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Broer, H.W.; Hoveijn, I.; Lunter, G.A.; Vegter, G.

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Resonances in a spring-pendulum: algorithms for equivariant singularity theory

H W Broer, I Hoveijn, G A Lunter† and G Vegter

University of Groningen, Department of Mathematics and Computer Science, PO Box 800, NL-9700 AV Groningen, The Netherlands

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Abstract. A spring-pendulum in resonance is a time-independent Hamiltonian model system for formal reduction to one degree of freedom, where some symmetry (reversibility) is maintained. The reduction is handled by equivariant singularity theory with a distinguished parameter, yielding an integrable approximation of the Poincaré map. This makes a concise description of certain bifurcations possible. The computation of reparametrizations from normal form to the actual system is performed by Gröbner basis techniques.

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1. Introduction

In the study of Hamiltonian systems, normal form theory is often used to distil dynamical information. In this paper we develop a two-stage normalization process to gain insight into the global organization of (part of) the phase space, the associated dynamics and some bifurcations. The first stage involves the standard Birkhoff normal form. It yields a rotational symmetry enabling a formal reduction to one degree of freedom. The second stage involves equivariant singularity theory with a distinguished parameter.

As a model system we consider a *spring-pendulum* (see section 2) in $1:2$ resonance, where some of the mechanical constants serve as parameters. This enables us to use intuitive descriptions of trajectories in terms of mechanical motions. As a starting-point for the calculations a generic $1:2$ resonant system is used, validating conclusions for all these systems. Many authors have contributed to this and related problems, see for instance [7, 16, 30–32, 34].

The present method gives an integrable approximation of the iso-energetic Poincaré map, namely a planar Hamiltonian vector field, to any order in both phase space variables and parameters. A simple‡ normal form is obtained, and we keep track of all transformations and reparametrizations. In this way we get quantitative information on certain bifurcations.

The planar Hamiltonian is simplified by applying equivariant singularity theory, as we look for normal forms under a suitable left–right equivalence. It turns out that the hyperbolic umbilic (D_4^+ in Arnol'd's classification [1]) plays a key role here. This part of the normal form computation involves repeatedly solving the *infinitesimal stability equation*, where techniques from Gröbner basis theory are used.

† Corresponding author: E-mail: Gerton@math.rug.nl

‡ Polynomial in the phase variables.

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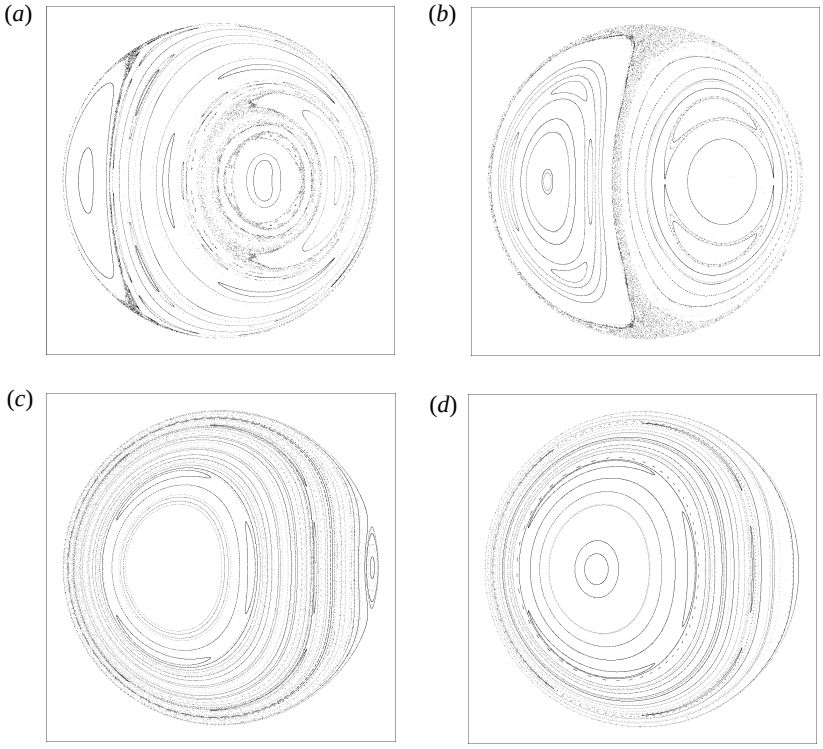

Figure 1. Orbits of iso-energetic Poincaré map of H^0 near 1 : 2 resonance, for various values of detuning parameter $1 - 2a_1$ (see proposition 5). (*a*) $a_1 = 0.40$, (*b*) $a_1 = 0.50$ (*c*) $a_1 = 0.68$, (*d*) $a_1 = 0.70$. For these pictures we used $a_2 = 0.07$, $a_3 = 0.001$, other coefficients zero and $H^0 = 0.2$.

Although we analyse the example system in some detail, the main emphasis lies on the *method*, which is applicable to many more systems. Therefore the theory is presented in greater generality than is needed for the present example. Sections 6, 8 and 9 provide tools applicable in the context of general (compact) symmetry groups, even though the $\mathbb{Z}_2$ group occuring in the example is simple enough to be handled by *ad hoc* methods.

1.1. Sketch of the results

The spring-pendulum lives in a four-dimensional phase space. It is customary to restrict to energy level sets, thereby reducing the dimension to three. A Poincaré section subsequently reduces the associated vector field to a planar symplectic map. Figure 1 shows a number of Poincaré map orbits. Coefficients are chosen such that the harmonic truncations of the constituting oscillators are in approximate 1 : 2 resonance. Our aim is to understand its structure, and to predict the parameter values for which bifurcations take place.

Using the Birkhoff normal form procedure we find an integrable vector field approximation to the Poincaré map. The associated planar Hamiltonian has a central singularity equivalent to the (symmetric) hyperbolic umbilic $x(x^2 + y^2)$, in the case of the 1 : 2 resonance. A versal deformation of this singularity, with corresponding bifurcation diagram, is shown in figure 2. The underlying question of this research is: How are figure 1 and 2 related?

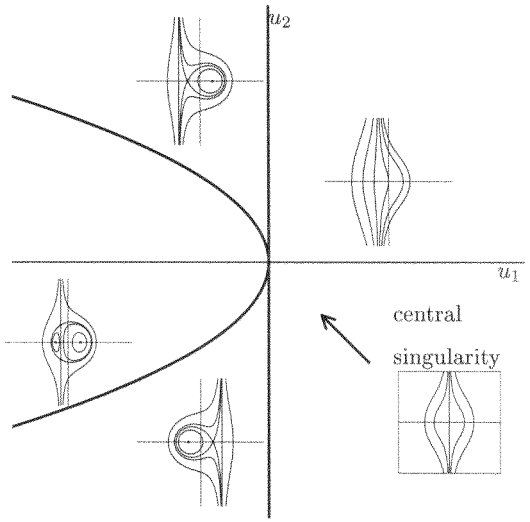

Figure 2. Bifurcation diagram of $x(x^2 + y^2) + u_1x + u_2y^2$. Across the bifurcation lines saddle-centre bifurcations occur. Across the parabola $u_1 + 3u_2^2 = 0$ Hamiltonian pitchfork bifurcations occur due to $\mathbb{Z}_2$ symmetry.

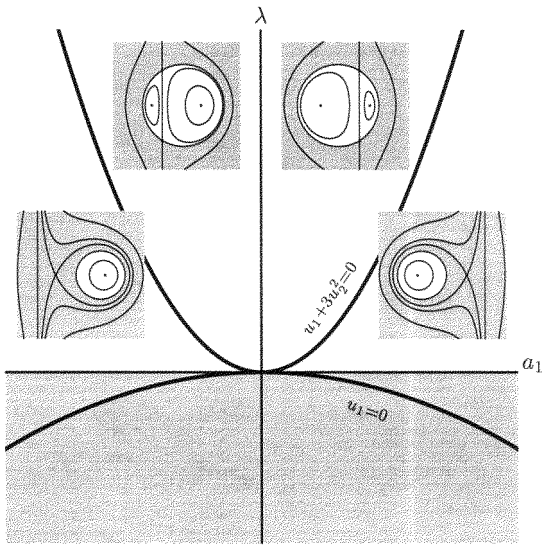

Figure 3. Bifurcation diagram of the planar reduced system H^r. Grey areas denote portions of phase or parameter space that do not correspond to phase points of the original system.

Figure 3 graphically presents a partial answer. It shows which parts of the parameter space (u_1, u_2) and phase space, are actually visited by the system. By purely topological arguments, it is *a priori* clear that some parts of phase and parameter space do not correspond to physical states or configurations. This is reflected in the normalizing transformations we compute.

We in fact obtain a quantitative answer, in the small-energy region, in the form of bifurcation equations in the original parameters. These results are checked against numerical estimates, with good agreement.

1.2. Overview of the reduction with distinguished parameters

One problem occurs in the naive application of equivariant singularity theory. The (formal) rotational symmetry of the Birkhoff normal form gives rise to a formal integral λ in the planar

Table 1. Overview of reductions and normalizations.

Section	2	3	3.2	4	5	6
Context	Original	Birkhoff normal form	Planar reduction	Central normal form	Versal deformation	BCKV normal form
System	H^0	H^n	H^r	H^c	H^u	H^B
Phase space	$\mathbb{R}^4$	$\mathbb{R}^4$	$\mathbb{D}^2$	$\mathbb{R}^2$	$\mathbb{R}^2$	$\mathbb{R}^2$
Coefficients	$a_i, i \geqslant 1$	$b_i, i \geqslant 2$	$b_i, i \geqslant 2$	d_i	—	a_i
Parameters	—	b_1	b_1, λ	c_i, λ	u_1, u_2	λ, b_1
Symmetry	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{S}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

system. This parameter is *distinguished*, in the sense that it is a function on the original phase space. It is natural not to allow reparametrizations to depend on this distinguished parameter, in contrast to ordinary equivariant singularity theory; see also remark 2. Second, the variable λ, which can be interpreted as an *angular momentum* variable, is always positive. This gives special significance to the value $\lambda = 0$, which we also want to preserve. Transformations respecting both the distinguished nature as the zero level of λ are called *BCKV-restricted reparametrizations* [6, 10].

Theorem 9 implements this restriction, and yields a versal deformation of the normalized system via BCKV-restricted reparametrizations:

$$x(x^2 + y^2) + (\lambda_1 + u_1)x + (\lambda_2 + u_2)y^2.$$

Here λ_i and u_i are distinguished and ordinary parameters, respectively. One consequence of the theorem is that a versal deformation requires at least two distinguished parameters, but we have only one at our disposal. This problem can be attacked by the path formalism. The resulting normal form, presented in theorem 13, involves coefficients that are functions of the available parameters. They describe the path traced out by the system through the parameter space of the versal deformation.

The BCKV normal form is built on the normal form in the ordinary context, together with the reparametrizations connecting it to the original system. See theorem 7 for these reparametrizations. The formal calculations involved make essential use of Gröbner basis techniques, and are a main focus of the present paper. It is dealt with in sections 8 and 9.

1.3. Outline of the procedure

To clarify the various contexts, phase spaces and systems from the outset, we give here a concise but detailed outline of the procedure leading to the BCKV normal form. This section is summarized in table 1.

The starting-point is the two degree-of-freedom Hamiltonian H^0 with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ spatio-temporal symmetry, depending on several coefficients a_i.

After Birkhoff normalizing, the system is renamed H^n and has acquired a formal $\mathbb{S}^1$ symmetry. It contains one of the $\mathbb{Z}_2$ symmetries as a subgroup, resulting in a $\mathbb{Z}_2 \times \mathbb{S}^1$ symmetry. Also, this step singles out a *detuning parameter*, measuring the deviation from the $1 : 2$ resonance, around which the Birkhoff procedure is performed. This parameter is called b_1, and can be controlled by changing, for example, the spring constant. For notational convenience, the other coefficients are renamed b_i, $i \geqslant 2$.

H^n has two independent integrals of motion: H^n itself, and the formal integral λ, the variable conjugate to the $\mathbb{S}^1$ symmetry. Trajectories lie on level sets of H^n, which, close to the elliptic equilibrium, are 3-spheres in $\mathbb{R}^4$.

After dividing out the $\mathbb{S}^1$ symmetry, on a section with λ nonzero and fixed, we get $\mathbb{S}^2$ (see section 7.1), which we represent by a disk with boundary. The boundary is an artefact of the singular coordinate transformation. It is the image of a single point, and is called the *singular circle*. The formally reduced system obtained is denoted by H^r. It has one $\mathbb{Z}_2$ symmetry left.

From here on we forget about the boundary introduced by the coordinate transformation, and consider the system in a full neighbourhood of the origin in $\mathbb{R}^2$. The system is now subjected to a transformation bringing the central singularity into a simple form: the $\mathbb{Z}_2$-symmetric hyperbolic umbilic. We are left with a deformation H^c of this singularity, in terms of the parameters b_1 and λ.

There exists a versal deformation of the hyperbolic umbilic with only two parameters. (In the nonequivariant case one finds three.) This deformation is denoted by H^u. In section 5 we find the reparametrizations that induce H^c from H^u. This step is computationally involved, indeed the second half of this paper is largely devoted to it. In this step we employ Gröbner basis techniques to efficiently compute the required morphisms.

Finally, we use the reparametrizations of section 5 to compute the BCKV-restricted normal form H^B of our system.

1.4. Formal aspects: a perturbation problem

The transformations performed on the system are all either conjugacies or equivalences (i.e. conjugacies modulo time scalings), except for one: the Birkhoff transformation. It provides a *formal* conjugacy.

By theorems of Borel and Schwarz (see [5, 21, 16]), this formal conjugacy can be lifted to a C^∞ transformation ϕ, uniquely defined modulo a flat perturbation. The normalized Hamiltonian $H^0 \circ \phi$ is therefore also defined up to a flat perturbation. This perturbation is generally not $\mathbb{S}^1$ symmetric, so that $H^0 \circ \phi$ only respects the acquired $\mathbb{S}^1$ symmetry up to flat terms.

These flat terms account for the differences between the integrable approximation (figure 3) and the numerical pictures (figure 1). Normalization reveals the dynamical *skeleton* of the iso-energetic Poincaré map, describing the actual system accurately for small energy, but disregarding details like transversality of stable and unstable manifolds (and chaos), and subharmonics. Subharmonics can be found by similar means though; see [9]. For more remarks on this flat perturbation problem, see [6, 8, 10].

1.5. Notation

Parameters and coefficients. The dynamical systems we investigate depend on a number of variables. Certain variables are supposed to be constant during the evolution of the system, for example the mass of a pendulum. Throughout, we reserve the name *coefficient* for a 'constant variable' that can take on arbitrary values, except possibly a few isolated ones that are excluded by nondegeneracy conditions. The name *parameter* is reserved for 'constant variables' that are *small*; for our system these are the distinguished parameter λ and the detuning parameter $b_1 = 1 - 2a_1$.

Hamiltonian contexts. The Hamiltonian system H we consider appears in many incarnations, depending on the context. We denote the appropriate context by a superscript, e.g. H^0 for the original Hamiltonian, H^n for the Birkhoff normal form.

Big-oh notation. We use the notation $\mathrm{O}(|x, y|^n)$ to denote terms of total order n and higher in x and y. In standard notation, this would be $\mathrm{O}(|x|^n + |y|^n) = \mathrm{O}((|x| + |y|)^n)$. Also,

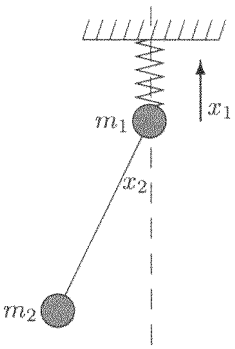

Figure 4. The spring-pendulum with its symmetry-axis.

e.g. $\mathrm{O}(|c_i, \lambda|^n)$ stands for $\mathrm{O}(|c_1|^n + |c_2|^n + \cdots + |\lambda|^n)$, when c is a vector of coefficients. This will be clear from the context.

2. The spring-pendulum system

This section introduces the system that is used as a leading example: a planar pendulum suspended by a spring constrained to move along the vertical axis. It is a typical two degree-of-freedom Hamiltonian system with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ (time-reversal and reflection) symmetry. Introducing this specific example allows us to describe the dynamics in terms of mechanical motion. The results obtained are more generally valid, however. This will be made precise; see also the remarks about proposition 1.

