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RESONANT ACOUSTIC OSCILLATIONS WITH DAMPING:

SMALL RATE THEORY

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RESONANT ACOUSTIC OSCILLATIONS WITH DAMPING: SMALL RATE
THEORY

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13. ABSTRACT

A gas in a tube is excited by a reciprocating piston operating at or near a resonant frequency. Damping is introduced into the system by three means: radiation of energy from one end of the tube, rate dependence of the gas, and boundary layer friction. A lumped damping coefficient is defined to account for the effects of all three. It is shown that in the small rate limit the signal in the periodic state suffers negligible distortion in one travel time, and hence its propagation according to acoustic theory is valid. The shape of the signal is determined by a nonlinear ordinary differential equation. The small rate condition provides a test of the applicability of the theory to given experimental conditions. When there is no damping, shocks are a feature of the flow for frequencies in the resonant band. For a given amount of damping an upper bound on the piston acceleration is given to ensure a shockless motion. The resonant band is analyzed for both damped and inviscid cases. The predictions of the theory are compared with experiment.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Resonance						
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1. Introduction

In this paper we discuss the periodic vibrations which
result when a column of gas in a Kundt's tube is driven by a
piston oscillating at a near resonant frequency. The basic
experimental observations are well documented, (Saenger and
Hudson (1960)). When the piston frequency is in a band about
a resonant frequency, the amplitude of the response is markedly
higher than the piston amplitude and shock waves appear in the
flow. These phenomena have been extensively investigated in
the recent literature, (Betchov (1958), Saenger and Hudson (1960),
Chu and Ying (1963), Chester (1964), Mortell (1971a,b), Collins
(1971)). Nevertheless, there are associated phenomena which
need further investigation, and some aspects of the various
analyses which need clarification. Two questions which are
resolved concern the range of validity of the usual modifica-
tions of acoustic theory used previously, and the existence of
a critical amount of damping which ensures a shockless motion.

The basis for the analysis given here is the fact that the
motion of the gas can be represented, to first order in the
amplitudes, as the superposition of two noninteracting simple
waves traveling in opposite directions, (see Mortell and Varley
(1970)). This implies that the travel time of a component wave
in the tube is determined by its own amplitude, and then any
distortion of a signal is self-induced. The essence of this
approximation, which is the first term in a regular perturbation
expansion, is that while the amplitudes remain small, $|u| << a_o$,

the acceleration in the gas, $\left|\frac{\partial u}{\partial t}\right|$, is not restricted. This contrasts with linear acoustic theory where both $|u| \ll a_o$ and $\left|\frac{\partial u}{\partial t}\right| \ll \frac{a_o^2}{L}$ are required for its validity. (Here, u is the particle velocity, a_o is the equilibrium sound speed and L is a typical length of the medium.) A nonlinear theory such as this has been used for many years in gas dynamics when the disturbance is generated by the passage of a single progressing wave (see Whitham (1952)). The noninteracting simple wave representation can be viewed as a generalization of Whitham's rule (1952) to wave motions having two components.

We consider the motion of a gas in a tube which is driven by a piston oscillating at one end. The other end is "partially open" in the sense that the system is considered to lose energy by radiation through this end into the adjacent medium. We seek the time-periodic response of the gas to these boundary condi-tions. Using the simple wave representation the problem of calculating the shape of the signal at a boundary is reduced to finding solutions of a nonlinear functional difference equation. The signal may distort as it travels, but its shape at any point in the body of the gas is determined by the simple wave repre-sentation. In the "small rate" limit, $\left|\frac{\partial u}{\partial t}\right| \ll \frac{a_o^2}{L}$, the non-linear functional equation which determines the signal in the periodic state can be reduced to a nonlinear ordinary differen-tial equation. Further, in this limit, the distortion of a signal in one travel time is negligible. The final differential equation only describes the signal at a boundary in the periodic

state, and cannot describe the cumulative distortion which led
to this state. It is evident that, for a given piston amplitude,
the small rate limit restricts the allowed values of the applied
frequency. In fact, for some experimentally used values of the
piston amplitude the restriction is surprisingly strong.

There are two basic phenomena in the model used here;
shocks due to nonlinearity and damping which can prevent shocks.
For a purely inviscid model, linear theory predicts an unbounded
amplitude in the periodic state for certain discrete (resonant)
frequencies. On the other hand, nonlinear inviscid theory pre-
dicts a bounded signal, which contains shocks, in a band about
the resonant frequencies. Since shocks act as a dissipative
mechanism they allow a balance of energy. This role of non-
linearity seems to be well-understood. To date, the most com-
prehensive investigation of the effect of damping on resonant
motions is due to Chester (1964). He investigated the effects
of compressive viscosity and boundary layer friction on the
motions. His conclusion was that the effect of the former is,
in most cases, negligible except in the interior of a shock. He
concurred with Betchov (1958) that damping could have a signifi-
cant effect, and was led in his analysis of boundary layer
friction to raise the question of the existence of a finite
critical value of damping which would ensure a continuous motion.
He concluded that it was an open question. We introduce three
different types of damping which can be treated within the same
theoretical framework. These are damping due to radiation of

energy from one end of the tube, rate dependence of the gas,
and boundary layer friction. We define a lumped damping co-
efficient which incorporates the effects of all three, and
show that these "lower order" damping mechanisms can prevent
the occurrence of shocks in the flow, i.e., for a given piston
motion there is a critical level of damping above which the
gas motion is continuous. B. Sturtevant, at the California
Institute of Technology, has carried out experiments in which
a hole is made in the closed end of the tube. He found that
for the particular conditions of an experiment there is a criti-
cal ratio of the area of the hole to the area of the tube end
at which shocks disappear. This points to the existence of a
critical amount of damping which ensures a shockless motion.

In section 3 there is a complete analysis of the resonant
band for both the inviscid and damped cases. This is achieved
by an examination of the integral curves of the governing dif-
ferential equation, using a condition on the mean of the flow
to fix the shock position. Explicit analytical results are
given for the inviscid case; some qualitative results for the
damped motion are found analytically while quantitative results
are determined numerically. In section 5 there is a comparison
between theory and experiment. The introduction of damping
improves the agreement. It suggests that if one is interested
in such gross features of the flow as the maximum or minimum
pressure, or the shock strength, then the lumped damping pro-
vides an adequate description.

2. Formulation

A column of gas, of length L in some reference (equilibrium) state, is contained in a pipe. One end of the pipe is closed while at the other end there is an oscillating piston. If pressure and density are measured from their values in the reference state (p_0, ρ_0) with the associated sound speed a_0, then in terms of the nondimensional variables $(a_0 u, \rho_0 a_0 p, \rho_0 \rho)$ and $(Lx, La_0^{-1}t)$ the governing equations in Lagrangian form are

$$[(1+e)^{-1}]_{,t} - u_{,x} = 0 , \qquad (2.1)$$

and

$$u_{,t} + p_{,x} = 0 , \qquad (2.2)$$

where $e(=\rho-1)$ is the condensation, γp the excess pressure ratio and u the nondimensional particle velocity. The equation of state of a polytropic gas in these variables is

$$\gamma p = (1+e)^{\gamma} - 1 . \qquad (2.3)$$

The end $x=0$ is considered to be 'near rigid' in the sense that we allow for the possibility of radiation of energy through this end of the tube, but do not consider the case when it is open. A boundary condition of this nature has been discussed by Mortell and Varley (1970). Across the boundary at $x=0$ both pressure and velocity are continuous, and so the disturbance must be compatible with the homogeneous boundary condition

$$u(0,t) = \cdot ip(0,t) , \qquad (2.4)$$

where $i/\gamma \ (\geq 0)$ is the impedance of the interface. Here the essential assumption is that the disturbance outside the tube is generated by the passage of a simple wave. Note that $i=0$ corresponds to a rigid end and $i=\infty$ to an open end. We examine the small amplitude, time-periodic response of the gas, governed by equations (2.1)-(2.3), to the boundary condition (2.4) at $x=0$ and a periodic piston displacement at $x=1$ of the form $\varepsilon h(\omega t)$. The amplitude of the displacement is $\varepsilon(\ll 1)$, and the period of h is normalized so that $h(y+1) = h(y)$. Then the piston velocity at $x=1$ is

$$u(1,t) = \varepsilon \omega h'(\omega t) = H(\omega t) . \qquad (2.5)$$

Since h is periodic, integration of (4) yields

$$\int_0^1 H(s)ds = 0 . \qquad (2.6)$$

Equations (2.1)-(2.3) are nonlinear and admit discontinuous solutions. However, it has been shown by Mortell and Seymour (1972b) that for time-periodic motions, be they continuous or discontinuous, the mean pressure and velocity do not vary from particle to particle. By choosing as the reference pressure, p_o, the constant mean of the periodic state, conditions (2.4) and (2.6) imply that the means of u and p are zero. The actual value of p_o can be determined only from an initial value problem.

2.1 Equation for the Periodic Motion

A representation derived by Mortell and Varley (1970) is

used to reduce the nonlinear boundary value problem defined by
(2.1)-(2.5) to a nonlinear difference, or functional, equation.
For a more restricted class of problems, this functional equa-
tion may be further reduced to a nonlinear ordinary differential
equation which determines the periodic motion of the gas.

It is convenient to reformulate equations (2.1)-(2.3) in
terms of the Riemann invariants and characteristic curves of
the system.   Upon defining

$$c(e) = \int_0^e a(s)(1+s)^{-1} ds = e[1 + (\tfrac{1}{2}M-1)e + 0(e^2)]$$

where

$$a^2(e) = (1+e)^2 \frac{dp}{de} = 1 + 2Me + 0(e^2) \, ,$$

and $M=\tfrac{1}{2}(\gamma+1)$, equations (2.1)-(2.3) define the Riemann in-
variants

$$2f(\beta) = u - c = u - p + 0(e^2) \tag{2.7}$$

and

$$- 2g(\alpha) = u + c = u + \neg + 0(e^2) \, . \tag{2.8}$$

The associated characteristics are given by

$$\left.\frac{dx}{dt}\right|_\alpha = a(e) \quad \text{and} \quad \left.\frac{dx}{dt}\right|_\beta = -a(e) \, . \tag{2.9}$$

