial magnetic field on the reference orbit.
We recall - as stated in section 1.2 - that the time dependence of the magnetic field is merely represented by a multiplying factor.
For times characteristic of the transverse and longitudinal motions, the quantity b(t) is very small. In case of the (synchro-)cyclotron $b(t) = 0$.

Before evaluating the Hamiltonian (1.13) or (1.16c) we notice that the quantity $p_{so}(t)$, defined in (1.11) - and of particular interest in case of time-dependent magnetic fields - depends on the constant term in the vector potential. This term is determined by the enclosed magnetic flux and we get for the different machines (see eq.(1.6) and $(1.6^{\dagger})$):

$$\begin{aligned} &\text{synchrotron} : \quad p_{so}(t) = eB_o\big(1 + b(t)\big)\rho \\ &\text{betatron} \quad\;\; : \quad p_{so}(t) = 0 \end{aligned} \tag{3.2}$$

whereas for the (synchro-)cyclotron the shape of the vector potential is of less importance because of the constant magnetic field (see (1.13)).
By this particular choice of p_{so} there are no first degree terms containing $\bar{x}$ in the square root of (1.13), which significantly simplifies the work of the expansion of this Hamiltonian.

We start with the case of a (synchro-)cyclotron and a synchrotron. The betatron will be briefly discussed in section 3.2.1.

We expand the Hamiltonian (1.13/1.16c) up to third degree in the variables. An expansion to higher degrees is more cumbersome but still possible.
The Hamiltonian - with the scaled time τ as the independent variable - is

$$\begin{aligned} \bar{\bar{H}} = \; & \frac{1}{2}\bar{\bar{p}}_x^2 + \frac{1}{2}(1-n)\bar{\bar{x}}^2 + \frac{1}{2}\bar{\bar{p}}_z^2 + \frac{1}{2}n\bar{\bar{z}}^2 \\ & + \frac{1}{2\gamma_o^2}\bar{\bar{p}}_\theta^2 + \bar{\bar{p}}_\theta\Big(1 + \frac{b}{\gamma_o^2}\Big) + \dot{b}\bar{\bar{\theta}} - \bar{\bar{p}}_\theta\bar{\bar{x}} \\ & - \frac{1}{6}(3-n)\bar{\bar{x}}^3 + \Big(1 + \frac{1-n}{2\gamma_o^2}\Big)\bar{\bar{p}}_\theta\bar{\bar{x}}^2 - \frac{1}{2}\bar{\bar{p}}_\theta\bar{\bar{p}}_x^2\Big(1 - \frac{1}{\gamma_o^2}\Big) \\ & - \frac{1}{2\gamma_o^2}\bar{\bar{p}}_\theta^3\Big(1 - \frac{1}{\gamma_o^2}\Big) - \frac{1}{\gamma_o^2}\bar{\bar{p}}_\theta^2\bar{\bar{x}} \\ & + \frac{n}{2\gamma_o^2}\bar{\bar{p}}_\theta\bar{\bar{z}}^2 - \frac{1}{2}\bar{\bar{p}}_\theta\bar{\bar{p}}_z^2\Big(1 - \frac{1}{\gamma_o^2}\Big) \end{aligned} \tag{3.3}$$

with $\dot{} = \frac{d}{d\tau}$.

The canonical variables are defined in (1.16a) with $P_o = eB_o\rho$ and as stated in section 1.3.1 the variables $\bar{\bar{s}}$ and $\bar{\bar{p}}_s$ are respectively written as $\bar{\bar{\theta}}$ and $\bar{\bar{p}}_\theta$.
The time-dependent magnetic field is written as $B(\tau) = B_o(1 + \dot{b}\tau)$ and the time derivative is taken into account in the lowest degree of the Hamiltonian only. A more detailed description gives some correction terms, all being small (see Cor82).

We notice that a coupling term between the radial and longitudinal motion appears in second degree of (3.3), i.e. the $\bar{\bar{p}}_\theta \bar{\bar{x}}$ term.
It is convenient to perform a transformation to new variables in order to separate the two modes of oscillation in the linearized case.
The generating function is

$$G = \tilde{p}_x \bar{\bar{x}} + \tilde{p}_z \bar{\bar{z}} + \tilde{p}_\theta \bar{\bar{\theta}} - \frac{1}{1-n} \tilde{p}_\theta \tilde{p}_x - \frac{\dot{b}}{1-n} \bar{\bar{x}}$$

so that

$$\begin{aligned} \bar{\bar{x}} &= \tilde{x} + \frac{\tilde{p}_\theta}{1-n} & \bar{\bar{p}}_x &= \tilde{p}_x - \frac{\dot{b}}{1-n} \\ \bar{\bar{z}} &= \tilde{z} & \bar{\bar{p}}_z &= \tilde{p}_z \\ \bar{\bar{\theta}} &= \tilde{\theta} + \frac{\tilde{p}_x}{1-n} & \bar{\bar{p}}_\theta &= \tilde{p}_\theta \quad . \end{aligned} \tag{3.4}$$

The Hamiltonian, expressed in the new variables, is

$$\begin{aligned} \tilde{H} = \; & \frac{1}{2} \tilde{p}_x^2 + \frac{1}{2}(1-n)\tilde{x}^2 + \frac{1}{2}\tilde{p}_z^2 + \frac{1}{2} n\tilde{z}^2 \\ & + \frac{1}{2} \tilde{p}_\theta^2 \Big(\frac{1}{\gamma_o^2} - \frac{1}{1-n}\Big) + \tilde{p}_\theta \Big(1 + \frac{b}{\gamma_o^2}\Big) + \dot{b}\tilde{\theta} \\ & - \frac{1}{6}(3-n)\tilde{x}^3 - \frac{1}{2} \tilde{p}_\theta \tilde{p}_x^2 \Big(1 - \frac{1}{\gamma_o^2}\Big) + \frac{1}{2} \tilde{p}_\theta^2 \tilde{x} \Big(\frac{1-3n}{(1-n)^2}\Big) \\ & + \tilde{p}_\theta^3 \Big(- \frac{1}{2\gamma_o^2}\Big(1 - \frac{1}{\gamma_o^2} + \frac{1}{1-n}\Big) + \frac{1}{6} \frac{3-5n}{(1-n)^3}\Big) \\ & - \frac{1}{2} \tilde{p}_\theta \tilde{x}^2 \Big(\frac{1+n}{1-n} - \frac{1-n}{\gamma_o^2}\Big) + \frac{n}{2\gamma_o^2} \tilde{p}_\theta \tilde{z}^2 - \frac{1}{2} \tilde{p}_\theta \tilde{p}_z^2 \Big(1 - \frac{1}{\gamma_o^2}\Big) \quad . \end{aligned} \tag{3.5}$$

The variables $\tilde{x}$, $\tilde{p}_x$ and $\tilde{z}$, $\tilde{p}_z$ describe the horizontal and vertical betatron oscillations and their corresponding oscillation "frequencies" are $Q_x = \sqrt{1-n}$ and $Q_z = \sqrt{n}$.
For the sake of completeness we mention that for (synchro-)cyclotrons the field index n varies from 0 in the centre to about 0.2 or 0.3 at extraction radius and in C.G. synchrotrons n will be about 0.5 or 0.6 (Liv61).

From $\dot{\tilde{\theta}} = \partial\tilde{H}/\partial\tilde{p}_\theta \approx 1$ we find that in first order approximation $\tilde{\theta}$ is equal to the scaled time τ.
The new longitudinal coordinate $\tilde{\theta}$ contains the radial motion via the momentum $\tilde{p}_x$ according to (3.4). As illustrated in figure 3.1 the

quantity $\tilde{p}_x$ is strongly related to the position of the centre of the orbit. The variable $\tilde{\theta}$ is based on the same idea as the so-called Central Position (CP) phase, first introduced by Schulte and Hagedoorn in their treatment of the motion of accelerated particles in cyclotrons using cartesian coordinates (Schu78, Schu80).

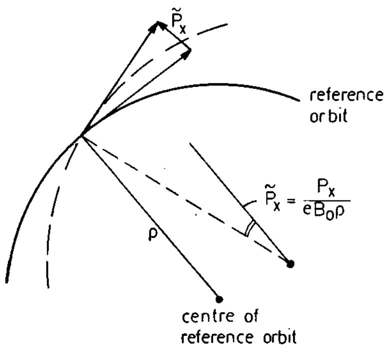


Figure 3.1
Schematic illustration of the relation between $\tilde{p}_x$ and the centre of the orbit in a homogeneous magnetic field.

Thus, when dealing with coupling between transverse and longitudinal motion it is necessary to perform a slight change of the longitudinal coordinate.

We accentuate the fact that for a perfect machine no coupling term $\bar{\bar{p}}_\theta\bar{\bar{z}}$ arises because of median plane symmetry. However, imperfections in the magnetic field may give rise to a term $\bar{\bar{p}}_\theta\bar{\bar{z}}$ in the Hamiltonian (3.3) and the longitudinal coordinate $\tilde{\theta}$ then also includes the vertical oscillations via $\tilde{p}_z$.

Finally we note that in case of an A.G. accelerator the relation between the coordinate $\tilde{\theta}$ and the centre position is much less evident because the reference orbit is no longer a real circle. But, as will be shown in section 3.5, it remains possible to define an equivalent CP phase based on the same idea, namely decoupling of both linear radial and longitudinal motions.

The longitudinal momentum $\tilde{p}_\theta$ is related to the deviation of the kinetic momentum P(t) of an arbitrary particle - moving in the time-dependent magnetic field on a radius $\rho + x$ - with respect to the kinetic momentum of a particle moving on radius ρ.
To illustrate this we recall the relation between P(t) and the original longitudinal momentum p_s (in case p_x=p_z=0):

$$P(t) = \frac{p_s}{1 + x/\rho} - eA_s(t) \ . \tag{3.6}$$

Substitution of successively $\bar{p}_s$ (defined in 1.12) and the vector potential (1.6) results in

$$\tilde{p}_\theta = \frac{\bar{p}_s}{P_o} \approx \frac{P(\tau) - eB(\tau)\rho}{P_o} = \frac{\Delta P(\tau)}{P_o} = \frac{\Delta W(\tau)}{P_o c \sqrt{1 - \frac{1}{\gamma_o^2}}} = \frac{\Delta W(\tau)}{R\omega_o P_o} \ . \tag{3.7}$$

In case of the cylindrical symmetry $R = \rho$.

The quantity $\tilde{p}_\theta/1-n$, subtracted from the $\bar{\bar{x}}$ coordinate in (3.4), is the relative change in the orbit radius due to a relative momentum deviation and $1/1-n$ is equal to the so-called momentum compaction factor α.

From $\dot{\tilde{p}}_\theta = - \partial\tilde{H}/\partial\tilde{\theta} = - \dot{b}$ and (3.7) we find in case of the C.G. or weak-focusing synchrotron

$$\frac{dP(\tau)}{d\tau} \approx 0, \tag{3.8}$$

in first order corresponding with the fact that the energy or momentum remains constant when no HF accelerating structures are present: the betatron action is (almost) zero in synchrotrons.

3.2.1 Betatron acceleration

A certain amount of betatron acceleration is represented in the Hamiltonian by taking a vector potential of the form

$$A_s = -B(t)\rho\{f_o + f(x,z)\} \tag{3.9}$$

where f_o is a constant ($0 \le f_o \le 1$) that "measures" the enclosed magnetic flux. In case of the betatron $f_o = 1$ (see also eq.($1.6^\dagger$), chapter 1).

After having carried out all transformations performed to obtain a Hamiltonian like (3.5), we find

$$\tilde{H} = \frac{1}{2}\tilde{p}_x^2 + \frac{1}{2}(1 - n)\tilde{x}^2 + \frac{1}{2}\tilde{p}_\theta^2\left(\frac{1}{\gamma_o^2} - \frac{1}{1-n}\right) + \tilde{p}_\theta\left(1 + \frac{b}{\gamma_o^2}\right) - (f_o - 1)b\dot{\tilde{\theta}} + \ldots . \tag{3.10}$$

For the betatron ($f_o=1$) the term $\dot{b}\tilde{\theta}$ is missing. Consequently $\tilde{p}_\theta$ is constant and for the reference particle $\tilde{p}_\theta = 0$.
As $(\tilde{x},\tilde{p}_x)=(0,0)$ is a solution of the Hamiltonian (3.10) we find (see (3.7))

$$P(\tau) = eB(\tau)\rho \quad . \tag{3.11}$$

The momentum (or energy) of the particle increases proportionally with the rate of change of the magnetic field (betatron action).
Of course eq.(3.11) can also be derived directly from (3.6) with $A_s(t)$ given in (3.9) and $\bar{p}_s = P_o\tilde{p}_\theta$.

3.3 The influence of longitudinal HF accelerating electric fields on the orbit motion

In general the longitudinal electric fields oscillate in time with a time-dependent frequency $\omega_{HF}(t)$.
For our description of the acceleration only the number of gaps or cavities and their positions are sufficient.

After application of the scale transformation (1.16a,b,c) the potential-like function (1.15) which represents the acceleration in the Hamiltonian becomes

$$eV_1(\bar{\bar{\theta}}) \cos \int \frac{\omega_{HF}(\tau)}{\omega_o} d\tau$$

with

$$V_1(\bar{\bar{\theta}}) = - \frac{R \int^{\bar{\bar{\theta}}} E_s d\bar{\bar{\theta}}'}{\left(1 - \frac{1}{\gamma_o^2}\right) W_o} \tag{3.12}$$

where we remind the reader that the variable $\bar{\bar{s}}$ of (1.16a) is written as $\bar{\bar{\theta}}$ (see section 1.3).

In this thesis we restrict ourselves - for simplicity - to homogeneous electric fields. In practice this may not be quite true (Lap65, Car65) and a substantial extension of the theory might be the description of e.g. radial electric fields or a variation of the accelerating voltage along the gap †, resulting in a representation of effects such as e.g. phase compression (Rus63, Mül70) or electric focusing (Dut75, Gor81a, Bot81).

Generally the frequency of the accelerating voltage is a multiple of the revolution frequency, indicated by the harmonic number h. Its time dependence is incorporated by writing

$$\omega_{HF}(\tau) = h\omega_o\bigl(1 + \delta(\tau)\bigr) = h\omega_o\bigl(1 + \dot{\delta}\tau\bigr) \tag{3.13}$$

in which $\delta(\tau)$ is of the same order of magnitude or even smaller than $b(\tau)$ (eq.(3.1)).

The Hamiltonian - including the acceleration term - expressed in the variables of (3.4) becomes

$$\begin{aligned}\tilde{H} = \frac{1}{2}\tilde{p}_x^2 + \ldots \text{ see (3.5) } \ldots + \\ eV_1(\tilde{\theta} + \frac{\tilde{p}_x}{1-n}) \cos \int h\bigl(1 + \delta(\tau)\bigr)\, d\tau \quad .\end{aligned} \tag{3.14}$$

To study resonance effects we are especially interested in slowly varying terms (see also chapter 1). Therefore we first subtract the fast time-dependence from $\tilde{\theta}$ - note that $\tilde{\theta} = \tau + ..$ see (3.5) - so that the new longitudinal coordinate $\tilde{\tilde{\theta}}$ varies slowly. The transformation is generated by the function

$$G = \tilde{x}\tilde{\tilde{p}}_x + \tilde{z}\tilde{\tilde{p}}_z + \tilde{\theta}\tilde{\tilde{p}}_\theta - \tilde{\tilde{p}}_\theta \cdot \int\bigl(1 + \delta(\tau)\bigr)d\tau$$

so that (3.15)

$$\tilde{\tilde{\theta}} = \tilde{\theta} - \int\bigl(1 + \delta(\tau)\bigr)d\tau$$

whereas all other variables remain unchanged.
The new coordinate $\tilde{\tilde{\theta}}$ is a deviation from a reference pointer rotating with $\omega_{HF}(\tau)/h$ and we should call $\tilde{\tilde{\theta}}$ a "phase".

The new Hamiltonian becomes

$$\tilde{\tilde{H}} = \tilde{H} + \partial G/\partial\tau = \tilde{H} - \bigl(1 + \delta(\tau)\bigr)\tilde{\tilde{p}}_\theta \tag{3.16}$$

and the potential-like function in $\tilde{\tilde{H}}$ is

---

† This theory may have its applications for the cyclotrons of GANIL (Grand Accélérateur National d'Ions Lourds) Caen, France; see Cha79 and Lap81.

$$eV_1\left(\tilde{\tilde{\theta}} + \frac{\tilde{\tilde{p}}_x}{1-n} + \tau^*\right)\cos h\tau^* \tag{3.17}$$

where we abbreviated $\int\left(1+\delta(\tau)\right)d\tau$ by τ^*.
To evaluate this function V_1 we subtract the purely time-dependent function $V_1(\tau^*)$ from $V_1(\tau^* + \tilde{\tilde{\theta}} + \tilde{\tilde{p}}_x/1-n)$, a similar procedure as given in Schu78. The result is a function $\bar{V}\cdot f(\tilde{\tilde{\theta}} + \tilde{\tilde{p}}_x/1-n;\tau^*)$ as sketched in figure 3.2 and 3.3 for the case of only one cavity in a synchrotron and for a one-Dee system, respectively. The width of the pulses of this function f depends on $\tilde{\tilde{\theta}}$ and $\tilde{\tilde{p}}_x$ and as the Hamiltonian must be expanded into its variables, it seems more appropriate to use this function f instead of V_1 of (3.17). This is allowed without further action because we subtracted a function depending on the independent variable τ only. Thus we write

$$\begin{aligned}\tilde{\tilde{H}} = \; & \frac{1}{2}\tilde{\tilde{p}}_x^2 + \ldots\text{see } (3.5)\ldots + \\ & - \left(1+\delta(\tau)\right)\tilde{\tilde{p}}_\theta \\ & + e\bar{V}\cdot f\left(\tilde{\tilde{\theta}} + \frac{\tilde{\tilde{p}}_x}{1-n};\tau^*\right)\cos h\tau^*\end{aligned} \tag{3.18}$$

where $e\bar{V}$ is the maximum fractional energy gain at the cavity or gapcrossing (see also (1.16c))

$$e\bar{V} = \frac{e\hat{V}}{\left(1-\frac{1}{\gamma_o^2}\right)W_o} \tag{3.19}$$

with $\hat{V}$ the peak voltage in the Dee or in the cavity.

The influence of the acceleration on the radial and longitudinal motion is now given by

$$\dot{\tilde{\tilde{x}}} = e\bar{V}\frac{\partial f}{\partial\tilde{\tilde{p}}_x}\cos h\tau^*$$

and $\qquad\qquad\qquad\qquad\qquad \dot{} = \frac{d}{d\tau} \qquad (3.20)$

$$\dot{\tilde{\tilde{p}}}_\theta = -e\bar{V}\frac{\partial f}{\partial\tilde{\tilde{\theta}}}\cos h\tau^*$$

where $\partial f/\partial\tilde{\tilde{p}}_x$ and $\partial f/\partial\tilde{\tilde{\theta}}$ both consist of delta pulses as illustrated in figure 3.2 and 3.3.

The next step in the discussion of the influence of the acceleration on the orbit motion is the examination of the function f.

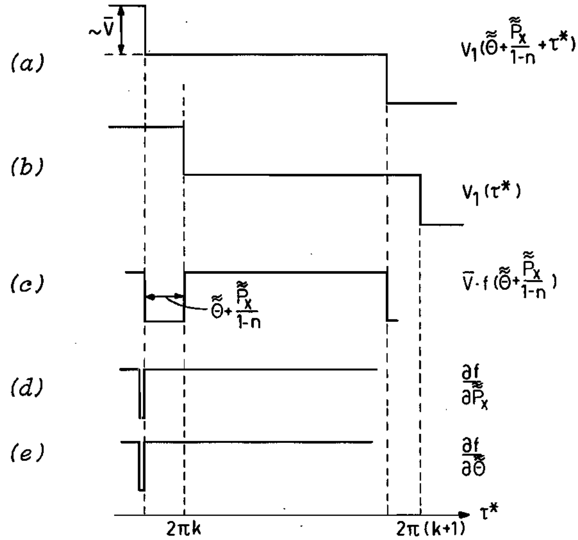


Figure 3.2

The potential-like function V_1 of (3.17) as a function of time τ^* for arbitrary value of the variables (a) and for τ^* as an argument (b) in case of one cavity. The difference (c) between these two functions is $\bar{V}\cdot f$. The derivatives of f to the variables are shown in (d) and (e).

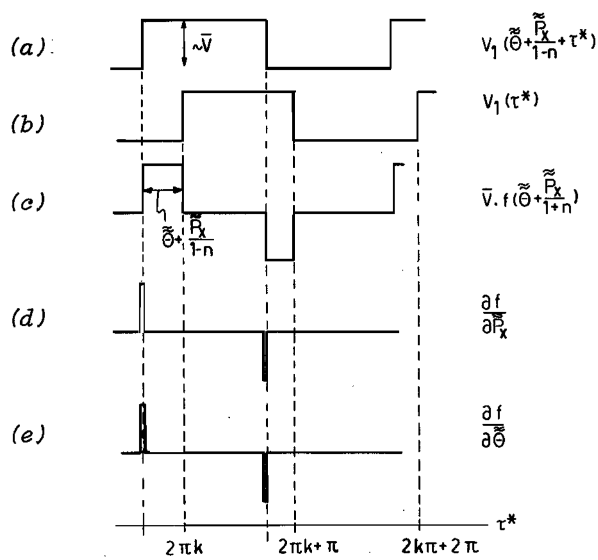


Figure 3.3

Similar to figure 3.2 but now for a one-Dee system or two equally spaced cavities with 0° phase difference between the electric fields.

The argument of this function $f(\tilde{\theta} + \tilde{p}_x/1-n)$ consists of a slowly variable $\tilde{\theta}$ and of the fast oscillating variable $\tilde{p}_x$. This means that the width of the pulses of f may vary rapidly over one "period" of τ^*. The variable $\tilde{p}_x$ is small compared to the usually assumed values of $\tilde{\theta}$ and in order to expand this function f into the canonical variables $\tilde{\theta}$ and $\tilde{p}_x$ we write f as a Taylor series:

$$f(\tilde{\theta} + \frac{\tilde{p}_x}{1-n}) = f(\tilde{\theta}) + \frac{\tilde{p}_x}{1-n}\frac{\partial f(\tilde{\theta})}{\partial\tilde{\theta}} + \frac{1}{2}\frac{\tilde{p}_x^2}{(1-n)^2}\frac{\partial^2 f(\tilde{\theta})}{\partial\tilde{\theta}^2} + \dots \qquad (3.21)$$

Subsequently, the function $f(\tilde{\theta})$ which is periodic in τ^* can be represented by a Fourier series:

$$f(\tilde{\theta}) = \sum_{p\geq 0}^{\infty} A_p(\tilde{\theta}) \cos p\tau^* + B_p(\tilde{\theta}) \sin p\tau^* \quad . \qquad (3.22)$$

After the calculation of these Fourier components and substitution into (3.21) the acceleration term - and thus the total Hamiltonian - is expanded into the canonical variables.