We now describe the system. Masses are attached to both ends of the rod, while both rod and spring are massless. The configuration is given by the displacement of the suspension point and the angle of the pendulum with the vertical axis, denoted by x_1 and x_2. The potential energy is $U(x_1, x_2) = -m_2 gl \cos x_2 + \frac{1}{2}a^2 x_1^2$ when the origin is suitably chosen. The m_i denote masses, $M^2 = m_1 + m_2$, g the gravitational acceleration, l the length of the pendulum and a^2 the spring coefficient. The Hamiltonian of the system, expressed in configuration coordinates x_i and their conjugates y_i reads

$$H(x, y) = \frac{1}{2}a^2 x_1^2 - m_2 gl \cos x_2 + \frac{l^2 m_2 y_1^2 + M^2 y_2^2 - 2lm_2 y_1 y_2 \sin x_2}{m_2 l^2 (2M^2 + m_2[\cos(2x_2) - 1])}. \tag{2.1}$$

This Hamiltonian exhibits two $\mathbb{Z}_2$-symmetries: a time-reversible symmetry denoted by T, and reflection symmetry in the vertical axis, denoted by S. Here

$$\begin{aligned} T &: (x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, -y_1, -y_2) \\ S &: (x_1, x_2, y_1, y_2) \mapsto (x_1, -x_2, y_1, -y_2). \end{aligned} \tag{2.2}$$

In the following, we use Cartesian canonical coordinates x_i, y_i as well as complex variables $z_i, \bar{z}_i$ and Hamiltonian polar coordinates L_i, ϕ_i. The relations between those coordinates

and the symmetry maps are as follows:

$$\begin{aligned}
&z_i = x_i + \mathrm{i}y_i = \sqrt{2L_i}\mathrm{e}^{\mathrm{i}\phi_i} && \bar{z}_i = x_i - \mathrm{i}y_i = \sqrt{2L_i}\mathrm{e}^{-\mathrm{i}\phi_i}\\
&\phi_i = \frac{1}{2i}\log\frac{z_i}{\bar{z}_i} = \arctan\frac{y_i}{x_i} && L_i = \frac{1}{2}z_i\bar{z}_i = \frac{1}{2}(x_i^2 + y_i^2)\\
&x_i = \sqrt{2L_i}\cos\phi_i = \frac{1}{2}(z_i + \bar{z}_i) && y_i = \sqrt{2L_i}\sin\phi_i = \frac{1}{2i}(z_i - \bar{z}_i)\\
&T : (z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto (\bar{z}_1, z_1, \bar{z}_2, z_2) && (L_1, \phi_1, L_2, \phi_2) \mapsto (L_1, -\phi_1, L_2, -\phi_2)\\
&S : (z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto (z_1, \bar{z}_1, -z_2, -\bar{z}_2) && (L_1, \phi_1, L_2, \phi_2) \mapsto (L_1, \phi_1, L_2, \phi_2 + \pi).
\end{aligned} \tag{2.3}$$

We now write H as a Taylor series in the x_i and y_i variables, and apply a rescaling of variables and time to tidy up the quadratic terms.

Proposition 1. *Provided that $m_2 \neq 0$ and $a \neq 0$, by a rescaling of variables and time we can bring the Hamiltonian (2.1) into the form*

$$\begin{aligned}
H^0(x, y) := \frac{x_1^2 + y_1^2}{2} + a_1\frac{x_2^2 + y_2^2}{2} - 8a_2x_2y_1y_2 - 16a_3x_2^4 + 16a_4x_2^2y_1^2 + 16a_5x_2^2y_2^2\\
+32a_6x_2^3y_1y_2 + 64a_7x_2^6 + 64a_8x_2^4y_1^2 + 64a_9x_2^4y_2^2
\end{aligned} \tag{2.4}$$

modulo $\mathrm{O}(|x, y|^7)$ *terms, with symplectic form* $\mathrm{d}x \wedge \mathrm{d}y$. *Here* $a_1 = \frac{\sqrt{g}M}{a\sqrt{l}}$ *and* $a_2 = \frac{1}{8al}$, *and* H^0 *is invariant under* S *and* T.

From now on, (2.4) is used as a starting point, with no conditions on the coefficients a_i. This system has the same qualitative form as the spring-pendulum system, in fact, for a proper choice of the coefficients a_i the latter is a high-order perturbation of (2.4) (modulo a rescaling).

The physical origin of the system imposes some constraints on the coefficients, for example $a_1 > 0$ and $a_2 > 0$. We will not use these. Instead, we keep an eye on the nondegeneracy conditions encountered during the calculations, allowing the a_i to otherwise take arbitrary values. Some of these conditions are implied by the physical constraints.

3. Formal normalization and reduction to one degree of freedom

This section discusses the application of the Birkhoff normal form procedure to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric Hamiltonian H^0 (see (2.4)) around the 1 : 2 resonance. Then we discuss the subsequent reduction of normalized Hamiltonian H^n to a one degree-of-freedom (i.e. planar) system H^r, depending on an extra, distinguished, parameter.

For introductions to the Birkhoff normal form procedure, see e.g. [7, appendix 7, 25, chapter VII]. For a historical overview see [34, section 3.5]. It is well known how to implement Birkhoff normal form computations in a computer [14, 15, 23] and we do not discuss this here, we only present a concise overview.

The normal form computation is done in the ring of formal power series; see section 1.4 for relevant remarks.

3.1. The normal form procedure for a general two degree-of-freedom system

Here we treat the Birkhoff normal form for general two degree-of-freedom Hamiltonian systems, with semisimple quadratic part, near resonance. After this we specialize to $\mathbb{Z}_2 \times \mathbb{Z}_2$ spatio-temporal symmetric systems.

Assume that a Hamiltonian H has a critical point at 0, and let H_2 denote its quadratic part. The adjoint action of H_2 is defined in terms of the Poisson bracket, namely $\mathrm{ad}_{H_2} := \{\cdot, H_2\}$. If the adjoint action is semisimple (which it is in the cases we consider), the Birkhoff normal form is in $\ker \mathrm{ad}_{H_2}$, implying that H_2 is conserved by the flow generated by the normalized Hamiltonian H^n. The kernel is an algebra with respect to ordinary addition and multiplication of formal power series, as ad_{H_2} is a derivation. A minimal set of generators for this algebra is called a *Hilbert-basis*.

$H^n \in \ker \mathrm{ad}_{H_2}$ also means that H^n is invariant under the $\mathbb{S}^1$-action generated by H_2, which is $A_\xi : (z_1, z_2) \mapsto (\mathrm{e}^{q\mathrm{i}\xi} z_1, \mathrm{e}^{p\mathrm{i}\xi} z_2)$, $\xi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. The Hilbert basis just mentioned is precisely the set of basic invariant polynomials for this action. If the group action acts diagonally there exist *monomial* generators.

Proposition 2. *Let $H_2 = iz_1\bar{z}_1 + i\omega z_2\bar{z}_2$, where $\omega > 0$, then a Hilbert-basis for the algebra* $\ker \mathrm{ad}_{H_2}$ *is given by:*

(1) $z_1\bar{z}_1$, $z_2\bar{z}_2$, if $\omega \notin \mathbb{Q}$,

(2) $z_1\bar{z}_1$, $z_2\bar{z}_2$, $z_1^p\bar{z}_2^q$, $\bar{z}_1^p z_2^q$, if $\omega = \frac{p}{q}$, $p, q > 0$, $\gcd(p, q) = 1$.

In our case, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ spatio-temporal symmetries lead to a smaller kernel.

Proposition 3 (Birkhoff normal form). *Let H^0 be a Hamiltonian on $\mathbb{R}^4$ with vanishing linear part, invariant under S and T as defined in (2.2). Let $H_2^0 = \mathrm{i}z_1\bar{z}_1 + \mathrm{i}\omega z_2\bar{z}_2$ be its quadratic part, and assume that $\omega = \frac{P}{Q} = \frac{p}{q}$ with Q even, $P, Q, p, q > 0$ and* $\gcd(P, Q/2) = \gcd(p, q) = 1$. *Then there exists a formal symplectic S- and T-equivariant coordinate transformation ϕ such that*

$$H^n := H^0 \circ \phi = H_2^0 + f_0(z_1\bar{z}_1, z_2\bar{z}_2, z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q),$$

where the Taylor series of $f_0(\zeta_1, \zeta_2, \zeta_3)$ starts off as $\alpha\zeta_3 +$ h.o.t. *The quadratic part H_2^0 is conserved under the flow of H^n, i.e. H^n is invariant under the $\mathbb{S}^1$-action $A_\xi : (z_1, z_2) \mapsto (\mathrm{e}^{q\mathrm{i}\xi} z_1, \mathrm{e}^{p\mathrm{i}\xi} z_2)$. This action is nondegenerate except on the axes $z_1 = 0$ and $z_2 = 0$ on which points have isotropy subgroup (stabilizor) $\mathbb{Z}_p$ and $\mathbb{Z}_q$ respectively.*

(In general the problem of finding basic invariant polynomials for a given group action is a difficult one; see e.g. [33].)

Proof. By proposition 2, a general element of $\ker \mathrm{ad}_{H_2}$ can be written in the form $f = g_1 + z_1^p\bar{z}_2^q g_2 + \bar{z}_1^p z_2^q g_3$, where $g_i = g_i(z_1\bar{z}_1, z_2\bar{z}_2, z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q, z_1^P\bar{z}_2^Q - \bar{z}_1^P z_2^Q)$. Suppose f is invariant under S and T. If q is even, $Q = q$, and we can choose $g_2 = g_3 = 0$. On the other hand, if q is odd, invariance of f implies that $g_2 = g_3 = 0$, by invariance of the g_i under S; see (2.3).

Invariance of g_1 under T implies that g_1 depends on the square of the argument $z_1^P\bar{z}_2^Q - \bar{z}_1^P z_2^Q$, which equals $(z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q)^2 - 4(z_1\bar{z}_1)^P(z_2\bar{z}_2)^Q$. This proves that $z_1\bar{z}_1$, $z_2\bar{z}_2$ and $z_1^P\bar{z}_2^Q + \bar{z}_1^P z_2^Q$ generate the S- and T-invariant part of $\ker \mathrm{ad}_{H_2}$. □

The resonance of the system in the proposition above is referred to as the $P : Q$ resonance. In particular we refer to the $2 : 2$ resonance instead of $1 : 1$.

3.2. Reduction to one degree of freedom

The system, in the incarnation H^n, now has acquired an additional (formal) $\mathbb{S}^1$-symmetry, with action $(z_1, z_2) \mapsto (\mathrm{e}^{\mathrm{i}p\xi} z_1, \mathrm{e}^{\mathrm{i}q\xi} z_2)$ for $\xi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and corresponding conserved quantity $H_2^0 = z_1\bar{z}_1 + \omega z_2\bar{z}_2$. This symmetry enables us to formally reduce to a one degree-of-freedom system.

We first express the normalized system in Hamiltonian polar coordinates L_i, ϕ_i; see (2.3):

$$H^n(L,\phi) = L_1 + \omega L_2 + f_1(L_1, L_2, L_1^{P/2} L_2^{Q/2} \cos(P\phi_1 - Q\phi_2)). \tag{3.1}$$

Here, and elsewhere in this section, the functions f_i are of the same form as f_0 in proposition 3, differing only by innocent linear changes of variables. Let $p = P/\gcd(P,Q)$, $q = Q/\gcd(P,Q)$, and let r, s be integers such that $pr - qs = 1$. Consider the following symplectic coordinate change:

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} = \begin{pmatrix} r & s \\ q & p \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \qquad \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} p & -q \\ -s & r \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The transformed system and symmetries now read:

$$H^n(\tilde{L},\tilde{\phi}) = \frac{1}{q}\tilde{L}_2 + f_2(\tilde{L}_1, \tilde{L}_2, (p\tilde{L}_1 - s\tilde{L}_2)^{P/2}(-q\tilde{L}_1 + r\tilde{L}_2)^{Q/2} \cos(\gcd(P,Q)\tilde{\phi}_1)),$$

$$T: \ (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (-\tilde{\phi}_1, -\tilde{\phi}_2),$$

$$S: \ (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (\tilde{\phi}_1 + q\pi, \tilde{\phi}_2 + r\pi),$$

$$\mathbb{S}^1\text{-action}: (\tilde{\phi}_1, \tilde{\phi}_2) \mapsto (\tilde{\phi}_1, \tilde{\phi}_2 + \xi/q),$$

from which it is manifest that $\tilde{L}_2$ is conserved (since the conjugate variable $\tilde{\phi}_2$ is cyclic); indeed, $\tilde{L}_2 = qH_2^0$. We now reduce to a planar system by dividing out the $\mathbb{S}^1$-symmetry generated by $\tilde{L}_2$, viewing $\tilde{L}_2$ as a *distinguished parameter* We denote the planar reduction of H^n we get in this way by H^r. Calling $\tilde{L}_2$ a parameter is justified if we consider only small deviations from the system's lower equilibrium, for then the system has little energy, so H^n, and therefore $\tilde{L}_2$, is small; see also remark 2. From here on, we write λ for the parameter $\tilde{L}_2$.

Next, we apply the translation $\bar{L}_1 = \tilde{L}_1 - \frac{s}{p}\lambda$, $\bar{\phi}_1 = \tilde{\phi}_1$. This is a symplectic transformation in the current planar context. The Hamiltonian becomes $H^r = \frac{\lambda}{q} + f_3(\bar{L}_1, \lambda, \bar{L}_1^{P/2}(\bar{L}_1 - \frac{\lambda}{pq})^{Q/2} \cos(\gcd(P,Q)\bar{\phi}_1))$.

Finally, we return to Cartesian coordinates. Dropping the constant and hence dynamically irrelevant term λ/q, we get the following.

Proposition 4. *Under the assumptions of proposition 3, let H^n be a Hamiltonian in Birkhoff normal form. There exist coordinates x, y, λ, ϕ on $\mathbb{R}^4$ such that λ is constant on orbits of H^n, and the projections of those orbits onto the (x,y)-plane coincide with those of a planar Hamiltonian system $H^r(x,y)$, with parameter λ and independent of ϕ, of the form*

$$P \text{ even: } H^r = f_4\left(x^2 + y^2, \lambda, (x^2+y^2)^{\frac{p}{2}-1}(x^2 - y^2)\left(x^2 + y^2 - \frac{2\lambda}{pq}\right)^{Q/2}\right);$$

$$P \text{ odd: } H^r = f_4\left(x^2 + y^2, \lambda, (x^2+y^2)^{\frac{p-1}{2}} x\left(x^2 + y^2 - \frac{2\lambda}{pq}\right)^{Q/2}\right),$$

where $f_4(\zeta_1, \zeta_2, \zeta_3) = b_1\zeta_3 + \text{h.o.t.}$

Remark 1 (singular circle). The coordinate transformation to Hamiltonian polar coordinates used in (3.1) is singular at the coordinate axes $L_1 = 0$ and $L_2 = 0$. These axes become $p\tilde{L}_1 - s\tilde{L}_2 = 0$ and $-q\tilde{L}_1 + r\tilde{L}_2 = 0$ in the transformed coordinates, and after translation $\bar{L}_1 = 0$ and $\bar{L}_1 = \lambda/pq$. The first singularity is removed by returning to Cartesian coordinates in the plane. The second singularity is called the *singular circle*. At this circle $L_2 = 0$ implying that the coordinate ϕ_2 is ill-defined, and therefore so is $\bar{\phi}_1 = p\phi_1 - q\phi_2$. In particular this implies that H^r is constant there; see also section 7.1.1.

Remark 2 (the parameter λ). The adjective *distinguished* refers to the fact that λ stems from the phase space of H^n, not from the coefficients a_i. If we are interested in the organization of level sets of H^n in $\mathbb{R}^4$ (i.e. including λ and its cyclic conjugate to the planar reduced phase space), we may not let reparametrizations of ordinary parameters depend on the distinguished parameter, see section 6. This should be contrasted to the point of view taken in section 5, where we merely classify the organization of level sets in $\mathbb{R}^2$, and where it is permissible to treat λ as an ordinary parameter. Note that in either setting we do not allow reparametrizations of λ to depend on phase variables, i.e. we regard it as a true parameter.

Remark 3 (symmetries). When q is even, the acquired $\mathbb{S}^1$ normal form symmetry group contains the reflection $\mathbb{Z}_2$-symmetry S as a subgroup. Before reduction the symmetry group is therefore $\mathbb{S}^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{S}^1 \times \mathbb{Z}_2$, depending on the parity of q, leading to a symmetry group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ for the reduced system.

3.3. Planar reduction of H^n around $1:2$ resonance

We now present the results of the normal form computations starting from the Hamiltonian (2.4) for the $1:2$ resonance.

The coordinate transformation in Hamiltonian polar coordinates takes the form

$$\bar{L}_1 = L_1, \qquad \lambda = 2L_1 + L_2, \qquad \bar{\phi}_1 = \phi_1 - 2\phi_2, \qquad \tilde{\phi}_2 = \phi_2$$

with singularities at $\bar{L}_1 = 0$ and $\bar{L}_1 = \lambda/2$, where λ is the distinguished parameter. Geometrically, the new coordinate $\bar{\phi}_1$ becomes constant in the unperturbed ($\omega = \frac{1}{2}$) linear flow. In complex coordinates, the transformation reads

$$z_1 = z_1' \frac{z_2'}{\bar{z}_2'}, \quad z_2 = z_2' \sqrt{1 - 2\frac{z_1'\bar{z}_1'}{z_2'\bar{z}_2'}} \tag{3.2}$$

where z_i are the old complex coordinates. The singular circle in these coordinates is $z_1'\bar{z}_1' = \frac{1}{2}z_2'\bar{z}_2'$.

