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Resonant excitation of motion perpendicular to galactic planes

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Summary. Resonant coupling between oscillations perpendicular to the equatorial plan of a galaxy and periodic changes in the force towards the Galactic Centre can cause stars to move far from the plane. The variation of the force to the Galactic Centre may be caused by a non-axisymmetric potential, but instability is possible even in the axisymmetric case. I discuss the circumstances under which these instabilities occur in highly flattened potentials and in nearly spherical systems. This work clarifies the restrictions one may place on the shapes of elliptical galaxies from observations of their nuclear discs. However, it appears that the most important instability is one experienced by stars that orbit at large radii in the disc of a galaxy which has a rotating central bar. Any barred galaxy will have an annulus of such stars. Complete analysis of the importance of this annulus requires that the theory of this paper be extended to include the disc's self-gravity. But the mechanism discussed here may help us understand (i) warps and corrugations in the gaseous discs of galaxies, (ii) the kinematics of stars in the solar neighbourhood, and (iii) the sharp edges possessed by many stellar discs.

1 Introduction

This paper is concerned with circumstances under which a star that initially keeps close to the fundamental plane of its galaxy may be caused to make progressively larger excursions out of the plane. The discussion, which is confined to *collisionless* motion of stars through a given galactic potential, focuses on resonant coupling between oscillations pependicular to the galactic plane and variations in the component of force on the star towards the centre of the system.

One may distinguish three classes of problem in which this type of coupling may be important: (a) The motion of stars on highly eccentric orbits in a strongly flattened potential; the latter may or may not be axisymmetric. (b) The motion of stars along nearly circular orbits in a strongly flattened, slightly non-axisymmetric potential whose figure rotates steadily. (c) The stability of planar orbits around a principal axis of a nearly spherical triaxial potential. In each of these cases one finds that when certain instability conditions are satisfied, stars may develop large oscillations perpendicular to the system's equatorial plane on a time-scale that is typically of the order of 10 rotation periods.

In an earlier paper (Binney 1978; Paper I), I discussed the stability of planar orbits in a nearly spherical potential in connection with galactic warps. This paper extends the discussion of Paper I to encompass highly flattened and rapidly rotating potentials, as well as to include the case of a star that is on a highly eccentric orbit.

In Section 2, I derive the linearized equation of motion of the coordinate of a star perpendicular to the equatorial plane of a given potential as the star orbits in the potential. This equation reduces to Mathieu's equation when either (a) the potential is axisymmetric, or (b) the underlying orbit is closed. Section 3 discusses the stability of highly eccentric orbits in an axisymmetric potential, both from the point of view of the linearized theory, and on the basis of full numerical calculations. Section 4 is concerned with the stability of closed orbits in strongly flattened, non-axisymmetric potentials. Section 5 discusses the stability of orbits in nearly spherical potentials. Section 6 sums up and discusses the significance of these results for both elliptical and disc galaxies.

2 An equation governing z-oscillation

A convenient starting point for a discussion of coupling between z-oscillations and periodic variation of the radial component of force on a star is the equation that is obtained by treating the motion of the star parallel to an equatorial plane in the usual epicyclic approximation. One then linearizes the equation of motion of the star's z-coordinate about this path. In this treatment one assumes that the star's path deviates from a circle by a small but finite amount in the radial direction, and by an infinitessimal amount in the z-direction.

Thus, let the star's motion be referred to an epicentre that moves in a circular path radius R_0 about the centre of a possibly triaxial potential $\Phi(R, \phi, z)$ with constant angular velocity ω. Here (R, ϕ, z) is a cylindrical coordinate system that rotates about the centre of the potential with constant angular velocity ω_p. Define a rectangular coordinate system (x, y, z) centred on the epicentre such that x is the radial component of the vector from the epicentre to the star, and set the speed ω of the revolution of the epicentre equal to Ω, the effective circular frequency at R_0. Ω is defined by

$$\Omega^2 \equiv \frac{1}{R_0} \frac{\partial \Phi}{\partial R}\bigg|_{(R_0, 0, 0)}. \tag{1}$$