In principle this treatment enables us to examine the influence of any Dee or cavity configuration. As an illustration we give the Fourier components of (3.22) for the two cases sketched in figure 3.2 and 3.3 :

- only one cavity in the ring

$$\begin{aligned} A_o &= -\frac{\tilde{\theta}}{2\pi} \\ A_p &= -\frac{1}{\pi}\frac{\sin p\tilde{\theta}}{p} \\ B_p &= \frac{1}{\pi}\frac{(1-\cos p\tilde{\theta})}{p} \end{aligned} \qquad p \geq 1 \qquad (3.23)$$

- the one-Dee system or two equally spaced cavities with a 0° phase difference between the electric fields

$$\begin{aligned} A_o &= 0 \\ A_p &= -\frac{1}{\pi}\{1-(-1)^p\}\frac{\sin p\tilde{\theta}}{p} \\ B_p &= \frac{1}{\pi}\{1-(-1)^p\}\frac{(1-\cos p\tilde{\theta})}{p} \quad . \end{aligned} \qquad p \geq 1 \qquad (3.24)$$

Substitution of the Taylor series (3.21) and the Fourier series (3.22) into (3.18) generally yields slowly and fast oscillating terms.

In fact, the fast oscillating terms must be transformed to higher degree. We will not carry out this procedure now, assuming they do not give any significant contributions (a procedure to eliminate fast oscillating terms is demonstrated extensively in chapter 4 for the one-dimensional betatron motion; see also Hag62).
Keeping the relevant terms, the Hamiltonian becomes (see (3.5) and (3.18)):

$$
\begin{aligned}
H = {} & \frac{1}{2}\, p_x^2 + \frac{1}{2}(1-n)x^2 + \frac{1}{2}\, p_\theta^2\left(\frac{1}{\gamma_o^2} - \frac{1}{1-n}\right) + \dot{b}\theta - \left(\delta - \frac{b}{\gamma_o^2}\right)p_\theta + \\
& \frac{1}{2}\, e\bar{V}A_h(\theta) + \\
& \frac{1}{2}\,\frac{e\bar{V}}{1-n}\; p_x \frac{\partial A_h}{\partial\theta} + \\
& \frac{1}{4}\,\frac{e\bar{V}}{(1-n)^2}\, p_x^2 \frac{\partial^2 A_h}{\partial\theta^2} + \;\ldots\ldots
\end{aligned}
\tag{3.25}
$$

For convenience we omitted the vertical motion, the third degree terms of (3.5) and the marks $\approx$ *above the variables.*

The term $p_x \cdot \partial A_h/\partial\theta$ has the effect of producing a displacement of the equilibrium orbit. This displaced orbit is sometimes called the "accelerated equilibrium orbit" abbreviated by AEO (Schu78).
The motion with respect to this AEO is described by the introduction of new variables, via the transformation

$$
G = x\bar{p}_x + \theta\bar{p}_\theta - \frac{e\bar{V}}{2(1-n)}\, x\, \frac{\partial A_h}{\partial\theta}
$$

$$
\begin{aligned}
\text{with } \bar{x} &= x \qquad & \bar{p}_x &= p_x + \frac{e\bar{V}}{2(1-n)}\,\frac{\partial A_h}{\partial\theta} \\
\bar{\theta} &= \theta \qquad & \bar{p}_\theta &= p_\theta + \frac{e\bar{V}}{2(1-n)}\, x\, \frac{\partial^2 A_h}{\partial\theta^2}
\end{aligned}
\tag{3.26}
$$

and no first degree terms in $\bar{x}$, $\bar{p}_x$ appear in the Hamiltonian:

$$
\begin{aligned}
\bar{H} = {} & \frac{1}{2}\, \bar{p}_x^2 + \frac{1}{2}(1-n)\bar{x}^2 + \frac{1}{2}\, \bar{p}_\theta^2\left(\frac{1}{\gamma_o^2} - \frac{1}{1-n}\right) + \dot{b}\bar{\theta} - \left(\delta - \frac{b}{\gamma_o^2}\right)\bar{p}_\theta + \\
& \frac{1}{2}\, e\bar{V}A_h(\bar{\theta}) + \\
& \frac{1}{4}\,\frac{e\bar{V}}{(1-n)^2}\bar{p}_x^2 \frac{\partial^2 A_h}{\partial\bar{\theta}^2} + \;\ldots\ldots
\end{aligned}
\tag{3.27}
$$

in which we neglected terms as $(e\bar{V})^2$, $\delta e\bar{V}$, $b e\bar{V}$ and $\dot{b} e\bar{V}$ because of their assumed smallness.
In fact, the coefficient $e\bar{V}$ defined in (3.19) is not small in the central region of cyclotrons since these machines accelerate ions, starting (in most cases) from nearly zero energy.

The increase of momentum (or energy) due to the gapcrossing is represented by the term in the second line of (3.27). The increase is ($\bar{x} = 0$, $\bar{p}_x = 0$):

$$\dot{\bar{p}}_\theta = -\frac{1}{2} e\bar{V} \frac{\partial A_h}{\partial \bar{\theta}} = \begin{cases} \frac{e\bar{V}}{2\pi} \cos h\bar{\theta} & \text{one cavity} \\ \frac{e\bar{V}}{2\pi}\{1 - (-1)^h\}\cos h\bar{\theta} & \text{one-Dee system .} \end{cases} \tag{3.28}$$

These relations can also be directly obtained from (3.20) by substitution of the Fourier series of a delta pulse at $\tau^* = 2\pi k - \tilde{\theta}$, see figure 3.2 (and at $\tau^* = (2k+1)\pi - \tilde{\theta}$ in case of the one-Dee system, see figure 3.3). Eq.(3.28) therefore fits into the physical idea we have of acceleration.

No effective acceleration occurs in the case of the one-Dee system (or in the case of two equally spaced cavities with 0° phase difference between the electric fields) if the harmonic number h is even because the particle is then alternatingly accelerated and decelerated. In case of two equally spaced cavities with a 180° phase difference between the electric fields, h must be even.

The radial oscillations are performed about an equilibrium orbit which shifts outwards in position at each gap crossing (see third line in (3.25) and transformation (3.26)). For an arbitrary particle the outward shift has an additional component depending on $\bar{p}_x$.
This component is represented by the last term in (3.27) and results in a change of the radial oscillation frequency. We will discuss this effect more closely in section 3.4.

To conclude this section we will consider the uncoupled longitudinal motion which is described by the Hamiltonian (3.27) with $\bar{x} = \bar{p}_x = 0$.
A few manupulations lead to the classical Hamiltonian for the description of the synchrotron oscillations.
The first degree term in $\bar{p}_\theta$ is removed by a transformation of the form:

$$G = \bar{\theta}\hat{p}_\theta + \frac{\delta - \frac{b}{\gamma_o^2}}{\frac{1}{\gamma_o^2} - \frac{1}{1-n}}\,\bar{\theta}$$

$$\hat{\theta} = \bar{\theta} \quad \text{and} \quad \hat{p}_\theta = \bar{p}_\theta - \frac{\delta - \frac{b}{\gamma_o^2}}{\frac{1}{\gamma_o^2} - \frac{1}{1-n}} \tag{3.29}$$

where $\hat{p}_\theta$ represents the relative deviation in kinetic momentum from a central particle known as the synchronous particle.

For the time being we restrict ourselves to the case of a one-Dee system and the new Hamiltonian becomes:

$$\hat{H} = \frac{1}{2}\,\hat{p}_\theta^2\left(\frac{1}{\gamma_o^2} - \frac{1}{1-n}\right) - \frac{e\bar{V}}{\pi}\,\frac{\sin h\hat{\theta}}{h} + \frac{\dot{\delta} - \frac{\dot{b}}{1-n}}{\frac{1}{\gamma_o^2} - \frac{1}{1-n}}\,\hat{\theta} \quad . \tag{3.30}$$

It is customary to define the phase in such a way that the rate of change of this phase is negative for particles rotating faster than the synchronous particle; therefore we change sign of $\hat{\theta}$.

Returning to real time units $t = \tau/\omega_o$ and to the generalized momentum $\Delta W/\omega_o = RP_o\hat{p}_\theta$ and $P_o = eB_o\rho$ (see (3.7)), the Hamiltonian must be changed to $H = -\,\omega_o P_o R\hat{H}$, resulting in

$$H = \frac{1}{2}\,\frac{\omega_o^2 K_o}{W_o}\left(\frac{\Delta W}{\omega_o}\right)^2 - \frac{e\hat{V}}{\pi}\,\frac{\sin h\hat{\theta}}{h} - \frac{W_o}{\omega_o K_o}\,\hat{\theta}\left\{\frac{1}{\omega_o}\,\frac{d\omega}{dt} - \frac{\alpha}{B_o}\,\frac{dB}{dt}\right\}$$

with

$$K_o = \frac{\alpha - 1/\gamma_o^2}{1 - 1/\gamma_o^2} \quad \text{and} \quad \alpha = \frac{1}{1-n} \tag{3.31}$$

where α is the so-called momentum compaction factor.

This Hamiltonian is similar to the one derived in the classical way (see e.g. Kol66) with the difference that the phase $\hat{\theta}$ or $\bar{\theta}$ is not the normally used HF phase of the particle with respect to the phase of the accelerating voltage ! The definition of $\bar{\theta}$ differs somewhat from that of the "Central Position" phase as introduced by Schulte and Hagedoorn (Schu79) †, but the basic ideas are essentially the same.

† We recall that Schulte and Hagedoorn studied the motion of accelerated particles in cyclotrons by using cartesian coordinates and splitting the particle motion into a circle and a centre motion and taking $Q_x \simeq 1$ (i.e. $n \simeq 0$); see also Schu78 and Schu80.

We define

$$\phi_{CP} = - h\bar{\theta} \tag{3.32}$$

and the relation between the CP phase and the HF phase is [†] (compare with (3.4))

$$\phi_{CP} = \phi_{HF} + \frac{h\bar{\bar{p}}_x}{1 - n} + \frac{h\dot{b}}{(1 - n)^2} \qquad \text{with} \quad \bar{\bar{p}}_x = \frac{p_x}{P_o} \tag{3.33}$$

Thus, when dealing with coupling between the radial and longitudinal motion, it is useful to extend the definition of the phase. Since the extra terms in (3.33) are proportional to h, the correction will be most pronounced for high harmonic number and only in the central regions of cyclotrons p_x/P_o may have a significant value.

In principle this definition of ϕ_{CP} is only correct for circular equilibrium orbits. A more general expression will be derived in section 3.5 which deals with the motion in an A.G. magnetic field structure.

A description of the synchrotron oscillations around the phase $\hat{\theta}_o$ of the synchronous particle is obtained by applying a transformation of the form

$$G = (\hat{\theta} + \hat{\theta}_o)p_\theta \qquad \text{with} \qquad \cos h\hat{\theta}_o = \frac{\pi}{e\bar{V}}\left(\frac{\dot{\delta} - \alpha\dot{b}}{\frac{1}{\gamma_o^2} - \alpha}\right) \tag{3.34}$$

and the new Hamiltonian is obtained by straightforward expansion of $\sin h(\theta - \hat{\theta}_o)$ in (3.30):

$$H = \frac{1}{2}\, p_\theta^2\left(\frac{1}{\gamma_o^2} - \alpha\right) + \frac{1}{2}\,\frac{Q_s^2}{\frac{1}{\gamma_o^2} - \alpha}\,\theta^2 + \tag{3.35}$$

$$\frac{1}{6}\,\frac{e\bar{V}}{\pi}\,h^2\theta^3 \cos h\hat{\theta}_o + \frac{1}{24}\,\frac{e\bar{V}}{\pi}\,h^3\theta^4 \sin h\hat{\theta}_o + \ldots.$$

with Q_s the synchrotron oscillation number, in case of the one-Dee system defined by

---

† The existence of this formula (3.33) - except for the factor $\dot{b}$ - has been reported recently by Gordon who derived it in a different way, not using the Hamilton formalism (Gor82).

$$Q_s^2 = -\left(\frac{1}{\gamma_o^2} - \alpha\right)\frac{he\bar{V}}{\pi}\sin h\hat{\theta}_o \quad . \tag{3.36}$$

The second line in (3.35) originates from the non-linear character of the synchrotron oscillations. Next, action and angle variables can be introduced and as pointed out in chapter 1, it is convenient to use these variables for the examination of resonances. This will be illustrated later in this chapter.

3.4 Transverse - longitudinal coupling in a cylindrical-symmetric magnetic field

The coupling between the transverse and the longitudinal motion is mathematically described by the Hamiltonian (3.27) plus the non-linear terms of (3.5). We distinguish two kinds of coupling terms: on the one hand coupling due to the magnetic field (see third degree terms in (3.5)) and on the other hand coupling due to the acceleration process (last term in (3.27)).

Possible third order synchro-betatron resonances excited by the magnetic field are (see (3.5)):

$$2Q_x \pm Q_s = 0,\ Q_x \pm 2Q_s = 0 \text{ and } 2Q_z \pm Q_s = 0 \ .$$

A closer examination shows that these resonances will hardly affect the particle motion.
A third order resonance of the type $Q_x - Q_z \pm Q_s = 0$ - which might be of interest in synchrotrons with nearly equal Q_x and Q_z - is not excited in an ideal cylindrical-symmetric field.

The radial-longitudinal coupling due to the acceleration process is represented by the Hamiltonian (see (3.27))

$$\bar{H} = \frac{1}{2}\,\bar{p}_x^2\left(1 + \frac{1}{2}\,\frac{e\bar{V}}{(1-n)^2}\cdot\frac{\partial^2 A_h}{\partial\bar{\theta}^2}\right) + \frac{1}{2}(1-n)\bar{x}^2 \tag{3.37}$$

and the result of the coupling is a change of the betatron number, given by

$$\Delta Q_x = \frac{1}{4}\,\frac{e\bar{V}}{Q_x^3}\,\frac{\partial^2 A_h}{\partial\bar{\theta}^2} \quad . \tag{3.38}$$

Corresponding the assumptions made before to evaluate the Hamiltonian we assumed ΔQ_x to be small. In fact, the result (3.38) is the limit where the coupling becomes weak.
Eq.(3.38) shows the tune change for a general Dee or cavity configuration. As an illustration we consider a one-Dee system, as this concerns a large number of cyclotrons in operation. Substitution of the Fourier component A_h of (3.24) yields

$$\Delta Q_x = -\frac{he\hat{V}\sin\phi_{CP}}{2\pi Q_x^3\left(1-\frac{1}{\gamma_o^2}\right)W_o} , \tag{3.38a}$$

or in the non-relativistic case, using $e\hat{V}/2E_{kin} = 1/4n_o$ with n_o the turn number:

$$\Delta Q_x = -\frac{h\sin\phi_{CP}}{8\pi n_o Q_x^3} . \tag{3.38b}$$

This phase-dependent effect of the acceleration process on the radial oscillations - due to the outward shift of the particle orbit at each gap crossing - was first noticed by Bolduc and Mackenzie in connection with design calculations on the TRIUMF cyclotron (Bol71) and analytically described by Schulte and Hagedoorn (Schu78, Schu80). They showed that the coupling may lead to radial instability in the centre of cyclotrons.
The effect is sometimes called "radial electric focusing" (Gor82), although the nature of the focusing force is basically different from that of the well-known vertical electric focusing.

For C.G. synchrotrons typical values of $e\bar{V}$ lie between 10^{-3} (just after injection) and e.g. 10^{-5} (at final energy) and the coupling effect will be of no importance.

In the subsequent sections we will extend the theory for the motion in A.G. accelerators, in which Q_s is not necessarily extremely small and Q_x (and also Q_z) may be much larger than 1.

3.5 Alternating gradient magnetic field structure

In this section we extend the preceding theory for the motion in an A.G. synchrotron or storage ring with a separated function lattice (see section 1.6).

The treatment is fundamentally the same as before and therefore only the most important stages are quoted. Finally we will obtain a Hamiltonian - similar to (3.25) - which will turn out to be an appropriate start to study coupling effects.
We will omit the time-dependence of the magnetic field and of the frequency of the accelerating voltage - i.e. $b(\tau) = 0$ and $\delta(\tau) = 0$ - as this has no essential consequences for the theory.

Substitution of the vector potential (1.6) into (1.13) and application of the scale transformation (1.16) results in a Hamiltonian being the analogue of (3.3).
As the coupling term $\bar{\bar{p}}_\theta \bar{\bar{x}}$ is most important we evaluate the Hamiltonian up to this quadratic term. To simplify the notation we omit the vertical motion and fringing fields and find:

$$\bar{\bar{H}} = \frac{1}{2}\,\bar{\bar{p}}_x^2 + \frac{1}{2}\left(\varepsilon^2(\bar{\bar{\theta}}) - n(\bar{\bar{\theta}})\right)\bar{\bar{x}}^2 + \frac{1}{2\gamma_o^2}\,\bar{\bar{p}}_\theta^2 + \bar{\bar{p}}_\theta - \varepsilon(\bar{\bar{\theta}})\bar{\bar{p}}_\theta\bar{\bar{x}} \quad . \tag{3.39}$$

The scaled time τ is again the independent variable and ε and n represent the "normalized" dipole and quadrupole field component both defined in (1.19). These components depend on the azimuthal position in the machine, i.e. on the coordinate $\theta = s/R$ which is, after the various transformations, written as $\bar{\bar{\theta}}$ (see section 1.3).
The elimination of the term $\varepsilon(\bar{\bar{\theta}})\bar{\bar{p}}_\theta\bar{\bar{x}}$ † will again lead to a new longitudinal coordinate which includes the horizontal betatron motion, similar to (3.4). The elimination is achieved by a transformation generated by the function

$$G = \bar{\bar{x}}\tilde{p}_x + \bar{\bar{\theta}}\tilde{p}_\theta - \eta(\bar{\bar{\theta}})\tilde{p}_\theta\tilde{p}_x + \eta'(\bar{\bar{\theta}})\tilde{p}_\theta\bar{\bar{x}} - \frac{1}{2}\,\eta(\bar{\bar{\theta}})\eta'(\bar{\bar{\theta}})\tilde{p}_\theta^2 \tag{3.40}$$

with $' = d/d\bar{\bar{\theta}}$
and the relation between the old (=) and the new (~) variables now becomes

† A term similar to $\bar{\bar{p}}_\theta\bar{\bar{x}}$ in (3.39) has already been noticed by Mills (Mil63). For a general calculation of damping times he used a Hamiltonian with the azimuth θ as the independent variable and with the time t and energy H as a canonical pair of variables. For the elimination of the term he suggested a new time variable by adding several terms due to the radial betatron oscillations and the field structure.

$$
\begin{aligned}
\bar{\bar{x}}' &= \tilde{x} + \eta\tilde{p}_\theta \\
\bar{\bar{p}}_x &= \tilde{p}_x + \eta'\tilde{p}_\theta \\
\bar{\bar{\theta}} &= \tilde{\theta} + \eta\tilde{p}_x - \eta'\tilde{x} \\
\bar{\bar{p}}_\theta &= \tilde{p}_\theta - \eta'\tilde{p}_\theta\tilde{p}_x + \eta''\tilde{p}_\theta\tilde{x} + \frac{1}{2}\left(\eta\eta'' - (\eta')^2\right)\tilde{p}_\theta^2
\end{aligned}
\tag{3.40}
$$

and the Hamiltonian up to the second degree in the variables becomes

$$
\tilde{H} = \frac{1}{2}\tilde{p}_x^2 + \frac{1}{2}\left(\varepsilon^2 - n\right)\tilde{x}^2 + \frac{1}{2}\tilde{p}_\theta^2\left(\frac{1}{\gamma_o^2} - \varepsilon\eta\right) + \tilde{p}_\theta \tag{3.41}
$$

with η the so-called off-momentum or dispersion function, which is the reduced displacement of the closed orbit per unity momentum deviation.†

The η-function is defined as that solution of the differential equation

$$
\eta'' + \left(\varepsilon^2 - n\right)\eta = \varepsilon \quad , \tag{3.42}
$$

which has the same periodicity as the linear magnetic guide field (see figure 2.3 for the non-reduced off-momentum function in IKOR). We recall that in case of the cylindrical-symmetric guide field $\varepsilon = 1$ and n = constant so that $\eta = 1/(1-n)$ and (3.40) reduces to (3.4).

Subsequently the acceleration must be included. This is achieved by following the procedure as outlined in section 3.3, resulting in a Hamiltonian which is in first order approximation given by

$$
\begin{aligned}
H = {} & \frac{1}{2}p_x^2 + \frac{1}{2}\left(\varepsilon^2 - n\right)x^2 + \frac{1}{2}p_\theta^2\left(\frac{1}{\gamma_o^2} - \varepsilon\eta\right) + \frac{1}{2}e\bar{V}A_h(\theta) + \\
& e\bar{V}\langle(\eta_c p_x - \eta_c' x)\{\frac{\partial A_p}{\partial\theta}\cos p\tau\cdot\cos h\tau + \frac{\partial B_p}{\partial\theta}\sin p\tau\cdot\cos h\tau\}\rangle + \\
& \frac{1}{2}e\bar{V}\langle(\eta_c p_x - \eta_c' x)^2\{\frac{\partial^2 A_p}{\partial\theta^2}\cos p\tau\cdot\cos h\tau + \frac{\partial^2 B_p}{\partial\theta^2}\sin p\tau\cdot\cos h\tau\}\rangle \; .
\end{aligned}
\tag{3.43}
$$

The variables are defined by (3.15) but for convenience we omitted the marks $\approx$ *above the variables.*

This Hamiltonian is the analogue of (3.25). The brackets $\langle\ \rangle$ indicate that we are only interested in the resonant or slowly varying terms

---

† Due to the use of reduced coordinates (see (1.16a)) the η-function is also a reduced dispersion function, i.e. the usual dispersion function divided by R.

and η_c and η_c' are the values of respectively the off-momentum function and its derivative at the position of the cavities.
We notice that the introduction of an "AEO" by a transformation similar to (3.26) has no direct consequences for our further discussion and therefore we will omit the transformation (compare (3.25) with (3.27)).

As the horizontal tune Q_x is usually not too close to an integer in an A.G. synchrotron, the coupling term in the second line of (3.43) may give resonant terms, only in case the oscillations in θ are not too slow, i.e. the synchrotron number Q_s not too small.

From the third line in (3.43) we will only briefly discuss the term with $p = h$, leading to a change in the radial (= horizontal) tune, similar as sketched in section 3.4.
In the next section we will return to the radial-longitudinal coupling effect due to a non-zero value of the dispersion function and its derivative in the cavity.

There is another resonance we are interested in, namely $Q_x - Q_z \pm Q_s = 0$. Electron storage rings often operate at nearly equal tunes Q_x and Q_z and consequently this resonance may be relevant. As noticed in section 3.4, this third order resonance is not excited in an "ideal" machine. But as we will see in the next section, the resonance can be excited by a so-called skew or rotated quadrupole field.

Before discussing coupling effects we recall that an equivalent CP phase can be defined for the motion in an A.G. accelerator with a separated function lattice.
Analogously to (3.32) and (3.33) we get:

$$\phi_{CP} = \phi_{HF} + h\eta\bar{\bar{p}}_x - h\eta'\bar{\bar{x}} \qquad (3.44)$$

where $\bar{\bar{x}}$ and $\bar{\bar{p}}_x$ are the reduced variables of (1.16a): $\bar{\bar{x}} = x/R$ and $\bar{\bar{p}}_x = p_x/P_o$. This very CP phase is a proper canonical variable to describe transverse-longitudinal coupling effects in an A.G. machine with a separated function lattice. The harmonic number h may be very large (e.g. h $\simeq$ 100) but $\bar{\bar{x}}$ and $\bar{\bar{p}}_x$ are usually very small.

3.6 Coupling effects between the transverse and longitudinal motions in A.G. synchrotrons and storage rings

A Hamiltonian like (3.43) will prove to be quite an appropriate start to study transverse-longitudinal coupling effects.

In illustration we will consider two specific cases namely the influence of a skew quadrupole field and of a non-zero off-momentum function and its derivative at the cavity. Both cases can result in synchro-betatron resonances.

It is not our intention to give an exhaustive analysis, but we will only discuss some features and properties of the resonances.