Proposition 5 (planar reduction). *After Birkhoff normalization and reduction to one degree of freedom, for the $1:2$ resonance (a_1 around $\frac{1}{2}$), up to $\mathrm{O}(|z_i, \bar{z}_i|^7)$ terms, the Hamiltonian (2.4) takes the form*

$$\begin{aligned} H^r = {} & b_1\zeta_1 + b_2\zeta_2 + \frac{1}{b_3^3}\zeta_3 + b_4\zeta_1^2 + b_5\zeta_1\zeta_2 + b_6\zeta_2^2 \\ & + b_7\zeta_1\zeta_3 + b_8\zeta_2\zeta_3 + b_9\zeta_1^3 + b_{10}\zeta_1^2\zeta_2 + b_{11}\zeta_1\zeta_2^2 + b_{12}\zeta_2^3 + b_{13}\zeta_3^2 \end{aligned}$$

where $\zeta_1 = x^2 + y^2$, $\zeta_2 = \lambda$, $\zeta_3 = x(x^2 + y^2 - \lambda)$, and the coefficients for the terms up to order four in the original phase coordinates are given by

$$\begin{aligned} & b_1 = \frac{1}{2} - a_1; \qquad b_2 = a_1; \qquad b_3 = \frac{1}{\sqrt[3]{a_2}}; \\ & b_4 = 8\left(\frac{a_2^2}{1+2a_1} - 3a_3 - a_4 + a_5\right); \qquad b_5 = 8(6a_3 + a_4 - 2a_5); \\ & b_6 = -8\left(\frac{a_2^2}{1+2a_1} + 3a_3 - a_5\right). \end{aligned}$$

The special form for the coefficient of ζ_3 was chosen for notational convenience, as will become apparent below. The coefficient b_1 vanishes at resonance ($a_1 = \frac{1}{2}$). It is considered small throughout, and is referred to as *detuning parameter*, measuring the deviation from the resonant frequency.

Remark 4. The first nondegeneracy condition is $1 + 2a_1 \neq 0$. If we continue to normalize to higher orders, more conditions of the form $a_1 \neq p/q$ are found, where $p/q \in \mathbb{Q}$.

4. Normalization using singularity theory

We have reduced the original system near 1 : 2 resonance to a planar Hamiltonian H^r depending on several coefficients b_i, a detuning parameter b_1 and one distinguished parameter λ. Because it is planar now, we may use general ($\mathbb{Z}_2$-symmetric) planar morphisms (as opposed to symplectic ones) to further normalize our system. The resulting normal form is not dynamically conjugate, but *equivalent* to the original system, that is, conjugate modulo a state-dependent reparametrization of time; see [10, 6] and in particular [8] for more details on this method.

The central singularity is defined by the parameter values $b_1 = 0$ (resonance) and $\lambda = 0$. At this central singularity the Hamiltonian still depends on the coefficients b_i, $i > 1$. In this section we bring the central singularity in the normal form $x(x^2+y^2)$, which is independent of the b_i. This singularity is the $\mathbb{Z}_2$-invariant hyperbolic umbilic (see [29]), in Arnol'd's classification denoted by D_4^+.

First, by a simple scaling transformation ϕ_1 we can achieve that the Hamiltonian takes the form $H^{r\prime} := H^r_{b_1=\lambda=0} \circ \phi_1 = x(x^2 + y^2) + \text{h.o.t.}$

Remark 5 (nondegeneracy conditions). This is possible provided that the coefficient of the third-order terms (in x, y) are nonzero. This translates into the condition $a_2 \neq 0$; see proposition 5.

Next, we look for a near-identity planar morphism ϕ removing the h.o.t. from $H^{r\prime}$. This morphism should respect the $\mathbb{Z}_2$-symmetry $(x, y) \mapsto (x, -y)$. By a generalization of [24, theorem III.5.2] that incorporates the symmetry group, $H^{r\prime}$ is isomorphic, by a $\mathbb{Z}_2$-equivariant morphism, to $x(x^2 + y^2)$; for details see appendix A.

Now that existence of ϕ is guaranteed, how do we *compute* it? We employ the following iterative approach. Set $\phi_1(x) = x$, and assume that

$$H^{r\prime} \circ \phi_k = x(x^2 + y^2) + \mathrm{O}(|x, y|^{k+3}) \tag{4.1}$$

for some k. To find $\phi_{k'}$ with $k' = k + 1$ we set $\phi_{k'} = \phi_k + \sum \alpha_i t_i$, where $\{t_i\}$ span the space of $\mathbb{Z}_2$-equivariant terms in x, y of degree k', and α_i the coefficients to be determined. In the present case, equation (4.1) for $k = k'$ is then satisfied for terms of degree $k' + 2$ and lower, and for degree $k' + 3$ it forms a, generally underdetermined, set of linear equations for the α_i. (The scaling transformation—replacing ϕ_1—can be found analogously, but then the equations are nonlinear.) Summarizing we have the following proposition.

Proposition 6. *There exists a coordinate transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H^c := H^r \circ \phi$ is of the form*

$$\begin{aligned} H^c = (1 + c_1)x(x^2 + y^2) + c_2(x^2 + y^2) + c_3x^4 + c_4x^2y^2 + c_5y^4 \\ +\lambda(d_1x + d_2x^2 + d_3y^2 + d_4x^3 + d_5xy^2 + d_6x^4 + d_7x^2y^2 + d_8y^4) \\ +\lambda^2(d_{12}x + d_{13}x^2 + d_{14}y^2) \end{aligned}$$

modulo terms of $O(|x, y|^5)$, $O(|c_i, \lambda|^3)$ *and* $O(|x, y|^3|c, \lambda|^2)$. *Here* $d_i = d_i(b_j)$ *are coefficients, and* $c_i = c_i(b_j)$ *are parameters, all of them polynomial expressions in the* b_i. *The* c_i *vanish at* $b_1 = 0$.

Proof. See appendix A.1. □

We say that H^c is in *central singularity reduced form*, i.e. at the central singularity $b_1 = \lambda = 0$ it reduces to the normal form $x(x^2 + y^2)$.

Remark 6 (dependence of ϕ on coefficients). We consider ϕ in the above proposition to be *fixed*, i.e. independent of parameters. It does depend on the coefficients $b_2, b_3, \ldots$ however, since $H^r|_{\lambda=b_1=0}$ also depends on those.

Some leading order parameters and coefficients are:

$$c_1 = -\frac{1}{96}b_1b_3^6b_4, \qquad c_2 = \frac{b_1b_3^2}{4},$$
$$d_1 = -\frac{1}{b_3^2}, \qquad d_2 = \frac{1}{3}b_3^2(b_4 + 3b_5), \qquad d_3 = b_3^2(b_4 + b_5).$$

5. Inducing H from a universal deformation

This section forms the core of the paper, in which we achieve our goal, namely finding the normalizing transformations explicitly. At this point our system H^0 is reduced to H^c, a deformation of the central singularity $x(x^2 + y^2)$, depending on parameters λ and c_i, and on a number of coefficients d_i. This singularity is of codimension two, with versal deformation $H^u := x(x^2 + y^2) + u_1x + u_2y^2$; see figure 2. We are to find transformations that induce H^c from the model H^u.

For the moment we disregard the distinguished nature of λ, treating it, like the c_i, as an ordinary parameter; see remark 2. The results of this section are used in section 6, where λ is treated as a distinguished parameter. For notational convenience we write $\lambda = c_0$ here.

In appendix C we show that $H^u(x, y, u_1, u_2) := x(x^2 + y^2) + u_1x + u_2y^2$ is a universal $\mathbb{Z}_2$-equivariant† deformation of the singularity $(x, y) \mapsto x(x^2 + y^2)$; see [26, 4]. It follows that there exists a pair of morphisms (transformations) (ϕ, ρ), where $\phi : \mathbb{R}^2 \times \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}^2$ is a parameter-dependent coordinate transformation, and $\rho : \mathbb{R}^c \times \mathbb{R}^d \to \mathbb{R}^2$ is a reparametrization from (c_i, d_j) to (u_1, u_2), such that

$$H^u(\phi(x, y, c_i, d_i), \rho(c_i, d_i)) = H^c(x, y, c_i, d_i).$$

These morphisms obey the following additional constraints: ϕ is $\mathbb{Z}_2$-equivariant, and both ϕ and ρ are trivial at the central singularity, i.e. $\phi(x, y, 0, d_i) = (x, y)$ and $\rho(0, d_i) = 0$.

In appendix C we give a necessary and sufficient condition for versality of a deformation. This is the well known *infinitesimal stability* equation‡ adapted to our equivariant context. For the particular case of H^u this condition boils down to: for every $\mathbb{Z}_2$-invariant germ g vanishing at the origin there should exist $\mathbb{Z}_2$-invariant germs $\alpha_i(x, y)$, $i = 1, 2, 3$ and real numbers β_1, β_2 such that

$$\alpha_1(x, y)x\frac{\partial f}{\partial x} + \alpha_2(x, y)y^2\frac{\partial f}{\partial x} + \alpha_3(x, y)y\frac{\partial f}{\partial y} + \beta_1x + \beta_2y^2 = g(x, y). \tag{5.1}$$

† Recall that the $\mathbb{Z}_2$-symmetry is given by $(x, y) \mapsto (x, -y)$.

‡ See [19]. Necessity of this condition is immediate by considering deformations of the form $H^u(x, y, 0, 0) + c_1g(x, y)$ for general ($\mathbb{Z}_2$-invariant) g; see [24, proposition IV.3.2].

Here f is the central singularity $x(x^2+y^2)$. For this f the condition is indeed satisfied (see appendix C.1). Starting from the infinitesimal stability condition, versality is proved by invoking the Mather–Malgrange preparation theorem [24, 28].

We are, however, interested not so much in existence but rather in actually *computing* the morphisms ϕ and ρ, up to a certain degree. There exists a rather straightforward algorithm to do so [22], which repeatedly uses the fact that we can solve equations of the form (5.1). The solutions α_i and β_i form the building blocks of ϕ and ρ. This algorithm is presented in section 8. It can be regarded as a constructive proof of the existence of a *formal* solution for ϕ and ρ.

Our ability to compute ϕ and ρ now rests on our ability to compute solutions to (5.1). This can be done efficiently using ideas from Gröbner basis theory. In section 9 we present the algorithm. Here we give the results.

Proposition 7. *Let H^c be a planar Hamiltonian depending on parameters c_i and coefficients d_i, with central singularity $x(x^2+y^2)$ at $c_0 = c_1 = \cdots = 0$, symmetric under the $\mathbb{Z}_2$-action $(x, y) \mapsto (x, -y)$. A versal deformation of this central singularity is given by $H^u := x(x^2+y^2) + u_1 x + u_2 y^2$, so that there exist ϕ and ρ such that*

$$H^c = H^u(\phi(x, y, c_i, d_i), \rho_1(c_i, d_i), \rho_2(c_i, d_i)) \tag{5.2}$$

with $\phi(x, y, 0, d_i) = (x, y)$, $\rho(0, d_i) = (0, 0)$. To compute;

(a) ϕ modulo $\mathrm{O}(|x, y|^A) + \mathrm{O}(|c_i|^B)$, it is sufficient to know H^c modulo $\mathrm{O}(|x, y|^{A+2}) + \mathrm{O}(|c_i|^B)$;

(b) ρ modulo $\mathrm{O}(|c_i|^B)$, it is sufficient to know H^c modulo $\mathrm{O}(|c_i|^B) + \mathrm{O}(|x, y|^3)$.

For system H^c of proposition 6, modulo $\mathrm{O}(|c_i, \lambda|^3)$ terms, and writing λ instead of c_0 again, the reparametrization ρ reads

$$\begin{aligned}
u_1 = {} & (-\tfrac{1}{3}c_2^2 + \mathrm{O}(c_i^3)) + \lambda(d_1 - \tfrac{1}{3}c_1 d_1 - \tfrac{2}{3}c_2 d_2 + \mathrm{O}(c_i^2)) \\
& + \lambda^2(d_{12} - \tfrac{1}{3}d_2^2 - \tfrac{1}{3}d_1 d_4 + \mathrm{O}(c_i)) + \mathrm{O}(\lambda^3) \\
u_2 = {} & (\tfrac{2}{3}c_2 - \tfrac{4}{9}c_2 c_1 + \mathrm{O}(c_i^3)) + \lambda(-\tfrac{1}{3}d_2 + d_3 + \tfrac{1}{9}c_3 d_1 - c_5 d_1 + \tfrac{2}{9}c_1 d_2 - \tfrac{2}{3}c_1 d_3 \\
& + \tfrac{5}{9}c_2 d_4 - c_2 d_5 + \mathrm{O}(c_i^2)) + \lambda^2(-\tfrac{1}{3}d_{13} + d_{14} + \tfrac{2}{9}d_2 d_4 + \tfrac{1}{3}d_3 d_4 - d_3 d_5 \\
& + \tfrac{1}{9}d_1 d_6 - d_1 d_8 + \mathrm{O}(c_i)) + \mathrm{O}(\lambda^3).
\end{aligned}$$

The coordinate transformation ϕ, modulo $\mathrm{O}(|x, y|^3) + \mathrm{O}(|c_i, \lambda|^2)$ terms, reads

$$\begin{aligned}
x &\mapsto \tfrac{1}{3}c_2 - \tfrac{1}{3}d_2\lambda + (1 + \tfrac{1}{3}c_1 + \tfrac{1}{3}d_4\lambda)x + (\tfrac{1}{3}c_3 - \tfrac{1}{3}d_6\lambda)x^2 + (c_5 + d_8\lambda)y^2 \\
y &\mapsto (1 - \tfrac{1}{6}d_4\lambda + \tfrac{1}{2}d_5\lambda + \tfrac{1}{3}c_1)y + (\tfrac{1}{2}c_4 - \tfrac{3}{2}c_5 - \tfrac{1}{6}c_3 + \tfrac{1}{2}d_7\lambda - \tfrac{3}{2}d_8\lambda - \tfrac{1}{6}d_6\lambda)xy.
\end{aligned}$$

Remark 7 (relevant degree for H^n). To compute ρ up to second order, it suffices to know H^c modulo $\mathrm{O}(|c_i, \lambda|^3) + \mathrm{O}(|x, y|^3)$ terms. In turn, for this, H^n modulo $\mathrm{O}(|x, y|^7)$ terms suffices, as λ is a quadratic polynomial on the phase space of H^n. To compute ϕ up to terms given in proposition 7, it suffices to know H^c modulo $\mathrm{O}(|x, y|^5) + \mathrm{O}(|c_i, \lambda|^2)$ terms, and again H^n modulo $\mathrm{O}(|x, y|^7)$ terms suffices.

Remark 8 (singular circle). In appendix D, the singular circle of H^u is defined as the circular level set that touches the two saddle points arising for $u_2 < 0$ (see figure 2). By a topological argument, its pullback by ϕ must coincide with the singular circle of H^c, defined as the set of singular points of (3.2). Up to the order in x, y, c_i and λ that we computed ϕ and ρ in, we verified that they indeed do.

Proof of proposition 7. The first part is proved by inspecting the algorithm described in section 8, and algorithm 18 (section 9.2). The fact that H^c is required up to order $A+2$

in order to compute ϕ only up to degree A is due to the first derivatives of the central singularity being of second degree. Similarly, in order to fix ρ, it is sufficient to compute H^c up to degree two in (x, y) as the deformation directions associated to ρ_1 and ρ_2 are of degree two or less (namely x and y^2 respectively).

A little computer algebra yields the second part. □

6. BCKV normal form

BCKV theory classifies the family of systems H^r as two degree-of-freedom systems. For a given member of the family (i.e. for certain values of the coefficients) it provides a normal form system, which is itself a two degree-of-freedom system. This should be contrasted to the deformation H^u, classifying H^r as a family of *planar* systems; see remark 2.

We now digress on the significance of planar unfolding H^u for the original two degree-of-freedom system H^r, with the perturbation problem mentioned in the introduction in mind. Essentially λ is a phase-space variable, hence it has to be viewed as a distinguished parameter. Then H^u no longer is a versal unfolding, as the related morphisms treat λ as an ordinary parameter. Indeed, in the distinguished-parameter setting the class of allowable morphisms (reparametrizations) shrinks, increasing the number of (equivalence classes of) normal forms.

It turns out to be possible to use H^u to construct a suitable unfolding H^B (see theorem 9 below), corresponding to a generic path (surface) in a more general parameter space. In this setting many more parameters are needed for versality. The path arises when the coefficients of such a versal normal form are expressed as functions of the available parameters. Moreover, these parameters will be expressed in the original (physical) constants of the system. This gives the natural set-up for the aforementioned perturbation problem; see [10] for a general discussion.

First we give the necessary definitions, in the context of a general (compact) symmetry group Γ. In section 6.3 we specialize to the example system.

6.1. BCKV theory—definitions and main theorem

We first give a heuristic motivation for the form of allowable morphisms. Suppose that $H^B(x, y, \lambda, u)$ is a normal form of the two-degree-of-freedom system H^n. Here x and y are phase space variables, λ the distinguished parameter and u the ordinary parameters. We require a morphism inducing H^n from H^B to respect the phase space and parameters. This means, as usual, that the u-reparametrization may not depend on x and y, but neither on λ. Furthermore, λ-reparametrizations are required to be independent of x and y, because λ is a constant of motion both for H^n and the normal form H^B.

Second, the distinguished parameter λ is physically interpreted as *angular momentum*. By its nature, it is non-negative. It is therefore natural to require the λ-reparametrization to respect the zero level. These ingredients lead to the following definition; see appendix C for the notation.