When only one component of the motion is excited, equations
(2.7)-(2.9) admit two exact solutions, simple waves, which
correspond to f=constant and g=constant.   When both components
of the motion are excited there is in general an interaction

between α-waves, moving to the right, and β-waves, moving to the left. However, it has been shown by Mortell and Varley (1970) that to first order, in the limit of <u>small amplitudes</u>, the waves do not interact as they pass through each other in the body of the gas. By this is meant that to first order the trajectory of an α-wave is determined only by the signal it carries and is not influenced by the β-waves through which it passes. Thus the motion of the gas may be represented as the superposition of two <u>noninteracting simple waves</u>. Then equations (2.7) and (2.8) imply that to first order

$$e = p = -f(\beta) - g(\alpha) \quad \text{and} \quad u = f(\beta) - g(\alpha) , \qquad (2.10)$$

while (2.9) integrate to give

$$\frac{\alpha}{\omega} = t - x - Mxg(\alpha) \quad \text{and} \quad \frac{\beta}{\omega} = t + x - 1 + M(x-1)f(\beta) , \quad (2.11)$$

where we have parametrized α and β by $\alpha = \omega t$ on $x=0$ and $\beta = \omega t$ on $x=1$. Upon using the boundary conditions (2.4) and (2.5), g is eliminated from (2.10) and (2.11) to yield the nonlinear functional difference equation to determine the signal, f , on the boundary $x=1$:

$$f(\eta) - kf(s) = H(\eta) \qquad (2.12)$$

where

$$\eta = s + 2\omega + \omega M(1+k)f(s) . \qquad (2.13)$$

In (2.12) and (2.13), $k = \frac{1-i}{1+i}$ is the reflection coefficient at $x=0$, where g is related to f by

$$g(\theta) = kf(\phi) , \qquad (2.14)$$

and

$$\theta = \phi + \omega + \omega Mf(\phi) . \qquad (2.15)$$

The governing differential equations (2.1)-(2.3) and the boundary conditions (2.4) and (2.5) have been reduced in the small amplitude limit to the functional difference equation (2.12) and (2.13). We now seek solutions to (2.12) and (2.13) which, like the piston motion, have unit period. Further, as a consequence of the representations (2.10) and (2.11), the boundary conditions (2.4) and (2.5) and the fact that u and p have zero mean over any period, f and g must satisfy

$$\int_0^1 f(s)\,ds = \int_0^1 g(s)\,ds = 0 . \qquad (2.16)$$

Since M is the ratio of second order to first order elastic constants, linear theory is recovered from (2.12) and (2.13) by formally setting $M=0$ to yield

$$f(\eta) - kf(\eta-2\omega) = H(\eta) . \qquad (2.17)$$

When $k=1$ there are no solutions of (2.17) with unit period; that is, when

$$\omega = \omega_n = \tfrac{1}{2}n , \quad n = 1,2,3,\dots . \qquad (2.18)$$

These are the linear resonant frequencies. Ultimately we will consider the time-periodic response of the system to frequencies near to those defined by (2.18) and consequently define

$$\omega = \omega_n(1+\delta) , \quad (|\delta| < \tfrac{1}{n}) . \qquad (2.19)$$

Then in terms of

$$F(y) = f(y) + \frac{\delta}{b} , \qquad (2.20)$$

where

$$b = \frac{1}{2}M(1+k)(1+\delta) \quad (=O(1)) , \qquad (2.21)$$

(2.12) and (2.13) become

$$F(\eta) - kF(s) = G(\eta) , \qquad (2.22)$$

and

$$\eta = s + n + nbF(s) , \qquad (2.23)$$

where

$$G(\eta) = \frac{\mu\delta}{b} + H(\eta) \quad \text{and} \quad \mu = 1 - k . \qquad (2.24)$$

Definition (2.20) now implies that the zero mean condition, (2.16) on f is replaced by

$$\int_0^1 F(s)\,ds = \frac{\delta}{b} . \qquad (2.25)$$

The approximations used to derive equations (2.22)-(2.24) are the small amplitude assumption, $|f|$, $|g| << 1$, and the fact that the impedance of the interface at $x=0$ is near zero, so that $0 \le k \le 1$. This latter assumption is required since whenever $1 + k = O(e)$ the next correction to the character- istics (at order e^2) is no longer negligible. For example, if the end of the tube is open $(k=-1)$ the nonlinear approxima- tion (2.11) to the characteristics leads to a linear difference equation which has no bounded periodic solution at a resonant

frequency. Consequently, for a problem involving an open end (or "nearly" open) the approximations, (2.11), for α and β must be improved. In fact, the approximations for α and β must contain terms in f^2 and g^2 so that now the motion in the tube is determined by the cubic term in the equation of state, with a resulting amplitude at $O(\varepsilon^{1/3})$. Note further that the difference equation (2.22)-(2.24) together with (2.20) determine the shape of the signal function, f , only on $x=1$. The velocity and pressure, u and p , are subsequently calculated at any particle x in the tube from the representations (2.10) and (2.11). This is particularly important when there is significant distortion of a wave in one travel time.

If we now make the additional small rate assumption

$$|nF'| \ll 1 ,$$

equation (2.23) implies

$$F(s) = F(\eta-n) - nbF(\eta-n)F'(\eta-n)[1+0(nF')] . \qquad (2.26)$$

Upon using (2.26), and since we seek solutions with unit period, the difference equation (2.22)-(2.24) can then be approximated by the nonlinear ordinary differential equation

$$\nu F(\eta)F'(\eta) + \mu F(\eta) = G(\eta) , \quad 0 \le \eta \le 1 , \qquad (2.27)$$

together with

$$F(\eta+1) = F(\eta) , \qquad (2.28)$$

where $\nu = nbk$. The small rate condition and the definition of

F give the further restriction

$$|\delta| \ll 1/n .\qquad(2.29)$$

Thus equation (2.27) is valid in the small rate limit only for periodic motions at frequencies in the neighborhood of linear resonant frequencies. In contrast to this, the difference equation (2.22)-(2.24) was derived with no restriction on the applied rates; it is valid for non-periodic phenomena (see Mortell and Varley (1970), Mortell and Seymour (1972a,b)), and the applied frequency is not restricted to lie near a linear resonant frequency.

Since the small amplitude restriction requires that $|f|\ll 1$, the small rate condition implies that the differential equation (2.27) is a good approximation to the difference equation only when n is at most $O(1)$ as $|f|\to 0$. In addition, the representations (2.11) for the characteristics imply that in this limit there will be no appreciable distortion of the waveform in one period, since

$$\beta = \omega(t+x-1) + O(nf) .$$

(Of course there will be a cumulative distortion of the signal until the periodic motion has been set up. This is not described by (2.27)). Hence, in the small rate limit, (2.10) and (2.11) can be replaced by the linear acoustic representation

$$-p = f(\omega[t+x-1]) + g(\omega[t-x]) \quad\text{and}\quad u = f(\omega[t+x-1]) - g(\omega[t-x]) ,$$
$$(2.30)$$

so that when the periodic motion has evolved, nonlinearity is of primary importance in determining the shape of the signal functions, but it is of secondary importance in determining how the signals propagate.

We point out that the small rate condition $|nF'| \ll 1$, which is necessary for the validity of both the differential equation (2.27) and the representation (2.30), is quite restrictive for usual experimental values of the parameters. The results of section 3 show that for $\varepsilon = .0147$, used by Sturtevant (1972), this restriction implies $n \ll 2$; for $\varepsilon = .0018$, used by Saenger and Hudson (1960), the restriction yields $n \ll 6$. In a sequel to this paper we shall examine periodic motions with no restrictions on the rates when some of the ideas introduced here are used to analyze the functional difference equation, (2.12) and (2.13), directly.

It may be of interest to note that equation (2.27) arises in other physical situations. It is a generalization of the equation which describes the motion of a viscously damped pendulum under a constant external moment, and also occurs in the study of the pull-out torque of a synchronous motor, see Stoker (1950) or Minorsky (1962). In Appendix I we show how equation (2.27) may be derived from the governing equations (2.1)-(2.6) by a regular perturbation procedure, and point out an extra limitation on μ imposed by that procedure. The analysis of equation (2.27), under the restriction (2.25), and the physical interpretation of the results constitute the remainder of this paper.

3. Determination of the Periodic Signal

Here we analyze the integral curves of the differential equation (2.27) for various ranges of the parameters μ and δ and then use these to construct the signal, F, of the periodic motion. When we have constructed F over one period, possibly by a composition of integral curves, it is continued periodically by (2.28). The signal function, F, may then have <u>discontinuities</u> representing a time periodic motion in the pipe containing shocks. The discontinuities in F arise in satisfying the mean condition (2.25). Acoustic theory allows discontinuities of either compression or rarefaction with no restriction on their strengths. However, to be physically acceptable a jump in a gas must be compressive. Here we consider only piston velocities which have three zeroes over one period, and then discontinuous solutions of (2.27), which satisfy the mean condition (2.25), additionally satisfy the appropriate weak shock relations. This is not strictly necessary within the acoustic approximation.

If $S(x)$ is the arrival time at x of a weak shock traveling in the negative x direction and $\beta^{\pm}(x)$ are the wavelets immediately ahead of and behind the shock, then the weak shock relations imply that

$$\frac{dS}{dx} = -1 - \frac{1}{2}\mu[f(\beta^{+}) + f(\beta^{-})] . \qquad (3.1)$$

A similar relation gives the speed of shocks moving to the right. However, since in the periodic state there is negligible

distortion, $\beta^{\pm}$ are independent of x and (3.1) can be integrated to give the travel time of a shock from $x=1$ to $x=0$. The boundary condition (2.14) then implies that the total travel time for the shock to return to $x=1$ is

$$T = 2 + \tfrac{1}{2}M(1+k)[f(\beta^+) + f(\beta^-)] . \qquad (3.2)$$

Since the periodicity requirement is that $T = \frac{n}{\omega}$, (3.2) implies that

$$f(\beta^+) + f(\beta^-) = -\frac{2\delta}{b} . \qquad (3.3)$$

On using (2.20), the definition of F, (3.3) becomes

$$F(\beta^+) + F(\beta^-) = 0 . \qquad (3.4)$$

The condition that only compressive shocks are allowed then requires that $F(\beta^+) > 0$.

On the other hand, integration of (2.27) over one period, assuming a discontinuity at $\eta = \beta$ yields

$$\tfrac{1}{2}\nu[F^2(\beta^+) - F^2(\beta^-)] + \mu \int_o^1 F(s)\,ds = \int_o^1 G(s)\,ds . \qquad (3.5)$$

Conditions (2.6), (2.24) and (2.25) then imply that (3.4) must hold at the discontinuity. Thus a solution of (2.27), containing a discontinuity, which satisfies the mean condition (2.25) and the restriction $F(\beta^+) > 0$ will necessarily satisfy the weak shock relations. Thus a shock is fitted into the solution by satisfying the mean condition. This analysis is for one shock per period of the piston, which gives n shocks in the tube at any time.

3.1 Special case of no damping, μ=0

When the boundary at x=0 is rigid and there is no radiation of energy through it, i=μ=0 and equation (2.27) is greatly simplified. It can then be integrated completely and the signal function, the width of the resonant band and the shock strength for these frequencies can be determined analytically in terms of the parameters of the problem. Further, the transition from a discontinuous motion inside the band to a continuous one outside is exhibited explicitly.

We wish to distinguish between the integral curves of equation (2.27) and the signal F which must additionally satisfy the mean condition (2.25). An integral curve is denoted by $Z(\eta)$. Notice that while F is defined only for $0 \leq \eta \leq 1$ and is then continued periodically, if F is continuous it must coincide with an integral curve Z which is both continuous and periodic for $-\infty < \eta < \infty$. Conversely, a continuous, periodic integral curve Z with unit period which satisfies the mean condition (2.25) is the required signal function F. When such an integral curve exists it is unique. When, for a particular frequency, no such curve exists, F is discontinuous and is composed of distinct integral curves.