We note that the coefficients of the terms in (3.43) which describe the unperturbed betatron and synchrotron oscillations still depend on the independent variable. How to handle these problems has been outlined in chapter 2 and leads e.g. to the introduction of the betatron functions β_x, β_z. As the treatment will not change fundamentally we omit the modulation of the β_x, β_z and η function and replace (ε^2-n), n and $\varepsilon\eta$ in the second degree of the Hamiltonian by Q_x^2, Q_z^2 and the momentum compaction factor α, respectively.†

3.6.1 The resonance $Q_x-Q_z\pm Q_s=0$ excited by skew quadrupole fields

The resonance $Q_x-Q_z\pm Q_s=0$ might be relevant in electron storage rings. This third order resonance is not excited in an ideal "linear" machine (see (3.5)), but it can be excited by a skew quadrupole field. This skew field may be due to e.g. rotational errors of the normal quadrupoles or is sometimes intentionally installed in an electron storage ring (see e.g. Bac79a).

The influence of a skew quadrupole component n_{skew}, defined by

$$n_{skew} = \frac{R^2}{B_o\rho}\left(\frac{\partial B_z}{\partial z}\right)_o \quad , \tag{3.45}$$

on the synchro-betatron resonance $Q_x-Q_z\pm Q_s=0$ is obtained by

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† see also section 4.2.3, chapter 4 for the relation between $\varepsilon\eta$ and α.

substituting the corresponding vector potential into the general Hamiltonian and following the procedure as outlined in this chapter. The relevant term - responsible for the excitation of the resonance - is $n_{skew}p_\theta xz$ and the Hamiltonian is (see also (3.35))

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}Q_x^2x^2 + \frac{1}{2}p_z^2 + \frac{1}{2}Q_z^2z^2 + \tag{3.46}$$
$$\frac{1}{2}p_\theta^2\left(\frac{1}{\gamma_o^2} - \alpha\right) + \frac{1}{2}\frac{Q_s^2}{\frac{1}{\gamma_o^2} - \alpha}\theta^2 + n_{skew}p_\theta xz + \ldots\ldots \quad .$$

We note that the skew field also gives rise to a term $n_{skew}xz$, exciting the resonance $Q_x-Q_z=0$. It is allowed to split both cases when the betatron oscillations in $n_{skew}xz$ give rise to terms which can be regarded as being fast with respect to the term coming from $n_{skew}p_\theta xz$. Sometimes it may be necessary to treat both cases simultaneously but this gives rise to complications with regard to the procedure as sketched in chapter 1. In this sub-section we will only take into account the term $n_{skew}p_\theta xz$.

The next step in examining the resonance is the use of action and angle variables. For the transverse motions these are defined in (1.24) and for the longitudinal motion we write

$$\theta = \sqrt{2J_s(\alpha - \frac{1}{\gamma_o^2})/Q_s}\cos\phi_s \tag{3.47}$$
$$p_\theta = \sqrt{2Q_sJ_s/(\alpha - \frac{1}{\gamma_o^2})}\sin\phi_s$$

in which we assumed to work above transition energy, i.e. $\alpha > \frac{1}{\gamma_o^2}$.

Retaining the constant and slowly varying terms in the Hamiltonian (3.46) only, the result is

$$K = Q_xJ_x + Q_zJ_z - Q_sJ_s + \frac{1}{16}h^2(\alpha - \frac{1}{\gamma_o^2})J_s^2 \tag{3.48}$$
$$\mp \kappa_o\sqrt{J_xJ_zJ_s}\sin(\phi_x - \phi_z \mp \phi_s)$$

in which the upper (lower) sign holds for the resonance $Q_x-Q_z+Q_s=0$ ($Q_x-Q_z-Q_s=0$). The excitation term κ_o is the average value of the function κ which is defined by

$$\kappa = \frac{1}{2}n_{skew}\sqrt{\frac{2Q_s}{Q_xQ_z(\alpha - \frac{1}{\gamma_o^2})}} \quad . \tag{3.49}$$

The actions J_x, J_z are related to the amplitude of the betatron oscillations (or to the emittances, see (2.16) in chapter 2), whereas J_s is related to the amplitude of the synchrotron oscillation i.e. to the momentum variation in the beam according to

$$J_s = \frac{\alpha - \frac{1}{\gamma_o^2}}{2Q_s} \left(\frac{\widehat{\Delta P}}{P_o}\right)^2 \quad . \tag{3.50}$$

The description of the six-dimensional problem represented by (3.48) is simplified by a reduction to a two-dimensional problem by two successive transformations of the type (1.32).
The final Hamiltonian becomes:

$$\begin{aligned}\bar{K} &= \Delta Q J_2 + \frac{1}{16} h^2 (\alpha - \frac{1}{\gamma_o^2}) J_2^2 + \\ &\quad + \kappa_o (J_o - J_1 \mp J_2)^{\frac{1}{2}} (J_1 \pm J_2)^{\frac{1}{2}} J_2^{\frac{1}{2}} \sin \phi_2\end{aligned} \tag{3.51}$$

with
$$\begin{aligned}\Delta Q &= \mp Q_x \pm Q_z - Q_s \\ J_2 &= J_s \\ \phi_2 &= \phi_s \pm (\phi_z - \phi_x) \\ J_o &= J_x + J_z \\ J_1 &= J_z \mp J_s \quad .\end{aligned}$$

The action variables J_o and J_1 are constants of the motion because ϕ_o and ϕ_1 do not appear in this Hamiltonian. Returning to the non-reduced amplitudes $\hat{x}$ and $\hat{z}$ of the horizontal and vertical betatron oscillations and substituting the amplitude of the synchrotron oscillation (3.50), the constants of the motion are

$$\begin{aligned}&\frac{Q_x \hat{x}^2}{R^2} + \frac{Q_z \hat{z}^2}{R^2} = \text{constant} \\ &\frac{Q_z \hat{z}^2}{R^2} \mp \frac{\alpha - \frac{1}{\gamma_o^2}}{Q_s} \left(\frac{\widehat{\Delta P}}{P_o}\right)^2 = \text{constant} \quad .\end{aligned} \tag{3.52}$$

The first relation shows that the sum of the transverse amplitudes remains constant, exactly the result which is also valid in case of the pure betatron resonance $Q_x - Q_z = 0$. However, in case of the resonance $Q_x - Q_z \pm Q_s = 0$ the energy exchange occurs via the synchrotron motion as represented by the second relation of (3.52).

It is interesting to know the amount of energy exchange. This exchange can be calculated by substituting the extreme values ± 1 for $\sin \phi_2$ in $\bar{K}$ (3.51). In most cases J_s is much larger than J_x and J_z and the relative change in J_s will be small. Starting from a vertical amplitude which is zero, i.e. $J_z = 0$, the energy exchange is given by

$$\frac{J_{z,max}}{J_o} = \frac{1}{1 + \left(\frac{\Delta Q}{\kappa_o}\right)^2 \cdot \frac{1}{J_s}} \tag{3.53}$$

and $J_{z,max}$ is proportional with the maximum value of the square of the vertical amplitude.

The strength of the skew quadrupole field - and thus κ_o - is usually determined by the advisable amount of betatron coupling via the resonance $Q_x - Q_z = 0$ (see e.g. Bry75, Gui76 and Bac79a). Typical values of κ_o lie in the order of 0.01 to 0.05 (see e.g. Bac79a). Substitution of $\kappa_o = 0.05$ and furthermore $\Delta Q = 0.01$ and $J_s = 2.5\ 10^{-6}$ (e.g. $\alpha=0.1$, $Q_s=0.02$ and $\Delta P/P_o = 10^{-3}$; see table 1.1 and Bac79a) leads to $J_{z,max}/J_o = 6.10^{-5}$.

Thus, the coupling effect via the resonance $Q_x - Q_z \pm Q_s = 0$ gives an appreciable amount of energy exchange in case of extremely small values of ΔQ - e.g. $\Delta Q \simeq 10^{-5}$ - only.

Bearing in mind that the energy exchange via the pure betatron coupling is e.g. $J_{z,max}/J_o = 0.5$ (Bac79a), it is obvious that the above-discussed synchro-betatron resonance can generally be neglected. Moreover the coupling is influenced by the non-linearity of the synchrotron oscillation, which changes Q_s with synchrotron amplitude thus giving a "limiting effect" (see chapter 5).

Finally we derive a relation from (3.51), satisfied by the fixed points, i.e. $\dot{J}_2 = 0$, $\dot{\phi}_2 = 0$:

$$2|\Delta Q| = |\kappa_o| \frac{1}{\sqrt{J_x J_z J_s}} |J_z J_s - J_x J_s \mp J_x J_z| \quad . \tag{3.54}$$

One has to be very careful with interpreting $2|\Delta Q|$ as being "the" stopband width, especially if one of the variables J_x, J_z or J_s approaches zero. We refer to chapter 5 for a description of this phenomenon and it will turn out a good insight in the resonance behaviour is obtained by phase plane considerations.

Substitution of typical values for the transverse emittances (see table 1.1), the momentum variation in the beam (3.50) and κ_o results in a ΔQ value of the order 10^{-5} to 10^{-4}.

3.6.2 Radial-longitudinal coupling due to the off-momentum function and its derivative in the cavity

The coupling between the radial and the longitudinal motion due to a non-zero value of the off-momentum or dispersion function and its derivative at the position of the cavity is described by the Hamiltonian (3.43).

The second line in the representation of the Hamiltonian (3.43) may have a considerable influence only when Q_s is not too small. Applying the transformation (3.34) this Hamiltonian can be written in the form (omitting third line of (3.43))

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} Q_x^2 x^2 + \frac{1}{2} p_\theta^2 \left(\frac{1}{\gamma_o^2} - \alpha\right) + \frac{1}{2} \frac{Q_s^2}{\frac{1}{\gamma_o^2} - \alpha} \theta^2 + \ldots.$$

$$+ \sum_k \left(\eta_c p_x - \eta_c' x\right) p^k \theta^k \{a_k \sin(p \pm h)\tau + b_k \cos(p \pm h)\tau\} \quad , \tag{3.55}$$

where k is a positive integer and p represents the p-th harmonic of the Fourier expansion (3.22). The coefficients a_k and b_k depend on the cavity configuration. They can be calculated from (3.43) and (3.22). Without going into detail we mention that a_k and b_k are proportional to $e\bar{V}/(k!)$.

As x, p_x oscillate with a frequency Q_x and θ with Q_s, the dispersion function and its derivative at the cavity can excite resonances of the type

$$Q_x \pm kQ_s = \pm(p \pm h) = \text{integer} \quad , \tag{3.56}$$

with h is the harmonic number defined in (3.13); we emphasize that p is not directly a multiple of the periodicity of the cavity configuration, e.g. for one cavity p = 1,2,3,.. (see (3.23)) and for two cavities with a phase difference of 0° between the electric fields p = 1,3,5,.. (see (3.24)).

As the strength of the excitation term is inversely proportional to k!, the resonance will be severe only for machines having a large value of Q_s.

Resonances excited by a non-zero value of the off-momentum function (and its derivative) at the position of the cavity have been observed in NINA (Cro71) and SPEAR II (SPE75, Cha75, SPE77) and indeed show the above-mentioned characteristics, i.e. in both cases Q_s was of the order of 0.1 (large !), whereas the strength of the resonance decreased with increasing energy (decreasing $e\bar{V}$, see (3.19)) and could be influenced by a rearrangement of the cavities (positions and number).

For electron storage rings the resonance will not be important. Q_x is not too close to an integer (e.g. Q_x = integer + 0.25) and Q_s is usually very small (e.g. $Q_s = 0.02$; see table 1.1). Moreover $e\bar{V}$ is of the order 10^{-3} to 10^{-4}.

In the past Piwinski, Wrülich and Chao gave the physical explanation of this coupling effect and investigated the resonance using matrix representations for the betatron and synchrotron oscillations (Piw76, Cha77, Piw78).

The excitation of the resonance can be calculated from the Hamiltonian (3.55) knowing the lattice and the HF parameters. From the equations of motion it turns out that (use action and angle variables, similar to the previous sub-section)

$$\frac{Q_x \hat{x}^2}{R^2} \pm \frac{\alpha - \frac{1}{\gamma_0^2}}{kQ_s} \left(\frac{\Delta \hat{P}}{P_o}\right)^2 = \text{constant}, \tag{3.57}$$

where ^ indicates the amplitude of the oscillation. The upper (lower) sign in (3.57) holds for $Q_x + kQ_s$ = integer ($Q_x - kQ_s$ = integer). Eq. (3.57) shows that the two modes exchange their energy for the sum resonance when working above transition energy, whereas both amplitudes can grow in case of the difference resonance, contrasting with pure betatron resonances (see section 1.5.2, chapter 1).

In an "ideal" machine no vertical dispersion exists and resonances $Q_z \pm kQ_s$ = integer do not appear. However, a non-zero vertical dispersion function can be produced by e.g. dipole imperfections (horizontal field component) or by a rotated (skew) quadrupole which causes linear coupling between the transverse motions.

Finally we consider the coupling effect which results in a change in Q_x, similar to section 3.4. This effect is described by the third line in (3.43). In a first order approximation, i.e. in the limit where the "radial electric focusing" is weak and considering the case of two equally spaced cavities with a 0^o phase difference between the electric fields (see (3.24)), the tune change is

$$\Delta Q_x = - \frac{he\hat{V}}{2\pi\left(1 - \frac{1}{\gamma_o^2}\right)W_o} \sin\phi_{CP}\left[Q_x\eta_c^2 + \frac{(\eta_c')^2}{Q_x}\right] . \tag{3.58}$$

In fact, this formula is the "alternating gradient" analogue of eq. (3.38a).

3.7 Discussion

In this chapter we presented a simultaneous treatment of the transverse and longitudinal motions in a circular accelerator and illustrated the theory with discussions of coupling effects due to the acceleration process and to a skew quadrupole field.

To show all possible resonances due to the various field components one has to use the total Hamiltonian (see Cor82). We will simply quote some possible synchro-betatron resonances not treated in this thesis:

- third order resonances excited by a sextupole field:
  $Q_x \pm 2Q_s = p$, $2Q_x \pm Q_s = p$ and $2Q_z \pm Q_s = p$,
- fourth order resonances excited by an octupole field:
  $3Q_x \pm Q_s = p$, $2Q_x \pm 2Q_s = p$, $Q_x \pm 3Q_s = p$, $\pm Q_x \pm 2Q_z \pm Q_s = p$ and $2Q_z \pm 2Q_s = p$,

in which p is a multiple of the periodicity of the magnetic field component that excites the resonance.

In general, the various field components will have their major effects on the betatron motion. Betatron resonances will be the subject of the following chapters.

CHAPTER 4 ONE-DIMENSIONAL NON-LINEAR BETATRON MOTION

4.1 Introduction

One of the first steps in designing an accelerator or storage ring is the choice of the pattern of the linear guide field, i.e. bending and focusing magnets. At this stage, non-linearities in the guide field are ignored. Paradoxically, when such a "linear" machine has been designed and built, there is often a need of non-linear magnetic fields to prevent high-intensity instabilities and sextupole and octupole magnets may have to be installed. Besides, these multipoles will also influence the betatron motion.

One of the intensity-limiting instabilities which may appear in a storage ring with a HF system is the so-called head-tail instability (Pel69, San69). Without mentioning the detailed mechanism, it is sufficient to know that its growth depends on the chromaticity. The chromaticity is the change of the betatron number due to a momentum deviation. The natural chromaticity, i.e. the chromaticity in the "linear" machine, is due to the fact that a particle with higher momentum (off-momentum) experiences a smaller focusing force than that experienced by the reference (= on-momentum) particle. In order to stabilize the beam against the head-tail effect, the chromaticity must be altered from its natural negative value into a value equal to zero or slightly positive (San69). An off-momentum particle moves on a new equilibrium orbit displaced from the reference orbit. Therefore it will - if sextupole terms are present in the magnetic field - experience an effective quadrupole term, proportional to this displacement. It will thus undergo a shift in Q with respect to the on-momentum particle. So the chromaticity may be modified by the deliberate inclusion of sextupole fields in the lattice. Generally, sextupole fields will be provided by special magnets rather than by pole-face windings in each dipole magnet. A realistic distribution of sextupoles is one placed close to an F-quadrupole and one close to a D-quadrupole. This arrangement allows almost independent control of horizontal and vertical chromaticities (Gen72, Bac79a).

For PAMPUS the positions of the sextupoles are indicated in figure 1.2: 4 sextupoles in each unit cell, all having a length of 0.25 m. The required sextupole fields to produce zero chromaticity vary from 2 T/m^2 at $Q_x \approx Q_z \approx 2.10$ to about 60 T/m^2 at $Q_x \approx Q_z \approx 6$ (see Bac79a).

Since no HF accelerating system has been planned in IKOR, the head-tail effect is not liable to occur. Nevertheless, sextupoles are planned. IKOR should be an isochronous ring (see Jül81) and as we will explore later, the main use of sextupoles in the compressor ring is to reduce the dependence of the transition energy on position to keep conditions uniform across the beam and to get the maximum benefit of working close to transition energy (Fis80).

With respect to the importance of non-linear magnetic fields, we mention the beneficial effect of octupoles on collective instabilities. Such an instability becomes dangerous when the growth rate is exceeding the damping rates of other mechanisms wich affect the coherent motion. One such mechanism is the Q-spread in the beam (Landau damping; Lan46, Her65). When the natural tune spread is small, one may get the required amount of Landau damping by putting sextupoles and/or octupoles in the machine. As sextupoles - which give a momentum dependent tune spread - may be needed to counteract the head-tail effect, they can usually not be used effectively for Landau damping and therefore octupoles are often installed to produce an amplitude-dependent Q-spread.

In this chapter we will first explain the purpose of extra sextupole fields in IKOR and their required strength will be calculated (section 4.2).

In section 4.3 we describe the effects of non-linear magnetic fields on the betatron motion. Besides the intentional or "lumped" multipoles, there are also non-linear fields due to imperfections in the linear guide field elements. The study of one-dimensional betatron resonances enables us to calculate allowed strengths of the correction elements or to fix an upper limit on permissible field tolerances.

The one-dimensional resonances are generally studied by ignoring the fast oscillating terms and retaining the resonant or slowly varying terms in the Hamiltonian only (see section 1.5, chapter 1). However, a precise treatment is obtained by application of transformations which eliminate the fast oscillating terms in the degree in question in the Hamiltonian.
The procedure to achieve this has already been sketched in Hag62, in which the third and fourth degree terms in case of AVF cyclotrons have been treated. In this chapter we will perform a general theory for an A.G. accelerator, starting with the introduction of the Twiss parameters via the transformations of section 2.3.
Further, an advanced examination of the consequences of the various transformations will be carried out. The final theory shows "first" as well as "second order" effects of non-linear magnetic fields on the betatron motion.
Examples will illustrate the theory in section 4.4.

4.2 Lumped sextupoles in IKOR

In this section we illustrate the role of lumped sextupoles in IKOR and calculate their required strength.

4.2.1 The isochronous operation of IKOR

IKOR is an option in the SNQ project in West-Germany (SNQ81I,II, III). The aim of the compressor ring IKOR is to have the opportunity to compress the 500 μs linac proton pulse to one of less than 1 μs, before sending it on a neutron production target.
IKOR is proposed to operate at high intensities (2.7 10^{14} ppp) and the relative beam losses must be kept small. So the design must ensure that beams can be injected, accumulated and ejected again almost loss-free. An azimuthal void in the beam is one of the requirements to achieve this. To maintain the void during accumulation (about 700 turns ≈ 500 μs) it is proposed to operate the machine near transition energy ($\gamma \approx \gamma_{tr}$). In that case the revolution period is almost independent on the particle energy (isochronous operation) and the shrinking of the void can be reduced to an acceptable level.

To optimize operation at highest intensity, the lattice should ensure that γ_{tr} as a function of the radial beam position (= momentum) can be controlled. As γ_{tr} depends on the position even in a "linear" machine, it is planned to use sextupoles to maintain the property of isochronous operation across the aperture.

4.2.2 Chromatic effects on γ_{tr}

Working at transition energy ($\gamma = \gamma_{tr}$) means that the revolution frequency ω of the particles does not depend on their momentum P. In general the dependence of ω on P consists of two contributions:

$$-\frac{P}{\omega}\frac{d\omega}{dP} = \frac{P}{L}\frac{dL}{dP} - \frac{P}{v}\frac{dv}{dP} = \alpha - \frac{1}{\gamma^2} \tag{4.1}$$

with L the length of the orbit of a particle with kinetic momentum P and velocity v; α is the momentum compaction factor and γ is the ratio of the total energy of the particle to its rest energy.

The reference particle which has momentum P_o moves along the orbit with length L_o, whereas a particle with a slightly different momentum $P = P_o + \Delta P$ moves on a new equilibrium orbit. The difference in orbit length due to a momentum deviation ΔP can be written as

$$\frac{\Delta L}{L_o} = \alpha_o \frac{\Delta P}{P_o}\left(1 + \alpha_1 \frac{\Delta P}{P_o} + \ldots\right) \tag{4.2}$$

so that for particles with a momentum $P = P_o + \Delta P$ holds

$$\frac{P}{L}\frac{dL}{dP} = \alpha_o\left(1 + (1 - \alpha_o + 2\alpha_1)\frac{\Delta P}{P_o}\right) \tag{4.3}$$

in which we retained only the first order terms in $\Delta P/P_o$.
Because

$$\frac{P}{L}\frac{dL}{dP} = \alpha = \frac{1}{\gamma_{tr}^2} \quad , \tag{4.4}$$

we can write eq. (4.3) as

$$\gamma_{tr}^2 = \gamma_{tr,o}^2\left(1 - (1 - \alpha_o + 2\alpha_1)\frac{\Delta P}{P_o}\right) \tag{4.5}$$

with

$$\gamma_{tr,o}^2 = \frac{1}{\alpha_o} \tag{4.6}$$

and this eq.(4.5) describes the behaviour of γ_{tr} with momentum.

Experiments at the CERN Proton Synchrotron have shown it to be useful to satisfy the condition $1/\gamma^2 - 1/\gamma_{tr}^2$ = constant, i.e. independent of the position (see Cap81). Because of space charge effects in IKOR this quantity is chosen slightly positive, namely equal to 0.01 (see Jül81).
As in first order in $\Delta P/P_o$ holds

$$\gamma^2 = \gamma_o^2\left(1 + 2\beta_o^2 \frac{\Delta P}{P_o}\right) \quad , \tag{4.7}$$

with $\beta_o = v_o/c$ (the subscript "o" refers to the reference particle), we conclude that the CERN "recommendation" implies the following condition to be fulfilled:

$$2\alpha_1 - \alpha_o + 1 + \frac{2\beta_o^2}{\gamma_o^2\alpha_o} = 0 \qquad \left(\text{for } \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2} = \text{constant}\right) . \tag{4.8}$$

When the quantities α_o and α_1 are known, eqs.(4.5) and (4.8) give the relevant behaviour. As we will see in the subsequent sub-section, the influence of sextupole fields appears in the quantity α_1.

4.2.3 Control of γ_{tr} with sextupole fields

To determine the role of sextupoles in the process of γ_{tr} control, expressions for the quantities α_o and α_1 are derived.

As mentioned before, the displacement x_e of the equilibrium orbit due to a momentum deviation is related to the off-momentum function η by (see also section 3.5)

$$x_e = \eta \frac{\Delta P}{P_o} \quad , \tag{4.9}$$

and the change in the orbit length is (from (1.2))

$$\Delta L = L - L_o = \int_o^{L_o} \left\{\frac{\eta}{\rho} \frac{\Delta P}{P_o} + \frac{1}{2}\left(\frac{d\eta}{ds}\right)^2 \left(\frac{\Delta P}{P_o}\right)^2\right\} ds \quad . \tag{4.10}$$

The displaced equilibrium orbit due to the momentum deviation - and thus η - can be determined by using the Hamiltonian (1.18) and substituting $P = P_o + \Delta P$ with $P_o = eB_o\rho$.