Definition 8 (BCKV-restricted morphisms). Let two deformations $F \in \mathcal{E}^\Gamma_{n+r+s}$ and $G \in \mathcal{E}^\Gamma_{n+r+t}$ of $f = F(\cdot, \cdot, 0) \in \mathcal{E}^\Gamma_{n+r}$ be given, such that $f(0, \lambda) = 0$. F is said to be *induced from G by Γ-equivariant BCKV-restricted morphisms* if there exist germs of Γ-equivariant mappings $\Psi : \mathbb{R}^{n+r+s} \to \mathbb{R}^{n+r+t}$, $\Phi : \mathbb{R}^{r+s} \to \mathbb{R}^{r+t}$ and $\Theta : \mathbb{R}^s \to \mathbb{R}^t$ such that the

following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R}^{n+r}\times\{0\}^s & \xrightarrow{\hookrightarrow} & \mathbb{R}^{n+r+s} & \xrightarrow{\pi_1} & \mathbb{R}^{r+s} & \xrightarrow{\pi_2} & \{0\}^r\times\mathbb{R}^s & \xrightarrow{\hookrightarrow} & \mathbb{R}^{r+s} \\
\mathrm{Id}\downarrow & & \Psi\downarrow & \searrow^{F} & \Phi\downarrow & & \Theta\downarrow & & \Phi\downarrow \\
 & & & \mathbb{R} & & & & & \\
 & & & \nearrow_{G} & & & & & \\
\mathbb{R}^{n+r}\times\{0\}^t & \xrightarrow{\hookrightarrow} & \mathbb{R}^{n+r+t} & \xrightarrow{\tilde{\pi}_1} & \mathbb{R}^{r+t} & \xrightarrow{\tilde{\pi}_2} & \{0\}^r\times\mathbb{R}^t & \xrightarrow{\hookrightarrow} & \mathbb{R}^{r+t}
\end{array}$$

On $\mathbb{R}^{n+r}$, $\mathbb{R}^{n+r+s}$ and $\mathbb{R}^{n+r+t}$ the action of Γ is defined by trivially extending it on $\mathbb{R}^n$.

In formulae, this amounts to: there exist $\phi : \mathbb{R}^{r+s} \to \mathbb{R}^r$, $\psi : \mathbb{R}^{n+r+s} \to \mathbb{R}^n$ such that $\Phi = (\phi, \Theta)$, $\Psi = (\psi, \phi, \Theta)$, $\psi(x, \lambda, 0) = x$, $\phi(\lambda, 0) = \lambda$, $\phi(0, u) = 0$, $\Theta(0) = 0$, and $F(x, \lambda, u) = G(\psi(x, \lambda, u), \phi(\lambda, u), \Theta(u))$. The ϕ, ψ and Θ are the analogues of ϕ and ρ_i of (5.2), but obey more restrictions. Morphisms (Ψ, Φ, Θ) as above are called *BCKV-restricted morphisms.*

We are looking for *versal* deformations, i.e. deformations such that every other deformation of the same germ can be induced from it. In [10, theorem 11], versal deformations with respect to BCKV-restricted morphisms are characterized; we give the Γ-equivariant version here.

Theorem 9 (BCKV-restricted versal deformations). *Let $f \in \mathcal{E}^{\Gamma}_{n+r}$ be a family of germs of Γ-equivariant germs depending on a distinguished parameter $\lambda \in \mathbb{R}^r$. Let $f_0 \in \mathcal{E}^{\Gamma}_n : x \mapsto f(x, 0)$ have codimension c. Then:*

(1) f has a universal deformation with respect to Γ-equivariant BCKV-restricted morphisms iff f, considered as a deformation of f_0, is versal with respect to ordinary Γ-equivariant morphisms.

(2) If $F(x, \lambda, u)$ is a (uni)versal deformation of f with respect to Γ-equivariant BCKV-restricted morphisms, then $F(x, 0, u)$ is a (uni)versal deformation of f_0 with respect to ordinary Γ-equivariant morphisms.

(3) If $f(x, \lambda)$ is a universal deformation of f_0 with respect to ordinary Γ-equivariant morphisms, then $r = c$ and $F : \mathbb{R}^{n+c+c} \to \mathbb{R}$ defined by

$$F(x, \lambda, u) = f(x, \lambda) + \sum_{j=1}^{c} u_j \frac{\partial f}{\partial \lambda_j}(x, 0)$$

is a universal deformation of f with respect to Γ-equivariant BCKV-restricted morphisms.

A universal deformation is a versal deformation with minimal number of parameters.

Proof. The proof for the nonequivariant case can be carried over to the present setting with obvious changes; see [10]. □

6.2. Path formulation

As the number of distinguished parameters is fixed, theorem 9 implies that when the central singularity f_0 has high codimension, there are no versal deformations with respect to Γ-equivariant BCKV-restricted morphisms.

However, we can view the system as a *subfamily* of a versally deformed system. The normal form then includes functions that describe the submanifold, embedded in the versal system's parameter space, that the system traces out. Bifurcations of the intersection of this submanifold with the bifurcation set yields additional information. This description

is usually called the path formulation, see [20, 3]. For this final reduction, we need the following.

Definition 10. *A BCKV-restricted reparametrization* is a mapping (ϕ, θ) with $\phi : \mathbb{R}^{r+s} \to \mathbb{R}^r$, $\theta : \mathbb{R}^s \to \mathbb{R}^s$ such that $\phi(0, u) = 0$, $\theta(0) = 0$.

Note that it is not required that $\phi(\lambda, 0) = \lambda$.

The following lemma is a slightly stronger version of [10, lemma 7], and is used in the proof of proposition 12 below.

Lemma 11 [10]. *Let $r \leqslant s$, let $\pi : \mathbb{R}^s \to \mathbb{R}^r$ be a projection onto some r-dimensional subspace of $\mathbb{R}^s$, and let $\tilde{h} : (\lambda, u) \in \mathbb{R}^{r+s} \to \mathbb{R}^s$ be a map (a 'normal form') such that $\tilde{h}(0, 0) = 0$ and the derivatives $D_\lambda(\pi \circ \tilde{h}(\lambda, u))|_{\lambda=u=0}$ and $D_u(\pi \circ \tilde{h}(\lambda, u))|_{\lambda=u=0}$ both have rank r. Then, for any $h \in \mathcal{E}(r + s, s)$ with $h(0, 0) = 0$ there exists a BCKV-restricted reparametrization $\Upsilon = (\phi, \theta)$ such that*

$$\pi(h(\lambda, u)) = \pi(\tilde{h}(\Upsilon(\lambda, u))).$$

Moreover, if $D_\lambda \pi \circ h(\lambda, u)$ and $D_u \pi \circ h(\lambda, u)$ both have rank r (at $\lambda = u = 0$), then Υ can be chosen invertible.

Proof. As $D_u \pi \circ \tilde{h}(0, u)$ has full rank, and $\pi \circ h(0, 0) = \pi \circ \tilde{h}(0, 0)$, by the inverse function theorem there exists a function $\theta(u)$ with $\theta(0) = 0$ such that $\pi \circ h(0, u) = \pi \circ \tilde{h}(0, \theta(u))$.

Now $D_\lambda \pi \circ \tilde{h}(\lambda, \theta(u))$ has full rank, and moreover $\pi \circ h(0, u) = \pi \circ \tilde{h}(0, \theta(u))$, for all u, so, applying the inverse function theorem again, we find a function $\phi(\lambda, u)$ with $\phi(0, u) = 0$, such that $\pi \circ h(\lambda, u) = \pi \circ \tilde{h}(\phi(\lambda, u), \theta(u))$.

The last remark follows by applying the lemma with the roles of h and $\tilde{h}$ interchanged. □

Proposition 12. *Let $g(x, \lambda, u) : \mathbb{R}^{n+r+s} \to \mathbb{R}$ be a generic Γ-invariant germ, and assume that $f(x, \sigma_1, \ldots, \sigma_s)$ is a universal deformation of $g(x, 0, 0)$ using unrestricted Γ-equivariant morphisms. Then there exists a BCKV-restricted reparametrization Υ such that for the normal form*

$$F(x, \lambda, u) := f(x, \lambda_1 + u_1, \ldots, \lambda_r + u_r, \tilde{\sigma}_{r+1}(\lambda, u), \ldots, \tilde{\sigma}_s(\lambda, u)),$$

where $\tilde{\sigma}_i$, $i = r + 1, \ldots, s$, are some functions, we have that g can be induced from $F \circ (\pi_x, \Upsilon)$ using BCKV-restricted Γ-equivariant morphisms. Here π_x denotes the projection $\pi_x : (x, \lambda, u) \mapsto x$.

Proof. Let $h(\lambda, u)$ be a reparametrization, and $\Phi(x, \lambda, u)$ a coordinate transformation, such that $f(\Phi(x, \lambda, u), h(\lambda, u)) = g(x, \lambda, u)$. Define $\tilde{h}_i(\lambda, u) := \lambda_i + u_i$ if $1 \leqslant i \leqslant r$ and $\tilde{h}_i(\lambda, u) := u_i$ if $r + 1 \leqslant i \leqslant s$, and set $\pi(\sigma_1, \ldots, \sigma_s) = (\sigma_1, \ldots, \sigma_r)$. The lemma now applies. By genericity we may assume that the relevant derivatives have rank r, so we find an invertible BCKV-restricted reparametrization Υ such that $h_i(\lambda, u) = \tilde{h}_i(\Upsilon(\lambda, u))$ for $i = 1, \ldots, r$, which means that for

$$F(x, \lambda, u) := f(x, \lambda_1 + u_1, \ldots, \lambda_r + u_r, h_{r+1} \circ \Upsilon^{-1}(\lambda, u), \ldots, h_s \circ \Upsilon^{-1}(\lambda, u)),$$

we have $g(x, \lambda, u) = F \circ (\pi_x, \Upsilon) \circ (\Phi, \pi_\lambda, \pi_u)$, where $\pi_\lambda : (x, \lambda, u) \mapsto \lambda$ and $\pi_u : (x, \lambda, u) \mapsto u$, proving the proposition. □

6.3. BCKV normal form of H^c

The constructive proofs of proposition 12 and lemma 11 provide an algorithm for computing the BCKV normal form. Using the reparametrizations of proposition 7, we choose for Υ the following:

$$\Upsilon(\lambda, c_i) = (u_2(\lambda, c_i) - u_2(\lambda, 0), c_1, u_2(0, c_i), c_3, c_4, \ldots),$$

which is invertible, and then $\tilde{\sigma}_2 := u_1 \circ \Upsilon^{-1}$. The result is as follows.

Theorem 13 (BCKV normal form). *The system H^c of proposition 6 is equivalent, modulo BCKV-restricted morphisms and reparametrizations, and modulo terms of order $\mathrm{O}(|c_i, \lambda|^3)$, to*

$$\begin{aligned} H^B(x, y, \lambda, c_i) = {} & x(x^2 + y^2) + y^2(\lambda + c_2) + x \\ & \times\Bigg(-\frac{c_2^2}{3} + \mathrm{O}(c_i^3) + \left(d_1 - \frac{c_1 d_1}{3} - \frac{2c_2 d_2}{3} + \mathrm{O}(c_i^2)\right)\beta\lambda \\ & + \left(d_{12} - \frac{d_2^2}{3} - \frac{d_1 d_4}{3}\right)\beta^2\lambda^2 \\ & + \left(\frac{d_1 d_{13}}{3} - d_1 d_{14} - \frac{2d_1 d_2 d_4}{9} - \frac{d_1 d_3 d_4}{3} + d_1 d_3 d_5 - \frac{d_1^2 d_6}{9} + d_1^2 d_8 + \mathrm{O}(c_i)\right) \\ & \times\beta^3\lambda^2 + \mathrm{O}(\lambda^3)\Bigg) \end{aligned}$$

where $\beta = 9(-3d_2 + 9d_3 + c_1(2d_2 - 6d_3) + c_2(5d_4 - 9d_5) + c_3 d_1 - 9c_5 d_1)^{-1} + \mathrm{O}(c_i^2)$. The coefficient of x expressed in the a_i reads

$$\begin{aligned} \tilde{\sigma}_2 = {} & -\frac{1}{48\alpha}(1 - 2a_1)^2 + \mathrm{O}((1 - 2a_1)^3) \\ & + \lambda\Bigg(\frac{-9a_2^2}{2\alpha\delta} + \frac{1}{4\alpha\delta^2}(2a_2^4 - 144a_3^2 + 5a_4^2 - 6a_3(a_4 - 16a_5) + 2a_4 a_5 \\ & - 16a_5^2 + a_2^2(-30a_3 + a_4 + 26a_5) + 3a_2 a_6)(1 - 2a_1) + \mathrm{O}((1 - 2a_1)^2)\Bigg) \\ & + \mathrm{O}(\lambda^2) \end{aligned}$$

where $\alpha = \sqrt[3]{2a_2^2}$ and $\delta = 2a_2^2 + 6a_3 - a_4 - 2a_5$.

Remark 9 (nondegeneracy conditions). The BCKV normal form is only well defined if β is, i.e. if $d_2 - 3d_3 \neq 0$. This translates into $a_2 \neq 0$ and $a_2^2 \neq (1 + 2a_1)(3a_3 + a_4 - a_5)$. For the spring-pendulum the first condition is trivial, the second one is not.

7. The planar system

In this section we regard the system as a *planar* system depending on the detuning parameter $1 - 2a_1$ and distinguished parameter λ. This gives an integrable approximation to the dynamics of the iso-energetic, or equivalently† iso-λ, Poincaré map.

In section 5 we found the planar versal normal form

$$H^u(x, y, u_1, u_2) = x(x^2 + y^2) + u_1 x + u_2 y^2$$

† See [10, section 4.2].

with saddle-centre and Hamiltonian pitchfork bifurcations occurring along the curves $u_1 = 0$ and $u_1 + 3u_2^2 = 0$ respectively; see figure 2. Plugging in the reparametrizations found in section 5 yields implicit equations for these bifurcation curves in the $(\lambda, 1 - 2a_1)$-plane. For practical reasons we choose to solve for λ in terms of $1 - 2a_1$. The result is as follows.

Proposition 14. *In the reduced system H^c of proposition 6, saddle-centre and Hamiltonian pitchfork bifurcations respectively occur along the following curves in parameter space:*

$$u_1 = 0: \qquad \lambda = -\frac{(1-2a_1)^2(4(5+8a_1)a_2^2 + (4a_1^2-1)(24a_3+5a_4-8a_5))}{3456(1+2a_1)a_2^4} + \mathrm{O}((1-2a_1)^3) \tag{7.1}$$

$$u_1 + 3u_2^2 = 0: \qquad \lambda = \frac{(1-2a_1)^2}{64a_2^2} + \frac{(a_2^2-a_4)(1-2a_1)^3}{128a_2^4} + \mathrm{O}((1-2a_1)^4). \tag{7.2}$$

Remark 10 (phantom bifurcation). The parameter λ is non-negative, and close to resonance ($a_1 \approx \frac{1}{2}$) the solution (7.1) is negative. In the system H^0, therefore, the corresponding bifurcation does not occur. This conclusion also follows from the observation that at the bifurcation (7.1) the singular circle disappears (see appendix D), whereas H^r exhibits this singularity for all parameter values (see remark 1).

The second solution does define a bifurcation, however. We continue with a description of it.

7.1. Bifurcations and dynamical implications

First we discuss the bifurcation of the reduced system H^c in the plane. If we let a_1 deviate sufficiently far from the resonant value $\frac{1}{2}$, the corresponding points in the (u_1, u_2)-plane in figure 2 will trace out a line that crosses the parabola twice, as u_1 is always negative.

Assume the parabola is crossed from below. Then at first the system has one maximum inside the singular circle, and a saddle point outside it. After the first Hamiltonian pitchfork bifurcation, two saddle points have formed on the singular circle, together with a minimum inside, with no critical points outside. The two saddle points have a heteroclinic connection because of the $\mathbb{Z}_2$-symmetry.

The second bifurcation destroys the maximum, leaving only a minimum inside the circle, and again a saddle outside of it.

7.1.1. Topological remarks. *A priori* the spring-pendulum lives on the fixed-energy submanifold in $\mathbb{R}^4$, in our case $\mathbb{S}^3$. This sphere is equivalent to $\mathbb{D}^2 \times \mathbb{S}^1$, modulo an identification on $\partial\mathbb{D}^2 \times \mathbb{S}^1$.

The normalized Hamiltonian H^n on $\mathbb{S}^3$ has a nondegenerate $\mathbb{S}^1$-symmetry, except on one $\mathbb{S}^1$-orbit where points have stabilizor $\mathbb{Z}_2$; see proposition 3. A model for this topology is the map $\mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$ given by

$$(x, y, \phi) \mapsto (\sqrt{1-r^2}\cos\phi, \sqrt{1-r^2}\sin\phi, x\cos 2\phi - y\sin 2\phi, x\sin 2\phi + y\cos 2\phi).$$

(Here $r^2 = x^2 + y^2$ and $\mathbb{D}^1 = \{r^2 \leqslant 1\}$.) This map is surjective, and injective on the interior of its domain. Fixing the $\mathbb{S}^1$-symmetry, it provides a correspondence between $\mathbb{S}^1$-symmetric functions on $\mathbb{S}^3$ and functions on $\mathbb{D}^2$ that are constant on $\partial\mathbb{D}^2$, that is, functions on $\mathbb{S}^2$. This justifies viewing the bifurcations described above on $\mathbb{S}^2$, and the remark in section 1.3 that $\mathbb{S}^3$ divided out by an $\mathbb{S}^1$-action gives $\mathbb{S}^2$. In this picture, the singular circle collapses to a single point on $\mathbb{S}^2$, and is referred to as the *pole*.