For the case $\mu=0$, the appropriate differential equation is

$$nbZ(\eta)Z'(\eta) = H(\eta) . \qquad (3.6)$$

There is no loss of generality in choosing the origin so that $H(0) = H(\eta_1) = H(1) = 0$, where $0<\eta_1<1$, with $H'(0) = H'(1) > 0$ and $H'(\eta_1) < 0$. Then in the (η,Z) plane the points $A_o = (0,0)$, $A_2 = (1,0)$ and $B_1 = (\eta_1,0)$ are isolated singular

points. A_o and A_2 are saddle points, while B_1 is a center. The two separatices through A_o are given by

$$Z^{\pm}(\eta) = \pm\left[\frac{2}{n\overline{b}} \int_o^{\eta} H(s)ds\right]^{\frac{1}{2}} , \qquad (3.7)$$

while the zero mean condition (5) implies that $Z^{\pm}(1) = 0$. Thus the separatices connect the two saddle points A_o , A_2 . The other integral curves which are defined for all η are given by

$$Z(\eta) = \pm\left[\frac{2}{n\overline{b}} \int_o^{\eta} H(s)ds + C\right]^{\frac{1}{2}} \qquad (3.8)$$

where $C=Z^2(0)$. These solutions are periodic, with unit period, for all η .

Since $Z^+(\eta) > 0$ for $0<\eta<1$, any solution $Z(\eta)$ with $Z(0) > 0$ is periodic in η with $Z(\eta) > Z^+(\eta)$ and therefore Z satisfies

$$\int_o^1 Z(s)ds > \int_o^1 Z^+(s)ds . \qquad (3.9)$$

Consequently the mean condition (2.25) implies that for an applied frequency $\omega = \frac{n}{2}(1+\delta)$ such that

$$\frac{\delta}{\overline{b}} > \int_o^1 Z^+(s)ds \qquad (3.10)$$

there exists a unique, <u>continuous</u>, periodic solution

$$Z_{\delta}(\eta) = \left[\frac{2}{n\overline{b}} \int_o^{\eta} H(s)ds + C(\delta)\right]^{\frac{1}{2}} = A(\eta) . \qquad (3.11)$$

The positive constant $C(\delta)$ is choosen so that $Z_{\delta}(\eta)$ satisfies the mean condition (2.25). A similar analysis shows there are

also continuous periodic solutions,

$$Z_\delta(\eta) = -A(\eta) \quad \text{for} \quad \frac{\delta}{b} < \int_0^1 Z^-(s)\,ds < 0 \ . \qquad (3.12)$$

Thus for the range of frequencies defined by (3.10) and (3.12) the signal function F is given by

$$F(\eta) = Z_\delta(\eta) \qquad (3.13)$$

and is continuous and periodic.

For frequencies such that

$$\int_0^1 Z^-(s)\,ds < \frac{\delta}{b} < \int_0^1 Z^+(s)\,ds \qquad (3.14)$$

no single integral curve will satisfy the mean condition (2.25) and the signal function will necessarily be <u>discontinuous</u>. The shock condition together with the fact that only compressive shocks are allowed then implies that the signal function F can only be constructed from the separatices $Z^\pm(\eta)$ with just one shock per period. The position of the shock at $\eta=\eta_s$ is chosen to satisfy the mean condition. The signal function F is then given by

$$F(\eta) = \begin{cases} Z^+(\eta) & 0 \le \eta < \eta_s \\ \\ Z^-(\eta) & \eta_s < \eta \le 1 \ , \end{cases} \qquad (3.15)$$

and

$$F(\eta+1) = F(\eta) \ ,$$

where condition (2.5) implies that $Z^\pm(\eta) = \pm\left[\frac{\varepsilon}{M}\{h(\eta)-h(0)\}\right]^{\frac{1}{2}}$.

$$(3.16)$$

The range of frequencies, defined by (3.14), for which the signal is <u>discontinuous</u>, is called the <u>resonant band</u>. If (3.14) is solved for δ the resonant band is given explicitly by

$$\delta^- < \delta < \delta^+$$

where

$$\delta^{\pm} = \frac{\pm(\epsilon M)^{\frac{1}{2}}\bar{h}}{1\mp(\epsilon M)^{\frac{1}{2}}\bar{h}} \tag{3.17}$$

and $\bar{h} = \int_0^1 [h(s)-h(0)]^{\frac{1}{2}} ds$. Notice that the amplitude of the response of the gas to an applied signal of $O(\epsilon)$ is $O(\epsilon^{\frac{1}{2}})$ and that the width of the resonant band

$$\frac{2(\epsilon M)^{\frac{1}{2}}\bar{h}}{1-\epsilon M\bar{h}^2}$$

is also $O(\epsilon^{\frac{1}{2}})$.

The above results are particularly simple for the important special forcing function

$$h(\eta) = -\cos 2\pi\eta . \tag{3.18}$$

Then

$$Z^{\pm}(\eta) = \pm \left(\frac{2\epsilon}{M}\right)^{\frac{1}{2}} \sin\pi\eta \tag{3.19}$$

and $\bar{h} = 2\sqrt{2}/\pi$. In addition the shock strength can be found explicitly in terms of δ ;

$$
\begin{aligned}
\text{Shock strength} &= Z^+(\eta_s) - Z^-(\eta_s) \\
&= 2\left(\frac{2\epsilon}{M}\right)^{\frac{1}{2}}(1-\phi^2(\delta))^{\frac{1}{2}} ,
\end{aligned}
\tag{3.20}
$$

where

$$\phi(\delta) = \frac{-\pi\delta}{2(1+\delta)(2\epsilon M)^{\frac{1}{2}}} \quad \text{and} \quad \cos(\pi\eta_s) = \phi(\delta) . \tag{3.21}$$

Thus for a given δ within the resonant band, the shock strength is given by (3.20) and the position of the shock by the second of (3.21). On using (3.17), it follows that

$$\phi(\delta^+) = - \phi(\delta^-) = - 1 \, ,$$

and so, by (3.20), the shock strength tends to zero as $\delta \to \delta^\pm$. Further, as $\delta \to \delta^+$, $\eta_s \to 1$, while as $\delta \to \delta^-$, $\eta_s \to 0$. The limiting solutions, when $\delta = \delta^\pm$, are given by $F(\eta) = Z^\pm(\eta)$, $0 \le \eta \le 1$. Then the signal F is continuous but has a <u>discontinuous slope</u> at $\eta = 0$ and $\eta = 1$. The resonant band is not symmetrically situated about the linear resonant frequencies $\omega = \omega_n$ since $|\delta^+| > |\delta^-|$, by (3.17). Note, further, that $F'(\eta_1) = 0$ and thus for the case $h = -\cos 2\pi\eta$ and $\omega = \omega_n$, $\eta_s = \eta_1 = \frac{1}{2}$, i.e., the maximum pressure equals the pressure immediately ahead of the shock, and the pressure immediately behind the shock is the minimum pressure.

The main point of the analysis given above for the undamped case is that all the results are obtained from the simple equation (3.6) together with the mean condition (2.25). By this we mean that for any frequency ω such that $|\omega - \omega_n| \ll 1$ (either inside or outside the resonant band) the signal function, F, is determined by (3.6) and (2.25). This is a consequence of the translation (2.20). The advantage of this approach is that it does not rely on having an integral of the governing equations to construct the resonant band, and hence can be generalized to the dissipative case, $\mu \neq 0$. The resonant band for the case $\mu = 0$, $h = -\cos 2\pi\eta$ has been treated previously by Chester (1964).

His analysis of the band involves finding, by an ad hoc procedure, an equation which is uniformly valid in the frequency parameter.

3.2 $\mu_c < \mu < 1$

When μ is nonzero the positions of the singular points in the (η, Z) plane now depend on both μ and δ and B_1 is no longer a center. We consider the variations in (μ, δ) in two parts. Here we show that for a given forcing function $H(\eta)$ there is a critical amount of damping, $\mu = \mu_c$, such that for $\mu > \mu_c$ the signal function is continuous for all frequencies. In section 3.3 we fix $\mu < \mu_c$ and consider variations in δ which will define the resonant band.

If we assume that the zeroes of $H(\eta)$ satisfy the conditions described in 3.1, the singular points of the equation

$$\nu Z(\eta) Z'(\eta) + \mu Z(\eta) = G(\eta) , \qquad (3.22)$$

where

$$G(\eta) = H(\eta) + \mu \frac{\delta}{D} , \qquad (3.23)$$

are the points $(\theta_i, 0)$, $i = 0, 1, 2$, such that $G(\theta_i) = 0$. (The periodicity of H ensures that $\theta_2 = \theta_0 + 1$.) We label them $A_0(\mu, \delta)$, $A_2(\mu, \delta)$ and $B_1(\mu, \delta)$ where $A_0(\mu, 0) = (0,0)$, $A_2(\mu, 0) = (1,0)$ and $B_1(\mu, 0) = (\eta_1, 0)$. Labeling them in this way is consistent with the notation of the previous section and ensures that A_0 and A_2 are again saddle points. The separatices through A_i (which we denote by $Z_i^\pm(\eta)$) have slopes

$$\lambda^{\pm}(\theta_i) = \left\{-\mu \pm [\mu^2 + 4\nu H'(\theta_i)]^{\frac{1}{2}}\right\} / 2\nu , \qquad (3.24)$$

where $\lambda^+(\theta_i) > 0 > \lambda^-(\theta_i)$, since $H'(\theta_i) > 0$, for $i=0,2$.
The slopes at B_1 are also given by (3.24), where $H'(\theta_1) < 0$.
When $I(\theta_1) > 0$, where

$$I(\eta) = \mu^2 + 4\nu H'(\eta) , \qquad (3.25)$$

B_1 is a node, while if $I(\theta_1) < 0$ B_1 is a focal point. Thus
for a given forcing function H (which is $O(\varepsilon)$) B_1 will be
a node if there is sufficient damping in the system. It may then
be possible to construct a continuous solution passing through
A_0 , B_1 and A_2 for any value of δ . Obviously the nodal
condition $I(\theta_1) > 0$ is <u>necessary</u> for the existence of such a
solution, however it may not be sufficient. Since the distortion
of the signal, and possible shock formation, depends on the
amplitude of $H'(\eta)$, one can expect the condition ensuring the
existence of a continuous solution to depend on a global property
of $H'(\eta)$. In fact

$$\mu^2 > \mu_c^2 = \max_{\eta} [-4\nu H'(\eta)] > 0 \qquad (3.26)$$

is a <u>sufficient</u> condition for the existence of a continuous,
periodic solution at all frequencies. The proof of this result
is given in Appendix 2. Thus for $\mu > \mu_c$,

$$F(\eta) = Z_0^+(\eta) , \qquad 0 \leq \eta \leq 1 , \qquad (3.27)$$

which is continuous and, by (3.5), satisfies the mean condition
(2.25), (see figure [4]).