The relevant Hamiltonian becomes:

$$\bar{\bar{H}} = - \varepsilon\bar{\bar{x}}\left[\frac{\Delta P}{P_o} - \left(\frac{\Delta P}{P_o}\right)^2\right] + \frac{1}{2}\bar{\bar{p}}_x^2 + \frac{1}{2}(\varepsilon^2 - n)\bar{\bar{x}}^2\left(1 - \frac{\Delta P}{P_o}\right) + \\ + \frac{1}{2}\varepsilon\bar{\bar{x}}\bar{\bar{p}}_x^2 - \frac{1}{6}S\bar{\bar{x}}^3 \tag{4.11}$$

in which the marks $=$ above the variables indicate the reduced variables (see 1.16a), whereas the field components ε, n, S are defined in (1.19).

The first degree term in $\bar{\bar{x}}$ in (4.11) indicates the presence of the new equilibrium orbit. Application of the transformation (1.20) fixes this equilibrium orbit by the requirement that first degree terms vanish. Writing

$$\eta = \eta_o + \eta_1 \frac{\Delta P}{P_o} \tag{4.12}$$

we obtain the relations

$$\begin{aligned} &\eta_o'' + (\varepsilon^2 - n)\eta_o = \varepsilon \\ &\eta_1'' + (\varepsilon^2 - n)\eta_1 = - \varepsilon + (2\varepsilon^2 - n)\eta_o + (\tfrac{1}{2}S - \varepsilon^3)\eta_o^2 + \frac{1}{2}\varepsilon(\eta_o')^2 \end{aligned} \tag{4.13}$$

in which the prime means differentiation to the azimuth θ: $' = d/d\theta$. We emphasize that η_o and η_1 are now reduced functions, i.e. the off-momentum function divided by R.
The terms $\varepsilon^2\eta_o$, $\varepsilon^3\eta_o^2$ and $\varepsilon(\eta_o')^2$ are of the order $1/\rho$ or $1/\rho^2$ (ρ is the radius of curvature) and can therefore be neglected for very large machines. In expressions for the off-momentum function in (Fau79) these terms do not appear. However, for IKOR these terms should not be neglected.
From (4.13) it is obvious that the influence of sextupoles only appears when the equilibrium orbit is solved up to second order in $\Delta P/P_o$.
Using the theory of section 2.3, the equations of (4.13) are simply soluble. The results are

$$\begin{aligned} &\tilde{\eta}_o = Q^2 \sum_p \frac{G_p^{(0)}}{Q^2 - (pN)^2} e^{ipN\psi} \\ &\tilde{\eta}_1 = - \eta_o + Q^2 \sum_p \frac{G_p^{(1)}}{Q^2 - (pN)^2} e^{ipN\psi} \end{aligned} \tag{4.14}$$

with Q the horizontal tune, N the machine periodicity and - in conformity with (2.22) and (2.25) - :

$$\tilde{\eta} = \sqrt{\frac{R}{\beta_x}}\,\eta \;, \quad d\psi = \frac{R d\theta}{\beta_x Q}$$

and furthermore

$$G^{(0)}(\psi) = \varepsilon \beta_x^{3/2} R^{-3/2} = \sum_p G_p^{(0)} e^{ipN\psi}$$

$$G^{(1)}(\psi) = \beta_x^{3/2} R^{-3/2} (2\varepsilon^2 - n)\eta_o + \beta_x^{3/2} R^{-3/2} (\tfrac{1}{2}S - \varepsilon^3)\eta_o^2 + \tag{4.15}$$

$$+ \frac{1}{2}\varepsilon \beta_x^{3/2} R^{-3/2} (\eta_o')^2$$

$$= \sum_p G_p^{(1)} e^{ipN\psi} \quad .$$

The quantities α_o and α_1 are obtained by substitution of the solutions $\tilde{\eta}_o$ and $\tilde{\eta}_1$ in (4.10) and this equation is subsequently set equal to (4.2):

$$\alpha_o = Q^3 \sum_p \frac{G_p^{(0)} G_{-p}^{(0)}}{Q^2 - (pN)^2} \tag{4.16}$$

$$\alpha_1 = -1 + \frac{Q^3}{\alpha_o} \sum_p \frac{G_{-p}^{(0)} G_p^{(1)}}{Q^2 - (pN)^2} + \frac{1}{4\pi\alpha_o} \int_o^{2\pi} (\eta_o')^2 \, d\theta \quad .$$

Of course the sextupole magnet is most effective at a position with large values of both β_x and η.
In IKOR this point of view leads to two positions: one near the F quadrupole, the other one just in front of the bending magnet, see figure 2.3. However, the latter one is less favourable because the long straight section is necessary for ejection (SNQ81III).

4.2.4 Results for IKOR

To control γ_{tr} in IKOR, a sextupole magnet with a length of 0.4m is placed in each unit cell between the F and D1 quadrupole, as indicated in figure 1.3.
The corresponding values of β_x and the non-reduced η function are (see figure 2.3): $\beta_x \simeq 27$ m/rad and $\eta \simeq 6.7$ m.

The Fourier components needed for the calculation of α_o and α_1 are given in (Cor81a) and the resulting values are (see also Cor81b):

$$\alpha_o = 0.202$$
$$\alpha_1 = 1.81 - 2.62\left(\frac{\partial^2 B_z}{\partial x^2}\right)_o \quad . \tag{4.17}$$

Substitution of these data into (4.5) yields the behaviour of γ_{tr} as a function of the sextupole strength. In figure 4.1 the quantity $1/\gamma_{tr}^2 - 1/\gamma^2$ is plotted versus the sextupole field for a relative momentum deviation of 5.10^{-3}.

This analytical result is compared with the result obtained with the AGS computer program, running at CERN (Kei75, Ris79).

The CERN "recommendation", working at constant value of $1/\gamma_{tr}^2 - 1/\gamma^2$,(4.8), requires a sextupole strength of about 1.15 T/m^2. This rather low value is a result of the large β_x- value at the position of the sextupole (see figure 2.3). A position near the D2 quadrupole (see figure 2.3) should require a strength which is about 50 times as large.

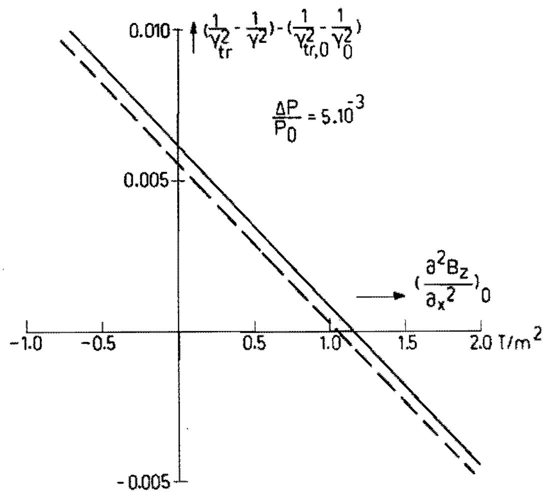


Figure 4.1

$(1/\gamma_{tr}^2-1/\gamma^2)-(1/\gamma_{tr,o}^2-1/\gamma_o^2)$ versus the sextupole field in IKOR for $\Delta P/P_o=5.10^{-3}$, using the analytical method (——) and the AGS program (--).

In the preceding sections we pointed out reasons why multipoles are often included in a storage ring. However, these multipoles also influence the non-collective particle motion. Their effects on the betatron motion will be described in the following sections.

4.3 Non-linear Hamilton theory

In this section we will develop a general theory to treat the effects of non-linearities on the one-dimensional betatron motion. The differences with existing theories have already been discussed in the introduction. The final theory enables us to examine "first" as well as "second order" effects of non-linear magnetic fields. As the treatment is similar for both transverse motions, only the horizontal one will be discussed. We will use the advance knowledge of section 1.5.1.

To describe non-linear effects we analyze the general Hamiltonian (1.19). After substitution of $\bar{\bar{z}}=0$, $\bar{\bar{p}}_z=0$ we write

$$H = \frac{1}{2}\, p_x^2 + \frac{1}{2}(\varepsilon^2 - n)x^2 + H^+(x,p_x;\theta) \quad . \tag{4.18}$$

For convenience we omitted the bars above the variables x and p_x. The non-linear part of the Hamiltonian is represented by H^+. The θ-dependence in the second degree in (4.18) is removed by the transformations (2.22) and (2.25) and the new Hamiltonian becomes

$$\begin{aligned}\bar{H}(\bar{x},\bar{p}_x;\psi) &= \frac{1}{2}\, Q\bar{p}_x^2 + \frac{1}{2}\, Q\bar{x}^2 + \frac{Q}{R}\, \beta_x H^+(\bar{x},\bar{p}_x) \\ &= \frac{1}{2}\, Q\bar{p}_x^2 + \frac{1}{2}\, Q\bar{x}^2 + \sum_{\substack{k,l \\ k+l=n\geq 3}} f_{k,l}(\psi)\bar{x}^k\bar{p}_x^l\end{aligned} \tag{4.19}$$

with k and l positive integers and n=k+l indicates the degree of the term in question in this Hamiltonian.† The functions $f_{k,l}(\psi)$ contain combinations of the Twiss parameters and non-linear fields, fringing fields etc. and are periodic in ψ.

Introduction of action and angle variables J and ϕ (see (2.27)) leads to a Hamiltonian which is apt to study non-linear effects. For convenience we use complex exponentials and (4.19) becomes

$$K = QJ + \sum_n \sum_m F_n^{(m)}(\psi)\; e^{im\phi}\; J^{n/2} \quad . \tag{4.20}$$

† In the rest of this chapter the symbol n is used to indicate the degree of a term in the Hamiltonian. This should not be confused with the normalized quadrupole component n, defined in (1.19).

The index m is limited by the order of the non-linearity, i.e. m takes the integral values $\pm n$, $\pm(n-2)$,... down to ± 1 for n odd and down to 0 for n even.
The function $F_n^{(m)}(\psi)$ in (4.20) is defined by

$$F_n^{(m)}(\psi) = \sum_{\substack{k,l \\ k+l=n}} a_{k,l}^{(m)} f_{k,l}(\psi)\, 2^{n/2} \tag{4.21a}$$

where the complex quantities $a_{k,l}^{(m)}$ are defined by the relations

$$\cos^k\phi \sin^l\phi = \sum_{\substack{m=\pm n, \pm(n-2),.. \\ \text{with } n=k+l}} a_{k,l}^{(m)} e^{im\phi} \quad . \tag{4.21b}$$

The functions $F_n^{(m)}(\psi)$ satisfy the relation

$$F_n^{(m)}(\psi) = \left[F_n^{(-m)}(\psi)\right]^* \tag{4.22}$$

in which * means the complex conjugate (c.c.).

As we are most interested in the sextupole and octupole fields we mention their contribution to the function $F_n^{(m)}$:
- sextupole field (n=3)

$$F_3^{(1)} = 3F_3^{(3)} = -\frac{1}{8}\sqrt{2}\,\beta_x^{5/2}R^{-5/2}QS$$

$$S = -\frac{R^3}{B_o\rho}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_o \text{ is the normalized sextupole field,} \tag{4.23}$$

- octupole field (n=4)

$$F_4^{(0)} = \frac{3}{2}F_4^{(2)} = 6F_4^{(4)} = -\frac{1}{16}\beta_x^3R^{-3}QO$$

$$O = -\frac{R^4}{B_o\rho}\left(\frac{\partial^3 B_z}{\partial x^3}\right)_o \text{ is the normalized octupole field.} \tag{4.24}$$

Other contributions to the functions $F_n^{(m)}(\psi)$ can be found from the Hamiltonian (1.19) and following the way as sketched in this section.

The next step in the development of the theory is the elimination of the ψ-dependence in the n-th degree terms of (4.20).

4.3.1 Non-linear transformations

For the elimination of the ψ-dependence in the n-th degree of (4.20) we consider

$$K = QJ + \sum_m F_n^{(m)}(\psi)\, e^{im\phi}\, J^{n/2} \quad , \text{ n fixed} \quad . \tag{4.25}$$

The first transformation is of the same type as (2.15), used in section 2.2. More generally we write:

$$\begin{aligned} G(\bar{J},\phi;\psi) &= -\bar{J}\phi - \bar{J}^{\frac{n}{2}} U_n(\phi,\psi) \\ J &= \bar{J}\left(1 + \bar{J}^{\frac{n}{2}-1} \frac{\partial U_n}{\partial\phi}\right) \\ \phi &= \bar{\phi} - \frac{n}{2} U_n \bar{J}^{\frac{n}{2}-1} \end{aligned} \tag{4.26}$$

where we assume that the correction terms $J-\bar{J}$ and $\phi-\bar{\phi}$ are small. Expressed in the new variables $\bar{J},\bar{\phi}$, the Hamiltonian (4.25) becomes:

$$\begin{aligned} \bar{K}(\bar{J},\bar{\phi};\psi) = \; & Q\bar{J} + g(\bar{\phi},\psi)\bar{J}^{\frac{n}{2}} + \\ & + \frac{n}{2}\bar{J}^{n-1}\left(\frac{\partial U_n}{\partial\bar{\phi}} \sum_m F_n^{(m)}(\psi) e^{im\bar{\phi}} - U_n(\bar{\phi},\psi) \frac{\partial g}{\partial\bar{\phi}}\right) \end{aligned} \tag{4.27}$$

with

$$g(\bar{\phi},\psi) = Q\frac{\partial U_n}{\partial\bar{\phi}} - \frac{\partial U_n}{\partial\psi} + \sum_m F_n^{(m)}(\psi) e^{im\bar{\phi}} \quad . \tag{4.28}$$

We suppose that the lowest order theory is applicable up to amplitudes equal to those of the fixed points (i.e. $\bar{J} = \bar{J}_{f.p.}$). To check the validity of this approach we retain the extra term $\bar{J}^{n-1}$.
In the derivation of this Hamiltonian (4.27) we used the approximations

$$g(\phi,\psi) = g(\bar{\phi}-\tfrac{n}{2}U_n\bar{J}^{\frac{n}{2}-1},\psi) = g(\bar{\phi},\psi) - \frac{n}{2} U_n \bar{J}^{\frac{n}{2}-1} \frac{\partial g}{\partial\bar{\phi}} \tag{4.29}$$

and

$$J^{\frac{n}{2}} = \bar{J}^{\frac{n}{2}} + \frac{n}{2}\frac{\partial U_n}{\partial\bar{\phi}} \bar{J}^{n-1} \quad . \tag{4.30}$$

The difference between ϕ and $\bar{\phi}$ is neglected in the 2(n-1)th degree terms in the second line in (4.27) because we will not carry out the calculations to a still higher degree. This second line in (4.27) is the direct result of the elimination of oscillating parts in the n-th degree. Now we will first analyze the n-th degree part in (4.27).

4.3.2 The n-th degree part: determination of $U_n(\bar{\phi},\psi)$

The elimination of the oscillating parts in the n-th degree of (4.27) is achieved with the transformation (4.26), if the function U_n satisfies the condition

$$\text{osc } g(\bar{\phi},\psi) = \text{osc}\left(Q\frac{\partial U_n}{\partial\bar{\phi}} - \frac{\partial U_n}{\partial\psi} + \sum_m F_n^{(m)}(\psi)\, e^{im\bar{\phi}}\right) = 0 \,, \tag{4.31}$$

in which osc $g(\bar{\phi},\psi)$ indicates the oscillating parts of the function g defined in (4.28).

To determine $U_n(\bar{\phi},\psi)$ we expand $F_n^{(m)}(\psi)$ in a Fourier series according to

$$F_n^{(m)}(\psi) = \sum_p F_{n,p}^{(m)}\, e^{ipN\psi} \quad , \tag{4.32}$$

in which N is the periodicity of the non-linearity in question.

In case of extra correction elements, this is often the periodicity of the linear guide field (i.e. number of unit cells). However, if the non-linearity is due to field imperfections, one generally has $N = 1,2,3,\ldots$.

The function U_n is periodic in $N\psi$ and referring to the requirement (4.31) we look for a solution of U_n of the form

$$U_n(\bar{\phi},\psi) = \sum_m \sum_p u_{m,p}\, e^{i(m\bar{\phi} + pN\psi)} \;. \tag{4.33}$$

Substitution of (4.32) and (4.33) into (4.31) leads to

$$u_{m,p} = -\frac{iF_{n,p}^{(m)}}{pN - mQ} \qquad \text{with} \quad pN \neq mQ \;. \tag{4.34}$$

The function $U_n(\bar{\phi},\psi)$ is known now (see (4.33)) and all oscillating terms in n-th degree part are transformed to higher degree $2(n-1)$ (see (4.27)). The relation $pN = mQ$ represents a resonance and terms with $pN-mQ \ll 1$ are called slowly varying or resonant terms which can not be transformed by (4.34).

4.3.3 Resonant terms

In this section we will examine the influence of the resonant terms. In the Hamiltonian (4.27) we keep these resonant terms, whereas all other terms are transformed to higher degree in the way explained above.

The last term in (4.28) is the only term which leads to a resonant n-th degree term. The values of p and m for which resonance occurs are denoted by p_r and m_r. Using the Fourier series of $F_n^{(m)}(\psi)$ (4.32), the Hamiltonian (4.27) becomes

$$\bar{K} = Q\bar{J} + \sum_{m_r}\sum_{p_r} F_{n,p_r}^{(m_r)} e^{i(m_r\bar{\phi} + p_r N\psi)} \bar{J}^{\frac{n}{2}} \tag{4.35}$$

where m_r and p_r occur in positive as well as in negative pairs. The higher degree terms are neglected now but we shall return to them in a subsequent section.

A special kind of "resonant" term appears for $m_r=0$, $p_r=0$. Strictly speaking this term is not a real resonant in the sense that the betatron number Q should have a specific value according to $m_rQ=p_rN$. The term with $m_r=p_r=0$ is always present in even degree (n even) - see (4.21) - and it is preferable in this case to speak of a constant term.
Moreover, we have the resonant terms with $m_r \neq 0$. After applying a transformation of the type (1.26), the ψ-dependence in (4.35) is removed and we find

$$\begin{aligned}\bar{\bar{K}}(\bar{\bar{J}},\bar{\bar{\phi}};\psi) = \left(Q - \frac{p_r N}{m_r}\right)\bar{\bar{J}} + \sum_{n \text{ even}} F_{n,0}^{(0)} \bar{\bar{J}}^{\frac{n}{2}} + \\ + \left(F_{n,p_r}^{(m_r)} e^{im_r\bar{\bar{\phi}}} + \text{c.c.}\right)\bar{\bar{J}}^{\frac{n}{2}}\end{aligned} \tag{4.36}$$

with $\bar{\bar{J}} = \bar{J}$ and $\bar{\bar{\phi}} = \bar{\phi} + \frac{p_r N}{m_r}\psi$.

Returning to a real representation, this Hamiltonian becomes

$$\bar{\bar{K}} = \left(Q - \frac{p_r N}{m_r}\right)\bar{\bar{J}} + \sum_{\substack{n \\ n \text{ even}}} F_{n,0}^{(0)} \bar{\bar{J}}^{\frac{n}{2}} + 2\left|F_{n,p_r}^{(m_r)}\right|\bar{\bar{J}}^{\frac{n}{2}} \cos(m_r\bar{\bar{\phi}} + \psi_F) \tag{4.37}$$

where m_r and p_r are now both positive integers and

$$F_{n,p_r}^{(m_r)} = \left|F_{n,p_r}^{(m_r)}\right| e^{i\psi_F} \quad . \tag{4.38}$$

The second term in the r.h.s. of (4.37) - which only appears in even degree - gives rise to an amplitude-dependent tune change ΔQ :

$$\Delta Q(\bar{\bar{J}}) = \frac{n}{2} F_{n,0}^{(0)} \bar{\bar{J}}^{\frac{n}{2}-1} \quad , \text{ n even} \quad . \tag{4.39}$$

This tune change can stabilize the resonant character because it can take the motion out of the resonance.

Examination of the flowlines in phase space gives a good insight in the (in)stability of the motion. For the unstable fixed points ($\dot{\bar{\bar{J}}} = 0$, $\dot{\bar{\bar{\phi}}} = 0$) we find

$$\bar{\bar{J}}_{fp}^{\frac{n}{2}-1} = \frac{Q - \frac{p_r N}{m_r}}{\pm\, n \left| F_{n,p_r}^{(m_r)} \right| - \frac{n}{2} F_{n,0}^{(0)}} \tag{4.40}$$

where the upper sign holds if $Q - \frac{p_r N}{m_r} + \Delta Q(\bar{\bar{J}}_{fp}) > 0$ and the lower sign if $Q - \frac{p_r N}{m_r} + \Delta Q(\bar{\bar{J}}_{fp}) < 0$.

Next we want to define the stopband width of the resonance. This can be done in several ways. A possible definition of the width is the range of Q for which the stable area is less than the beam emittance. Other definitions are obtained when using the areas of the circles with radius $\sqrt{2\bar{\bar{J}}_{min}}$ or $\sqrt{2\bar{\bar{J}}_{fp}}$ as sketched in figure 4.2. These areas are obvious when we remind the reader that the beam is represented by a circle when the non-linear field is turned off. The variations in the definitions are not too important as long as one is consistent in their use.
In this thesis we will use the definition with the area $2\pi\bar{\bar{J}}_{min}$. Thus, a beam emittance which exceeds the area $2\pi\bar{\bar{J}}_{min}$ is "not allowed". Substitution of this requirement into (4.40) leads to the following sufficient condition for stable motion:

$$\left| Q - \frac{p_r N}{m_r} + \frac{n}{2} c_n F_{n,0}^{(0)} \left(\frac{\varepsilon_x}{R}\right)^{\frac{n}{2}-1} \right| > n c_n \left| F_{n,p_r}^{(m_r)} \right| \left(\frac{\varepsilon_x}{R}\right)^{\frac{n}{2}-1} \tag{4.41}$$

where c_n is defined considering the phase plane for $F_{n,0}^{(0)} = 0$ (see fig. 4.2):

$$c_3 = \sqrt{2} \qquad c_4 = 1.21 \tag{4.42}$$

and $\pi\varepsilon_x$ is the beam emittance as defined in section 2.1, chapter 2.

Applications will be given later in section 4.4, after having discussed the higher degree terms, i.e.terms of 2(n-1)th degree in the Hamiltonian (4.27).

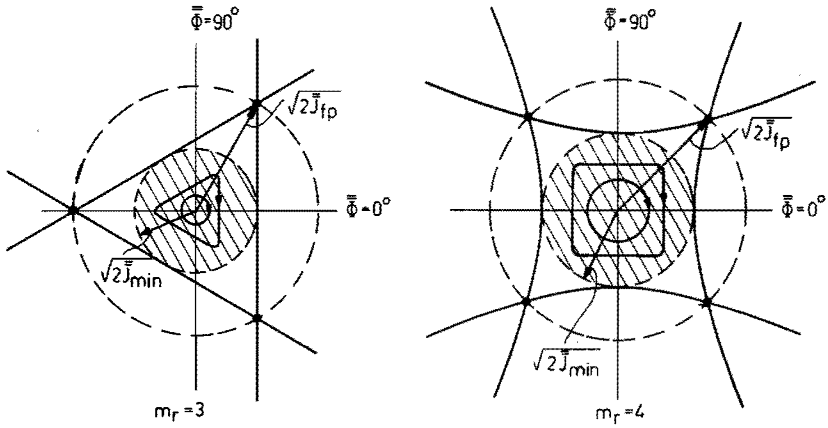


Figure 4.2

Phase plane for third (m_r=3) and fourth order resonances (m_r=4) with $F^{(0)}_{n,0}$ = 0. When $F^{(0)}_{n,0} \neq 0$ the trajectories are slightly different (Hag62). In this thesis the stopband width is defined by taking the shaded area as the maximum allowed "normalized" emittance.