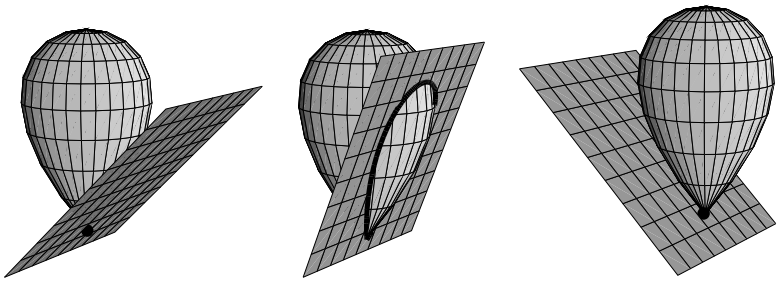

Figure 5. Intersections of the reduced phase space with level sets of H^n (here depicted as planes) through the singular point, for several values of the parameter a_1.

More precise information can be obtained by exploiting the algebraic structure. The normalized Hamiltonian can be written as $H^n = f(\xi_1, \xi_2, \xi_3, \xi_4)$ where $\xi_1 = z_1\bar{z}_1$, $\xi_2 = z_2\bar{z}_2$, $\xi_3 = z_1\bar{z}_2^2 + \bar{z}_1 z_2^2$ and $\xi_4 = (z_1\bar{z}_2^2 - \bar{z}_1 z_2^2)/\mathrm{i}$ are the basic (real) invariant polynomials. These basic invariants are not *free*, but satisfy the relation $\xi_1\xi_2^2 = (\xi_3^2 + \xi_4^2)/4$. Moreover, reality conditions imply $\xi_1 \geqslant 0$ and $\xi_2 \geqslant 0$. The quadratic part H_2 is an integral of H^n, and without loss of generality we may reduce to $H_2 = 2\xi_1 + \xi_2 = \epsilon$, where ϵ is some small positive number. Then, the relation between the invariants defines a two-dimensional manifold, the *reduced phase space*, in $\mathbb{R}^3$, namely $(\epsilon - 2\xi_1)^2\xi_1 = \frac{1}{4}(\xi_3^2 + \xi_4^2)$. Topologically it is a sphere, but has a cone-like singularity at $\xi_2 = 0$; see figure 5). This singularity has dynamical significance: it is always a fixed point.

We now interpret the bifurcations on this (topological) sphere. Levels of H^n are surfaces in $\mathbb{R}^3 \ni (\xi_1, \xi_3, \xi_4)$ and intersect the reduced phase space in a curve; again, see figure 5. As in the previous section, suppose we traverse the (u_1, u_2)-plane on the left of the u_2-axis crossing the parabola of Hamiltonian pitchfork bifurcations twice. First, the Hamiltonian has one maximum somewhere on $\mathbb{S}^2$, and a minimum at the pole. The heteroclinic connection appearing in the planar normal form after the first bifurcation corresponds to a level curve passing through the pole. In this situation, the pole is no longer a minimum. At the second bifurcation the heteroclinic connection disappears, implying that the pole is an extremum again, now a maximum.

7.1.2. Dynamics of the spring-pendulum At the pole, L_1 is a maximum, corresponding to the pendulum moving vertically without swinging ($x_2 \equiv 0$, see figure 4). This periodic trajectory corresponds to the one with nontrivial stabilizor under the global $\mathbb{S}^1$-action, in other words, its period is half that of other periodic trajectories. Outside the parabola in figure 2 this motion is stable, corresponding to a minimum or a maximum of H^n, but close to resonance it is unstable.

In the latter situation, the spring-pendulum exhibits two stable periodic trajectories. The lower mass traces out a $\cup$-shaped and $\cap$-shaped path, respectively. Far away from resonance, one of these turned into the now-stable vertical motion, while the other turned into a purely swinging motion ($x_1 \equiv$ constant in normalized coordinates). In ordinary coordinates this motion is special in that m_1 oscillates with the same frequency as m_2, instead of roughly twice that in the general situation.

7.2. Comparison with numerical simulations

To check the results above, we integrated H^0 numerically, and plotted the iso-energetic Poincaré section $\phi_2 = 0$ for varying values of the energy and detuning parameter a_1. The

Table 2. Comparison of bifurcation values, found numerically and analytically.

H	a_1	a_2	a_3	$\lambda_{\text{measured}}$	$\lambda_{\text{predicted}}$
0.01	0.538 5	0.07	0.001	0.020	0.018
0.01	0.463	0.07	0.001	0.020	0.018
0.001	0.512 55	0.07	0.001	0.002 00	0.001 98
0.001	0.487 6	0.07	0.001	0.002 00	0.001 99
0.001	0.536 5	0.2	0.001	0.002 00	0.002 01
0.001	0.465	0.2	0.001	0.002 00	0.001 98

resulting pictures, shown in figure 1, are similar to those found by computation and are shown in figure 3. The differences (chaotic regions, subharmonics) are caused by the flat perturbation between the normalized H^0 and H^n, destroying integrability in H^0; see section 1.4.

To check (7.2), we located some bifurcation points, by varying the detuning parameter a_1 for fixed H, a_2 and a_3. Other a_i were set to zero. The results are given in table 2. For these values of the energy, $\lambda = 2H$ to good approximation. The final column gives the bifurcation value of λ given by (7.2) in each situation. The agreement with the measured value of λ is very good, especially for small H, as expected.

8. Computing universal deformation morphisms

This section is devoted to describing an algorithm that computes morphisms inducing a given deformation from some universal deformation. First we duscuss the case without symmetry. This algorithm was first described in [22], where it was taken for granted that the infinitesimal stability equation could effectively be solved. In section 9 we solve this equation using Gröbner basis techniques. At the end we discuss the modifications to the algorithm that incorporate the effects of a symmetry group. We note that Gröbner bases appear more often in the context of dynamical systems, commonly in relation to finding fundamental invariant polynomials; see e.g. [11, 17, 18].

Suppose that a deformation $F(x, \mu_j)$ of some germ $f(x) \in \mathcal{E}_n$ is given, that is, $F(x, 0)$, $x \in \mathbb{R}^n$. Suppose further that a versal deformation of $f(x)$ has been found. In practice this means that some polynomials P_j exist such that

$$\left\langle \frac{\partial f}{\partial x_i} \right\rangle_{\mathcal{E}_n} + \text{span}_{\mathbb{R}}\{P_j\} = \mathcal{E}_n, \tag{8.1}$$

(but see appendix C for a general definition) and then $G(x, u) := f(x) + \sum_j u_j P_j$ is a versal deformation. With this explicit form in hand, it is possible to investigate the bifurcation set. This gives a catalogue of all bifurcations that may occur in the original deformation F.

This does not, however, give information about *which* bifurcations actually do occur in F, and for which parameter values, nor about the *location* of critical points in phase space. For this kind of information we need the morphism inducing F from G. This section describes an algorithm that computes this morphism.

8.1. An algorithm computing deformation morphisms

The algorithm presented here was taken from [22]. It is an iterative algorithm, computing the solution degree by degree in the *parameters*. It constitutes a constructive proof of the

existence of a formal solution, and an explicit algorithm for finding such a solution up to any desired degree.

The assumption that $G = f(x) + \sum \mu_j P_j(x)$ is a versal deformation implies that (8.1) holds. It is equivalent to the statement that for any germ $g \in \mathcal{E}_n$, we can find germs $\alpha_i(x) \in \mathcal{E}_n$ and real numbers β_i that solve the *infinitesimal stability equation* (see [19]):

$$\sum_i \alpha_i(x) \frac{\partial f}{\partial x_i} + \sum_i \beta_i P_i(x) = g(x). \tag{8.2}$$

In the case of our model $G = x(x^2 + y^2) + u_1 x + u_2 y^2$, this becomes (5.1). The result of this section is the following.

Proposition 15 (computing versal deformation morphisms). *Let $F(x, \mu_j)$, $x \in \mathbb{R}^n$ and $j = 1, \ldots, k$ be a deformation of $f(x) = F(x, 0)$, and let $G(x, u_j)$, $j = 1, \ldots, d$ be a versal deformation of $f(x)$. Assume that we have an algorithm that, for given f, P_i and g solves the infinitesimal stability equation (8.2) modulo $\mathrm{O}(x^d)$ terms. Then the algorithm presented below computes, for any m, a diffeomorphism $\phi(x, \mu_j)$ and a reparametrization $\rho(\mu_j)$ such that $\phi(x, 0) = x$, $\rho(0) = 0$ and*

$$F(x, \mu_j) = G(\phi(x, \mu_j), \rho(\mu_j)) + \mathrm{O}(x^d) + \mathrm{O}(\mu^m). \tag{8.3}$$

The morphism (ϕ, ρ) is said to induce F from G.

An algorithm to solve the inifinitesimal stability equation is presented in section 9.

We make the inessential assumption that the versal deformation G is of the form $G(x, u_j) = f(x) + \sum_{j=1}^{d} \mu_j P_j(x)$. The algorithm can be easily adapted to cope with more general deformations, but allowing these does not yield stronger results, and does clutter the notation.

So we are to find a deformation morphism inducing F from G. The line of attack is to expand ϕ and ρ as formal power series in the *parameters* μ_j, and to solve (8.3) iteratively for increasing order in μ. We define

$$\phi(x, \mu) := \sum_{i \geqslant 0} \phi_i(x, \mu), \qquad \rho(\mu) := \sum_{i \geqslant 0} \rho_i(\mu)$$

where ϕ_i and ρ_i are homogeneous of degree i in the parameters μ, and denote the solutions up to and including order p in μ by superscripting the mappings by p:

$$\phi^p(x, \mu) := \sum_{i=0}^{p} \phi_i(x, \mu), \qquad \rho^p(\mu) := \sum_{i=0}^{p} \rho_i(\mu).$$

Now assume (8.3) has been solved up to μ-order p, that is,

$$F(x, \mu_j) = G(\phi^p(x, \mu_j), \rho^p(\mu_j)) + \mathrm{O}(\mu^{p+1}) + \mathrm{O}(x^d).$$

(For $p = 0$ this is true if we set $\phi^0(x, \mu) := x$ and $\rho^0(\mu) := 0$.) To solve (8.3) up to order $p + 1$ we add $(p + 1)$st-order terms in μ:

$$\begin{aligned} G(\phi^{p+1}, \rho^{p+1}) &= G(\phi^p + \phi_{p+1}, \rho^p + \rho_{p+1}) = G(\phi^p, \rho^p) + D_x G(\phi^p, \rho^p) \cdot \phi_{p+1} \\ &\quad + D_\mu G(\phi^p, \rho^p) \cdot \rho_{p+1} + \mathrm{O}(|\phi_{p+1}|^2 + |\rho_{p+1}|^2) \\ &= G(\phi^p, \rho^p) + D_x f(x) \cdot \phi_{p+1} + D_\mu G(x, \mu)|_{\mu=0} \cdot \rho_{p+1} + \mathrm{O}(\mu^{p+2}). \end{aligned}$$

To obtain the last equality we used the estimates $\phi^p(x, \mu) = x + \mathrm{O}(\mu)$, $\phi_{p+1}(x, \mu) = \mathrm{O}(\mu^{p+1})$, $\rho^p(\mu) = \mathrm{O}(\mu)$ and $G(x, \mu) = f(x) + \mathrm{O}(\mu)$. Using this we write (8.3) up to

μ-order $p+1$ in the following way, expanding inner products of vectors in terms of their component functions, and using that $G(x,u) = f(x) + \sum_k u_k P_k(x)$:

$$F(x,\mu_j) - G(\phi^p(x,\mu_j),\rho^p(\mu_j)) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x)\cdot\phi_{p+1,k}(x,\mu_j) + \sum_{k=1}^{d} P_k(x)\cdot\rho_{p+1,k}(\mu_j) + \mathrm{O}(|\mu_j|^{p+2}). \tag{8.4}$$

The left-hand side of (8.4) does not contain terms of order less than $p+1$ in μ, as we assumed that ϕ^p, ρ^p solved (8.3) up to order p. We can solve (8.4) by equating coefficients of $\mu^\sigma = \mu_1^{\sigma_1}\cdots\mu_d^{\sigma_d}$ left and right, where $\sigma_1+\cdots+\sigma_d = p+1$. For each term μ^σ we obtain an equation of the form (8.2), and by the universality condition each of those equations can be solved. This proves existence of a formal solution to (8.3).

Section 9 presents an algorithm to compute a solution to (8.2), up to any desired degree in x.

8.2. Universal deformations with symmetry

In the presence of a symmetry group Γ, the versal deformation condition (8.1) changes into

$$T^\Gamma(f) + \mathbb{R}\{P_j\} = \mathcal{E}_n^\Gamma. \tag{8.5}$$

Here $T^\Gamma(f)$ is the Γ-equivariant tangent space to f. It is the ideal generated by $\{\boldsymbol{v}_i f\}$, where the $\boldsymbol{v}_i$ are the generators of the $\mathcal{E}_n^\Gamma$-module of Γ-equivariant vector fields on $\mathbb{R}^n$ (see appendix A).

When $\Gamma = \mathbb{Z}_2$ with action $(x,y)\mapsto(x,-y)$, the generators $\boldsymbol{v}_i$ are $\boldsymbol{v}_1 = \frac{\partial}{\partial x}$ and $\boldsymbol{v}_2 = y\frac{\partial}{\partial y}$. The discussion above can be copied almost verbatim, and in the end we have to solve several instances of an equation of the form

$$\sum_{k=1}^{N}\alpha_i(x)(\boldsymbol{v}_i f)(x) + \sum_{k=1}^{d}\beta_i P_i(x) = g(x). \tag{8.6}$$

Here N is the number of generators $\boldsymbol{v}_i$. The functions $\boldsymbol{v}_i f$, P_i and g are all Γ-invariant, and we are to find Γ-invariant α_i's and β_i's that solve (8.6)

9. Solving the infinitesimal stability equation using singularity Gröbner bases

In section 8 we reduced the algorithmic problem of finding deformation morphisms to solving the infinitesimal stability equation (8.2) several times over. Solving this equation is similar to ordinary division. We present an efficient algorithm that uses ideas from Gröbner basis theory. Similar algorithms have been proposed in, e.g. [12, 27]. First the case without symmetry is discussed. In section 9.5 symmetry is incorporated.

To start, we reformulate our problem in a slightly more general way. The infinitesimal stability equation can be cast in the form

$$\sum_{i=1}^{k}\alpha_i f_i + \sum_{i=1}^{m}\beta_i r_i = g, \tag{9.1}$$

where we want to find $\alpha_i \in \mathcal{E}_n$ and $\beta_i \in \mathbb{R}$ in terms of a given $g\in\mathcal{E}_n$. The $f_1,\ldots,f_k,r_1,\ldots,r_m\in\mathcal{E}_n$ are considered fixed, and we suppose that

$$\langle f_i\rangle_{\mathcal{E}_n} + \mathrm{span}_{\mathbb{R}}\{r_1,\ldots,r_m\} = \mathcal{E}_n, \tag{9.2}$$

implying existence of a solution. We suppose, for clarity of exposition, that the number m of complementing functions r_i is minimal.

In practice we are interested in the solution up to some given degree; say in (9.1) we want to know α_i and β_i modulo degree d terms. Then the question reduces to linear algebra: if V denotes the finite-dimensional vector space of truncated power series, the map $(\alpha_i, \beta_i) \to \sum \alpha_i f_i + \sum \beta_i r_i$ is a linear map from $V \oplus \cdots \oplus V \oplus \mathbb{R}^m \to V$, surjective by assumption. Although it is possible to find a solution by straightforward Gaussian elimination, this is laborious. We therefore seek a more efficient algorithm.

Below we give an algorithm that effects the splitting

$$g = \sum_i \alpha_i f_i + r + \mathrm{O}(x^d). \tag{9.3}$$

The first term lies in the ideal $I = \langle f_i \rangle$. The algorithm takes g and a set of generators of I as input, and produces truncated power series α_i and r. The output r of the algorithm depends on the particular set of generators used, and on their ordering. Specifically, r does not *a priori* lie in an m-dimensional vector space, but generally in a larger one.

By adding certain functions to our initial set of generators $\{f_1, \ldots, f_k\}$, but in such a way that the ideal they generate remains the same, the output r can be forced to lie in an m-dimensional vector space. Moreover, it becomes uniquely determined, in a sense explained below. A set of generators that make our algorithm behave in this nice way is called a *singularity Gröbner basis*. The name is taken from [12].

Finally, a small computation brings the output in the desired form (9.1).

Remark 11 (fixing the zero level). In our application, g of (9.1) vanishes at the origin, as we deal with *potential* deformations. This generally renders one of the deformation directions r_i redundant. Apart from this detail, the discussion remains applicable to our situation without change.

9.1. Definitions

In order to write down the division algorithm we need the following concepts. They are adapted from [13].

9.1.1. Monomial ordering.

Definition 16. A *monomial ordering* on $\mathbb{Z}^n_{\geqslant 0}$, or equivalently on monomials x^α, is an ordering $<$ such that:

(i) $<$ is a linear ordering, i.e. for every α, β exactly one of $\alpha = \beta$, $\alpha < \beta$, $\beta < \alpha$ holds.

(ii) If $\alpha < \beta$ then $\alpha + \gamma < \beta + \gamma$.

(iii) $<$ is a well-ordering, i.e. every nonempty subset of $\mathbb{Z}^n_{\geqslant 0}$ has a smallest element under $<$.