3.3 $0 < \mu < \mu_c$

In this case there is not enough damping to produce a shockless solution for all frequencies. Thus for $\delta=0$

$$Z_0^+(\theta_1) > 0 \ , \quad Z_2^-(\theta_1) < 0$$

while

$$Z_0^+(\eta_0) = Z_2^-(\eta_2) = 0$$

where $0<\eta_2<\theta_1<\eta_0<1$ (see figure [1]) . Hence the separatices do not connect the saddle points A_0 and A_2 . As δ is increased through the resonant band, so that $\int_0^1 G(s)\,ds > 0$, there exists a unique frequency, given by $\delta=\delta^+$, such that

$$Z_0^+(\eta) \equiv Z_2^+(\eta) \ , \quad \theta_0 \leq \eta \leq \theta_2 \ .$$

That is, for $\delta=\delta^+$, the positive separatrix connects the saddle points A_0 and A_2 , (see figure [2]). Further, for $\delta>\delta^+$ there exists a <u>unique</u>, continuous periodic solution $Z=Z_\delta(\eta)>0$, (see figure [3]). Similarly there exists $\delta=\delta^-<0$ for which $Z_0^- \equiv Z_2^-$ and such that when $\delta<\delta^-$ there is a <u>unique</u>, continuous periodic solution $Z=Z_\delta(\eta)<0$. These results can be inferred from the results of Amerio (1949,1950). Whereas for the case $\mu=0$ explicit values have been given for $\delta^\pm$ (see equation (3.17)), when $0<\mu<\mu_c$ this is not possible. However, for a particular forcing function, $\delta^\pm$ can easily be found numerically by varying δ until a solution is found such that $Z_0^+(\theta_2)=0$ or $Z_2^-(\theta_0)=0$. Since these limiting solutions are continuous they satisfy the mean condition (2.25) and hence we can give the im-

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plicit conditions for the edge of the band as

$$\int_{\theta_0}^{\theta_2} Z_0^+(s,\delta^+)ds = \frac{\delta^+}{b(\delta^+)} \quad \text{and} \quad \int_{\theta_0}^{\theta_2} Z_2^-(s,\delta^-)ds = \frac{\delta^-}{b(\delta^-)} \ .$$

There have been several attempts to obtain analytical bounds on $\delta^\pm$. Hayes (1953) and Böhm (1953) found bounds for $h=-\cos 2\pi\eta$ while Lillo and Seifert (1955) used similar techniques to find bounds for a general forcing function. Further reference can be found in Sansone and Conti (1964)

By equation (3.5) the <u>unique continuous</u> solutions $Z_\delta(\eta)$ automatically satisfy the mean condition (2.25) and hence for $\delta>\delta^+$ or $\delta<\delta^-$

$$F(\eta) = Z_\delta(\eta) \ , \qquad \theta_0 \leq \eta \leq \theta_2 \ . \tag{3.28}$$

We contrast the case of no damping, $\mu=0$, when the continuous periodic solutions $Z_\delta(\eta)$ were not unique, but the mean condition uniquely determined the signal function F .

When $\delta^-<\delta<\delta^+$ there are no continuous periodic solutions of (3.22). Again, like the case $\mu=0$, we construct the signal function F by a composition of integral curves, (see figure [1]). The discontinuous signal function F must satisfy both the mean condition (2.25) and the weak shock condition (3.4). However it has been shown that if the mean condition is satisfied the shock condition is automatically satisfied. The condition that a shock is compressive then implies that we choose

$$F(\eta) = \begin{cases} Z_0^+(\eta) & \theta_0 \leq \eta \leq \eta_s \\ Z_2^-(\eta) & \eta_s < \eta \leq \theta_2 \end{cases} \qquad (3.29)$$

where $\eta = \eta_s$ is the position of the shock. It is shown in Appendix 2 that it is always possible to choose an η_s to combine $Z_0^+(\eta)$ and $Z_2^-(\eta)$ so that the mean condition (2.25) is satisfied. Hence F, as given by (3.29), is the required signal function. By equation (3.22), $F'=0$ at $\eta = \eta_{max}$, η_{min} where $F = G/\mu$. Then, in general, when $\omega = \omega_n$ the maximum pressure exceeds the pressure immediately ahead of the shock and the pressure immediately behind the shock exceeds the minimum pressure.

It is clear from the structure of the integral curves for the inviscid case that if the piston frequency is an even multiple of the fundamental, then a possible continuous solution is

$$F(\eta) = Z_0^+(\eta), \qquad \theta_0 \leq \eta \leq \theta_2$$

with

$$F(\eta+1) = -F(\eta),$$

which is a "subharmonic" solution. From the preceeding analysis, this solution is unstable to perturbations in both damping and frequency.

Finally we note that when $k=0$ ($\mu=1$) the impedances at $x=0$ are 'matched'. Then there is no reflected wave, so that by (2.14) $g \equiv 0$. The differential equation (2.27), together with (2.20) and (2.24) then yields that on $x=1$,

$$f(\eta) = H(\eta) \quad \text{for all } \eta.$$

4. Lumped Damping; Critical Acceleration Level

In the preceding we have analyzed resonant oscillations
when the only damping in the system is due to radiation of
energy away from one end of the pipe. Here we discuss two other
forms of damping which fit into the same theoretical framework.
These are damping due to internal dissipation and to wall fric-
tion. It is shown how to define a lumped damping coefficient,
$\bar{k}$, which allows one to incorporate the effect of the three
damping mechanisms mentioned into an equation of the form (2.27).
The coefficient is referred to as "lumped" since on the basis
of experiments it can be used as a measure of the effective
damping in the system without being able to assess the effect
of the individual contributions.

It has been shown by Mortell and Seymour (1972a) that the
representation (2.10) and (2.11) can be extended, in the high
frequency limit, to include the effect of internal dissipation
of the transmitting media (specifically, there, for a viscoelastic
rod). For a gas such dissipation would result from the excita-
tion of any of the internal degrees of freedom, e.g. vibrational
excitation or molecular dissociation. If it is assumed that only
one rate dependent process is of significance and that this can
be represented by the relaxation variable $\sigma(X,t)$, then the
rate of adjustment of σ may be described by

$$\frac{\partial \sigma}{\partial t} = \psi(p,e,\sigma) .$$

(4.1)

ψ will then define a rate parameter or relaxation time $\tau (>0)$

proportional to $\psi,_\sigma(p_o,e_o,\sigma_o)$. A periodic disturbance is considered of high frequency, or is near-frozen, if its period is small compared with τ , so that $L(a_o\omega)^{-1} \ll \tau$ (note that in our variables $L(a_o\omega)^{-1}$ represents the dimensional piston frequency). For such disturbances 'a_o' should now be interpreted as the frozen rather than equilibrium sound speed (more details of nonlinear wave propagation in a relaxing gas are given by Blythe (1969)). In this limit a disturbance in the gas can be represented as two noninteracting, modulated simple waves traveling in opposite directions (see Mortell and Seymour (1972a)) (modulated simple waves in rate-dependent media are discussed in detail by Seymour and Varley (1970)). The appropriate representation corresponding to equations (2.10) and (2.11) is then

$$e = p = -f(\beta)\ e^{d(x-1)} -g(\alpha)\ e^{-dx}, \quad u = f(\beta)\ e^{d(x-1)} - g(\alpha)\ e^{-dx} ,$$

$$(4.2)$$

$$\frac{\alpha}{\omega} = t - x - Mg(\alpha)\ d^{-1}(1-e^{-dx}) \qquad (4.3)$$

and

$$\frac{\beta}{\omega} = t + x - 1 + Mf(\beta)\ d^{-1}(e^{d(x-1)} - 1) , \qquad (4.4)$$

where

$$d = L/a_o\tau \ll \omega . \qquad (4.5)$$

On eliminating g from (4.2) and (4.3) through the boundary conditions (2.4) and (2.5), the functional difference equation (2.12) and (2.13) is recovered with the parameters k and b replaced by $\bar{k}$ and $\bar{b}$ where

$$\bar{k} = ke^{-2d} \quad \text{and} \quad \bar{b} = \tfrac{1}{2}M(1+\epsilon)(1+ke^{-d})(1-e^{-d})d^{-1} . \qquad (4.6)$$

In the small rate limit, when $\omega=O(1)$, the condition (4.5) implies $d<<1$. Under these circumstances the procedure of section 2 then leads to the nonlinear ordinary differential equation (2.27) with the parameters μ and ν replaced by $\bar{\mu}$ and $\bar{\nu}$ where

$$\bar{\mu} = 1 - \bar{k} \quad \text{and} \quad \bar{\nu} = n\bar{b}\bar{k} . \qquad (4.7)$$

The rate independent case is recovered in the limit $\tau\to\infty$ when $d\to0$. The parameter $\bar{k}$ in (4.6) consists of two factors; the first, k , is the attenuation of the signal at the interface $x=0$ due to transmission of energy into the neighboring medium; the second, e^{-2d} , is the attenuation of the signal over one cycle due to internal damping. The latter acts continuously throughout the gas, whereas the former only acts at the interface. The role of both in preventing shock formation is the same.

In the high frequency limit, internal dissipation is a _lower order damping_ in the sense that the appropriate linearized equation satisfied by u is

$$u_{tt} - u_{xx} + 2d\, u_t = 0 . \qquad (4.8)$$

In contrast, a _higher order damping_, introduced for instance by compressive viscosity, in represented in the linear case by

$$u_{tt} - u_{xx} - \nu_o\, u_{xxt} = 0 .$$

Higher order damping will only structure a shock in a resonant oscillation (see Chester (1964)), while lower order damping can prevent shock formation even for $d \ll 1$. A form of damping which is always present to some degree in a tube is that due to the viscous effects of the boundary layer at the tube wall. Its effect can be modeled as in Chester (1964), as a weighted integral of F in the equivalent of (2.27), or, equivalently by adding a body force term in the momentum equation. The latter procedure was used by Rayleigh (1945) for periodic oscillations in a circular pipe and yields a lower order damping of the form given by equation (4.8) with the parameter 'd' replaced by

$$R = \left(\frac{\kappa \omega L}{a_o r^2} \right)^{\frac{1}{2}}$$

where κ is the kinematic viscosity of the gas and r is the radius of the tube. The effect is to produce a damping coefficient

$$k_R = e^{-2R} .$$

Thus the lumped damping coefficient to account for these three damping mechanisms takes the form $ke^{-2(a+R)}$. The damping coefficient, k, in (2.22), and the corresponding μ in (2.27) can always be interpreted in this broader sense. The effect of $k<1$ is to decrease the shock strength in a resonant oscillation. Then, the effective damping in the system can be measured by adjusting k to ensure agreement between the theoretical and experimental shock strengths. A purely inviscid model predicts the shock strength with an error of about 30% at the fundamental

linear resonant frequency.

Since $A_p(t) = \omega H'(\omega t)$ represents the acceleration of the piston, we can interpret condition (2.26) as follows: for a given amount of damping, k, there is a critical acceleration level of the piston such that for applied accelerations below this the motion of the gas is shockless at all frequencies. By (3.26), for a known μ the gas motion is shockless provided the <u>piston</u> acceleration, satisfies

$$|A_p(t)| < \frac{\omega \bar{\mu}^2}{4\bar{\nu}} = \frac{\bar{\mu}^2}{4M\bar{k}(1+\bar{k})} \tag{4.9}$$

on using (2.18), (2.19) and (2.21). In the limit as the damping tends to zero, $\bar{k} \rightarrow 1$ and $\bar{\mu} \rightarrow 0$ which implies $|A_p(t)| \rightarrow 0$ for a shockless motion. Thus when there is no damping present there is always a shock at resonance.