4.3.4 The higher degree terms

The terms of 2(n-1)th degree ($\bar{J}^{n-1}$) in (4.27) are of interest because:

- these terms may give rise to a change in Q, depending on the amplitude. This effect will influence the stability criterion when Q is near a resonant value,
- these terms may contribute to resonance effects. It turns out that the resonances with m_r = n, n-2,... are not the only possible resonances that may arise from a n-th degree term in the initial Hamiltonian (4.19).

The higher degree, $\bar{J}^{n-1}$, in (4.27) consists of two parts. For simplicity we shall treat them separately.

First we examine the term $U_n \partial g/\partial\bar{\phi}$.
If there is no resonance in the n-th degree, g=0 because of (4.28) and (4.31) and $U_n \partial g/\partial\bar{\phi}$ gives no contribution to the higher degree term $\bar{J}^{n-1}$.
If there is a resonance m_rQ=p_rN in n-th degree, all oscillating

parts in g are made zero by using the function U_n (see (4.31)). Thus g only contains a constant or slowly varying term. Since the terms with $m = \pm m_r$ are excluded from the function U_n (see (4.33) and (4.34)), this function U_n consists of rapidly oscillating terms. Consequently the function $U_n \partial g/\partial\bar{\phi}$ contains no constant or slowly varying parts and will not contribute to an amplitude-dependent tune change or a resonance effect.

Secondly we consider the other term of higher degree in (4.27). This term is periodic in ψ and $\bar{\phi}$ and can be expanded in a Fourier series. Using (4.33):

$$
\begin{aligned}
\frac{\partial U_n}{\partial\bar{\phi}} \sum_{m_2} F_n^{(m_2)}(\psi)\, e^{im_2\bar{\phi}} &= \\
&= \sum_{p_1}\sum_{m_1}\sum_{m_2} im_1 u_{m_1,p_1}\, F_n^{(m_2)}(\psi)\, e^{i\{(m_1+m_2)\bar{\phi} + p_1 N\psi)\}} \qquad (4.43)\\
&= \sum_m \sum_p A_{m,p}\, e^{i(m\bar{\phi} + pN\psi)}
\end{aligned}
$$

with $m = \pm 2n, \pm(2n-2),\ldots$ and $p = 0, \pm 1, \pm 2,\ldots$.
After substitution of the Fourier series for $F_n^{(m)}(\psi)$((4.32)) and the expression for u_{m_1,p_1} ((4.34)) into (4.43), $A_{m,p}$ can be written as

$$A_{m,p} = \sum_{m_1}\sum_{p_1} m_1 \frac{F^{(m_1)}_{n,p_1}\, F^{(m-m_1)}_{n,p-p_1}}{p_1 N - m_1 Q} \qquad (4.44)$$

with $m = \pm n, \pm(n-2),\ldots$ and $p = 0, \pm 1, \pm 2,\ldots$.
In this summation resonant terms in the n-th degree are excluded ($m_1 \neq |m_r|$, $p_1 \neq |p_r|$; $m_1 \neq -|m_r|$, $p_1 \neq -|p_r|$).

The final contribution to the higher degree term in (4.27) is now

$$\frac{n}{2}\sum_m\sum_p A_{m,p}\, e^{i(m\bar{\phi} + pN\psi)}\, \bar{J}^{n-1} . \qquad (4.45)$$

This term leads to an amplitude-dependent tune change when m=0, p=0. When there is a resonance $m_r Q = p_r N$ in n-th degree, we have to look for slowly varying terms in this 2(n-1)th degree term (4.45), which correspond to the resonant Q-value.
When there is no resonance in n-th degree, this higher degree term might lead to new resonance effects.

The higher degree term (4.45) is of second degree in the Fourier components of the field that excites the resonance (see (4.44)), whereas the original constant or resonant terms are all of first degree in the Fourier components (see (4.37)). In future we shall speak of "first order" effects when dealing with terms up to the first power in the Fourier components. So-called "second order" effects (second degree in the Fourier components) are thus caused by the higher degree terms which are a direct consequence of the transformation (4.26). These concepts "first" and "second order" should not be confused with the order of a resonance (see section 1.5, chapter 1).

4.3.5 "Second order" tune change

Besides the first order tune change (4.39) which only comes from even degree, a second order tune change exists, caused by the higher degree term (4.45), for which there is no restriction for n. This second order amplitude-dependent tune change is given by (from (4.45) with m=0, p=0)

$$\Delta Q(\bar{J}) = \frac{n}{2}(n-1)A_{0,0}\bar{J}^{n-2}$$

with

$$A_{0,0} = \sum_m \sum_p m \frac{F^{(m)}_{n,p} F^{(-m)}_{n,-p}}{pN - mQ} \quad . \tag{4.46}$$

Thus, sextupoles (n=3) which do not produce a first order amplitude-dependent tune change produce a second order one, proportional to the square of the betatron amplitude (and thus to $\bar{J}$). Octupoles (n=4) produce a first as well as a second order tune change, proportional respectively to the square and to the fourth power of the betatron amplitude ($\bar{J}$ and $\bar{J}^2$).
Numerical results will be given in section 4.4.

4.3.6 Resonance effects due to the higher degree term

The higher degree term (4.45) can give rise to new resonances. We consider the case for which there is no resonance in n-th degree, but there is a resonance $m_r Q = p_r N$ in the 2(n-1)th degree.

The total Hamiltonian is then:

$$\bar{\bar{K}} = \left(Q - \frac{p_r N}{m_r}\right)\bar{\bar{J}} + \left(F^{(0)}_{2n-2,0} + \frac{n}{2} A_{0,0}\right)\bar{\bar{J}}^{n-1} + \qquad (4.47)$$

$$+ \sum_{m_r} \sum_{p_r} \left(F^{(m_r)}_{2n-2,p_r} + \frac{n}{2} A_{m_r,p_r}\right) e^{i m_r \bar{\bar{\phi}}}\, \bar{\bar{J}}^{n-1}$$

with $\bar{\bar{J}} = \bar{J}$ and $\bar{\bar{\phi}} = \bar{\phi} + \frac{p_r N}{m_r}\psi$

and where m_r and p_r occur in positive and negative pairs.
The term in (4.47) with coefficient $F^{(m_r)}_{2n-2,p_r}$ is the original resonant term in 2(n-1)th degree, whereas A_{m_r,p_r} is the contribution from the lower degree n.
This means that e.g. third degree terms in the initial Hamiltonian (4.19) can also excite fourth (m_r=4) and second order resonances (m_r=2) (both "second order" effects).
In a first order effect sextupoles produce a resonance of the type $3Q=p_rN$ but moreover they can excite a resonance of the type $4Q=p_rN$.
An illustration of this resonance effect will be given in the following section.

4.4 Applications

In this section we give some results of applications of the preceding theory on IKOR and PAMPUS.

4.4.1 Third order resonance $3Q=p_rN$ excited by sextupole fields

An important source for the excitation of third order resonances is a sextupole field. Besides the lumped sextupoles for chromaticity or γ_{tr} control, there will also be imperfections of sextupolar nature e.g. in dipole magnets.

For stable motion near the resonance $3Q=p_rN$ the following condition should be fulfilled ((4.41) with $F^{(0)}_{n,0} = 0$):

$$\left|Q - \frac{p_r N}{3}\right| \geq 3c_3 \left|F^{(3)}_{3,p_r}\right| \sqrt{\frac{\varepsilon_x}{R}}$$

with (4.48)

$$F^{(3)}_{3,p_r} = \frac{N}{2\pi} \int_0^{2\pi/N} (\sqrt{2}/24) Q \beta_x^{5/2} \frac{R^{1/2}}{B_o \rho} \left(\frac{\partial^2 B_z}{\partial x^2}\right)_o e^{-i p_r N \psi}\, d\psi \ .$$

This eq.(4.48) gives the "required" distance to the resonance when the sextupole strength is known (in case of the lumped elements). When the strength is not known (in case of imperfections) - and a working point is chosen - the considerations will lead to a maximum allowed sextupole strength, i.e. to tolerances for the magnetic field. Results for PAMPUS and IKOR are listed in table 4.1.

For PAMPUS (8 unit cells) we considered the influence of the chromaticity-sextupoles. Besides the 8-fold periodicity, there will also be a strong 16-fold periodicity, as can be seen in figure 1.2. As the PAMPUS working point had not been fixed, we examined both cases, i.e. 3Q=8 and 3Q=16. For the calculation of the Fourier components we used the sextupole fields as given in Bac79a:
at $Q \approx 8/3$: $(\partial^2 B_z/\partial x^2)_{SF} = 3.5\ T/m^2$, $(\partial^2 B_z/\partial x^2)_{SD} = -4.7\ T/m^2$ and
at $Q \approx 16/3$: $(\partial^2 B_z/\partial x^2)_{SF} = 20\ T/m^2$, $(\partial^2 B_z/\partial x^2)_{SD} = -45\ T/m^2$.
The subscripts SF ans SD indicate the positions of the sextupoles: SF is located near the F-quadrupole whereas SD is located near the D-quadrupole (see figure 1.2).

For IKOR we considered the resonance 3Q=11 due to the sextupoles for the control of γ_{tr} (strength 1.15 T/m^2, see section 4.2.4) and the resonances 3Q=10 and 3Q=11 excited by sextupolar imperfections in the dipole magnets. In principle the imperfections affect all dipoles, resulting in a strong systematic component (N=11, p_r=1, 3Q=11) but moreover there will be a component due to randomly distributed imperfections. As IKOR is proposed to operate at $Q_x \approx 3.25$ we investigated the resonance 3Q=10 ($Q-10/3 \simeq -0.083$) excited by a 10th harmonic of a sextupole component in only one dipole magnet (i.e. p_r=10 and N=1).
The considerations concerning the imperfections result in tolerances for the magnetic field.

In this section we have only considered the contribution of sextupole fields to the influence on the resonance effect. In principle there are more terms contributing to the relevant Fourier component $F^{(3)}_{3,p_r}$, as can be seen from (1.19). However, these can be treated in quite a similar way and moreover they are, in general, small compared to the sextupole contribution (see Cor80a).

Table 4.1

Requirements made by the resonance $3Q=p_rN$ excited by a sextupole field. For PAMPUS with 3Q=8 and 3Q=16 we used respectively $\varepsilon_x=5.0\ 10^{-5}$ m.rad and $\varepsilon_x=4.7\ 10^{-6}$ m.rad. For IKOR $\varepsilon_x=1.5\ 10^{-4}$ and the proposed Q-value is 3.25 so that Q-10/3=-0.083 and Q-11/3=0.417. $B'' = (\partial^2 B_z/\partial x^2)_o$ is the sextupole field in T/m^2.

resonance	p_r	N	sextupole field	$\lvert F^{(3)}_{3,p_r}\rvert$	required distance $\lvert Q-p_rN/3\rvert$	tolerance for $\lvert Q-10/3\rvert=0.083$ $\lvert Q-11/3\rvert=0.417$
PAMPUS						
3Q = 8	1	8	lumped	0.12	0.001	-
3Q = 16	2	8	lumped	14.4	0.036	-
IKOR						
3Q = 11	1	11	lumped	7.1	0.065	-
3Q = 10	10	1	dipole imperf.	$3.6\lvert B''\rvert$	-	$\lvert B''\rvert \le 2.5\ T/m^2$
3Q = 11	1	11	dipole imperf.	$43.2\lvert B''\rvert$	-	$\lvert B''\rvert \le 1.1\ T/m^2$

Because of the small value of the required distance to the resonance, the lumped sextupoles will not lead to problems with respect to the one-dimensional resonance $3Q=p_rN$.

A sextupolar imperfection in the dipoles must be smaller than 1.1 T/m^2 (in IKOR). In practice one should prefer to stay well below this value. We can measure $\Delta B_z(z=h)$ as a function of x, where ΔB_z is the deviation from the magnetic field B_o (field in the dipole on the reference orbit). When we assume $\Delta B_z(x=0,z=0)=0$ and the field imperfection at x=d to be entirely due to a sextupole term, we may write

$$\left.\frac{\Delta B_z}{B_o}\right|_{z=0,x=d} = \frac{d^2}{2B_o}\left(\frac{\partial^2 B_z}{\partial x^2}\right) \quad . \tag{4.49}$$

For a good field region of 20 cm (see Jül81), we get an allowed imperfection $\Delta B_z/B_o = 4.10^{-3}$ at the "edge" of the dipole (d = 0.1m). In practice a value of 2.10^{-4} can be reached without too excessive costs.

4.4.2 "First order" tune change and fourth order resonance $4Q=p_r N$

Terms of fourth degree in the Hamiltonian (4.19) lead to a tune change which depends on the betatron amplitude and can excite fourth order resonances (see section 4.3.3).

Tune change

First of all there is a tune change caused by the kinematical terms in the original Hamiltonian (1.19), which we call the inherent tune change. Additionally there is a tune change due to octupole fields e.g. lumped Landau octupoles or imperfections in the guiding elements.

In principle, the tune change depends on both horizontal and vertical amplitude. Strictly speaking one must consider the coupled betatron motion which will be done in chapter 5. In order to illustrate the theory we consider the horizontal motion only. Especially in electron storage rings the neglect of the vertical motion is not too serious because of the maximum vertical amplitude being much smaller than the maximum horizontal one.

The function $F_4^{(0)}$ - that determines the tune change, see (4.39) - thus consists of two terms. The contribution of the kinematical term $\frac{1}{8}\, p_x^4$ (see (1.19)) can be calculated with (4.19) and (4.20), whereas the contribution of the octupole field is given in (4.24).
Thus - for n=4 - we find for the "first order" tune change:

$$\Delta Q = 2\, F_{4,0}^{(0)}\, \bar{\bar{J}} \tag{4.50}$$

with

$$F_{4,0}^{(0)} = \frac{R}{32\pi}\left(\int_0^{L_o} 3\gamma_x^2\, ds + \int_0^{L_o} \frac{1}{B_o\rho}\left(\frac{\partial^3 B_z}{\partial x^3}\right)_o \beta_x^2\, ds\right) .$$

The first term represents the inherent tune change, the second one shows the influence of the octupole field. γ_x and β_x are the well-known Twiss parameters, defined in chapter 2, eq.(2.12).

The maximum inherent tune change for PAMPUS is given in figure 4.3 as a function of the horizontal tune. We note that this result is obtained by substituting $\bar{\bar{J}} = \frac{1}{2}\varepsilon_x/R$ in (4.50).

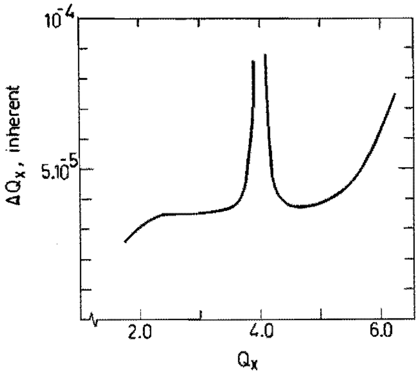


Figure 4.3
Inherent tune change for PAMPUS as a function of the tune Q_x, for a particle with a betatron amplitude corresponding with the relation $\bar{\bar{J}} = \frac{1}{2}\,\varepsilon_x/R$. The order of magnitude is characteristic of common electron storage rings.

For IKOR there holds $\varepsilon_x = 3\varepsilon_z$ (see table 1.1) and in order to calculate the maximum inherent tune change one should deal with coupled betatron motion. Nevertheless, to get the order of magnitude we considered a particle with a vertical amplitude equal to zero and a horizontal one corresponding to $\varepsilon_x = 1.5\ 10^{-4}$ m.rad (table 1.1).
Eq(4.50) results in $F^{(0)}_{4,0} = 16.8$ and consequently

$$\Delta Q_{inherent,max} = 8.10^{-5} \qquad \text{(IKOR)} \ . \tag{4.51}$$

Generally, the inherent tune spread in the beam is rather small and octupoles are often used in the accelerator or storage ring to provide a tune spread in order to prevent instability of the beam. Because the amplitude is proportional with the square root of the β-function (see (2.7)) it is obvious that the octupoles are most effective at positions with a high β_x-value.

As an example we take the positions of the octupoles in PAMPUS being the same as the positions of the sextupoles (see fig.1.2) and having a length of 0.25 m . As we consider only the horizontal motion here, the octupole in front of the F-quadrupole is mainly of interest because of the large β_x value. From (4.50) one can calculate the needed octupole fields to achieve a certain tune spread in the beam. The result for PAMPUS is plotted in figure 4.4 (see also Cor80a).

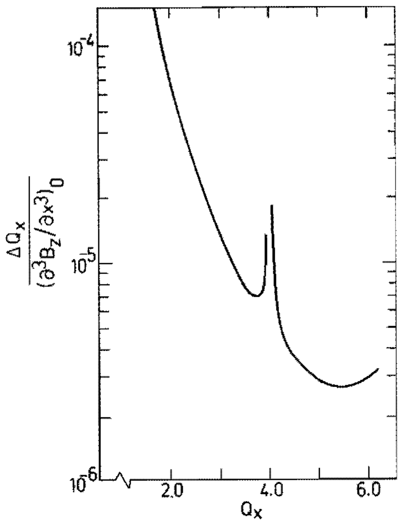


Figure 4.4
Maximum tune change per unit of the octupole strength for PAMPUS. Only one octupole in each unit cell, at the position of the sextupole SF near the F-quadrupole (fig. 1.2) with a length of 0.25 m is taken into account. The order of magnitude is characteristic of common electron storage rings.

The required tune spread in the beam strongly depends on the characteristics of the instability which has to be prevented. A common number for the tune spread is of the order 10^{-3} to 10^{-2} (Hüb72, Rug76). As an example, a maximum tune change $\Delta Q = 2.10^{-3}$ at $Q_x = 3.25$ for PAMPUS would require an octupole strength of about 200 T/m^3. The order of magnitude is typical of a common electron storage ring.

The discussion whether or not octupoles should be included in IKOR to provide an adjustable tune spread, has not been finished yet. As an illustration, we assume an octupole with a length of 0.4 m in each unit cell at the position of the sextupole magnet (see fig.2.3 : $\beta_x \approx 27$ m/rad). The octupole contribution to the Fourier component $F^{x(0)}_{4,0}$ is then $170(\partial^3 B_z/\partial x^3)_o$. The tune change with amplitude is known from (4.50) and the maximum tune change becomes (substituting $\bar{\bar{J}} = \frac{1}{2}\frac{\varepsilon_x}{R}$ with $\varepsilon_x = 1.5\ 10^{-4}$ m.rad and R = 32.18 m):

$$\Delta Q = 8.10^{-4}\left(\frac{\partial^3 B_z}{\partial x^3}\right)_o \qquad \text{(IKOR)} \quad . \qquad (4.52)$$

A more accurate result is obtained by studying the coupled betatron motion in chapter 5.

Resonance effects

Considering the fourth order resonances $4Q=p_rN$, the excitation term due to the octupole fields will, in general, be the most important one in storage rings. As noticed in section 4.3.3, the amplitude-dependent tune change will affect the stability limits.
According to (4.41), the following condition must be fulfilled:

$$|Q - \frac{p_r N}{4} + 2F_{4,0}^{(0)} c_4 \left(\frac{\varepsilon_x}{R}\right)| \geq 4c_4 |F_{4,p_r}^{(4)}| \left(\frac{\varepsilon_x}{R}\right)$$

with $c_4 = 1.21$ (4.53)

and
$$F_{4,p_r}^{(4)} = \frac{N}{2\pi} \int_0^{2\pi/N} (1/96) Q\beta_x^3 \frac{R}{B_o\rho} \left(\frac{\partial^3 B_z}{\partial x^3}\right)_o e^{-ip_r N\psi} \, d\psi \quad .$$

Given the distance to the resonance, the allowed octupole field can be calculated. For IKOR ($Q_x \simeq 3.25$) we consider as examples:

- the influence of lumped octupoles - one in each unit cell at the position of the sextupole magnet, see fig. 1.3 - on the resonance $4Q=11$ $(Q-11/4\simeq 0.5)$,
- the influence of a 13th harmonic component of an imperfection in only one of the lumped octupoles on the resonance $4Q=13$ $(Q-13/4\simeq 0)$,
- the influence of an octupolar imperfection in all F-quadrupoles on the resonance $4Q=11$.

The results are listed in table 4.2.

Table 4.2
Requirements for the octupole fields $B'''=(\partial^3 B_z/\partial x^3)_o$ in IKOR due to the resonances $4Q=11$ $(Q-11/4=0.5)$ and $4Q=13$ $(Q-13/4\simeq 0)$; δ is a relative octupole field deviation.

resonance	p_r	N	octupole field	$F_{4,0}^{(0)}$	$F_{4,p_r}^{(4)}$	field requirements B''' in T/m^3
4Q = 11	1	11	lumped	$170B'''$	$29B'''$	$B'''<-395$ or $B'''>-194$
4Q = 13	13	1	imperfect. in lumped	$170B'''+15\delta B'''$	$2.6\delta B'''$	$\delta<-17.3$ or $\delta>-8.5$
4Q = 11	1	11	imperfect. in F-quads	$215B'''$	$36B'''$	$B'''<-310$ or $B'''>-154$

The result from the upper line in table 4.2 leads, together with (4.52), to an achievable tune shift $\Delta Q < -0.32$ and $\Delta Q > -0.15$. In practice the needed Q spread in the beam will be of the order of 10^{-3} to 10^{-2} and the octupoles will not give any problems with respect to the resonance $4Q=11$.

The tolerances in the lumped octupoles are so weak that this aspect hardly needs any attention (second result table 4.2).

The requirements for the field quality of the F-quadrupoles with respect to the octupole component is not stringent either: the field on the pole tip should fulfil the condition

$$\left|\frac{B_{oct,pole}}{B_{quad,pole}}\right| \simeq \frac{1}{6}\, a^2 \left|\left(\frac{\partial^3 B_z}{\partial x^3}\right)_{oct} / \left(\frac{\partial B_z}{\partial x}\right)_{quad}\right| \leq 0.09 \qquad (4.54)$$

with $2a=0.21$ m the diameter of the quadrupole aperture and $(\partial B_z/\partial x)_{quad} = 3$ T/m (see table 1.1).
It is clear that in practice this requirement (4.54) can be fulfilled without any problems.

4.4.3 "Second order" tune change

An amplitude-dependent Q change affects the stability limits and is in first order theory predicted as caused by terms of even degree only. However, also terms of odd degree produce a Q change depending on the amplitude. Eq.(4.46) shows that the second order tune change is proportional to the square of the Fourier component $F^{(m)}_{n,p}$.
In general the function $F^{(m)}_n$ contains sextupole (n=3) or octupole fields (for n=4) but also terms arising from the bending magnets, fringing fields etc. (see (1.19)). Taking into account all these contributions leads to mixed terms and subsequently to an expression which is quite unmanageable. As we are particularly interested in the role of sextupole and octupole magnets, only these terms are considered in detail.

When there is no resonance in third degree, the second order Q change due to sextupoles can be written as (using (4.23) and (4.46))

$$\Delta Q = 54\ Q\bar{J} \sum_p F^{(3)}_{3,p}\ F^{(3)}_{3,-p} \left(\frac{1}{(pN)^2 - 9Q^2} + \frac{1}{(pN)^2 - Q^2}\right) \qquad 4.55)$$

with the Fourier components $F^{(3)}_{3,p}$ defined in (4.48).