In our computations we use the following ordering: $x^\alpha < x^\beta$ if either the total degree of x^α is smaller than the total degree of x^β, or the total degrees are equal and x^α precedes x^β in lexicographic ordering. For example, $x^2y < xy^2$ as xxy occurs before xyy in a dictionary. More precisely, the relation $\alpha < \beta$ holds if:

- $|\alpha| < |\beta|$, or
- $|\alpha| = |\beta|$, and for some j we have $\alpha_i = \beta_i$ for $i = 1, \ldots, j-1$ and $\alpha_j > \beta_j$.

Here $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

9.1.2. Lowest term of a power series. The following concepts are the 'opposite' of the LT, LM and LC used with ordinary Gröbner bases [13]. In that context they stand for *leading term*, *leading monomial* and *leading coefficient* respectively, and they refer to the greatest monomial occurring in a polynomial. In the context of truncated power series, the concept of *greatest* monomial is not well defined. However, the *smallest* monomial is, and turns out to be useful.

Definition 17.

(i) $\mathbf{MM}(f)$ *is the minimal monomial occurring in* f*, with respect to the monomial ordering.*

(ii) $\mathbf{MC}(f)$ *is the coefficient associated to the monomial* $\mathbf{MM}(f)$.

(iii) $\mathbf{MT}(f)$ *is the term associated to* $\mathbf{MM}(f)$*, that is,* $\mathbf{MT}(f) = \mathbf{MC}(f) \cdot \mathbf{MM}(f)$.

(iv) multideg(f) *is the exponent of the monomial* $\mathbf{MM}(f)$*, an element of* $\mathbb{Z}^n_{\geqslant 0}$.

(v) A monomial x^α *is said to divide a monomial* x^β*, denoted by* $x^\alpha | x^\beta$*, if* $\beta - \alpha$ *is a vector with non-negative entries, and then* $x^\beta / x^\alpha := x^{\beta-\alpha}$.

9.2. The division algorithm

Let a degree bound d be given. The following algorithm is a first step towards solving (9.1) modulo terms of degree d or higher in g.

Algorithm 18 (division of g through $\{f_i\}$).

Input: integer d, power series $g, f_1, \ldots, f_k$ truncated at degree d.

Output: power series $r, q_1, \ldots, q_k$ truncated at degree d, such that

$$g = \sum_{i=1}^{k} q_i f_i + r \qquad \text{modulo terms of degree } d \text{ and higher} \tag{9.4}$$

$$r \in \operatorname{span}_{\mathbb{R}}\{x^\alpha : \mathbf{MM}(f_i) \not| \, x^\alpha \forall i\}. \tag{9.5}$$

Algorithm:

$h \leftarrow g$
Reduce h modulo terms of degree d or higher
$r \leftarrow 0$
$q_i \leftarrow 0 \qquad (i = 1, \ldots, k)$
While $h \neq 0$ do the following.
    If $\mathbf{MM}(f_i) | \mathbf{MM}(h)$ for some i, then
        $q_i \leftarrow q_i + \mathbf{MT}(h)/\mathbf{MT}(f_i)$
        $h \leftarrow h - (\mathbf{MT}(h)/\mathbf{MT}(f_i))\, f_i$
        Reduce h modulo terms of degree d or higher
    Else
        $r \leftarrow r + \mathbf{MT}(h)$
        $h \leftarrow h - \mathbf{MT}(h)$
    Endif
Endwhile.

(The symbol $\leftarrow$ is the assignment operator and should be read as 'becomes'.)

Proof. Termination is guaranteed because each pass through the While-loop removes the minimal term from h and does not introduce smaller terms. As there are only finitely many

monomials of degree less than d, this implies termination. The body of the While-loop maintains the following invariant:

$$h + \sum_i q_i f_i + r = g \qquad \text{modulo terms of degree } d \text{ and higher}$$

implying (9.4) when the algorithm is finished. □

The reductions modulo degree-d terms are necessary if the monomial ordering is not compatible with the degree, for in that case $\mathbf{MM}(h)$ might be of degree d or higher even if there are still terms of lower degree present in h, and the algorithm might not terminate.

We know *a priori* that the output r lies in the vector space (9.5). The following example shows that this not necessarily determines r uniquely.

9.2.1. Example. Taking $f_1 = x^2 + y^2$, $f_2 = xy$ and the monomial ordering we described before, then the input $g = y^3$ gives output $q_1 = q_2 = 0$, $r = y^3$. On the other hand, we have

$$y^3 = yf_1 - xf_2$$

showing that $q_1 = y$, $q_2 = x$ and $r = 0$ also satisfy the output criteria. It follows that for these f_i the output criteria do not determine r uniquely.

A very similar problem is encountered when trying to solve the polynomial ideal membership problem. This problem is solved using Gröbner bases. Our problem can be solved in much the same way, using a modification of Gröbner basis ideas; see [12, 27].

9.3. Singularity Gröbner bases

We define a singularity Gröbner basis (SGB), to be a set of generators for a given ideal that makes the division algorithm above behave nicely.

Definition 19. $G = \{f_1, \ldots, f_k\}$ is an SGB for an ideal I if $I = \langle G \rangle$, and the output r of algorithm 18 is uniquely determined by (9.4) and (9.5), for all g.

For example, $\{x^2 + y^2, xy\}$ is not an SGB, as we saw above. The set $\{x^2 + y^2, xy, y^3\}$ is, but we cannot prove this yet; see appendix E.

We now give an intrinsic characterization of SGBs. To do so we need one definition.

Definition 20. Let $I \in \mathcal{E}_n$ be an ideal. $\mathbf{MM}(I) := \langle \{\mathbf{MM}(f) : f \in I\} \rangle$.

Proposition 21 (intrinsic characterization of SGBs). *$G = \{g_1, \ldots, g_k\}$ is an SGB for the ideal $I = \langle G \rangle$ iff $\langle \mathbf{MM}(g_1), \ldots, \mathbf{MM}(g_k) \rangle = \mathbf{MM}(I)$.*

We check the example above. G is not an SGB, and indeed: $y^3 \in \langle G \rangle$ because $y^3 = y(x^2+y^2) - x(xy)$, so $y^3 \in \mathbf{MM}(\langle G \rangle)$, but $y^3 \notin \langle \mathbf{MM}(x^2 + y^2), \mathbf{MM}(xy) \rangle = \langle x^2, xy \rangle$.

Proof of proposition 21.

(1) Assume that G is an SGB. Let $g \in \langle G \rangle$, and apply the division algorithm to g and the generators g_i. Because the output r is uniquely determined, it must be zero. This implies that the algorithm never executed the Else part of the If statement. In particular, in the first pass this means that $\mathbf{MT}(g_i) | \mathbf{MT}(g)$ for some i, in other words $\mathbf{MM}(g) \in \langle \mathbf{MM}(g_1), \ldots, \mathbf{MM}(g_k) \rangle$, or, as g is a general element of $\langle G \rangle$, $\mathbf{MM}(\langle G \rangle) \subset \langle \mathbf{MM}(g_1), \ldots, \mathbf{MM}(g_k) \rangle$. The reverse inclusion is immediate. This proves the first part.

(2) Assuming that $\mathbf{MM}(\langle G \rangle) = \langle \mathbf{MM}(g_1), \ldots, \mathbf{MM}(g_k) \rangle$, we want to show that r is uniquely determined by (9.4) and (9.5). Suppose it is not, and write $g = \sum \alpha_i g_i + r =$

$\sum \alpha'_i g_i + r'$, or $\sum(\alpha_i - \alpha'_i)g_i = r' - r$ where $r - r' \neq 0$. We have $r' - r \in \langle G\rangle$, implying that $\mathbf{MM}(r' - r) \in \mathbf{MM}(\langle G\rangle)$, and invoking the assumption we find that $\mathbf{MM}(r' - r) \in \langle \mathbf{MM}(g_1), \ldots, \mathbf{MM}(g_k)\rangle$. This in turn implies that $\mathbf{MM}(g_i)|\mathbf{MM}(r' - r)$ for some i, contradicting (9.5), so $r = r'$. □

Every ideal has an SGB. In appendix E we give a constructive proof of this fact: an algorithm that adds elements to a given set of generators $G = \{g_1, \ldots, g_k\}$ so that it becomes an SGB. Each of these new elements g'_j are elements of the ideal $\langle G\rangle$, and in fact the algorithm can be adapted to supply γ_{ij} that explicitly express the new elements in the old ones: $g'_j = \sum_i \gamma_{ij} g_i$. With these γ_{ij} the output of the division algorithm,

$$g = \sum_i \alpha_i g_i + \sum_j \alpha'_j g'_j + r,$$

can be rewritten in the form

$$g = \sum_i \left(\alpha_i + \sum_j \gamma_{ij}\alpha'_j\right) g_i + r. \tag{9.6}$$

9.4. Solution of (9.1)

Our last task is to rewrite (9.6) in the form (9.1), that is, using the given r_i, instead of a remainder r of the form (9.5).

Remark 12. If we may choose the form of the versal deformation ourselves, we can control the r_i that occur in (9.1). If we choose $P_i = x^{\alpha_i}$, where the set of x^{α_i} forms the monomial basis of the vector space in (9.5), we get $r_i = x^{\alpha_i}$ and the output of the algorithm is automatically in the desired form.

Assume we have an SGB $\{g_1, \ldots, g_t\}$ for the ideal $\langle f_1, \ldots, f_k\rangle$, and let $\gamma_{ij} \in \mathcal{E}_n$ be such that $g_i = \sum_j \gamma_{ij} f_j$. As noted above, these γ_{ij} can be computed while computing the SGB; see appendix E.

Denote the vector space (9.5) by $G^\perp$, and its monomial basis by $\{x^\alpha : \alpha \in A\}$. Divide all r_i through the SGB, and use the γ_{ij} to get expressions

$$r_i = \sum_j c_{ij} f_j + \sum_{\alpha\in A} d_{i\alpha} x^\alpha.$$

Because $\langle f_i\rangle_{\mathcal{E}_n} + \mathrm{span}_{\mathbb{R}}\{r_1, \ldots, r_m\} = \mathcal{E}_n$, we know that the canonical projections of the r_i span $G^\perp$, that is, the matrix $d_{i\alpha}$ has rank $\#A$. So, with some linear algebra we can find matrices $c'_{\alpha j}$ and $d'_{\alpha j}$ such that

$$x^\alpha = \sum_j c'_{\alpha j} f_j + \sum_j d'_{\alpha j} r_j.$$

Now divide the g through the SGB. Using the γ_{ij} we get

$$g = \sum_i \alpha_i f_i + \sum_\alpha \beta_\alpha x^\alpha = \sum_i \left(\alpha_i + \sum_\alpha \beta_\alpha c'_{\alpha i}\right) f_i + \sum_j \left(\sum_\alpha \beta_\alpha d'_{\alpha j}\right) r_j$$

which is in the form (9.1) as desired.

9.5. The infinitesimal stability equation with symmetry

We now discuss how to modify the algorithm to make it applicable to the symmetric case. We assume that we are dealing with a compact symmetry group. This implies the existence of a finite Hilbert basis of invariants, and a normalized Haar-measure $\mathrm{d}\gamma$. Using this measure the operator $A : \mathcal{E}_n \to \mathcal{E}_n^\Gamma$ is defined as follows:

$$A(f) := \int_\Gamma f \circ \gamma \mathrm{d}\gamma.$$

(Here $f \circ \gamma$ is the composition of f with the linear action of γ on $\mathbb{R}^n$.) The operator A is called the Reynolds, or *averaging* operator; see [13]. It is the identity operator on $\mathcal{E}_n^\Gamma \subset \mathcal{E}_n$. For finite groups the integral reduces to a finite sum.

To state the symmetric division algorithm, some extra notation is useful. Let $f_i \in \mathcal{E}_n^\Gamma$, $i = 1, \ldots, k$ be the given divisors. Algorithm 37 extends this set to an SGB $f_i \in \mathcal{E}_n$, $i = 1, \ldots, m$. Note that the f_i need no longer be symmetric. The algorithm also yields γ_{ij} that express the f_i with $i > k$ in the original f_i.

Let R denote the $\mathbb{R}$-vector space, defined in (9.5), of possible r-outputs of algorithm 18. Finally, let R^Γ denote the $\mathbb{R}$-vector space of averaged rests $A(r)$, where r is the rest by division through $f_1, \ldots, f_m$ by algorithm 18 of the elements of $A(R)$. In general $R^\Gamma \subset A(R)$.

Algorithm 22 (symmetric division of $g \in \mathcal{E}_n^\Gamma$ through $\{f_1, \ldots, f_k\} \subset \mathcal{E}_n^\Gamma$.).

Input: integer d, power series g, f_i truncated at degree d.

Output: power series $r, q_1, \ldots, q_k \in \mathcal{E}_n^\Gamma$ truncated at degree d, such that

$$g = \sum_{i=1}^{m} q_i f_i + r \qquad \text{modulo terms of degree } d \text{ and higher} \tag{9.7}$$

$$r \in R^\Gamma. \tag{9.8}$$

Moreover, r only depends on the restclass of g modulo $\langle f_1, \ldots, f_k \rangle_{\mathcal{E}_n^\Gamma}$, and if $g \in R^\Gamma$ then $r = g$.

Algorithm:

Compute Gröbner basis $\{f_1, \ldots, f_m\}$ of $\langle f_1, \ldots, f_k \rangle_{\mathcal{E}_n^\Gamma}$, with γ_{ij}, using algorithm 37.
Regard g as element of $\mathcal{E}_n$, and apply algorithm 18 yielding r and $q_1, \ldots, q_m$.
$q_i \leftarrow q_i + \sum_{j=k+1}^{n} q_j \gamma_{ji} f_i \qquad (i = 1, \ldots, k)$
$q_i \leftarrow A(q_i) \qquad (i = 1, \ldots, k)$
$r \leftarrow A(r)$.

Proof. After the second step, we have $r \in R$ and $g = \sum_{i=1}^{m} q_i f_i + r$. As $f_i = \sum_{j=1}^{k} \gamma_{ij} f_j$, after the third step we have $g = \sum_{i=1}^{k} q_i f_i + r$. Taking the average on the left-hand side has no effect as $g \in \mathcal{E}_n^\Gamma$. We have $A(h_i f_i) = A(h_i) f_i$ because $f_i \in \mathcal{E}_n^\Gamma$, $i = 1, \ldots, k$. This implies $g = \sum_{i=1}^{k} A(h_i) f_i + A(r)$, proving (9.7).

The output r of algorithm 18 only depends on the restclass of g modulo $\langle f_1, \ldots, f_m \rangle_{\mathcal{E}_n}$ because $\{f_1, \ldots, f_m\}$ is a Gröbner basis. This restclass is uniquely defined by the restclass of g modulo $\langle f_1, \ldots, f_k \rangle_{\mathcal{E}_n^\Gamma}$, proving the first remark.

For any $r \in R^\Gamma$, let g_0 be an input yielding r as rest. As r and g_0 are in the same restclass, applying the algorithm to $g = r$ yields r again, by the first remark. This proves the second remark.

As R is the vector space of rest classes modulo $\langle f_i \rangle_{\mathcal{E}_n}$, $A(R)$ contains all rest classes modulo $\langle f_1, \ldots, f_k \rangle_{\mathcal{E}_n^\Gamma}$. By the first remark again, the set R^Γ is exactly the set of possible r-outputs of the algorithm. This proves (9.8) and the algorithm. □

9.5.1. Example. This example shows that $R^\Gamma \subsetneq A(R)$ in general. The dimension of R^Γ is important, as it determines the codimension of the singularity. In practice, the spanning vectors of R^Γ can be obtained by applying the symmetric division algorithm to the averaged basis elements of R (whenever these are nonzero), with some linear algebra to identify dependent elements.

Let $\Gamma = \mathbb{Z}_2$ with action $(x, y) \mapsto (y, x)$, and let $f_1 = x + y$ and $f_2 = xy$. A Gröbner basis for this ideal over $\mathcal{E}_n$, with a degree-lexicographic ordering and $x < y$, is $\{f_1, f_2, y^2\}$, and the space of rests is spanned by $\{1, y\}$. We find $A(R) = \text{span}_{\mathbb{R}}\{A(1), A(y)\} = \text{span}_{\mathbb{R}}\{1, \frac{1}{2}(x + y)\}$. The element $\frac{1}{2}(x + y)$ lies in the ideal, and applying the symmetric division algorithm we find $R^\Gamma = \text{span}_{\mathbb{R}}\{1\}$, properly included in $A(R)$.

9.5.2. Symmetric division in 1 : 2 *and* 2 : 2 *resonance cases.* The symmetry groups considered in this paper are $\Gamma = \mathbb{Z}_2$ and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, with actions $(x, y) \mapsto (x, \pm y)$ and $(x, y) \mapsto (\pm x, \pm y)$, related to the 1 : 2 and 2 : 2 resonance case respectively.

As it turns out, these cases are particularly easy. Both the Gröbner basis algorithm 37 and the division algorithm 18 automatically produce Γ-invariant outputs, and there is no need for further averaging. Moreover, the equality $R^\Gamma = A(R)$ holds.

The averaging procedure sends terms $x^a y^b$ with b odd (or either a or b odd, respectively) to zero. These terms never appear during execution of the algorithms. Averaging leaves all other terms untouched.

10. Conclusions

In this paper we analysed Hamiltonian systems using Birkhoff normal forms and equivariant singularity theory in the plane. Algorithms that compute both normal form and the associated coordinate transformations are well known for the Birkhoff case [14, 23, 25]. For the singularity case the algorithm of Kas and Schlessinger [22] is available. We managed to efficiently apply this using SGB techniques. These results are applicable to a large class of resonant Hamiltonian systems near equilibrium.