When $i \ll 1$, $a \ll 1$, then (4.9) reduces to

$$|MA_p(t)| < \frac{1}{2}(i+a)^2 \tag{4.10}$$

as the condition for a shockless solution for a resonant <u>forced</u> motion. In contrast, for a <u>transient</u> or "standing wave" motion in the same system the condition for a shockless motion is (see Mortell and Seymour (1972a))

$$|MA(t)| < (i+a) , \tag{4.11}$$

where $A(t)$ here is the acceleration level defined by the initial conditions. The formation of a shock is determined by the induced acceleration level in the gas flow. For the shockless

transient motion the applied and induced accelerations have the same order of magnitude. However, for the resonant forced motion, the induced acceleration has the same order of magnitude as the <u>square root</u> of the applied acceleration. With this observation the results (4.10) and (4.11) are in harmony. In a study of a radiating gas, Eninger (1971) found numerically a critical damping which prevents shocks at resonance. For his analysis damping enters as a weighted integral of F in the equivalent of (2.27).

When the resonant motion is shockless, linear theory is a uniformly good approximation to the nonlinear theory provided the piston acceleration is sufficiently small, (see figure [4]). By setting $\nu=0$ in (2.27), the linear solution is given by

$$F_L(\eta) = G(\eta)/\mu \tag{4.12}$$

and hence, by (2.27),

$$\left| \frac{F_L(\eta)}{F(\eta)} - 1 \right| = \frac{\nu}{\mu} |F'(\eta)| \quad . \tag{4.13}$$

By differentiating (2.27) and setting $F''(\eta)=0$ we find that $|F'(\eta)| \leq \lambda^+(\eta)$ where $\lambda^+(\eta)$ is given by (3.24). If now the applied acceleration is small, in the sense that

$$\left| \frac{4\nu H'(\eta)}{\mu^2} \right| \ll 1 \quad , \tag{4.14}$$

then

$$\left| \frac{F_L(\eta)}{F(\eta)} - 1 \right| = \frac{\nu}{\mu} \left| F'(\eta) \right| \leq \frac{\nu}{\mu} \left| \lambda^r(\eta) \right|$$

$$= \frac{1}{2} \left| \left(1 + \frac{4\nu H'}{\mu^2} \right)^{\frac{1}{2}} - 1 \right|$$

$$= \left| \frac{\nu}{\mu^2} H' \right| + O\left(\frac{\nu}{\mu^2} H' \right)^2$$

$$\ll 1 \text{ , by (4.14)}.$$

The inequality (4.9) defines a critical acceleration level which provides a sufficient condition on the applied rate to ensure a response of the gas which is continuous. Numerical integration of the equations shows that shockless solutions exist for piston accelerations greater than the critical one, so that inequality (4.9) is conservative, as may be expected.

5. Comparison with Experiments

The theoretical predictions of the analysis presented are
compared with some experimental measurements made by Sturtevant
(1972). His set up consists of a tube of length 132.5 inches
with an inside diameter of 3.0 inches which contains air ($\gamma=1.4$).
At one end of the tube is a piston which is displaced sinusoidally
with an amplitude, normalized against the length of the tube, of
.0147 . The experiments we are concerned with here have the
two configurations

    i)   the far end of the tube is closed;

    ii) the far end of the tube has a hole in it whose area

          is small compared to the rea of the end.

For case i) we are concerned with measurements of the following
quantities, at the closed end, for values of the piston fre-
quency around the fundamental:

    a)   the absolute maximum and minimum of a normalized

          pressure waveform,

    b)   the pressure immediately before and after the shock

          jump.

As a consequence of these readings the values of the frequency
corresponding to the lower and upper ends of the resonant band
are available.

Figure [5] shows the comparison between inviscid theory
($i=0$) and experimental "response curve" of Sturtevant. In this
case, our theory is equivalent to that in section 4 of Chester
(1964). In computing the theoretical curves in Figure [6] a

value of $i = \dfrac{1-\bar{k}}{1+\bar{k}}$ ($i=.08$) is chosen so that the shock strength
exactly at resonance ($\delta=0$) is equal to the observed strength.
The theory predicts that, for $\delta=0$, the pressure immediately
before the shock is the negative of the pressure immediately
afterwards, which is not the experimental result. Thus the
theoretical and experimental curves do not coincide at $\delta=0$,
even with our choice of $\bar{k}$. We should also bear in mind that
for the conditions of the experiment the small rate condition
is only marginally satisfied (see the comment at the end of
section 2). The experiments show that at resonance the maximum
pressure exceeds the pressure ahead of the shock and the pres-
sure behind the shock exceeds the minimum pressure. This is
not predicted by inviscid theory, but is a property of the
solution of the equation with damping. Another point to note
is that the absolute maximum of the pressure occurs about 10%
to the right of the resonant frequency while the absolute min-
imum occurs about 5% to the left. For the experimental conditions
here boundary layer damping has little effect. Nevertheless,
if the other damping mechanisms were absent, boundary layer
damping could prevent a shock if the radius of the tube and the
amplitude of the piston were appropriately adjusted.

An interesting point is that the amount of damping required
to get the shock strength correct for $\delta=0$ has a negligible
effect on the width of the resonant band. This might seem sur-
prising since damping decreases the shock strength which in turn
determines the resonant band. The result can be understood,

qualitatively, by considering equation (2.27) and bearing in mind the definition of $G(\eta)$ given by (2.24). When there is damping, $\mu \neq 0$, the system defined by (2.27) is being driven by the forcing function G whose mean is non-zero for $\delta \neq 0$. The increased damping is then counteracted by the increased amplitude of the effective driver G.

For case ii) it is observed that for particular experimental conditions there is a critical area ratio at which the shock in the tube disappears for all frequencies. If we interpret the presence of a small hole in the end of the tube as a means of introducing damping into the system then the prediction of the theory agrees qualitatively with experiment. It cannot be expected that the impedance condition (2.4), as introduced in the theory, will account for the detailed motion of the gas near the orifice. Nevertheless, it seems to be useful in predicting the gross features of the motion.

Curves of shock strength, $S(i)$, versus impedance, i, were plotted for various values of the piston amplitude, ε, Figure [7] shows there is a linear relation between $S(0)-S(i)$ and i, which is independent of ε for $0 \leq i < .2$ when $.01 \leq \varepsilon \leq .02$. A corresponding plot of shock strength versus area ratio would give a measure of the effective impedance (or effective damping). The linear relationship indicates that shock strength is a good measure of damping.

This result can be understood from a rough analysis of the energy balance. When $i \ll 1$, the results of section 3 indicate

that if the amplitude of the piston is ε , the amplitude of
the response is $O(\varepsilon^{\frac{1}{2}})$, while the shock strength S is $O(\varepsilon^{\frac{1}{2}})$.
The balance between the input of energy due to the piston and
the loss due to the shock and radiation from the end is

$$A\varepsilon^{\frac{3}{2}} = S^3(i) + iB\varepsilon , \qquad (5.1)$$

where A, B are constants. Since $S = O(\varepsilon^{\frac{1}{2}})$, (5.1) can be
interpreted, dividing through by ε , as

$$S(i) - S(0) = iB'$$

where $S(0) = A'\varepsilon^{\frac{1}{2}}$, and A', B' are constants. The linear
relationship is lost when shock dissipation is no longer a
major effect. As the piston amplitude (ε) decreases, the
point at which the curves bifurcate moves towards the origin,
so that the linear relation holds for a smaller range of the
impedance.

Appendix 1.

We sketch a regular perturbation procedure for the derivation of equation (2.27) directly from the governing equations (2.1)-(2.5). The procedure is a generalization of that used by Mortell (1971b). We assume an expansion of the form

$$u(t,x;\epsilon) = \epsilon^{\frac{1}{2}} u_1(t,x) + \epsilon\, u_2(t,x) + \ldots$$

$$e(t,x;\epsilon) = \epsilon^{\frac{1}{2}} e_1(t,x) + \epsilon\, e_2(t,x) + \ldots$$

$$\omega = \omega(\epsilon) = \omega_0 + \epsilon^{\frac{1}{2}} \omega_1' + \epsilon\omega_2 + \ldots$$

$$i = i(\epsilon) = \epsilon^{\frac{1}{2}} i_1 + \epsilon i_2 + \ldots \;,$$

where the perturbation parameter is the amplitude of the response. On noting that under the above expansion the problem at $O(\epsilon^{\frac{1}{2}})$ is homogeneous, we find

$$u_1 = f(\eta+\omega_0 x) - f(\eta-\omega_0 x) \;; \quad - e_1 = f(\eta+\omega_0 x) + f(\eta-\omega_0 x)$$

where

$$\eta = \omega t \;, \quad \omega_0 = \tfrac{1}{2} n \;, \quad n = 1,2,3,\ldots \;,$$

and $f(\eta)$ is an arbitrary function with period $\dfrac{2\omega_0}{n}$. According to the expansion, ω_0 is $O(1)$, so that $\epsilon^{\frac{1}{2}} n \ll 1$, which is the small rate condition.

The problem at $O(\epsilon)$ is non-homogeneous, but it can be integrated to give

$$2u_2(t,x) = g_2^- + f_2^+ + 2\omega_1'x(f_+' + f_-') + \frac{3-\gamma}{4}(f_+^2 - f_-^2)$$

$$+ (\gamma+1)\left[\omega_o x(f_- f_-' + f_+ f_+') + \frac{1}{2}f_-' \int^{\eta+\omega_o x} f\, ds - \frac{1}{2}f_+' \int^{\eta-\omega_o x} f\, ds\right]$$

$$2e_2(t,x) = g_2^- - f_2^+ + 2\omega_1'x(f_-' - f_+') + (3-\gamma)\left[\frac{1}{4}(f_+^2 + f_-^2) + f_+ f_-\right]$$

$$+ (\gamma+1)\omega_o x(f_- f_-' + f_+ f_+') + \frac{\gamma+1}{2}\left[f_-' \int^{\eta+\omega_o x} f\, ds + f_+' \int^{\eta-\omega_o x} f\, ds\right]$$

where the subscripts $+$, $-$ indicate the argument of the function is $\eta+\omega_o x$, $\eta-\omega_o x$ respectively, prime denotes differentiation with respect to the argument, and g_2 and f_2 are arbitrary functions associated with the general solution of the homogeneous wave equation. The boundary conditions at this order are

$$u_2(t,0) = -i_1 e_1(t,0) \quad \text{and} \quad u_2(t,1) = \omega_o h'(\eta) \ .$$

Upon using the conditions that f and g_2 have period $\frac{2\omega_o}{n}$, and that f has zero mean over this period, these boundary conditions imply that f satisfies

$$\omega_o(\gamma+1)f(\eta)f'(\eta) + 2\omega_1'f'(\eta) + 2i_1 f(\eta) = \omega_o h'(\eta-\omega_o) \ .$$

If we now note that

$$\omega_o = \frac{1}{2}n \ , \quad \frac{\gamma+1}{2} = M$$

and define

$$\omega_1' = \omega_o \omega_1 \quad \text{and} \quad F = f + \omega_1/M \ ,$$

the equation for F is

$$nMF(\eta)F'(\eta) + 2i_1F(\eta) = \frac{2i_1\omega_1}{M} + \frac{1}{2}nh'(\eta-\omega_0) . \qquad (A1.1)$$

This agrees with equation (2.27) except for the phase of h . This is accounted for by noting that the parametrization of the β-wave is chosen differently here (for convenience in the integration) than in the body of the text. The perturbation scheme is predicated on the assumption that the amplitude of the response is $O(\epsilon^{\frac{1}{2}})$. There is then an implicit assumption on the amount of damping present, and hence the restriction $i=O(\epsilon^{\frac{1}{2}})$. The derivation of (2.27) avoids this limitation on its range of applicability. There is no prior assumption on the final amplitude and hence none on the impedance, i . Consequently, from (2.27) it is seen that if $i=0$, nonlinearity dominates and the resulting amplitudes are $O(\epsilon^{\frac{1}{2}})$, while if $i=O(1)$, i.e., damping dominates, then the amplitude of the response is $O(\epsilon)$. In this sense equation (2.27) is uniformly valid in the damping parameter, whereas (A1.1) is not.