When there is a resonance $3Q=p_rN$ in the third degree, the resonant terms must be excluded in the expression of (4.55) and the result becomes

$$\Delta Q = 54\ Q\bar{\bar{J}} \sum_{p\neq\pm p_r} F^{(3)}_{3,p}\, F^{(3)}_{3,-p}\left(\frac{1}{(pN)^2 - 9Q^2} + \frac{1}{(pN)^2 - Q^2}\right) +$$

$$+\ 18\ \bar{\bar{J}}\ F^{(3)}_{3,p_r}\ F^{(3)}_{3,-p_r}\left(\frac{6Q}{(p_rN)^2 - Q^2} - \frac{1}{p_rN + 3Q}\right) \quad . \qquad (4.56)$$

The sextupoles for chromaticity control in PAMPUS lead to a maximum second order Q change of about $\Delta Q=7.10^{-6}$ at $Q_x=2.25$ and $\Delta Q=-8.10^{-4}$ at $Q_x=6.25$ (see Cor 80a). The order of magnitude of these numbers will be characteristic of the majority of common electron storage rings.

For IKOR the sextupoles (strength 1.15 T/m^2) give a maximum second order tune change of $\Delta Q=-1.6\ 10^{-3}$. This rather "large" value is both due to the high β_x value at the sextupole position and to the large beam emittance.

There is no such second order Q change due to sextupoles in the vertical plane, since we have only one orientation of sextupoles in the machine. So-called skew sextupoles will lead to such a Q change vertically and of course there is also a vertical tune change due to coupling effects.

Quite similarly one can derive the second order tune change due to the octupoles. Using (4.46) with n=4 and (4.24), we find:

$$\Delta Q = 192\ Q\bar{J}^2 \sum_{p} F^{(4)}_{4,p} F^{(4)}_{4,-p}\left(\frac{1}{(pN)^2 - 16Q^2} + \frac{4}{(pN)^2 - 4Q^2}\right) \qquad (4.57)$$

where the Fourier components $F^{(4)}_{4,p}$ are defined in (4.53).
When there is a resonance, the resonant terms must again be excluded, similar to (4.56).

For PAMPUS this second order tune change is about 10^3 times smaller than the first order one. In common electron storage rings and synchrotrons only the first order effects of the octupoles have to be considered.

4.4.4 Resonance 4Q=N excited by a sextupole field

Higher order effects of sextupoles or octupoles may manifest themselves in e.g. resonance effects in higher degree terms (see section 4.3.6). As an illustration we consider the influence of sextupole fields on the fourth order resonance 4Q=N. Furthermore, we assume no resonance in the third degree.
The Hamiltonian of interest is ((4.47) with n=3):

$$\bar{\bar{K}} = \left(Q - \frac{N}{4}\right)\bar{\bar{J}} + \left(F^{(0)}_{4,0} + \frac{3}{2}A_{0,0}\right)\bar{\bar{J}}^2 + \sum_{m_r}\sum_{p_r}\left(F^{(m_r)}_{4,p_r} + \frac{3}{2}A_{m_r,p_r}\right)e^{im_r\bar{\bar{\phi}}}\,\bar{\bar{J}}^2 \quad . \tag{4.58}$$

$$m_r=4, p_r=1; \quad m_r=-4, p_r=-1$$

When we take into account the octupole fields only in the original fourth degree term in the Hamiltonian, the Fourier components $F^{(0)}_{4,0}$ and $F^{(4)}_{4,\pm1}$ are defined in (4.50) and (4.53). The term A_{m_r,p_r} in (4.58) originates from the lower degree terms and contains the sextupole field via the relations

$$A_{\pm4,\pm1} = \sum_p \pm 3F^{(3)}_{3,p}\,F^{(3)}_{3,\pm1-p}\left(\frac{3}{pN \mp 3Q} + \frac{1}{pN \mp Q}\right) \tag{4.59}$$

with the Fourier components $F^{(3)}_{3,p}$ defined in (4.48).

Examination of the Hamiltonian (4.58) again leads to a condition to avoid unstable motion, similar to the procedure outlined in section 4.3.3.
The influence of the sextupoles and octupoles on the resonance 4Q=N is studied by comparing the magnitudes of the coefficients $A_{\pm4,\pm1}$ and $F^{(\pm4)}_{4,\pm1}$. Assuming an octupole strength that produces a tune change of 2.10^{-3} for PAMPUS at $Q \simeq 2$, the influence of the chromaticity-sextupoles on the resonance 4Q=8 is about a factor 10^2 smaller than the used Landau octupoles (see Cor80a).
In common electron storage rings the second order effects of the sextupoles will be of minor importance.

4.4.5 Remarks on the "second order" effects

We may conclude that the "second order" effects of the non-linear fields are, in general, small and negligible for machines like IKOR and common electron storage rings.

It seems justified to treat the coupled betatron motion by keeping the resonant terms in the original Hamiltonian only, i.e. transformations to remove the fast oscillating parts are not performed.

Finally, we note that non-linear fields which are larger by an order of magnitude may be necessary for stability control in futuristic accelerators (Don77). In that case one is obliged to consider effects which are of "second order" in the strength of these non-linear fields.

It is not allowed to apply this theory when the "second order" effects are comparable or even larger than the "first order" ones.

CHAPTER 5 TWO-DIMENSIONAL NON-LINEAR BETATRON MOTION

5.1 Introduction

The results of the preceding chapter showed that in the majority of common synchrotrons and storage rings, the one-dimensional betatron resonances can be studied accurately by only taking into account the constant and slowly varying terms in the initial Hamiltonian.
This procedure will be applied in this chapter to study non-linear coupled betatron motions in synchrotrons and storage rings.
A treatment of the resonances should again result in requirements for the distance to the resonance line or for the strength of the non-linearity, in order to "guarantee" stable motion.

Guignard has recently worked out a general treatment of sum and difference resonances (Gui76, Gui78). He remarked that the phase space representation is not suitable for the discussion of two-dimensional betatron resonances. Therefore he generalized the method with the so-called "resonance curves" as introduced earlier by Hagedorn and Schoch for the description of one-dimensional resonances (Hag57, Scho57).
However, the remedy to get round the "difficulty" of the complicated four-dimensional phase space has already been mentioned in chapter 1: the Hamiltonian can be simplified to one describing a one-dimensional problem by the transformation (1.32). The resulting Hamiltonian enables us to study trajectories in a phase plane and this will lead to a good insight in the resonances $m_1Q_x + m_2Q_z = p_rN$.

This theory offers the possibility to evaluate the importance of a non-linear magnetic field, to judge the necessity to compensate it and finally to calculate at what distance from resonance the working point should be, in order to keep the motion stable or to keep the beam blow-up within given limits. Further expressions for the tune changes - depending on the betatron amplitudes - are given.

We should like to report in advance that in some special cases - namely the resonances with $m_1 = 1$ or $m_2 = 1$ - the results of our treatment show qualitative discrepancies with the theory of Guignard. We will return to this subject in due course.

In section 5.2 and 5.3 we derive a general Hamiltonian which is suitable for the study of two-dimensional betatron resonances. In section 5.3 some aspects of Guignard's theory are discussed too. Our description, using phase plane representations, is illustrated in section 5.4 and 5.5 for the third order sum resonances and in section 5.6 results for IKOR are given.
The amplitude-dependent change of the tunes, due to octupole fields, is examined in section 5.7 and moreover, the limiting effect of octupole fields on the resonant character will be shown in section 5.8. Finally, we treat the fourth order difference resonance $2Q_x-2Q_z=p_rN$. This resonance with $p_r=0$ might be of interest in electron storage rings which are often operated with nearly equal tunes. Usually the difference resonances are ignored because the oscillation amplitudes always remain finite. However, the amplitudes may grow to unwanted values. A formula is derived, which indicates a condition or recommendation in order to avoid these large amplitudes (section 5.9).

5.2 Hamiltonian for transverse motions

To study the betatron resonances we start with the time-independent Hamiltonian. Generally, its non-linear part is a polynomial of higher than second degree in the variables x, p_x, z, p_z with coefficients which are functions of θ (see (1.19)). The kinematical terms are usually small (see also chapter 4) and since we are especially interested in the effects of sextupole and octupole fields we only take these into account. In that case the non-linear part of the Hamiltonian only contains the variables x and z. With this assumption the notation is much more simplified and it is not a restriction of the subsequent theory (see Cor80b).

We write the Hamiltonian (1.19) now as

$$H = \frac{1}{2}\,p_x^2 + \frac{1}{2}(\varepsilon^2 - n)x^2 + \frac{1}{2}\,p_z^2 + \frac{1}{2}\,nz^2 + \sum_{\substack{k,l \\ k+l\geq 3}} h_{k,l}(\theta)x^k z^l \qquad (5.1)$$

For simplicity we omitted the bars above the variables.

Next, the elimination of the θ-dependence in the quadratic part of (5.1) is achieved by the transformation to action and angle variables according to (2.30).

To study the resonance $m_1Q_x+m_2Q_z=p_rN$, we restrict ourselves to the low frequency part of the Hamiltonian. On the one hand this includes all terms with zero frequency (constant terms, they appear only in even degree) and on the other hand the terms with very low frequency associated with the condition $m_1Q_x+m_2Q_z=p_rN$.
Using complex exponentials, the Hamiltonian is

$$K = Q_xJ_x + Q_zJ_z + \sum_{\substack{k\ \text{even} \\ l\ \text{even}}} F^{(0,0)}_{k,l,0}\, J_x^{\frac{k}{2}} J_z^{\frac{l}{2}} + \left(F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}\, e^{i(m_1\phi_x+m_2\phi_z+p_rN\theta)} + \text{c.c.}\right) J_x^{\left|\frac{m_1}{2}\right|} J_z^{\left|\frac{m_2}{2}\right|} \tag{5.2}$$

in which the non-linear field is expanded in a Fourier series:

$$F^{(m_1,m_2)}_{k,l,p} = \frac{N}{2\pi}\int_0^{2\pi/N} F^{(m_1,m_2)}_{k,l}(\theta)\, e^{-ipN\theta}\, d\theta$$

with

$$F^{(m_1,m_2)}_{k,l}(\theta) = a^{(m_1)}_{k,0}\, a^{(m_2)}_{l,0} \left(\frac{2}{R}\right)^{\frac{|m_1|+|m_2|}{2}} \beta_x^{\frac{k}{2}}\beta_z^{\frac{l}{2}}\, h_{k,l}(\theta) e^{i(m_1\Psi_x+m_2\Psi_z)} \tag{5.3}$$

with $\Psi_{x,z}(\theta)$ defined in (2.28) and analogously to (4.21):

$$\cos^k\phi = \sum_m a^{(m)}_{k,0}\, e^{im\phi} \quad .$$

In analogy with the one-dimensional case there is an amplitude-dependent tune change, provided k and l are both even numbers. These tune changes satisfy the relations:

$$\Delta Q_x = \sum_{\substack{k,l\ \text{even} \\ k\geq 2}} \frac{k}{2} F^{(0,0)}_{k,l,0}\, J_x^{\frac{k-2}{2}} J_z^{\frac{l}{2}}$$

$$\Delta Q_z = \sum_{\substack{k,l\ \text{even} \\ l\geq 2}} \frac{l}{2} F^{(0,0)}_{k,l,0}\, J_x^{\frac{k}{2}} J_z^{\frac{l-2}{2}} \quad . \tag{5.4}$$

From the equations of motion, $dJ_{x,z}/d\theta=\partial K/\partial\phi_{x,z}$ and $d\phi_{x,z}/d\theta=-\partial K/\partial J_{x,z}$, it can be shown that these tune changes may tend to limit the build-up of amplitude, thus "stabilizing" or "limiting" the resonant behaviour.

In section 5.8 we will illustrate this limiting effect due to octupole fields. In the next section we evaluate the resonant Hamiltonian.

5.3 Resonances in non-linear coupled betatron oscillations

The resonant Hamiltonian (5.2) can now be transformed to an equivalent one, describing a one-dimesional problem. This type of transformation has been illustrated in chapter 1 (see (1.32) and (1.33)). The resulting Hamiltonian is

$$K = \delta Q J_2 + 2\left|F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}\right| \left(J_1 + \frac{m_1}{m_2}J_2\right)^{\left|\frac{m_1}{2}\right|} J_2^{\left|\frac{m_2}{2}\right|} \cos(m_2\phi_2+\theta_F)$$

$$+ \sum_{\substack{k,l \\ \text{even}}} F^{(0,0)}_{k,l,0} \left(J_1 + \frac{m_1}{m_2}J_2\right)^{\frac{k}{2}} J_2^{\frac{l}{2}}$$

with (5.5)

$$\delta Q = \frac{1}{m_2}(m_1 Q_x + m_2 Q_z - p_r N)$$

$$J_1 = J_x - \frac{m_1}{m_2}J_z \qquad \phi_1 = \phi_x + Q_x\theta$$

$$J_2 = J_z \qquad \phi_2 = \phi_z + \frac{m_1}{m_2}\phi_x + \frac{p_r N}{m_2}\theta$$

and $F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r} = \left|F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}\right| e^{i\theta_F}$.

As it has no essential consequences for the subsequent description we fix $\theta_F=0$.

Since we are mostly interested in the growth of the oscillation amplitudes and do not care so much about the absolute value of the phases, we haven chosen a new relative phase ϕ_1 which is a cyclic variable: $dJ_1/d\theta = 0$. The constant J_1 is completely determined by initial conditions.

The meaning of δQ in (5.5) is illustrated in figure 5.1.

Using (5.5), trajectories in a phase plane with as polar coordinates e.g. $\sqrt{2J_2}$ and ϕ_2 can be calculated from the equations of motion. In general, there will be a stable and an unstable region. Once the separatrix has been found, the required distance to the resonance can be evaluated by putting a circle within the boundary, analogously to the treatment in chapter 4. We remind the reader that the beam is represented by a circle in this phase plane when the non-linear field is turned off.

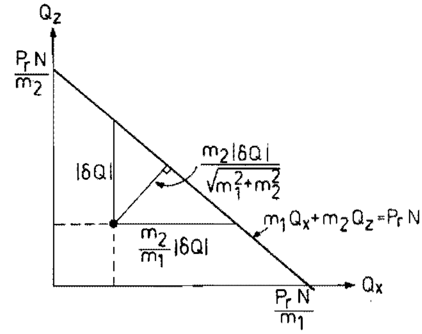


Figure 5.1
Illustration of the meaning of $\delta Q = (m_1 Q_x + m_2 Q_z - p_r N)/m_2$ in the (Q_x, Q_z) working diagram.

Before starting the examination of the phase plane, we want to emphasize a consequence of the choice of the new variables in (5.5). Because of the definition in (5.5), the constant of the motion J_1 can be negative in case of a sum resonance only.
Since

$$J_x = J_1 + \frac{m_1}{m_2} J_2 \geq 0 \ , \qquad (5.6)$$

a negative value of J_1 results in an unphysical region in the phase plane, namely the area inside the circle with radius $\sqrt{2J_2} = \sqrt{-2\frac{m_2}{m_1} J_1}$. Particles on this circle have a horizontal oscillation amplitude equal to zero (J_x=0).
In case of a difference resonance, J_1 is always positive and (5.6) results in an unphysical region outside the circle with radius $\sqrt{2J_2} = \sqrt{-2\frac{m_2}{m_1} J_1}$. Both cases are sketched in figure 5.2.

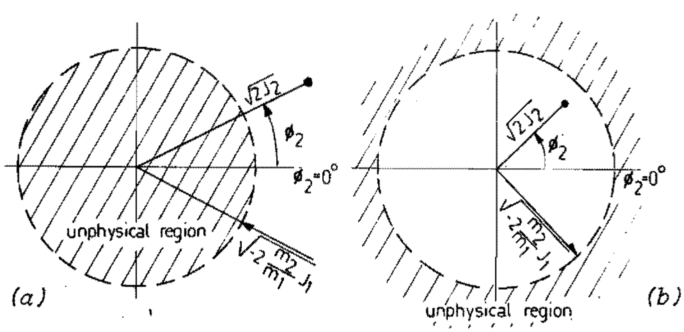


Figure 5.2 Physical and unphysical regions in the phase plane: (a) sum resonance, (b) difference resonance.

It will turn out to be very convenient to define a parameter q for the description of the resonance problem in the phase plane:

$$q = \frac{\delta Q}{\left|F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}\right|} \quad . \tag{5.7}$$

The fixed points in the phase plane may again provide some understanding. They are characterized by the conditions $\dot{J}_2=0$ and $\dot{\phi}_2=0$ ($'=d/d\theta$). Ignoring the constant terms in the Hamiltonian (5.5) and returning to the variables J_x and J_z, we find the following condition fulfilled by the fixed points:

$$2|m_1Q_x+m_2Q_z-p_rN| = 2\left|F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}\right| J_x^{\left|\frac{m_1}{2}\right|-1} J_z^{\left|\frac{m_2}{2}\right|-1} \{m_1^2J_z+m_2^2J_x\} \quad . \tag{5.8}$$

Substitution of the horizontal and vertical emittances ε_x and ε_z for the variables J_x, J_z according to $J_{x,z}=\frac{1}{2}\varepsilon_{x,z}/R$ - see (2.16) - results in the formula which Guignard has defined as "the" stopband width. This derivation seems more transparent than the one given by Guignard (Gui76, Gui78).
However, we emphasize that (5.8) yields both stable and unstable fixed points. This may indeed have important consequences for the stopband definition. A good insight in the behaviour and the positions of the stable and unstable fixed points in the phase plane is required.
As a consequence of not discriminating between the stable and the unstable fixed points, the formula of Guignard shows an inaccuracy in some special cases: for $m_1=1$ (or $m_2=1$) the stopband width increases infinitely when J_x (J_z) approaches zero whilst J_z (J_x) is fixed. This means that particles in the central area of a beam can then become "more unstable" compared to those in the outer area, which is hard to understand. This peculiar result has recently been reported in (Cor81c) and also by Ohnuma (Ohn81).

5.4 $2Q_x + Q_z = p_rN$ excited by skew sextupole fields

As an illustration of the study of coupled resonances, we treat the resonance $2Q_x+Q_z=p_rN$ (i.e. $m_1=2$ and $m_2=1$). It may be of interest for IKOR since $Q_x\simeq3.25$, $Q_z\simeq4.4$ and $N=11$ so that $2Q_x+Q_z-11\simeq-0.1$. The above-mentioned resonance can be excited by a x^2z-term in the

Hamiltonian (5.1) which can arise from skew sextupole fields $(\partial^2 B_x/\partial x^2)_o$. These fields may be due to imperfections in the guiding field or, for instance, due to a rotational error χ of the normal sextupoles $(\partial^2 B_z/\partial x^2)_o$ which are usually installed in a machine:

$$\left(\frac{\partial^2 B_x}{\partial x^2}\right)_o = -4\chi\left(\frac{\partial^2 B_z}{\partial x^2}\right)_o \quad . \tag{5.9}$$

The third degree Hamiltonian is now (see (5.5)):

$$K = \delta Q J_2 + 2\left|F^{(2,1)}_{2,1,p_r}\right|(J_1 + 2J_2)J_2^{\frac{1}{2}} \cos\phi_2$$

$$\text{with } J_1 = J_x - 2J_z \ , \quad J_2 = J_z \tag{5.10}$$

$$\delta Q = 2Q_x + Q_z - p_r N$$

$$\text{and} \quad F^{(2,1)}_{2,1,p_r} = \frac{N}{2\pi}\int_0^{2\pi/N} \frac{1}{8}\sqrt{2}\ \beta_x \beta_z^{\frac{1}{2}} R^{-3/2}\ S_{skew}\ e^{i(2\Psi_x + \Psi_z - p_r N\theta)}\ d\theta$$

in which S_{skew} is the "normalized" skew sextupole field defined by

$$S_{skew} = \frac{R^3}{B_o\rho}\left(\frac{\partial^2 B_x}{\partial x^2}\right)_o \quad . \tag{5.11}$$

A return to cartesian coordinates simplifies the description of the behaviour of the fixed points. The qualitative behaviour of these points in the phase plane with polar coordinates $\sqrt{2J_2}$ and ϕ_2 is schematically sketched in figure 5.3.
When $q^2 < 24J_1$ there is neither an unstable nor a stable fixed point. For $q^2 = 24J_1$ there is one fixed point, splitting up when J_1 decreases (or q increases). Another two fixed points appear when $q^2 < -8J_1$. These latter points both lie on the circle with radius $\sqrt{-J_1}$, being the boundary of the unphysical region (see also fig.5.2).

Corresponding trajectories in the phase plane are shown in figure 5.4 for a fixed value of q and different values of J_1. From these considerations it is obvious that a stable region exists if and only if

$$-\frac{1}{8}q^2 < J_1 < \frac{1}{24}q^2 \quad . \tag{5.12}$$

Analogously to the one-dimensional case we are now interested in the required value of δQ or in the allowed field strength (value of $F^{(2,1)}_{2,1,p_r}$) in order to avoid unstable motion.

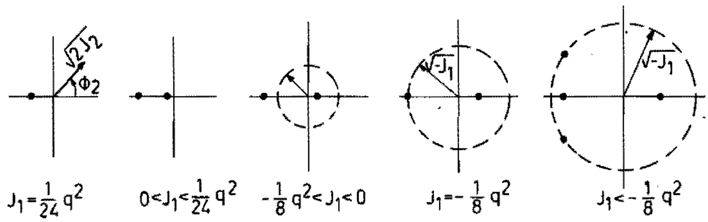


Figure 5.3 Behaviour of the fixed points for $2Q_x+Q_z=p_rN$. The dashed circles indicate the boundary of the unphysical region.

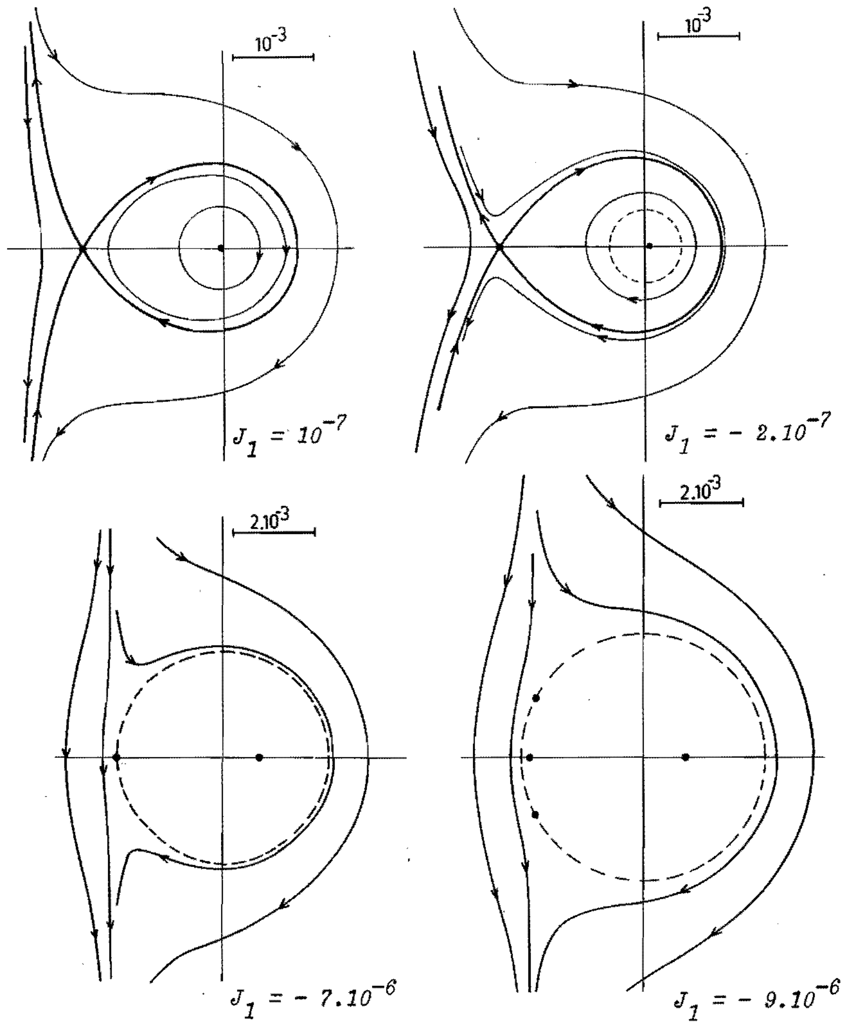


Figure 5.4 Flowlines in the $(\sqrt{2J_2},\phi_2)$ phase plane for the resonance $2Q_x+Q_z=p_rN$ for different values of J_1 and $q=7.5\ 10^{-3}$ ($\delta Q=7.5\ 10^{-2}$ and $F^{(2,1)}_{2,1,p_r}=10$).