By the BCKV normal form we found the right setting for the perturbation problem, which largely predicts the dynamics and its bifurcations quantitatively in terms of the original physical quantities. As a motivating example we applied the method to a spring-pendulum model near 1 : 2 resonance. In this example the intermediate unfolding H^u turned out to be the (equivariant) hyperbolic umbilic (D_4^+).

Elsewhere we intend to compare our method with Duistermaat's [16], who uses a slightly different equivalence relation between Hamiltonians, resulting in quite different codimensions for certain unfoldings. It is therefore interesting to compare results obtained in both ways.

Our overall aim remains to develop symbolic algorithms to compute the normalizing transformations, in order to get quantitative information on the bifurcations and the organization of the phase space, similar to the approach of the current paper.

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Appendix A. Germ isomorphy

We give sufficient conditions for germs to be isomorphic, with respect to morphisms respecting a symmetry. The case without symmetry is well known, see e.g. [24], and the present results are straightforward generalizations of this case; see also [35]. The section ends with an application that was used in section 4.

Let Γ be a compact group with a faithful linear action on $\mathbb{R}^n$. Group elements $\gamma \in \Gamma$ are identified with their corresponding linear action. In this paper we only consider the groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, with action $(x, y) \mapsto (x, \epsilon_1 y)$ and $(x, y) \mapsto (\epsilon_1 x, \epsilon_2 y)$ ($\epsilon_i = \pm 1$) respectively.

Definition 23. $\mathcal{E}_n^\Gamma := \{f : \mathbb{R}^n \to \mathbb{R} : f(\gamma x) = f(x) \forall \gamma \in \Gamma\}/\sim$, where $\sim$ is the germ-equivalence $f \sim g \Leftrightarrow \exists B \ni 0$, open, such that $f(x) = g(x) \forall x \in B$.

$\mathcal{E}_n^\Gamma$ is the ring of Γ-invariant germs of functions on $\mathbb{R}^n$.

Definition 24. $\boldsymbol{V}_n^\Gamma$ denotes the $\mathcal{E}_n^\Gamma$ module of Γ equivariant germs of vector fields on $\mathbb{R}^n$.

In the case of $\Gamma = \mathbb{Z}_2$, the module $\boldsymbol{V}_n^\Gamma$ is generated over $\mathcal{E}_n^\Gamma$ by $\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$.

Definition 25. $\boldsymbol{V}_n^{0,\Gamma} := \{X \in \boldsymbol{V}_n^\Gamma : X(0) = 0\}$.

$\boldsymbol{V}_n^{0,\Gamma}$ is the $\mathcal{E}_n^\Gamma$-module of Γ-equivariant vector fields that vanish at the origin. In the case of $\Gamma = \mathbb{Z}_2$ again, its generators are $x\frac{\partial}{\partial x}$, $y^2\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$.

Definition 26. $T^\Gamma(f) := \{Xf : X \in \boldsymbol{V}_n^\Gamma\}$, $T_0^\Gamma(f) := \{Xf : X \in \boldsymbol{V}_n^{0,\Gamma}\}$.

These are *tangent spaces* to f, ideals of $\mathcal{E}_n^\Gamma$. The generators of these ideals are found by applying the generators of the modules $\boldsymbol{V}_n^\Gamma$ and $\boldsymbol{V}_n^{0,\Gamma}$ respectively to f. The subscript 0 in T_0^Γ indicates that the isomorphism between germs must fix the origin.

Definition 27. $f \sim_\Gamma g$ iff there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(\gamma x) = \phi(x) \forall \gamma \in \Gamma$ and $\phi(0) = 0$ and $f \circ \phi = g$.

If $f \sim_\Gamma g$ then f and g are called *isomorphic* (as Γ-invariant germs).

Definition 28. $\mathfrak{m}_k^\Gamma := \{f \in \mathcal{E}_n^\Gamma$:the taylor polynomial of f at 0 vanishes up to and including order $k-1\}$.

$$\mathfrak{m}^\Gamma := \mathfrak{m}_1^\Gamma = \{f \in \mathcal{E}_n^\Gamma : f(0) = 0\}.$$
$$j^k \text{ is the projection } \mathcal{E}_n^\Gamma \to \mathcal{E}_n^\Gamma/\mathfrak{m}_{k+1}^\Gamma.$$

In words, $j^k(f)$ is the Taylor-polynomial of f up to, and including, order k.

At this point we quote [24] for the case that $\Gamma = \{\mathrm{id}\}$. To stress this special case we drop the Γ.

Proposition 29. [24, III.4.2] *Let $f, g \in \mathcal{E}_n$, and assume that $g - f \in \mathfrak{m}_k$, i.e. $j^{k-1}(g-f) = 0$.*
(a) If $T_0(f) \supset \mathfrak{m}_k$ then $g \sim f$ provided that $j^k(g - f)$ is small enough.
(b) If $\mathfrak{m} \cdot T_0(f) \supset \mathfrak{m}_k$ then $g \sim f$.

Note that $T_0(f) = \mathfrak{m} \cdot T(f)$, or in the notation of [24] $T_0(f) = \mathfrak{m} \cdot J(f)$ where $J(f)$ denotes the Jacobian ideal $\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle_{\mathcal{E}_n}$. Also, note that in this case, $\mathfrak{m}_k = \mathfrak{m}^k$. The analogous result for germs with symmetry is as follows.

Proposition 30. *Let $f, g \in \mathcal{E}_n^\Gamma$, and suppose that $g - f \in \mathfrak{m}_k^\Gamma$, i.e. $j^{k-1}(g - f) = 0$. Let M denote the finite-dimensional vector space $\mathfrak{m}_k^\Gamma/(\mathfrak{m}^\Gamma \cdot \mathfrak{m}_k^\Gamma)$, and set $M_m := M \cap (\mathfrak{m}_m^\Gamma/\mathfrak{m}_{m+1}^\Gamma) = \{h \in M : h$ is homogeneous of degree $m\}$.*

(a) Suppose that $T_0^\Gamma(f) \supset \mathfrak{m}_k^\Gamma$ then $g \sim_\Gamma f$ provided that the projection of $g - f$ into M is sufficiently small.

(b) Suppose that $\mathfrak{m}^\Gamma \cdot T_0^\Gamma(f) \supset \mathfrak{m}_k^\Gamma$ then $g \sim_\Gamma f$.

(c) Suppose that $T_0^\Gamma(f) \supset \mathfrak{m}_k^\Gamma$. Suppose further that the projection of f into M is an element of M_k. Then $g \sim_\Gamma f$ provided that $j^k(g - f)$ is sufficiently small.

For a proof, see appendix B.

A.1 Application

Proposition 31. *$x(\alpha x^2 + \beta y^2) +$* h.o.t. *is $\mathbb{Z}_2$-isomorphic to $x(x^2 + y^2)$, if $\alpha \neq 0$ and $\beta \neq 0$. Here $\mathbb{Z}_2$ has $\mathbb{R}^2$-action generated by $(x, y) \mapsto (x, -y)$.*

Proof. First of all we apply a linear transformation, so that we can assume that the nonzero α and β in fact equal 1. The tangent space $T_0^\Gamma(f)$ is generated by $x\frac{\partial f}{\partial x} = 3x^3 + xy^2$, $y^2\frac{\partial f}{\partial x} = 3x^2y^2 + y^4$ and $y\frac{\partial f}{\partial y} = 2xy^2$. Equivalently, $T_0^\Gamma(f) = \langle x^3, xy^2, y^4 \rangle_{\mathcal{E}_2^\Gamma} \supset \mathfrak{m}_3^\Gamma$. (In fact, they are equal.)

We now apply proposition 30(c), with $k = 3$. As f is homogeneous of degree 3 all conditions are satisfied, and we conclude that all g of the form $g = x(\alpha x^2 + \beta y^2) +$ h.o.t. where α and β are sufficiently close to 1 but with arbitrary h.o.t. are isomorphic to f. □

Appendix B. Proof of proposition 30

We need the following version of Nakayama's lemma, quoted without proof from [24, chapter 1].

Lemma 32 (Nakayama). *Let K and L be $\mathcal{E}^\Gamma$-modules, then*

$$K + \mathfrak{m}^\Gamma L \supset L \Rightarrow K \supset L.$$

We also need the following lemma. It is a symmetric version of the fundamental geometric lemma. See [24] for a proof.

Lemma 33 (symmetric geometric lemma). *Let $F(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a t-dependent family of Γ-invariant functions, defined on a neighbourhood of $(t, x) \in [0, 1] \times \{0\}$, and suppose there exists a vector field $X \in \boldsymbol{V}_n^{0,\Gamma}$ of the form*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(t, x)\boldsymbol{v}_i^0$$

(where X_i are Γ-invariant families of functions and $\boldsymbol{v}_i^0$ are generators of $\boldsymbol{V}_n^{0,\Gamma}$ as a module over $\mathcal{E}_n^\Gamma$), defined on a neighbourhood $(t, x) \in [0, 1] \times \{0\}$, such that $XF = 0$. Then there exists a Γ-equivariant germ of a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(0, \phi(x)) = F(1, x)$ and $\phi(0) = 0$.

Proof of proposition 30. Parts (a) and (b) are based on [24, IV.4.2]. We first introduce some notation. Let l be the integer such that $M = M_k \oplus \cdots \oplus M_l$. Let π denote the projection $\pi : \mathfrak{m}_k^\Gamma \to M$. Let $\alpha_{im} \in \mathfrak{m}^\Gamma$ be homogeneous germs such that α_{im} is of degree m, and such that the set $\{\pi\alpha_{im}\}_{im}$ forms a basis of M. The generators of $\boldsymbol{V}_n^{0,\Gamma}$ are $\boldsymbol{v}_i^0$, in particular $T_0^\Gamma(f) = \langle \boldsymbol{v}_i^0(f)\rangle_{\mathcal{E}_n^\Gamma}$.

We write $g = f + h$, where g is the germ that is supposedly isomorphic to f. We have $h \in \mathfrak{m}_k^\Gamma$ by hypothesis.

(a, first part) The first part consists of proving that $T_0^\Gamma(f + th) \supset \mathfrak{m}_k^\Gamma$ for $t \in [0, 1]$. By hypothesis, $T_0^\Gamma(f) \supset \mathfrak{m}_k^\Gamma$, so we can find λ_{ijm} such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \boldsymbol{v}_j^0(f).$$

Next, define the linear operator H on M by

$$H\alpha_{im} := \pi \sum_j \lambda_{ijm} \boldsymbol{v}_j^0(h).$$

Using this we find

$$\begin{aligned} \pi T_0^\Gamma(f + th) &\supset \operatorname{span}_{\mathbb{R}} \left\{ \sum_j \lambda_{ijm} \boldsymbol{v}_j^0(f + th) \right\}_{im} = \operatorname{span}_{\mathbb{R}}\{(I + tH)\alpha_{im}\}_{im} \\ &= \operatorname{span}_{\mathbb{R}}\{\alpha_{im}\}_{im} = M. \end{aligned} \tag{B.1}$$

The penultimate equality holds, for $t \in [0, 1]$, if $I + tH$ is invertible for these values of t, which is true if $\pi h = \pi(g - f) \in M$ is small enough.

(B.1) can also be written as

$$T_0^\Gamma(f + th) + \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma \supset \mathfrak{m}_k^\Gamma.$$

and, by Nakayama, this implies $T_0^\Gamma(f + th) \supset \mathfrak{m}_k^\Gamma$, proving the first part.

(a, second part) As $h \in \mathfrak{m}_k^\Gamma$, the statement $T_0^\Gamma(f + th) \supset \mathfrak{m}_k^\Gamma$ implies that, for any $\tau \in [0, 1]$, we can find germs $X_i(t, x) \in \mathcal{E}_{1+n}^\Gamma$ defined on some neighbourhood of $(t, x) = (\tau, 0)$, so that

$$\sum_i X_i \boldsymbol{v}_i^0(f + th) = -h.$$

Now write $F(t, x) = f(x) + th(x)$, and define the vector field $X := \frac{\partial}{\partial t} + \sum_i X_i(t, x)\boldsymbol{v}_i^0$, then $XF = 0$.

By compactness of $[0, 1]$ we can find a finite number of such vector fields that can be combined to one defined on the entire interval. Lemma 33 now provides the required isomorphism between $F(0, \cdot) = f$ and $F(1, \cdot) = f + h = g$.

(b) The hypothesis $\mathfrak{m}^\Gamma \cdot T_0^\Gamma(f) \supset \mathfrak{m}_k^\Gamma$ implies that there exist $\lambda_{ijm} \in \mathfrak{m}^\Gamma$ such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \boldsymbol{v}_j^0(f).$$

As $h \in \mathfrak{m}_k^\Gamma$ we also have $\boldsymbol{v}_j^0(h) \in \mathfrak{m}_k^\Gamma$, so $\lambda_{ijk}\boldsymbol{v}_j^0(h) \in \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma$. But $T_0^\Gamma(f + th) = \langle \alpha_{im} + t\sum_j \lambda_{ijm}\boldsymbol{v}_j^0(h)\rangle_{\mathcal{E}_n^\Gamma}$, that is, $T_0^\Gamma(f + th) + \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma \supset \mathfrak{m}_k^\Gamma$, and by Nakayama this implies $T_0^\Gamma(f + th) \supset \mathfrak{m}_k^\Gamma$. The rest of the proof is the same as the second part of (a).

(c) We assume that the $\boldsymbol{v}_i^0$ are homogeneous. (If not, note that $\boldsymbol{V}^{0,\Gamma}/(\mathfrak{m}^\Gamma \cdot \boldsymbol{V}^{0,\Gamma})$ is finite dimensional, and write $\boldsymbol{v}_i^0 = \sum_j \boldsymbol{v}_{ij}^0 + \boldsymbol{v}_{i,\text{rest}}^0$ where $\boldsymbol{v}_{ij}^0$ are finitely many homogeneous terms, and $\boldsymbol{v}_{i,\text{rest}}^0$ is an element of $\mathfrak{m}^\Gamma \boldsymbol{V}^{0,\Gamma}$, so that $\langle \boldsymbol{v}_{ij}^0\rangle_{\mathcal{E}_n^\Gamma} + \mathfrak{m}^\Gamma \cdot \boldsymbol{V}^{0,\Gamma} = \boldsymbol{V}^{0,\Gamma}$. Now use Nakayama to conclude that the $\boldsymbol{v}_{ij}^0$ generate $\boldsymbol{V}^{0,\Gamma}$ over $\mathcal{E}_n^\Gamma$; then use these $\boldsymbol{v}_{ij}^0$ instead of the $\boldsymbol{v}_i^0$.)

Write f_k for the homogeneous kth degree part of f. We will prove the equivalence $f_k \sim_\Gamma g$. The same argument with $g = f$ then proves $f_k \sim_\Gamma f$, completing the proof.

First we prove that $T_0^\Gamma(f_k) \supset \mathfrak{m}_k^\Gamma$. By hypothesis $h := f - f_k \in \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma$, so we can write $h = h_1 h_k$ with $h_i \in \mathfrak{m}_i^\Gamma$. $\boldsymbol{v}_i^0$ maps $\mathfrak{m}_j^\Gamma$ into itself, so $\boldsymbol{v}_i^0(h) = h_1 \boldsymbol{v}_i^0(h_k) + \boldsymbol{v}_i^0(h_1) h_k \in \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma$, or $\boldsymbol{v}_i^0(f) \in T_0^\Gamma(f_k) + \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma$. So we have

$$\mathfrak{m}_k^\Gamma \subset T_0^\Gamma(f) = \langle \boldsymbol{v}_i^0(f) \rangle_{\mathcal{E}_n^\Gamma} \subset T_0^\Gamma(f_k) + \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma.$$

Applying Nakayama we find $T_0^\Gamma(f_k) \supset \mathfrak{m}_k^\Gamma$. This inclusion implies the existence of λ_{ijm} such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \boldsymbol{v}_j^0(f_k)$$

and, as g_{im}, $\boldsymbol{v}_j^0$ and f_k are homogeneous, we may assume that the λ_{ijm} are too.