Appendix 2.

Here we prove two results used in section 3:

(i) A sufficient condition for the existence of a continuous, periodic solution of equation (3.22), for all δ , is that

$$\mu^2 > \max_{\eta} \ [-4\nu H'(\eta)] \ . \tag{A2.1}$$

(ii) Given $\mu \epsilon (0, \mu_c)$, $\delta \epsilon (\delta^-, \delta^+)$ and F defined by (3.29), there exists an $\eta_s \epsilon (\eta_2, \eta_o)$ such that

$$\int_{\theta_o}^{\theta_2} F(s) \, ds = \frac{\delta}{b} \ . \tag{A2.2}$$

Proof of (i)

Using the notation of section 3 we must show that when (A2.1) holds the separatices Z_o^+ and Z_2^- pass through the node B_1 , i.e. $Z_o^+(\theta_1) = Z_2^-(\theta_1) = 0$. We will prove the result for Z_o^+ ; the argument for Z_2^- is similar.

Firstly, since the curve $G(\eta)/\mu$ is the isocline $Z'(\eta)=0$ and $Z(\eta) \to 0^+$ yields the isocline $Z'(\eta) \to +\infty$, for $\theta_o < \eta < \theta_1$, the separatrix Z_o^+ is continuous and differentiable in (θ_o, θ_1) and satisfies

$$0 < Z_o^+(\eta) \leq \max_{\eta} \ [\frac{G(\eta)}{\mu}] \ .$$

In particular $Z_o^+(\theta_1) \geq 0$. We show that $Z_o^+(\theta_1) = 0$ by bounding $Z_o^+(\eta)$ above by a function $Y(\eta)$ which has the properties $Y(\eta) > 0$ for $\theta_o < \eta < \theta_1$ and $Y(\theta_1) = 0$. Such a curve bounds

Z_o^+ above if $\frac{dZ_o^+}{d\eta} < Y'(\eta)$, for all $\eta \in [\theta_o, \theta_1]$, whenever $Z^+ = Y$.
The curve $Y(\eta) = \frac{2\dot{G}(\eta)}{\mu}$ has these properties whenever (A2.1)
holds. For when $Z_o^+ = Y$,

$$\frac{dZ_o^+}{d\eta} = \frac{G - \mu Y}{\nu Y} = \frac{-\mu}{2\nu} < \frac{2H'(\eta)}{\mu} = Y'$$

which holds whenever condition (A2.1) does. Hence since
$Y(\theta_1) = 0$, $Z_o^+(\theta_1) = 0$.

Proof of (ii)

Defining

$$y(\eta_s) = \int_{\theta_o}^{\theta_2} F(s)\,ds = \int_{\theta_o}^{\eta_s} Z_o^+(s)\,ds + \int_{\eta_s}^{\theta_2} Z_2^-(s)\,ds$$

we wish to show that, for a given δ , there is a value of
$\eta_s \in (\eta_2, \eta_o)$ such that $y(\eta_s) = \frac{\delta}{b}$, where $\theta_o < \eta_2 < \theta_1 < \eta_o < \theta_2$, and
$Z_o^+(\eta_o) = Z_2^-(\eta_2) = 0$. We first note that, for a given δ , y
is a continuous function of η_s . Since Z_o^+ is continuous in
(θ_o, η_o) , integration of (3.22) yields

$$\int_{\theta_o}^{\eta_o} Z_o^+(s)\,ds = \frac{1}{\mu} \int_{\theta_o}^{\eta_o} G(s)\,ds .$$

Then consider $y(\eta_o) = \int_{\theta_o}^{\eta_o} Z_o^+(s)\,ds + \int_{\eta_o}^{\theta_2} Z_2^-(s)\,ds$

$$= \frac{1}{\mu}\int_{\theta_o}^{\theta_2} G(s)\,ds + \int_{\eta_o}^{\theta_2}\left(Z_2^-(s) - \frac{G(s)}{\mu}\right)ds .$$

Thus, since $Z_2^-(\eta) \geq \frac{G(\eta)}{\mu}$ for $\theta_1 < \eta < \theta_2$,

$$y(\eta_o) \geq \frac{1}{\mu}\int_{\theta_o}^{\theta_2} G(s)\,ds = \frac{\delta}{b} .$$

Similarly $y(\eta_2) \leq \frac{\delta}{6}$. Therefore by the continuity of y , there is an $\eta_s \epsilon(\eta_2, \eta_o)$ such that

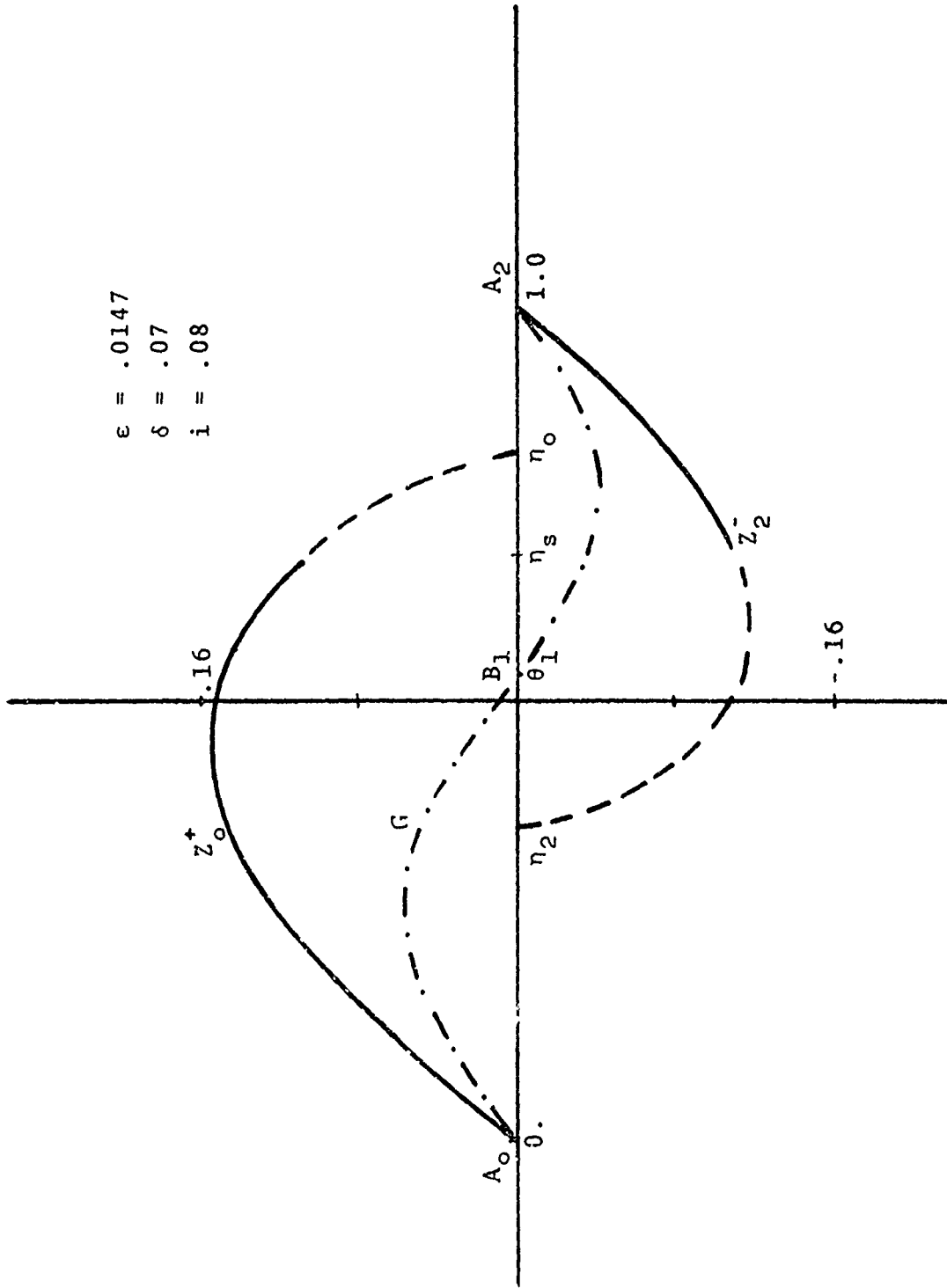$$y(\eta_s) = \int_{\theta_o}^{\theta_2} F(s) \, ds = \frac{\delta}{6} \; .$$