Since the separatrix is known, the maximum allowed emittance - i.e. the "beam circle" must lie entirely inside the stable region - can be expressed in the coordinates of the unstable fixed point now being a function of J_1 and q. This procedure results in curves in a J_x, J_z diagram with q as the parameter. For the resonance considered the result is plotted in figure 5.5. When the horizontal and vertical emittances (corresponding to maximum transverse amplitudes) are given, the required value of q can easily be found.

As an illustration we consider IKOR with $\varepsilon_x=3\varepsilon_z=1.5\ 10^{-4}$ m.rad leading to $J_x=3J_z=2.3\ 10^{-6}$ and consequently $q \geq 1.06\ 10^{-2}$.
In the foregoing consideration we examined a case with q positive. A similar argument also serves for negative q and in principle the condition $|q| \geq 1.06\ 10^{-2}$ should be fulfilled in IKOR. In section 5.6 we will work out this result and its consequences for the allowed rotational errors or field imperfections.

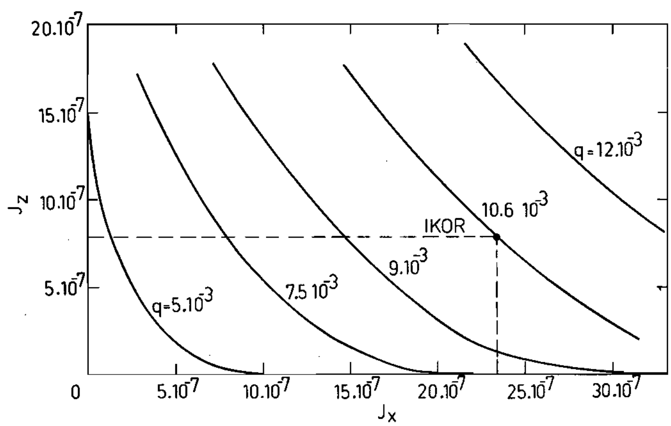


Figure 5.5 *Curves for constant values of q in the J_x, J_z diagram for the resonance $2Q_x+Q_z=p_rN$. For IKOR there must hold $q \geq 10.6\ 10^{-3}$.*
Note that $J_x(J_z=0) < \frac{1}{24}\, q^2$ and $J_z(J_x=0) < \frac{1}{16}\, q^2$.

Last but not least we emphasize the fact that the value of the minimum required q diminishes monotonically when J_z decreases ($J_z \to 0$) and J_x is constant, which is in accordance with the physical point of view. The difference with the formula of Guignard can be explained by the fact that in his definition of stopband width (see (5.8)) for constant

ε_x and decreasing ε_z, the nature of the fixed point changes from unstable to stable.

5.5 $Q_x + 2Q_z = p_r N$ excited by normal sextupole fields

The resonance $Q_x+2Q_z=p_rN$ ($m_1=1$, $m_2=2$) can be excited by normal sextupole fields in the accelerator (xz^2-term in the Hamiltonian (5.1)) and is examined in a similar way as the previous one.

For this case the Hamiltonian (5.5) is

$$K = \delta Q J_2 + 2|F^{(1,2)}_{1,2,p_r}|(J_1 + \tfrac{1}{2}J_2)^{\frac{1}{2}} J_2 \cos 2\phi_2$$

with $J_1 = J_x - \frac{1}{2}J_z$, $J_2 = J_z$ (5.13)

$$\delta Q = \frac{1}{2}(Q_x + 2Q_z - p_r N)$$

and $F^{(1,2)}_{1,2,p_r} = \frac{N}{2\pi}\int_0^{2\pi/N} \frac{1}{8}\sqrt{2}\, \beta_x^{\frac{1}{2}}\beta_z R^{-3/2}\, S\, e^{i(\Psi_x + 2\Psi_z - p_r N\theta)}\, d\theta$

in which S is the "normalized" sextupole field defined in (1.19).

For the resonance $Q_x+2Q_z=p_rN$ the trajectories in the phase plane with polar coordinates $\sqrt{2J_x}$ and $2\phi_2$ are qualitatively the same as the trajectories for $2Q_x+Q_z=p_rN$ in the phase plane with polar coordinates $\sqrt{2J_2}=\sqrt{2J_z}$ and ϕ_2 which are sketched in figure 5.4.
Examination of the phase plane shows that a stable region exists if and only if

$$-\frac{1}{12}q^2 < J_1 < \frac{1}{4}q^2 \quad . \tag{5.14}$$

Analogously to the previous section we determine the curves for constant q-values in the J_x,J_z diagram. The result is depicted in figure 5.6 and in principle the problem of the required distance to the resonance or allowed sextupole strength is solved. As an example the situation for IKOR is indicated. As a same reasoning holds for both negative and positive values of q, in IKOR the condition $|q| \geq 4.9\ 10^{-3}$ should be fulfilled.
In the next section we will investigate the consequences of this requirement.

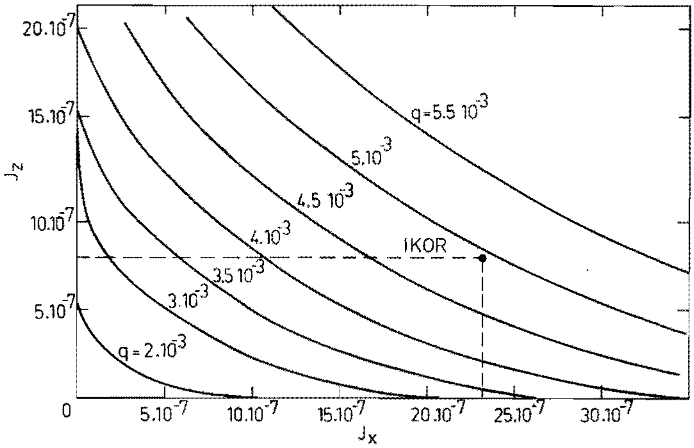


Figure 5.6 *Curves for constant values of q in the J_x,J_z diagram for the resonance $Q_x+2Q_z=p_rN$. For IKOR there must hold $q \geq 4.9\ 10^{-3}$.*
Note that $J_x(J_z=0)<\frac{1}{4}\,q^2$ and $J_z(J_x=0)<\frac{1}{6}\,q^2$.

5.6 Applications

In this section the consequences of the results of the preceding sections are calculated for the lattice of IKOR.

We examine the effects of the normal sextupoles used for the control of γ_{tr} (strength 1.15 T/m^2 , see section 4.2) and of skew sextupole fields due to a rotational error of a lumped sextupole and due to imperfections in non-ideal dipoles.
The results of the figures 5.5 and 5.6 lead to a required δQ-value or to tolerances for the magnetic field. These results are listed in table 5.1.

Table 5.1

Effects of sextupole fields - normal and skew - on the third order sum resonances for IKOR : $Q_x \approx 3.25$, $Q_z \approx 4.4$.

$2Q_x + Q_z = 11 + \delta Q$: $\delta Q = -0.1$ *and* $|q| \geq 10.6 \; 10^{-3}$.

$Q_x + 2Q_z = 11 + 2\delta Q$: $\delta Q = 0.525$ *and* $|q| \geq 4.9 \; 10^{-3}$.

The normal sextupole field is indicated by B'' *and the skew sextupole field by* B''_{skew}.

resonance	p_r	N	excitation	$F^{(m_1,m_2)}_{\lvert m_1\rvert,\lvert m_2\rvert,p_r}$	required δQ	tolerance B'' in T/m^2
$Q_x+2Q_z=11$	1	11	lumped sext. 1.15 T/m^2	2.1	0.01	-
$Q_x+2Q_z=11$	1	11	dipole imperf.	$14.6B''$	-	$\lvert B''\rvert < 7.2$
$2Q_x+Q_z=11$	11	1	rot. error χ	2.2χ	-	$\chi < 4$ rad
$2Q_x+Q_z=11$	1	11	dipole imperf.	$40.0B''_{skew}$	-	$\lvert B''\rvert_{skew} < 0.25$

The lumped sextupoles - both the strength and a rotational error - will not give rise to problems with respect to the resonances considered.

However, a systematic skew sextupole component in each dipole may be harmful. When

$$\Delta B_z(x,z) = \left(\frac{\partial^2 B_x}{\partial x^2}\right) xz \quad ,$$

with ΔB_z the vertical field deviation from the nominal field, the last requirement of table 5.1 results in an allowed imperfection of 10^{-4} T at x=4 cm, z=1 cm.

It might be useful to place an extra skew sextupole in the machine to cancel this critical dipole effect. A possible position for a correction element is between the bending magnet and the F-quadrupole (see fig.1.3, free space 0.5 m). Assuming only one such element in the whole machine with a length of 0.3 m, its strength should be about 100 times larger than the skew sextupole component in the dipoles. When the dipole imperfection is completely due to a skew field and is e.g. 2.10^{-4} T at x=4 cm, z=1 cm, the needed skew sextupole should have a strength of about 50 T/m^2.

A pole diameter of 20 cm then leads to a field of 0.25 T on the poles.

Finally, we calculate the required value of δQ for the upper case of table 5.1 using the formula of Guignard. Eq.(5.8) with $m_1=1$, $m_2=2$, $F^{(1,2)}_{1,2,1} = 2.1$ and substitution of $J_{x,z} = \frac{1}{2}\epsilon_{x,z}/R$ leads to $\delta Q = 0.007$. This somewhat smaller value of δQ compared to the result of table 5.1 is due to Guignard's definition of the stopband width: it is related to the fixed points, whereas in our definition the beam must lie entirely within the stable region. See, in illustration of this, fig.4.2 for the one-dimensional analogue.

5.7 Amplitude-dependent tune change due to octupole fields

Fourth degree terms in the original Hamiltonian (5.1) produce an amplitude-dependent tune change (when k and l are even) which is (see (5.4)):

$$\begin{aligned} \Delta Q_x &= 2F^{(0,0)}_{4,0,0}\, J_x + F^{(0,0)}_{2,2,0}\, J_z \\ \Delta Q_z &= 2F^{(0,0)}_{0,4,0}\, J_z + F^{(0,0)}_{2,2,0}\, J_x \quad . \end{aligned} \tag{5.15}$$

Taking only into account the octupole fields, the relevant Fourier components are given by (see (5.3)):

$$F^{(0,0)}_{k,l,0} = [-\tfrac{1}{4}][\tfrac{1}{16}]\left(\frac{R^2}{B_o\rho}\right)\int_0^{2\pi} \beta_x^{\frac{k}{2}}\beta_z^{\frac{l}{2}}\left(\frac{\partial^3 B_z}{\partial x^3}\right)_o d\theta \tag{5.16}$$

in which the factor 1/16 between the square brackets holds for k=4 or l=4 and the factor -1/4 for k=l=2.
From these equations above, it is obvious that the octupoles are most effective on positions with large amplitude functions $\beta_{x,z}$.

In electron storage rings - in which the maximum vertical amplitude is often much smaller than the maximum horizontal one - thus $J_z \ll J_x$ - the coupling between the transverse motions is more pronounced for the vertical tune change.
Consequently the results for PAMPUS of figure 4.5 (horizontal tune change) do not change much due to the coupling. Numerical results for the vertical tune change are given in (Cor80b).

As an illustration for IKOR, we assume two octupoles with a length of 0.4 m in each unit cell: one between the quadrupoles F and D1 (same position as the sextupole for γ_{tr} control) and another one just in front of the D2 quadrupole (see lattice in fig.1.3).
The values of $\beta_{x,z}$ at these positions are (see fig.2.3):
β_x = 27 m/rad and β_z = 2.5 m/rad between F and D1, whereas
β_x = 4.1 m/rad and β_z = 37 m/rad just in front of D2.
For simplicity the $\beta_{x,z}$ functions are assumed to be constant in the octupoles.

Substitution of the emittances into (5.15), i.e. $J_{x,z}=\frac{1}{2}\varepsilon_{x,z}/R$, leads to the maximum tune change in the beam as a function of the octupole fields. In matrix form the result is

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_z \end{pmatrix} = \begin{pmatrix} 73 & -9 \\ -15 & 17 \end{pmatrix} \cdot 10^{-5} \begin{pmatrix} B_F''' \\ B_{D2}''' \end{pmatrix} \tag{5.17}$$

in which B_F''' and B_{D2}''' represent the octupole field (in T/m^3) at the two positions, respectively.
By placing an octupole at a position with high β_x-value and another one at high β_z-value, the horizontal and vertical tune change can be controlled almost separately.
The required strength of the octupoles, to achieve a certain tune spread in the beam, can easily be obtained by inverting (5.17).
We mention that a maximum horizontal and vertical tune change of 10^{-2} requires the octupole fields to be B_F'''=23 T/m^3 and B_{D2}'''=80 T/m^3.
It is obvious that the sign of the tune change - which might be of interest in relation to other resonances - has a considerable influence on the needed octupole fields.

Finally, we make two remarks.
Firstly, the result for ΔQ_x in (5.17) agrees quite well with the result in chapter 4 - in (4.54) the betatron coupling has been ignored - when we omit the octupole near the quadrupole D2.
Secondly, the tune change ΔQ_x hardly depends on the octupole near the quadrupole D2 as long as its strength does not exceed the strength of the octupole between the F and D1 quadrupole.

As reported before, these tune changes may limit the beam blow-up when resonance occurs. This "limiting" effect will be discussed now.

5.8 Limiting effect of octupole fields

In the discussion on the third degree resonances we only regarded the excitation term $F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}$. This term might give rise to an amplitude growth. We are going to discuss the feature of an amplitude-dependent tune change limiting this increase of the amplitude. An illustration for a one-dimensional case is given in (Hag62).

Considering the resonance $m_1Q_x+m_2Q_z=p_rN$, the Hamiltonian in the variables J_2 and ϕ_2 is (see (5.5))

$$K = \delta Q J_2 + 2\left|F^{(m_1,m_2)}_{|m_1|,|m_2|,p_r}\right| \left(J_1 + \frac{m_1}{m_2}J_2\right)^{\left|\frac{m_1}{2}\right|} J_2^{\left|\frac{m_2}{2}\right|} \cos m_2\phi_2 +$$

$$+ V_{m_1,m_2} J_1 J_2 + W_{m_1,m_2} J_2^2$$

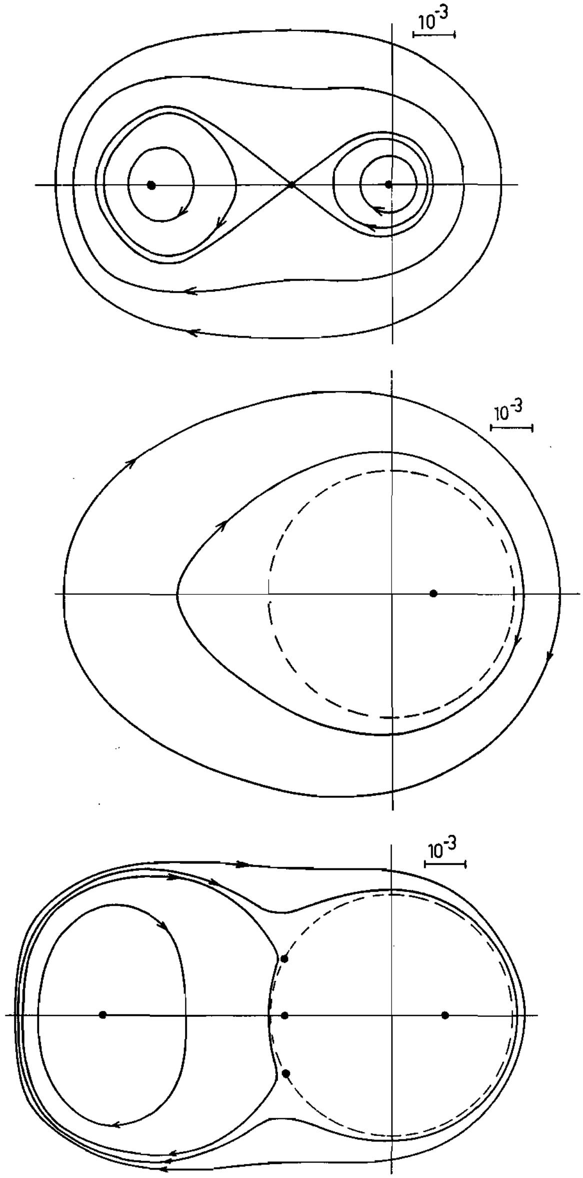
$$\text{with } V_{m_1,m_2} = 2\frac{m_1}{m_2}F^{(0,0)}_{4,0,0} + F^{(0,0)}_{2,2,0} \tag{5.18}$$

$$W_{m_1,m_2} = \frac{m_1^2}{m_2^2}F^{(0,0)}_{4,0,0} + \frac{m_1}{m_2}F^{(0,0)}_{2,2,0} + F^{(0,0)}_{0,4,0}$$

and $F^{(0,0)}_{k,1,0}$ defined in (5.16).

The term in (5.18) with coefficient V_{m_1,m_2} results in a change of the effective value of δQ, whereas the term W_{m_1,m_2} causes the limiting effect. In spite of the presence of octupole fields, there is the unfortunate case of no limiting effect at all in case $W_{m_1,m_2} = 0$.

We treat the influence of the octupoles on $2Q_x+Q_z=p_rN$. This resonance has already been discussed in section 5.4 by omitting the fourth degree terms. Results for different values of $V_{2,1}$ and $W_{2,1}$ are plotted in figure 5.7. There is no longer an unstable region in the sense that the amplitudes can grow infinitely. The upper result in fig.5.7 can be compared with the first plot in fig. 5.4. A certain value of δQ should be preserved also when octupoles are present because amplitudes may undergo a large (although limited) growth. The second and third plot in fig. 5.7 can be compared with the unstable situation of the last plot in fig. 5.4. It is obvious that the values of $V_{2,1}$ and $W_{2,1}$ play an important role on the detailed shape of the phase plane figures.



$J_1 = 10^{-7}$
$V_{2,1} = 0$
$W_{2,1} = 5.10^3$

$J_1 = -\ 9.10^{-6}$
$V_{2,1} = 0$
$W_{2,1} = 5.10^3$

$J_1 = -\ 9.10^{-6}$
$V_{2,1} = 5.10^3$
$W_{2,1} = 5.10^3$

Figure 5.7 *Illustration of the limiting effect of octupoles on the resonance $2Q_x+Q_z=p_rN$ with $q=7.5\ 10^{-3}$ in the $(\sqrt{2J_2}, \phi_2)$ phase plane. The various cases can directly be compared with plots of figure 5.4, in which no octupoles were present. The region within the dashed circle is unphysical.*

5.9 The fourth order difference resonance $2Q_x-2Q_z=p_rN$

As a last example of the study of two-dimensional betatron resonances, we investigate the difference resonance $2Q_x-2Q_z=p_rN=0$. Usually the difference resonances are considered to be stable because of the limited amplitudes, but nevertheless we are interested in it as too large amplitudes may occur.

The resonance $2Q_x-2Q_z=0$ may be important in e.g. electron storage rings which are often operated at nearly equal tunes and the Landau octupoles can excite this resonance. In that case the second order resonance $Q_x-Q_z=0$ - excited by skew quadrupole fields - may be important too. Since this latter resonance has already been discussed by several authors (Bry75, Gui78, Bac79a) it will not be treated here.

The relevant Hamiltonian becomes (from (5.5)):

$$K = \delta Q J_2 + 2|F^{(2,-2)}_{2,2,0}|J_2(J_1 - J_2)\cos 2\phi_2 + V_{2,-2}J_1J_2 + W_{2,-2}J_2^2$$

with $J_1 = J_x + J_z$, $J_2 = J_z$

$$\delta Q = Q_z - Q_x \tag{5.19}$$

$$F^{(2,-2)}_{2,2,0} = -\frac{1}{32\pi}\int_0^{2\pi} \beta_x\beta_z \frac{R^2}{B_o\rho}\left(\frac{\partial^3 B_z}{\partial x^3}\right)_o e^{i(2\Psi_x - 2\Psi_z)}\, d\theta$$

and $V_{2,-2}$, $W_{2,-2}$ both defined in (5.18).

For the moment the examination of the resonance is simplified by putting $V_{2,-2}=0$ and $W_{2,-2}=0$. At the end of this section we will then generalize the results for the case with $V_{2,-2}\neq 0$ and $W_{2,-2}\neq 0$.

The first step in the investigation of the resonance is the calculation of the behaviour of the fixed points. Their qualitative behaviour in the $\sqrt{2J_2},\phi_2$ phase plane is schematically sketched in figure 5.8. The region outside the dashed circles indicates the unphysical region as already explained in section 5.3 (see also figure 5.2).

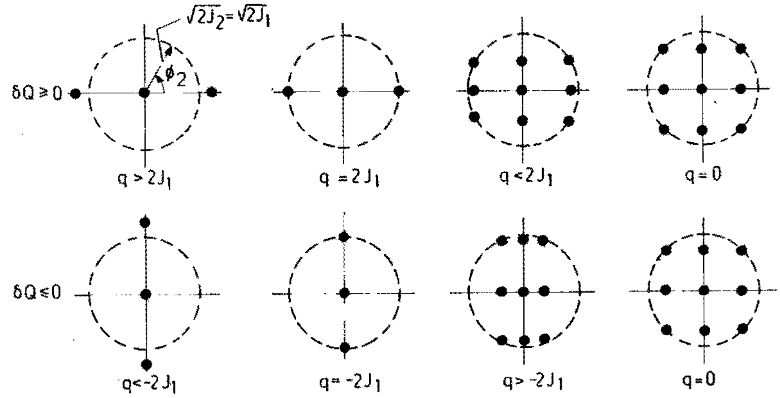


Figure 5.8 *Qualitative behaviour of fixed points in the $\sqrt{2J_2}, \phi_2$ phase plane for the resonance $2Q_x - 2Q_z = 0$. The area outside the dashed circle with radius $\sqrt{2J_2} = \sqrt{2J_1}$ is unphysical; $J_1 = J_x + J_z$, $\delta Q = Q_z - Q_x$ and $q = \delta Q/F$.*

Phase plane trajectories belonging to the situations depicted above are plotted in figure 5.9 for different (positive) values of δQ and constant values of J_1 and $F^{(2,-2)}_{2,2,0}$. The value $J_1 = 9.10^{-7}$ corresponds with the emittances of PAMPUS at $Q_x \approx Q_z \approx 3.25$. The value $|F^{(2,-2)}_{2,2,0}| = 250$ is obtained from Landau octupoles, providing $\Delta Q_x = 3.25\ 10^{-3}$.