Now write $g = f_k + h_k + h_{>k}$, where h_k is homogeneous of degree k, and $h_{>k}$ only contains terms of degree $k+1$ and higher. We define the operators H_k and $H_{>k}$ on M by

$$H_{(>)k} \alpha_{im} := \pi \sum_j \lambda_{ijm} \boldsymbol{v}_j^0 h_{(>)k}.$$

We now prove that $H_{>k}$ is nilpotent. Let $\deg(f)$ denote the total degree of a homogeneous germ f, $\operatorname{sdeg}(f)$ the smallest total degree of terms of f, and set $\deg(0) = \operatorname{sdeg}(0) = \infty$. Then

$$\begin{aligned}\operatorname{sdeg}(H_{>k}\alpha_{im}) &\geqslant \min_j(\deg(\lambda_{ijm}) + \operatorname{sdeg}(\boldsymbol{v}_j^0(h_{>k}))) > \min_j(\deg(\lambda_{ijm}) + \deg(\boldsymbol{v}_j^0(f_k))) \\ &= \deg(\alpha_{im}) = m,\end{aligned}$$

so $H_{>k}$ maps M_m into $M_{m+1} \oplus M_{m+2} \oplus \cdots \oplus M_l$, so it is nilpotent, say $H_{>k}^{j_0} = 0$. The operator $I + t(H_k + H_{>k})$ is invertible, for $t \in [0, 1]$, if H_k is small enough, i.e. if $\pi(f_k - g) = \pi(f - g)$ is small enough. Indeed, the inverse is given by the sum

$$(I + t(H_k + H_{>k}))^{-1} = \sum_{j=0}^{\infty} (-t(H_k + H_{>k}))^j, \tag{B.2}$$

and nilpotency of $H_{>k}$ allows us to derive the inequality $\|(H_k + H_{>k})^j\| \leqslant C \|H_k^{j - j_0}\|$, where C is some constant, so that for small H_k, (B.2) converges. We have now

$$\begin{aligned}\pi T_0^\Gamma(f + t(h_k + h_{>k})) &\supset \operatorname{span}_{\mathbb{R}} \left\{ \sum_j \lambda_{ijm} \boldsymbol{v}_j^0(f + t(h_k + h_{>k})) \right\} \\ &= \operatorname{span}_{\mathbb{R}} \{(I + t(H_k + H_{>k}) g_{im}\}_{im} = \operatorname{span}_{\mathbb{R}} \{g_{im}\} = M \qquad (t \in [0, 1])\end{aligned}$$

where we used that $I + t(H_k + H_{>k})$ is invertible. Now apply Nakayama to conclude that $T_0^\Gamma(f + th) \supset \mathfrak{m}_k^\Gamma$, where $h = h_k + h_{>k}$. The rest of the proof is the same as the second part of (a). □

Appendix C. Universal deformations

Here we present a necessary and sufficient condition for deformations to be versal. The definitions and results are straightforward generalizations of the nonsymmetric case. Our main source is [24, ch XI]. Other good references are [19, 35]. See appendix A for some of the notation used in this section.

Definition 34.

(a) The ring of germs of Γ-invariant functions on $\mathbb{R}^n \times \mathbb{R}^r$, where Γ acts trivially on $\mathbb{R}^r$, is denoted by $\mathcal{E}^\Gamma_{n+r}$.

(b) $F \in \mathcal{E}^\Gamma_{n+r}$ is called a *deformation* of $f \in \mathcal{E}^\Gamma_n$ if $F(x, 0) = f(x)$.

(c) A map ϕ on $\mathbb{R}^n$ is called Γ-*equivariant* if $\phi(\gamma x) = \gamma\phi(x)$ for all $\gamma \in \Gamma$.

(d) Two deformations $F, G \in \mathcal{E}^\Gamma_{n+r}$ of the same germ $f \in \mathcal{E}^\Gamma_n$ are called *isomorphic*, notation $F \simeq_\Gamma G$, if there exists a germ of a parameter-dependent Γ-equivariant map $\phi(x, u)$ with $\phi(x, 0) = x$ such that

$$F(x, u) = G(\phi(x, u), u).$$

(e) A deformation $G \in \mathcal{E}^\Gamma_{n+q}$ is said to be *induced* from a deformation $F \in \mathcal{E}^\Gamma_{n+r}$ if there exists a germ of a reparametrization $h : \mathbb{R}^q \to \mathbb{R}^r$ such that

$$G(x, v) \simeq_\Gamma F(x, h(v)).$$

(Note that the isomorphism depends on v, not on $u = h(v)$.)

(f) A deformation F of $f(x) = F(x, 0)$ is said to be *versal* if any deformation G of f can be induced from F.

Proposition 35. *Let $F \in \mathcal{E}^\Gamma_{n+r}$ be a deformation of $f = F(x, 0) \in \mathcal{E}^\Gamma_n$. A necessary and sufficient condition for F to be versal is that*

$$T^\Gamma(f) + \text{span}_\mathbb{R}\{\dot{F}_1, \ldots, \dot{F}_r\} = \mathcal{E}^\Gamma_n.$$

Here $\dot{F}_i := \frac{\partial F}{\partial u_i}|_{u=0}$ are called the initial speeds *of the deformation F. See definition 26 for $T^\Gamma(f)$.*

The codimension of the germ f is, by definition, the codimension of $T^\Gamma(f)$ in $\mathcal{E}^\Gamma_n$, and is equal to the minimum number of parameters of a versal deformation.

C.1. Application: versal deformation of $x(x^2 + y^2)$

We apply proposition 35 to the case $f = x(x^2 + y^2)$, and symmetry group $\Gamma = \mathbb{Z}_2$ acting by $(x, y) \mapsto (x, -y)$.

The tangent space $T^\Gamma(f)$ is $\langle 3x^2 + y^2, 2xy^2\rangle_{\mathcal{E}^\Gamma_2} = \langle 3x^2 + y^2, 2xy^2, x^3, y^4\rangle_{\mathcal{E}^\Gamma_2}$, so that $T^\Gamma(f) + \text{span}_\mathbb{R}\{1, x, y^2\} = \mathcal{E}^\Gamma_2$. Therefore, $F(x, u_0, u_1, u_2) = x(x^2 + y^2) + u_0 + xu_1 + y^2u_2$ is a versal deformation of f.

In our application we only consider deformations that fix the zero level, so that the parameter u_0 can be dispensed with; see remark 11.

Appendix D. Bifurcation analysis

The model in terms of which our system was described is the universal deformation $G(x, y, u_1, u_2) = x(x^2 + y^2) + u_1x + u_2y^2$. This deformation has the following critical points:

$$(x, y) = \left(\pm\sqrt{-\frac{u_1}{3}}, 0\right) \qquad \text{and} \qquad (x, y) = \left(-u_2, \pm\sqrt{-u_1 - 3u_2^2}\right).$$

Bifurcations therefore occur along the curves $u_1 = 0$ and $u_1 + 3u_2^2 = 0$; see figure 2.

The level sets of G are organized by a special level set that factorizes into first- and second-degree algebraic curves crossing in the points $(x, y) = (-u_2, \pm\sqrt{-u_1 - 3u_2^2})$. The curves are level sets for the level $G = -u_2(u_2^2 + u_1)$, and are given by the equations

$$x = -u_2 \qquad \text{and} \qquad (x - \tfrac{1}{2}u_2)^2 + y^2 = -u_1 - \tfrac{3}{4}u_2^2.$$

For parameter values for which the curves cross, the second equation defines a circle that separates compact level curves from unbounded ones. (Note that for $u_1 > 0$ the second equation has no real solutions.) This circle is referred to as the *singular circle*. The reason is that it is the image of singular points of the transformation (3.2) we employ in our application.

Appendix E. Constructing an SGB

This section addresses the problem of, given a set of generators for an ideal I, how to add elements from the ideal to this set so that it becomes an SGB. (See section 9.3 for the definition of an SGB.) The algorithm to accomplish this closely follows the Buchberger algorithm for ordinary Gröbner bases, see [13, chapter 2]. As a corollary we obtain a necessary and sufficient condition for a set of generators to be an SGB.

Definition 36 (S-function).

$$S(f,g) := \frac{\text{l.c.m.}(\mathbf{MM}(f),\mathbf{MM}(g))}{\mathbf{MT}(f)} f - \frac{\text{l.c.m.}(\mathbf{MM}(f),\mathbf{MM}(g))}{\mathbf{MT}(g)} g.$$

The S-function of f and g is the simplest combination of f and g such that their minimal monomials cancel. For example, $S(x^2+y^2, 2xy) = \frac{x^2y}{x^2}(x^2+y^2) - \frac{x^2y}{2xy}2xy = y^3$.

Algorithm 37 (construction of an SGB with basis transformation).

Input: $f_1, \ldots, f_k$.

Output: An SGB $(g_1, g_2, \ldots)$ for the ideal $\langle f_1, \ldots, f_k\rangle$, and γ_{ij} such that

$$a_{ij} f_j, \qquad i = 1, \ldots, \#G.$$

Algorithm:

$G \leftarrow (f_1, \ldots, f_k)$
$\gamma_{ij} \leftarrow (1 \text{ if } i = j, 0 \text{ if } i \neq j)$
For every distinct pair $(g_i, g_j) \in G \times G$, $i < j$, do the following.
  Compute r and α_i resulting from dividing $S(g_i, g_j)$ through G
  If $r \neq 0$ then
    $G \leftarrow G \cup (r)$
    $\gamma_{\#G,m} \leftarrow \frac{l.c.m.(\mathbf{MM}(g_i),\mathbf{MM}(g_j))}{\mathbf{MT}(g_i)}\gamma_{im}$
      $- \frac{l.c.m.(\mathbf{MM}(g_i),\mathbf{MM}(g_j))}{\mathbf{MT}(g_j)}\gamma_{jm}$
      $- \sum_{n=1}^{\#G-1} \alpha_n \gamma_{nm} \qquad (m = 1, \ldots, k)$
  Endif
Endfor.

The main loop is over every ordered pair of (nonequal) elements of G. When elements are added to G in the body of the loop, the number of pairs to be considered increases accordingly.

The algorithm does the bookkeeping necesssary to write the output SGB in terms of the input f_i; a short calculation verifies that the invariant

$$g_i = \sum_{j=1}^{k} \gamma_{ij} f_j, \; i = 1, \ldots, \#G$$

is maintained over the While-loop. If the γ's are not required, the statements involving them may be removed.

Proof.

Termination The successive sets G give rise to an ascending chain of ideals $\langle \mathbf{MM}(g_1), \ldots \rangle$. By the Hilbert basis theorem, this chain stabilizes. Since, for nonzero r, $\mathbf{MM}(r) \notin \langle \mathbf{MM}(g_1), \ldots \rangle$ by (9.5), this implies termination.

Correctness $\langle G \rangle = \langle f_1, \ldots, f_k \rangle$ throughout the algorithm. In the end, we know that every $S(g_i, g_j)$, $g_i, g_j \in G$, has remainder zero upon division through G. Let $f \in \langle G \rangle$. We must show $\mathbf{MM}(f) \in \langle \mathbf{MM}(g_1), \ldots, \mathbf{MM}(g_t) \rangle$.

We can write $f = \sum_{i=1}^{t} h_i g_i$, and define $\delta = \min_i(\text{multideg}(h_i g_i))$. Choose the h_i such that δ is maximal. If $\delta = \text{multideg}(f)$ we are done, so assume $\delta < \text{multideg}(f)$. Define $m(i) := \text{multideg}(h_i g_i)$, and split the sum as follows:

$$f = \sum_{m(i)=\delta} \mathbf{MT}(h_i) g_i + \sum_{m(i)=\delta} (h_i - \mathbf{MT}(h_i)) g_i + \sum_{m(i)>\delta} h_i g_i. \tag{E.1}$$

Define $c_i = \mathbf{MC}(h_i g_i)$ and $p_i = \mathbf{MT}(h_i) g_i / c_i$. Without loss of generality we now assume that $\mathbf{MC}(g_i) = 1$, implying that $\mathbf{MC}(p_i) = 1$, and furthermore we assume that the g_i are ordered in such a way that $\text{multideg}(h_i g_i) = \delta$ for $i = 1, \ldots, t'$, and $\text{multideg}(h_i g_i) > \delta$ for $i = t'+1, \ldots, t$. The first sum appearing in (E.1) can be rewritten as

$$\begin{aligned} \sum_{i=1}^{t'} \mathbf{MT}(h_i) g_i = {} & c_1(p_1 - p_2) + (c_1 + c_2)(p_2 - p_3) + \cdots + (c_1 + \cdots + c_{t'-1})(p_{t'-1} - p_{t'}) \\ & + (c_1 + \cdots + c_{t'}) p_{t'}. \end{aligned} \tag{E.2}$$

The second and third sum of (E.1) only contain terms of multidegree strictly larger than δ. Because $\mathbf{MT}(p_i) = x^\delta$ so that $\text{multideg}(p_i - p_{i+1}) > \delta$, and $\text{multideg}(f) > \delta$, the coefficient of p_t in the last term of (E.2) must vanish.

Define $\gamma_i = \text{l.c.m.}(\mathbf{MM}(g_i), \mathbf{MM}(g_{i+1}))$, $i = 1, \ldots, t'-1$. $\mathbf{MM}(g_i)$ divides $\mathbf{MM}(h_i g_i) = x^\delta$, so $x^{\delta - \gamma_i}$ is a monomial. Now

$$x^{\delta-\gamma_i} S(g_i, g_{i+1}) = \frac{g_i x^\delta}{\mathbf{MT}(g_i)} - \frac{g_{i+1} x^\delta}{\mathbf{MT}(g_{i+1})} = \frac{g_i \mathbf{MT}(h_i)}{\mathbf{MC}(g_i h_i)} - \frac{g_{i+1} \mathbf{MT}(h_{i+1})}{\mathbf{MC}(g_{i+1} h_{i+1})} = p_i - p_{i+1}.$$

Since $\mathbf{MT}(p_i) = x^\delta$, this implies that $\text{multideg}(x^{\delta-\gamma_i} S(g_i, g_{i+1})) > \delta$.

Next, we use that $S(g_i, g_{i+1})$ has remainder zero upon division through G, i.e. they can be written as

$$S(g_i, g_{i+1}) = \sum_{j=1}^{t} a_{ij} g_j$$

where $\text{multideg}(a_{ij} g_j) \geqslant \text{multideg}(S(g_i, g_{i+1}))$. The latter inequality follows from the division algorithm. If we set $b_{ij} = x^{\delta-\gamma_i} a_{ij}$, we find that $\text{multideg}(b_{ij} g_j) > \delta$. Equation (E.2) now becomes

$$\sum_{i=1}^{t'} \mathbf{MT}(h_i) g_i = c_1 \sum_{j=1}^{t} b_{1j} g_j + \cdots + c_{t'-1} \sum_{j=1}^{t} b_{t'-1,j} g_j$$

where each term on the right has multidegree $> \delta$. If this is substituted back into (E.1), it follows that we have written f in the form $\sum h_i g_i$ where each $\text{multideg}(h_i g_i) > \delta$, contradicting the choice of δ. □

E.1. Condition for a set to be an SGB

Corollary 38. *A set $G = \{g_1, \ldots, g_k\}$ is an SGB iff $S(g_i, g_j)$ reduces to zero upon division through G, for all $1 \leqslant i < j \leqslant k$.*

Proof. If all $S(g_i, g_j)$ reduce to zero, algorithm 37 does not add any element to the set G, so that G itself was an SGB to start with. To prove the converse, note that every $S(g_i, g_j)$ is an element of $\langle G \rangle$, meaning that each of them can be written in the form $S(g_i, g_j) = \sum_k q_k^{ij} g_k + r$, where $r = 0$. As G is an SGB by assumption, the rest upon division through G is unique, so it must be 0. □

E.2. Examples

The central singularity in the 1 : 2 resonance case is $f = x(x^2 + y^2)$, and generators of the tangent module $T^\Gamma(f)$ (with symmetry group $\Gamma = \mathbb{Z}_2$) are $g_1 := \frac{\partial f}{\partial x}$ and $g_2 := y\frac{\partial f}{\partial y}$, that is $g_1 = 3x^2 + y^2$ and $g_2 = 2xy^2$. To turn $\{g_1, g_2\}$ into an SGB, we apply algorithm 37, with the degree-lexicographic ordering with $x < y$. $S(g_1, g_2) = (x^2y^2 + \frac{1}{3}y^4) - (x^2y^2) = \frac{1}{3}y^4$, and this is not further reduced by dividing through $\{g_1, g_2\}$, so we set $g_3 := y^4$. $S(g_1, g_3) = (x^2y^4 + \frac{1}{3}y^6) - (x^2y^4) = \frac{1}{3}y^6$, which reduces to zero by division through g_3. Also, $S(g_2, g_3) = (xy^4) - (xy^4) = 0$ right away. We conclude that

$$\{3x^2 + y^2, 2xy^2, y^4\}$$

is an SGB for the ideal generated by $\{g_1, g_2\}$.

For the 2 : 2 resonance case we find the central singularity $f = x^4 + ax^2y^2 + y^4$. (The coefficient a here is a *modulus*, and different values for a give nonisomorphic germs.) We are now in a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric setting, and generators for the tangent module $T^\Gamma(f)$ are given by

$$g_1 = x\frac{\partial f}{\partial x} = 2x(2x^3 + axy^2) \qquad \text{and} \qquad g_2 = y\frac{\partial f}{\partial y} = 2y(ax^2y + 2y^3).$$

This is not yet an SGB. $S(g_1, g_2) = \frac{1}{4}y^2g_1 - \frac{1}{2a}x^2g_2 = \frac{a^2-4}{2a}x^2y^4$. This expression can be divided through $\{g_1, g_2\}$, because $\mathbf{MM}(g_2) = x^2y^2|x^2y^4$. This results in the remainder $(\frac{4}{a^2} - 1)y^6$. Further combinations do not lead to new SGB generators, and we conclude that

$$\{2x(2x^3 + axy^2), 2y(ax^2y + 2y^3), y^6\}$$

forms an SGB for $T^\Gamma(f)$ if $a \notin \{0, \pm 2\}$.

If $a = 0$, an SGB is given by $\{x^4, y^4\}$, and $T^\Gamma(f)$ has the same codimension as in the generic case. For the exceptional values $a = \pm 2$ the tangent space $T^\Gamma(f)$ has infinite codimension. An SGB in these cases is $\{g_1, g_2\}$ as $S(g_1, g_2) = 0$, and indeed the vector space (9.5) is then infinite dimensional.

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