Acknowledgment

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Figure 1. Construction of solution, F , using integral curves Z_o^+ , Z_2^- .
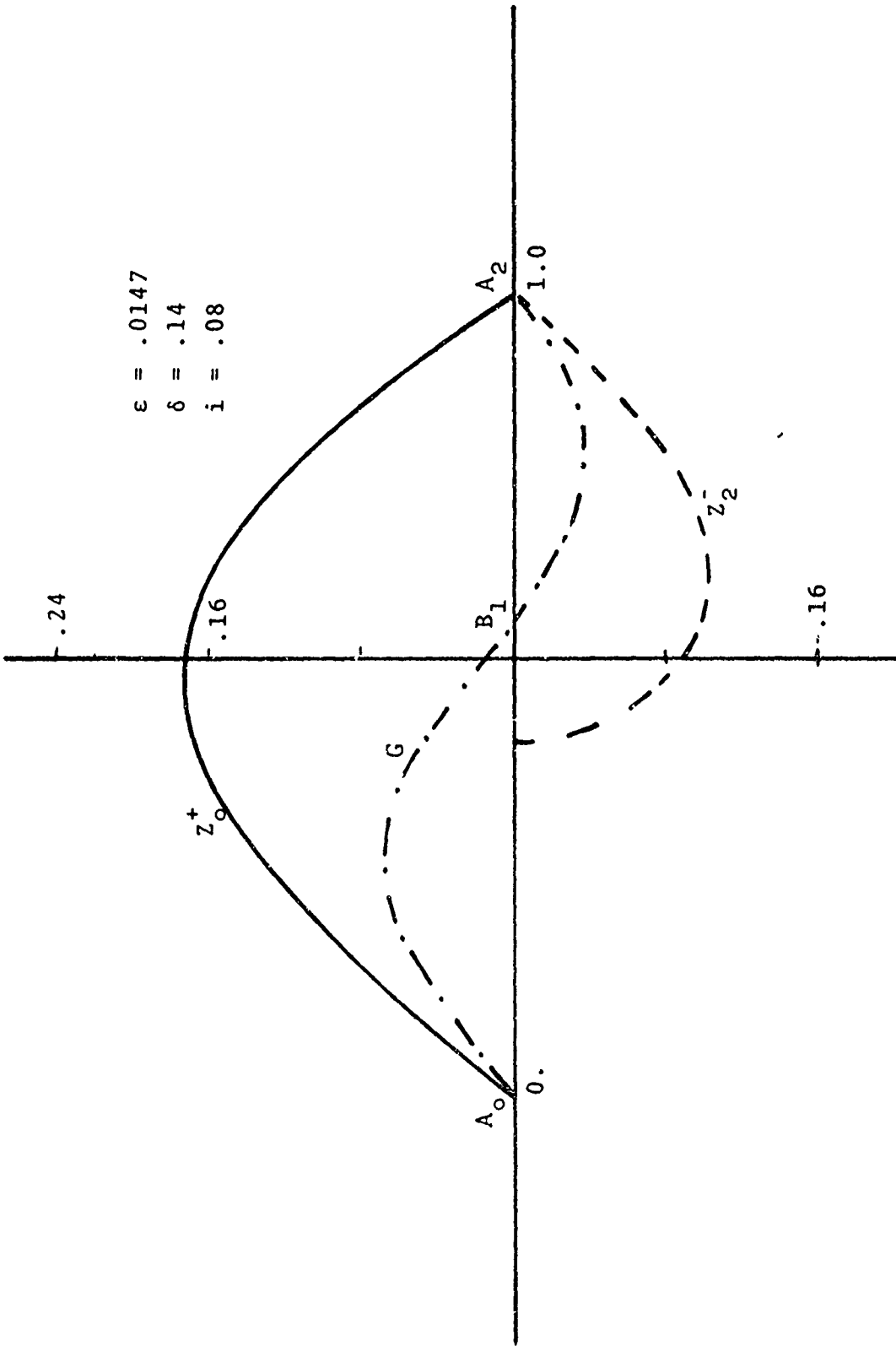Shock is at η_s .

Figure 2. $F = Z_o^+$ is solution at edge of resonant band, $\delta^+ = .14$.
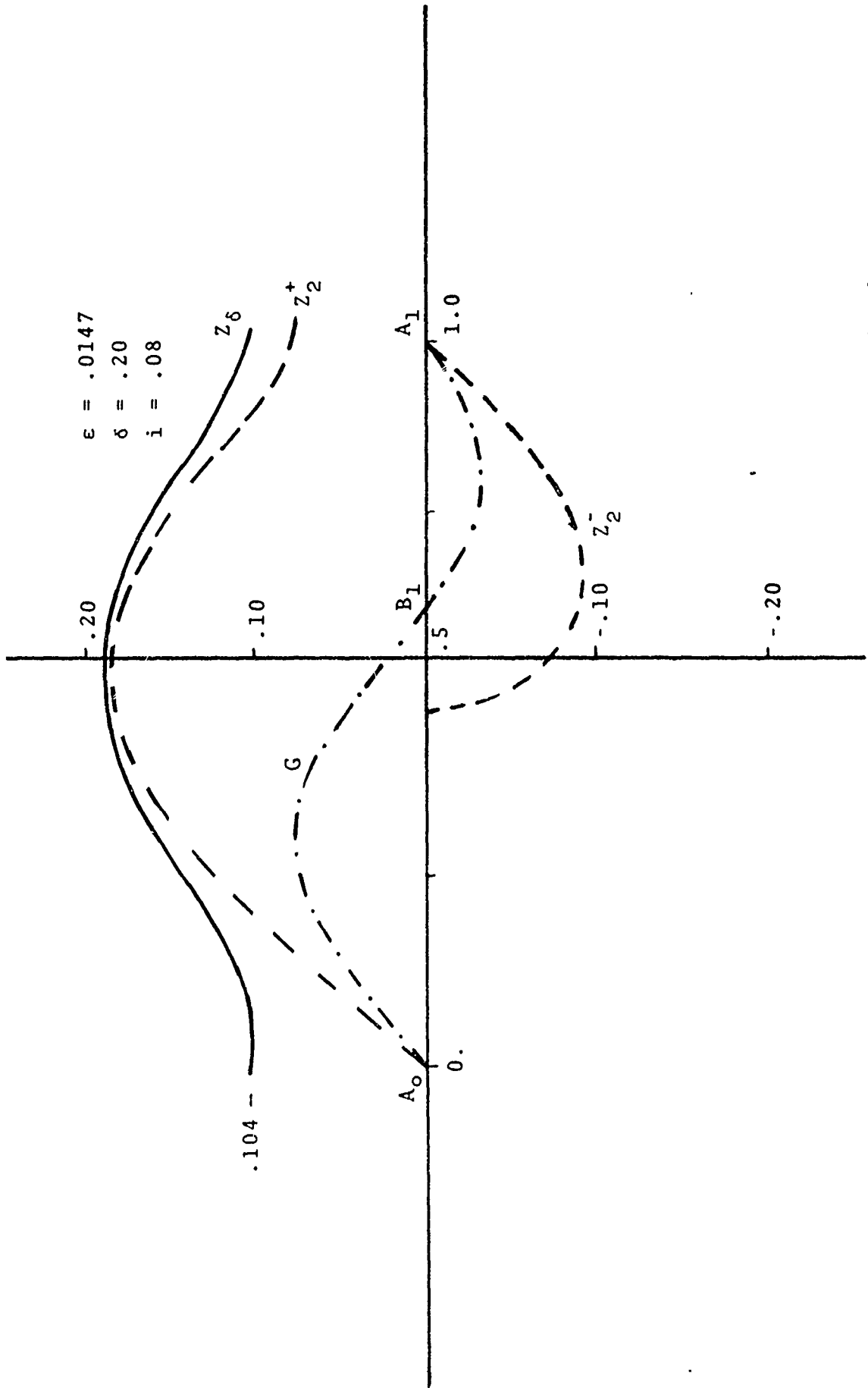
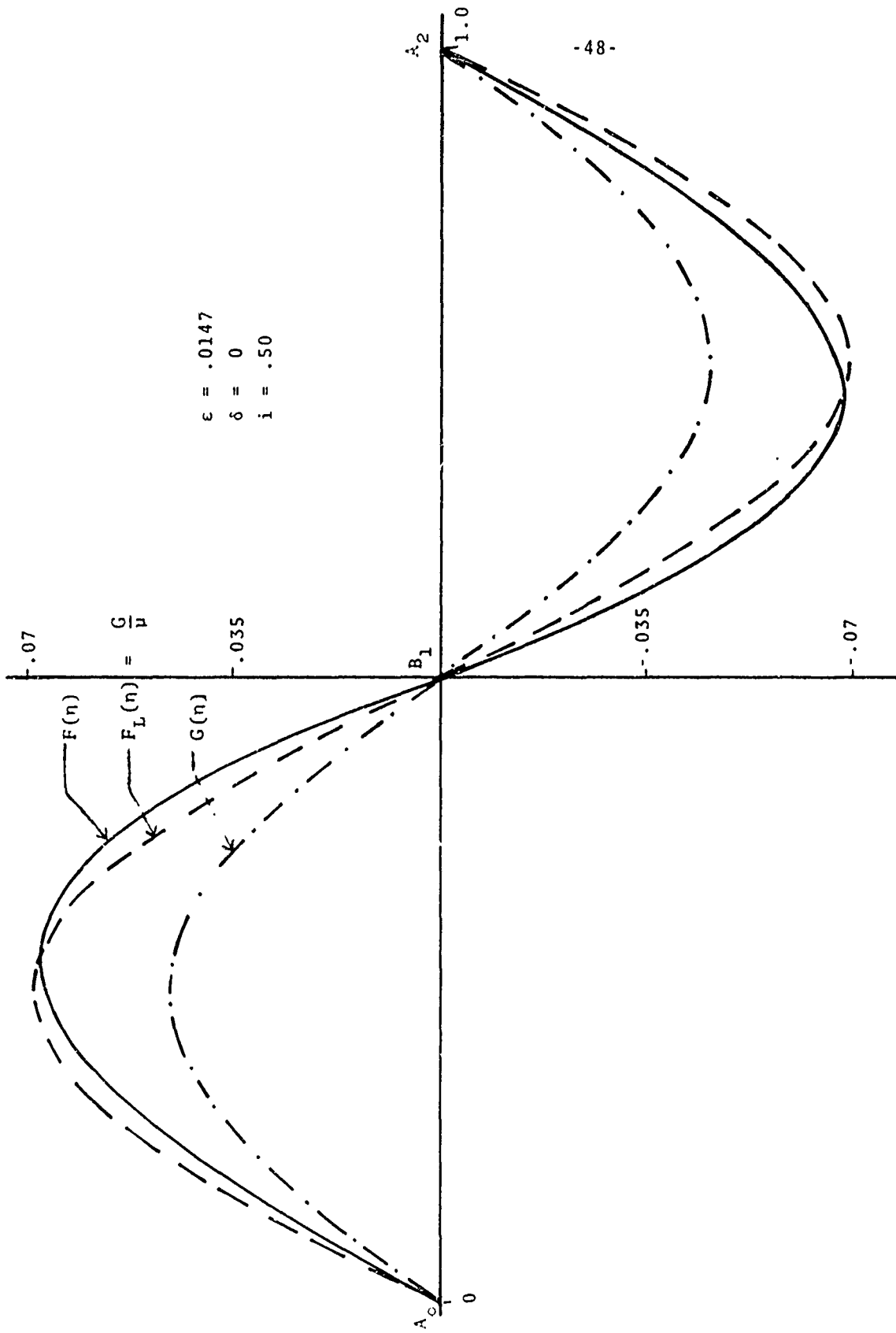Figure 3. $F = Z_6$ is continuous solution outside resonant band.