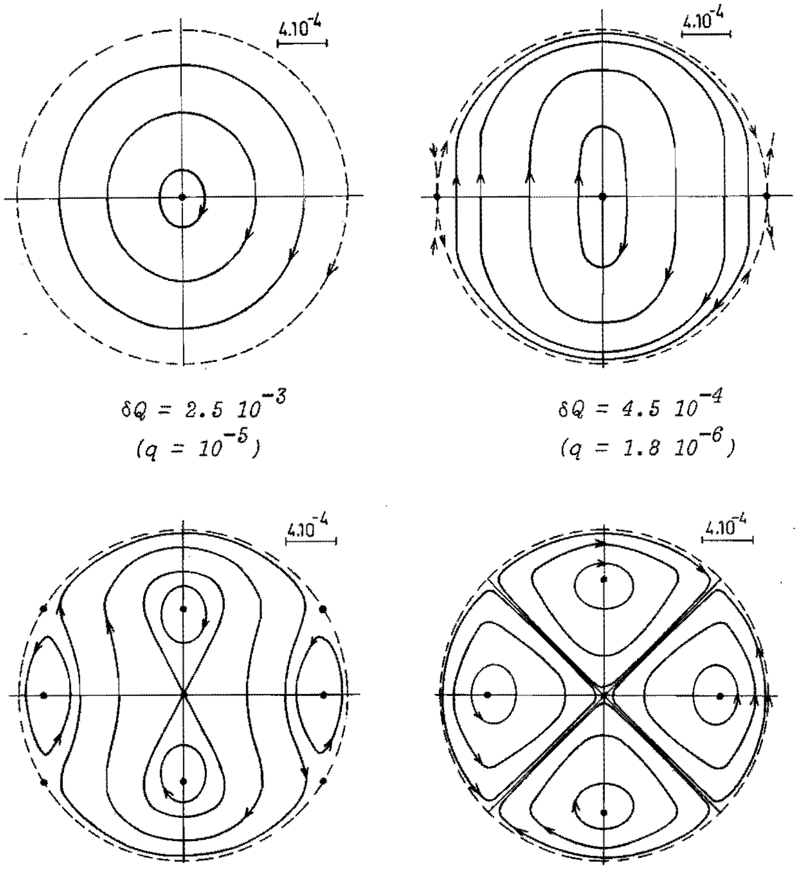
Considering the flowlines in fig. 5.9, it is clear that the two lower cases are less favourable because of the very large amount of energy exchange between the transverse motions. Consequently we recommend a situation with

$$|q| \geq 2J_1 \tag{5.20}$$

The energy exchange - strictly speaking the exchange of amplitude squares - is determined by substituting the extreme values ±1 for the cosine function into the invariant K of (5.19). We obtain the relation:

$$r^2 \mp \left(\frac{|q| + 2J_1}{2J_{z,max}}\right) r \pm \left(\frac{|q| - 2J_1}{2J_{z,max}} + 1\right) = 0 \tag{5.21}$$

with $r = J_{z,min}/J_{z,max}$ and the upper (lower) sign holds for $\delta Q > 0$ ($\delta Q < 0$). $J_{z,min}$ and $J_{z,max}$ are related to respectively the minimum and maximum vertical oscillation amplitude.



$\delta Q = 2.5\ 10^{-3}$ $(q = 10^{-5})$ | $\delta Q = 4.5\ 10^{-4}$ $(q = 1.8\ 10^{-6})$

$\delta Q = 2.0\ 10^{-4}$ $(q = 8.0\ 10^{-7})$ | $\delta Q = 0$ $(q = 0)$

Figure 5.9 Flowlines in the $(\sqrt{2J_2}, \phi_2)$ *phase plane for the resonance* $2Q_x - 2Q_z = 0$; $J_1 = 9.10^{-7}$ *and* $|F^{(2,-2)}_{2,2,0}| = 250$.

As an example the ratio r for PAMPUS at $Q_x \approx Q_z \approx 3.25$ with $J_1 = 9.10^{-7}$ is given in figure 5.10.

In case $J_{z,max} \to 0$ the ratio $r \to \frac{|q| - 2J_1}{|q| + 2J_1}$.

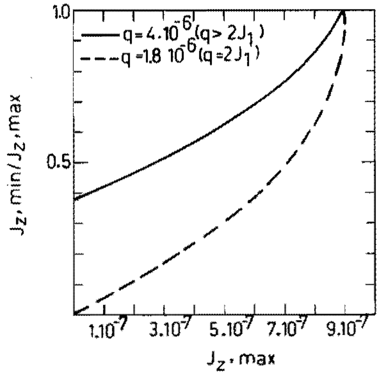


Figure 5.10

The ratio $J_{z,min}/J_{z,max}$ *for PAMPUS at* $Q_x \simeq Q_z \simeq 3.25$, $J_1 = J_x + J_z = 9.10^{-7}$, *for two different values of q:* $q = 4.0\ 10^{-6}$ *and* $q = 1.8\ 10^{-6}$.

As mentioned in the beginning of this section, we now generalize the previous considerations for the case $V_{2,-2} \neq 0$ and $W_{2,-2} \neq 0$. The recommendation - analogous to (5.20) - in order to avoid too large energy exchange between the transverse oscillation modes now becomes

$$\left| q + \frac{V_{2,-2}\ J_1}{|F^{(2,-2)}_{2,2,0}|} + \frac{2W_{2,-2}\ J_1}{|F^{(2,-2)}_{2,2,0}|} \right| \geq 2J_1 \quad . \tag{5.22}$$

The energy exchange itself can again be obtained by taking the extreme values ± 1 for the cosine function in (5.19)(see Cor80b).

CHAPTER 6 CONCLUDING REMARKS

Here we give a few summarizing and concluding remarks from the theory and calculations presented in this thesis.

- To describe the particle motion in circular accelerators, the HF accelerating electric field and a time-dependent magnetic field are incorporated into the general Hamilton function via a time-dependent vector potential consisting of a fast and slowly varying part.
  This approach is of interest particularly in case of an odd number of cavities in the accelerator.

- The analytical formulae of the Twiss parameters - expressed in the Fourier components of the linear guide field, see (2.19) - have a simple shape for not too large a field modulation (see (2.21)).
  In the considered cases in which the β-function in a lattice varies from 3 to 30 m/rad, the expressions yield satisfactory results.

- The simultaneous description of the transverse and longitudinal motion shows various coupling effects. Principally, the synchro-betatron resonances can be studied in a way similar to the procedure to study coupled betatron resonances (see chapter 5).

- A "Central Position" phase - instead of the well-known HF phase - turns out to be a proper canonical variable for the description of the synchrotron motion (chapter 3). The difference between the CP and HF phase is most pronounced in central regions in case of high harmonic acceleration.
  The results correspond with results derived by Schulte and Hagedoorn in the case of cyclotrons with $Q_x \approx 1$ (Schu78, Schu80) and with results derived by Gordon (Gor82), both with different treatments.
  The definition of the CP phase is extended to the case of motion in a time-dependent magnetic field with an alternating gradient structure.

- The resonance $Q_x - Q_z \pm Q_s = 0$ - excited by a skew quadrupole field- will be relevant only in case of an extremely small "distance" to the resonance. For common electron storage rings ($Q_x \approx Q_z$) we find a significant energy exchange between the transverse motions via the longitudinal motion for e.g. $|Q_x - Q_z \pm Q_s| \leq 10^{-5}$.

- The off-momentum function and its derivative at the position of the cavity can excite resonances of the type $Q_x \pm kQ_s$=integer. The resonance can lead to a substantial effect only in accelerators with a large Q_s value and large value of the ratio of the peak voltage over the kinetic energy, i.e. in the beginning of the acceleration process.

- Working above transition energy, the sum (difference) of the transverse and longitudinal oscillation amplitudes remains constant in case of a sum (difference) synchro-betatron resonance. Below transition energy the behaviour is similar to coupled betatron resonances.

- The control of γ_{tr} in IKOR can be achieved by including a sextupole magnet in each unit cell. The analytical results correspond quite good to results of the AGS computer program running at CERN (see section 4.2).

- Terms of m-th degree in the Hamilton function ($m \geq 3$) can contribute to resonances of higher order than m. Fortunately, these so-called "second order" effects on the betatron motion are negligible in the majority of common synchrotrons and storage rings.

- The effects of non-linear coupled betatron resonances $m_1 Q_x + m_2 Q_z = p_r N$ become clear by an examination of flowlines in a phase plane. The treatment of chapter 5 leads, in case $m_1 = 1$ or $m_2 = 1$ to results which differ qualitatively from results of the general approach of Guignard (Gui76, Gui78). We obtain Guignard's results when not discriminating between the stable and unstable fixed points in phase plane.

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SUMMARY

This thesis deals with a general theory for the description of resonance and coupling effects in circular particle accelerators. For this we use the Hamilton formalism. The theory is mainly applied to the proposed proton accumulator ring IKOR in West Germany and to an electron storage ring which is characteristic of existing synchrotron radiation facilities (PAMPUS; this project has been dismissed meanwhile by the Dutch government).

For the description of coupling effects between the transverse (=betatron) and longitudinal (=synchrotron) motions a general theory is developed. For that purpose the HF accelerating electric field and the time-dependent magnetic field are incorporated in the Hamilton function via a time-dependent vector potential consisting of a fast and a slowly varying part. The final theory offers the possibility to examine the influence of so-called synchro-betatron resonances on the particle motion. The theory is applicable to any circular accelerator with an arbitrary Dee or cavity configuration.
The concept "Central Position" phase (CP phase) arises and is based on the same idea as the CP phase introduced by Schulte and Hagedoorn. To make the theory suitable for any circular accelerator, i.e. for cyclotrons, synchrotrons and storage rings, the concept CP phase is extended for machines with an A.G. field structure and a time-dependent magnetic field.
In case of weak coupling between the radial and the longitudinal motion the theory shows a change in the radial tune Q_x caused by the acceleration process. This corresponds with results given by Schulte and Gordon both using different treatments.
The effect of the synchro-betatron resonances $Q_x - Q_z \pm Q_s = 0$ is generally of minor importance unless the distance to the resonance becomes extremely small: in common electron storage rings if $|Q_x - Q_z \pm Q_s| \leq 10^{-4}$ or even $\leq 10^{-5}$.
Resonances of the type $Q_x \pm kQ_s = p$ (k and p integers), excited by the off-momentum function and its derivative in the cavities, will be significant only in case of large Q_s values ($Q_s \sim 0.1$) since the effect decreases fast with the order of the resonance.

Coupling effects between the transverse motions are studied by ignoring the acceleration process (betatron theory). Application of

canonical transformations in linear betatron theory, used to achieve a well-suited Hamilton function for further study, leads to analytical formulae for the Twiss parameters α, β and γ. In the cases considered, the representation of the β-function produces satisfactory results rather quickly. For β-values between 3 and 30 m/rad in a lattice, only a restricted number of terms of a series have to be taken into account to obtain results with a relative deviation within a few per cent compared with results of a matrix code.

The one-dimensional betatron theory shows "first" as well as "second" order non-linear effects caused by e.g. sextupole and octupole magnets. The first (second) order effects are described by expressions which are of first (second) degree in the field components.
A sextupole magnet seems necessary in IKOR to obtain the maximum benefit of working close to transition energy. For the control of γ_{tr} a sextupole with a length of 0.4 m is planned in each unit cell. The needed strength turns out to be about 1.15 T/m^2 and this result corresponds quite good with results of the AGS computer program running at CERN.
Octupole magnets can be used to enlarge the tune spread in the beam. In case octupole fields will be used in IKOR their strengths will be at most 10^2 T/m^3.
In the cases considered, the "second order" non-linear effects are a factor 10^2 to 10^3 smaller than the "first order" effects.

The treatment of the coupled betatron resonances $m_1Q_x+m_2Q_z=p$ leads to stopband widths or field tolerances using phase plane considerations. In this context we mention that the resonance $2Q_x+Q_z=11$ imposes a strong requirement upon the dipole quality in IKOR ($Q_x \approx 3.25$, $Q_z \approx 4.4$). The skew sextupole component in these dipoles must be smaller than 0.25 T/m^2. To cancel this critical effect, one skew sextupole with a strength of 50 T/m^2 and a length of 0.3 m, placed at a well-defined position in the machine, seems to be sufficient.
For the resonances with $m_1=1$ or $m_2=1$ there are qualitative discrepancies with the general theory of Guignard. These can be explained when not dicriminating between the stable and unstable fixed points in the phase plane.

SAMENVATTING

Dit proefschrift handelt over een algemene theorie voor de beschrijving van de invloed van resonantie- en koppelingseffecten op de deeltjesbeweging in circulaire deeltjesversnellers. De theorie wordt hoofdzakelijk toegepast op de voorgestelde protonenring IKOR in West-Duitsland en een elektronenopslagring die karakteristiek is voor de huidige synchrotronstralingsbronnen (PAMPUS; dit project is inmiddels door de Nederlandse overheid afgewezen).

Voor het opstellen van een algemene theorie is gebruik gemaakt van het Hamilton formalisme. Na het toepassen van een aantal geschikte transformaties worden Hamilton functies verkregen, waarin de verschillende effecten zichtbaar zijn.

Om de koppelingseffecten tussen de transversale (=betatron) en longitudinale (=synchrotron) bewegingen te beschrijven zijn het hoogfrequente versnellende elektrische veld en het tijdsafhankelijke magnetische veld ingebouwd in de Hamilton functie. Daartoe is een tijdsafhankelijke vectorpotentiaal geïntroduceerd bestaande uit een snel en een langzaam variërend deel. De theorie blijkt toepasbaar op elke circulaire versneller met een willekeurige Dee- of trilholte-configuratie en leidt tot de invoering van een "Central Position" fase (CP-fase). Deze CP-fase is op dezelfde idee gebaseerd als de CP-fase zoals die geïntroduceerd is door Schulte en Hagedoorn. Om de theorie geschikt te maken voor elke circulaire versneller, zoals voor cyclotrons, synchrotrons en opslagringen, is het begrip CP-fase uitgebreid voor een tijdsafhankelijk magnetisch veld en een "alternating gradient" veldstructuur. Uit de theorie volgt onder andere dat bij zwakke koppeling tussen de radiale en longitudinale beweging een verandering in het radiale betatrongetal Q_x optreedt die wordt veroorzaakt door het versnelproces. Dit is in overeenstemming met resultaten gegeven door Schulte en Gordon die beiden een andere methode gebruikten.
De theorie is ook geschikt om zogenaamde "synchro-betatronresonanties" te bestuderen. Het effect van de resonanties $Q_x - Q_z \pm Q_s = 0$ blijkt in het algemeen van weinig belang voor de deeltjesbeweging, behalve wanneer de afstand tot de resonantie uiterst klein wordt: in gewone elektronen-opslagringen kunnen de effecten slechts dan relevant zijn als

$|Q_x - Q_z \pm Q_s| \leq 10^{-4}$ of zelfs $\leq 10^{-5}$. Resonanties van het type $Q_x \pm kQ_s = p$ (k en p gehele getallen), die geëxciteerd worden door de "off-momentum" functie en zijn afgeleide op de plaats waar de versnelling plaats vindt, zullen slechts belangrijk zijn als Q_s groot is ($Q_s \sim 0.1$) aangezien het effect snel afneemt met de orde van de resonantie.

Koppelingseffecten tussen de transversale (betatron) bewegingen onderling, worden bestudeerd door het versnelproces buiten beschouwing te laten. Het toepassen van kanonische transformaties, nodig om de Hamilton functie in een geschikte vorm te krijgen voor de verdere studie, leidt in de lineaire betatrontheorie tot analytische formules voor de Twiss parameters α, β and γ. De uitdrukking voor de β-functie geeft in de beschouwde gevallen tamelijk snel bevredigende resultaten: slechts een beperkt aantal termen van een reeks is nodig om voor β-waarden die in een rooster variëren van 3 tot 30 m/rad resultaten te verkrijgen met een relatieve afwijking van hooguit een paar procent vergeleken met resultaten van een matrixcomputerprogramma.
De één-dimensionale betatrontheorie beschrijft zowel "eerste orde" als "tweede orde" niet-lineaire effecten die veroorzaakt worden door onder andere sextupool- en octupoolmagneten.
Een sextupoolmagneet lijkt bijvoorbeeld nodig in IKOR om maximaal profijt te trekken van het werken dicht bij de overgangsenergie. Wanneer in elke eenheidscel zo'n sextupoolmagneet op een goed gedefinieerde positie geplaatst wordt, is bij een lengte van 0.4 m een sterkte van 1.15 T/m^2 nodig. Dit resultaat komt goed overeen met dat van het AGS programma van CERN.
Octupoolmagneten kunnen gebruikt worden om de spreiding in de betatrongetallen Q_x en Q_z in de bundel te vergroten. Als die in IKOR nodig zijn zal hun sterkte maximaal 10^2 T/m^3 bedragen.
In de beschouwde gevallen bleken de "tweede orde" effecten, dat wil zeggen effecten die beschreven worden door uitdrukkingen die van de tweede graad zijn in de veldgrootheden, tengevolge van de multipoolmagneten een factor 10^2 tot 10^3 kleiner te zijn dan de "eerste orde" effecten.
De theorie voor de bestudering van de twee-dimensionale (gekoppelde) betatronbeweging en in het bijzonder van de resonanties $m_1 Q_x + m_2 Q_z = p$ leidt, door gebruik te maken van faseruimtebeschouwingen, tot stopbandbreedtes of toleranties voor het magnetisch veld. Zo stelt de resonantie $2Q_x + Q_z = 11$ hoge eisen aan de kwaliteit van de dipoolmagneten

in IKOR ($Q_x \approx 3.25$, $Q_z \approx 4.4$). De "skew" sextupoolcomponent in deze dipolen moet kleiner zijn dan 0.25 T/m^2. Om dit effect op te heffen lijkt één "skew" sextupoolmagneet met een lengte van 0.3 m en een sterkte van 50 T/m^2, geplaatst op een goed gedefinieerde positie, voldoende. Voor de resonanties met $m_1=1$ of $m_2=1$ zijn er duidelijke kwalitatieve verschillen met de resultaten van de theorie van Guignard. Deze verschillen kunnen worden verklaard wanneer geen onderscheid gemaakt wordt tussen de stabiele en instabiele vaste punten in de faseruimte.

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In 1979 we got involved in the design study of the proton accumulator ring IKOR (West-Germany). I recall with pleasure the meetings of the IKOR Study Group and the nice cooperation with its members from the institutes KFA Jülich, KfK Karlsruhe and CERN Geneva and from the Swedish firm Scanditronix.
Particularly I wish to thank dr. K.H. Reich of the PS Division at CERN for the discussions on γ_{tr}-control in IKOR and for giving me the opportunity to carry out the γ_{tr}-calculations with the AGS program running at CERN.

It is a pleasure to acknowledge the discussions I had with prof.dr. Lapostolle of the GANIL Institute in Caen and which led to interesting suggestions.

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NAWOORD

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Aanvankelijk had het onderzoek betrekking op de ontwerpstudie van een elektronenopslagring in Nederland, bedoeld voor de opwekking van synchrotronstraling. In dit verband wil ik dr.ir. W.H. Backer noemen met wie ik gedurende het eerste deel van dit werk plezierig heb samengewerkt. Hij heeft mij wegwijs gemaakt in de theorie van elektronenopslagringen.

In 1979 werden we betrokken bij de ontwerpstudie van de protonenopslagring IKOR in West-Duitsland. Ik denk met plezier terug aan de bijeenkomsten van de IKOR Studie Groep en aan de fijne samenwerking met de leden hiervan, afkomstig van de instituten KFA Jülich, KfK Karlsruhe, CERN Genève en van de Zweedse firma Scanditronix. In het bijzonder wil ik dr. K.H. Reich van de PS-afdeling van CERN bedanken voor de discussies omtrent de controle van γ_{tr} in IKOR en voor het feit dat hij me de gelegenheid heeft gegeven in zijn groep berekeningen uit te voeren met het AGS-computerprogramma.

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Levensloop

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vanaf 1 augustus 1979	wetenschappelijk ambtenaar THE, FOM groep A V

STELLINGEN

1. Analytische uitdrukkingen kunnen op eenvoudige wijze een goed kwantitatief inzicht verschaffen in het verloop van de Twiss parameters die het lineaire gedrag van deeltjesbanen in circulaire versnellers beschrijven.

   Dit proefschrift, hoofdstuk 2.

2. In opslagringen met een grote waarde van het synchrotrongetal Q_s (bijvoorbeeld van de orde 0.1) heeft een waarde van het radiale betatrongetal Q_x beneden een geheel getal de voorkeur boven een waarde boven een geheel getal.

   Dit proefschrift, hoofdstuk 3.

3. De door Guignard afgeleide formule voor de stopbandbreedte voor somresonanties $m_1Q_x+m_2Q_z=p$ geeft voor $m_1=1$ of $m_2=1$ een niet-fysisch beeld; dit wordt veroorzaakt door het feit dat geen onderscheid wordt gemaakt tussen stabiele en instabiele vaste punten in de faseruimte.

   Dit proefschrift, hoofdstuk 5.

4. In een verdere uitwerking van de door Buys en De Jonge gegeven uitdrukking voor de relaxatietijd van resonante soliton-fonon-verstrooiing zou het verschil in dimensie van beide excitaties tot uiting moeten komen.

   J.A.H.M. Buys en W.J.M. de Jonge,
   European Conference Abstracts 6A(1982)208.

5. Als bij een interpretatief taalsysteem de monitor in de taal zelf is geschreven, biedt het grote voordelen als de eigenlijke interpretator als 're-entrant'-systeemprocedure aangeroepen kan worden. Dit kan in het bijzonder tot een goed bibliotheekmechanisme leiden.

   P.W.E. Verhelst en N.F. Verster, PEP: An interactive programming system with an ALGOL-like programming language,
   Mathematisch Centrum Amsterdam, IW 172/81 (1981).

6. Voor de berekening van de activiteit van $^{81}Rb/^{81m}Kr$-generatoren op het moment van gebruik mag de extra produktie van ^{81}Rb vanuit ^{81m}Rb, bij korte bestralingstijden en indien het ijken van de generator kort na het bestralen geschiedt, niet verwaarloosd worden.
   E. Acerbi et al., Int. J. of Appl. Rad. a. Isotopes 32(1981)465.

7. De berekening van de invloed van een niet ideale 2π-π-intersectie in een opslagring op de transversale deeltjesbeweging met behulp van een matrixvoorstelling van veldvrije ruimten en dunne lenzen verloopt eenvoudiger en sneller dan de methode van Autin en Verdier.
   B. Autin en A. Verdier, CERN ISR-LTD 76-14 (1976).

8. De techniek van pseudo-random-gecorreleerde looptijdmetingen kan aanzienlijk worden vereenvoudigd wanneer bundelmodulatie wordt verkregen met optisch pompen in plaats van met een mechanische chopper.

9. Voor het maken van een protonenmicrobundel met lage-energie cyclotrons is het gewenst bij het ontwerp van dergelijke cyclotrons voorzieningen te treffen die bij een voldoend grote externe bundelstroom (bijvoorbeeld 10 μA) een voldoend kleine energiespreiding in de geëxtraheerde bundel bewerkstelligen (bijvoorbeeld enkele tienden tot een promille).

10. Bij al het doemdenken moeten we niet vergeten dat ook de toekomst eens 'die goeie ouwe tijd' zal kunnen zijn.

11. Met betrekking tot het probleem van de verwerking van chemisch afval is in het verleden de uitdrukking 'zand erover' soms al te letterlijk gehanteerd.

Eindhoven, 17 september 1982 C.J.A. Corsten