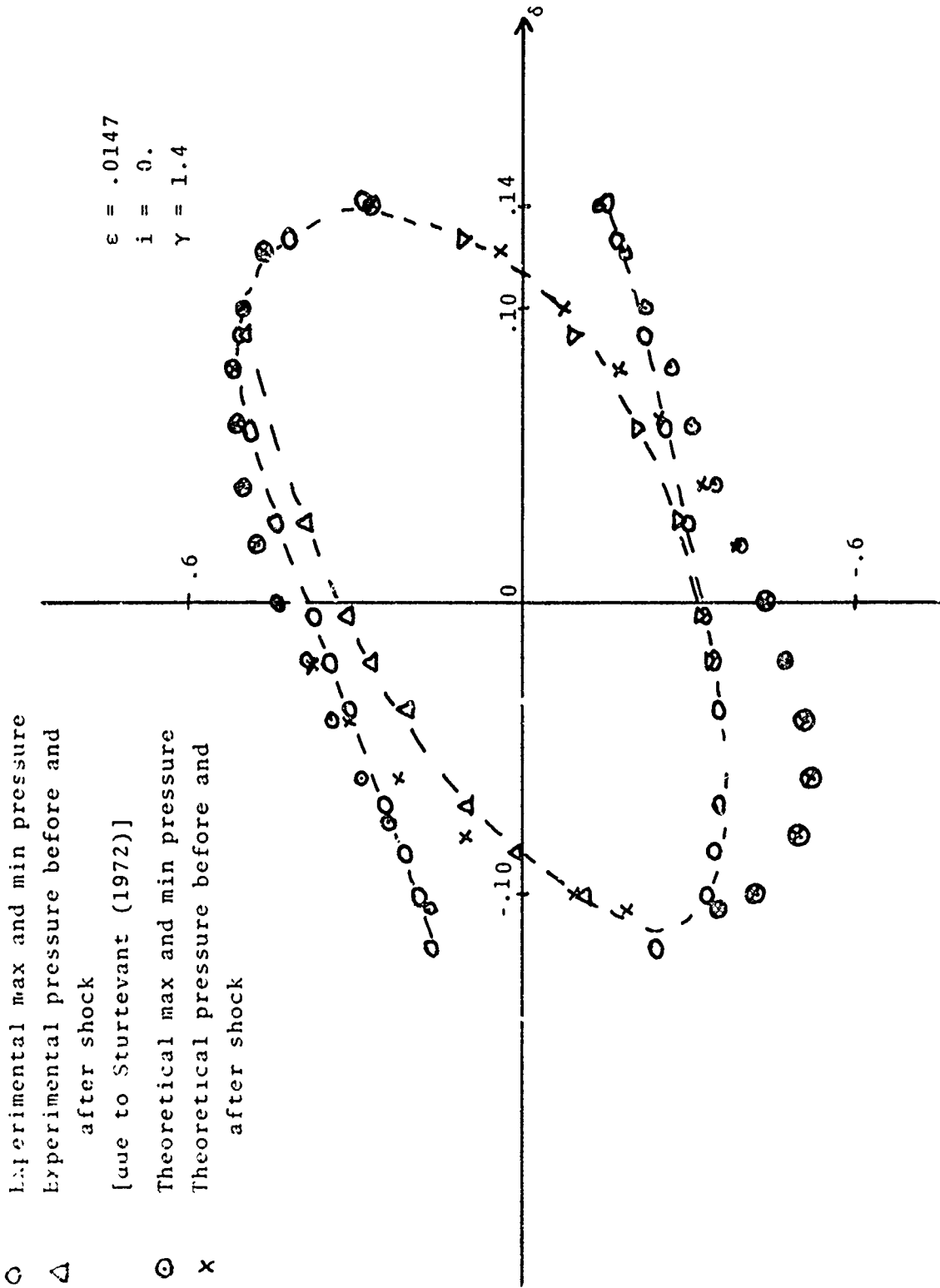Figure 4. Highly damped case; linear theory a good approximation.

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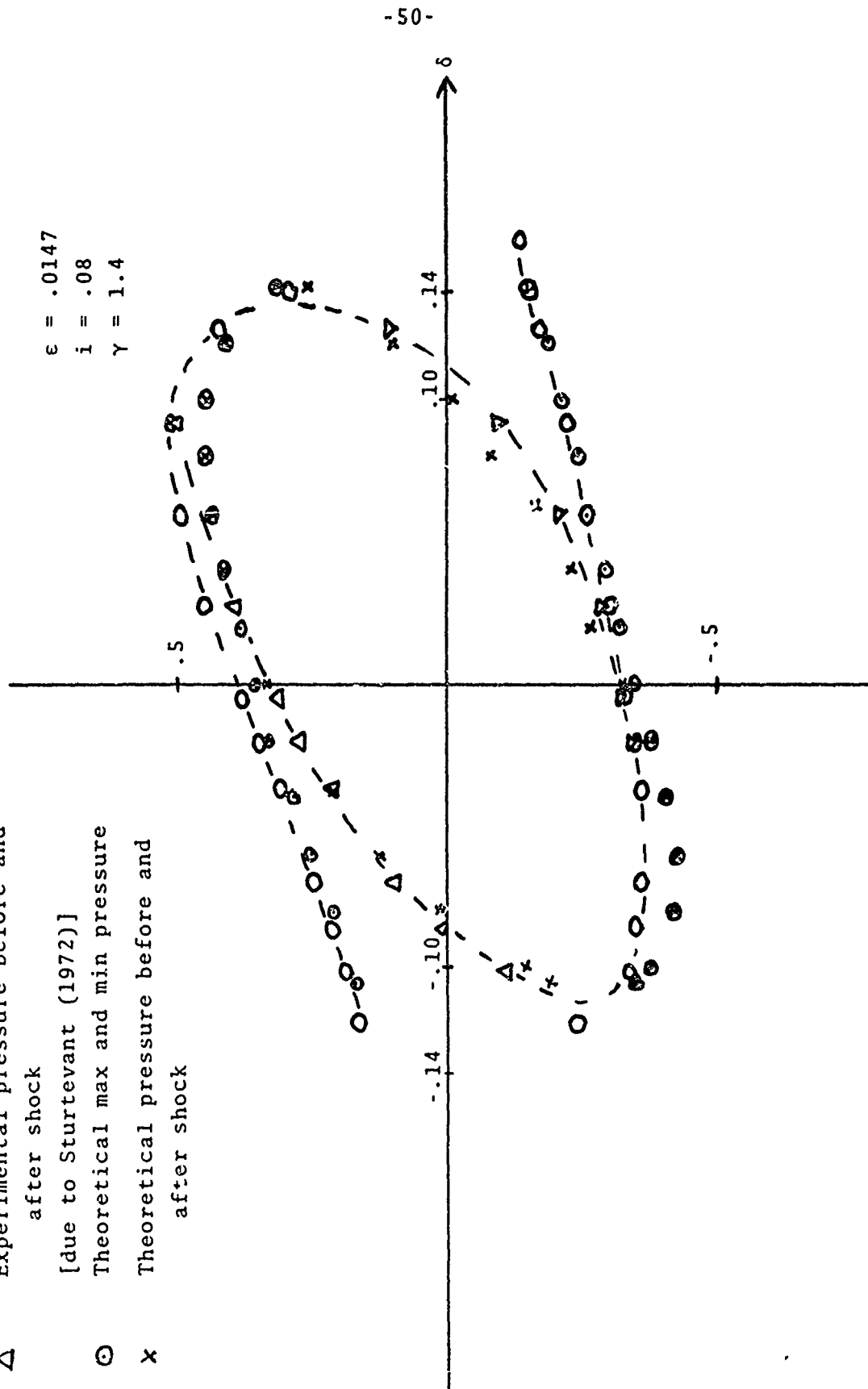Figure 5. Theoretical and experimental response curves: no damping.

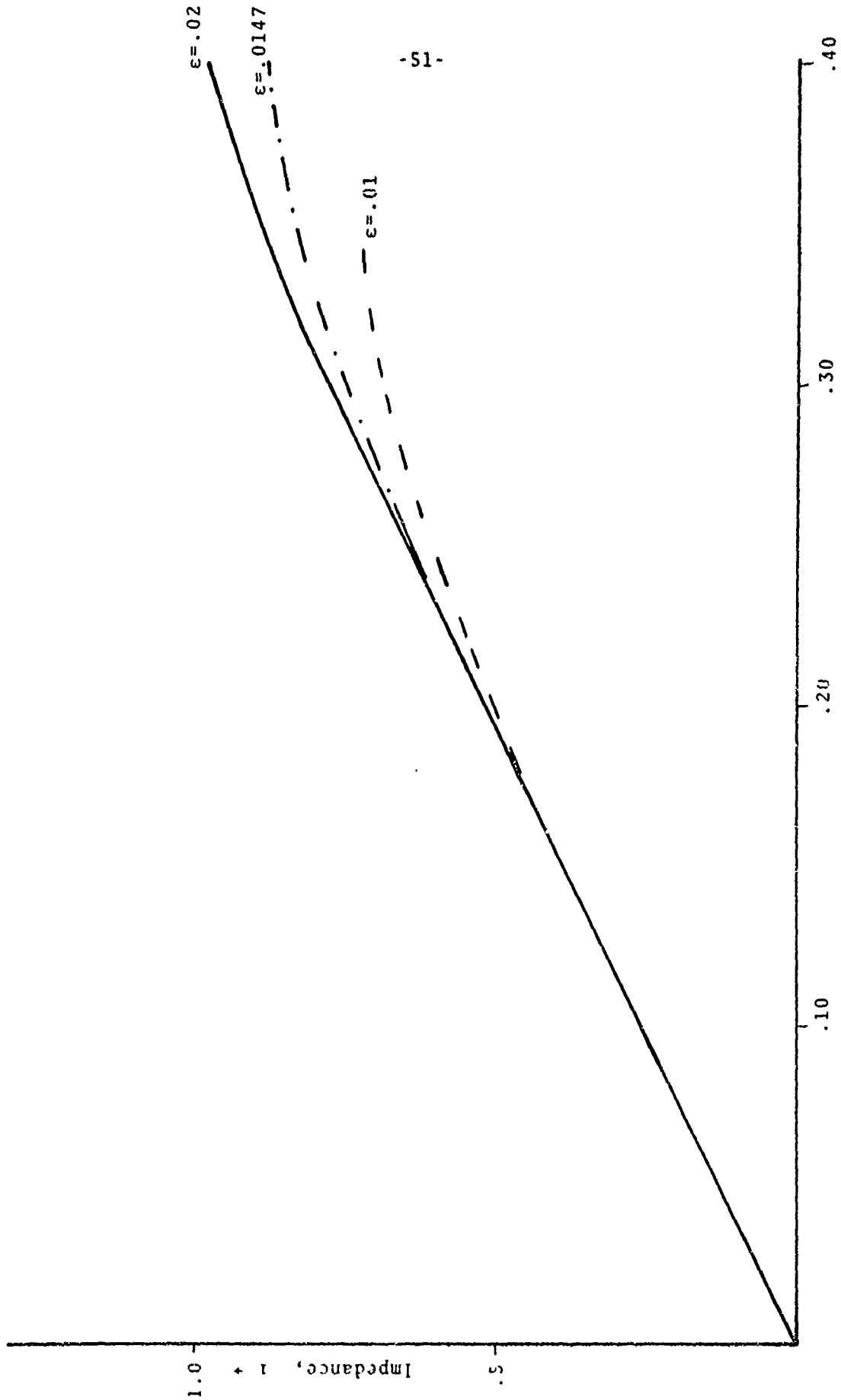Figure 6. Theoretical and experimental response curves: damped case.

Figure 7. Shock amplitude for $i=0$ minus shock amplitude for $i \neq 0$