Then to first order in x, y and z, the equations of motion of the star become

$$\ddot{x} - 2\Omega \dot{y} = -x\left(\frac{\partial^2 \Phi}{\partial R^2}\bigg|_{R_0} - \Omega^2\right), \tag{2a}$$

$$\ddot{y} + 2\Omega \dot{x} = -\frac{1}{R_0} \frac{\partial \Phi}{\partial \phi}, \tag{2b}$$

$$\ddot{z} = -\frac{\partial \Phi}{\partial z}. \tag{2c}$$

Now expand the ϕ-dependence of Φ as a Fourier series in 2ϕ;

$$\Phi(R, \phi, z) = \Phi_0(R, z) + \Phi_1(R, z) \cos 2\phi + \ldots \tag{3}$$

and set $\phi \approx (\Omega - \omega_p)\, t$, $R \approx R_0$ and $z \approx 0$ in the second and higher terms of this series. Then equations (2a) and (2b) may be combined to a single equation for $x(t)$ whose general solution is

$$x(t) = A \cos(\kappa t + \phi_0) + B \cos[2(\Omega - \omega_p)t], \tag{4}$$

where

$$\kappa = \left[\left.\frac{\partial^2 \Phi_0}{\partial R^2}\right|_{R_0} + 3\Omega^2\right]^{1/2} \tag{5}$$

is the epicyclic frequency, A and ϕ_0 are arbitrary constants and

$$B = \frac{-2\Omega\Phi_1}{R_0(\Omega - \omega_p)[\kappa^2 - 4(\Omega - \omega_p)^2]}. \tag{6}$$

Now consider equation (2c) for the evolution of the z-coordinate of a star whose orbit is characterized by finite, though small eccentricity and an initially infinitesimal deviation z from the equatorial plane. Expanding $\partial\Phi/\partial z$ up to first order in z and deleting all terms involving higher powers than the first in x or y, one has

$$\frac{\partial \Phi}{\partial z} \equiv g = \frac{\partial g}{\partial z} z + \frac{\partial^2 g}{\partial z \partial x} xz + \frac{\partial^2 g}{\partial z \partial y} yz + \dots$$

$$= \left[\frac{\partial^2 \Phi}{\partial z^2} + x \frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial z^2}\right) + y \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial z^2}\right)\right] z. \tag{7}$$

Furthermore, from equation (3) one has

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial^2 \Phi_0}{\partial z^2} + \frac{\partial^2 \Phi_1}{\partial z^2} \cos 2\phi \equiv \nu_0^2 + \nu_1^2 \cos 2\phi, \tag{8a}$$

$$x \frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial z^2}\right) = x \frac{\partial \nu_0^2}{\partial R} + x \frac{\partial \nu_1^2}{\partial R} \cos 2\phi, \tag{8b}$$

$$y \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial z^2}\right) = y \frac{\partial}{\partial y}(\nu_1^2 \cos 2\phi). \tag{8c}$$

In general $|\nu_1| \lesssim |x| \approx |y|$, so the second term on the right-hand side of equation (8b) and the only term on the right of equation (8c) may be neglected when $|x|$ and $|y|$ are small. With this assumption, substitution of equations (7) and (8) into equation (2c) yields

$$\ddot{z} + \{\nu_0^2 + 2q'_A \cos(\kappa t + \phi_0) + 2q'_B \cos[2t(\Omega - \omega_p)]\} = 0, \tag{9}$$

where

$$q'_A = \tfrac{1}{2} A \frac{\partial \nu_0^2}{\partial R} \tag{10a}$$

$$q'_B = \tfrac{1}{2}\left\{\nu_1^2 - \frac{2\Omega\Phi_1\, \partial\nu_0^2/\partial R}{R_0(\Omega - \omega_p)[\kappa^2 - 4(\Omega - \omega_p)^2]}\right\}. \tag{10b}$$

Equation (9) reduces to Mathieu's equation

$$\frac{d^2 z}{d\tau^2} + [a + 2q \cos(2\tau)]z = 0 \tag{11}$$

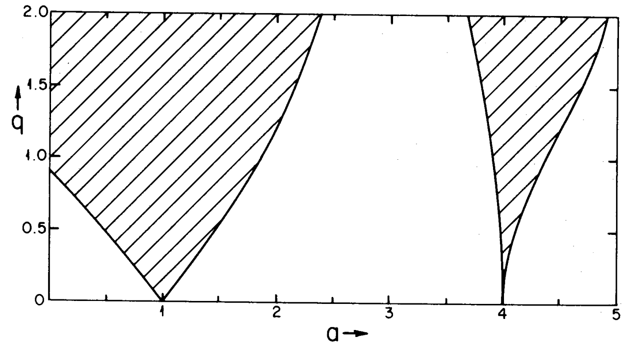

Figure 1. Stability diagram for Mathieu's equation. When the point (a, q) lies in one of the shaded regions, equation (11) possesses an exponentially growing solution.

in two cases: (a) if the potential is axisymmetric, with the result that $q'_B = 0$; or (b) if the underlying orbit in the equatorial plane is closed, that is when A and therefore q'_A are zero. In the following I shall discuss these two special cases in some detail.

Equation (11) has periodic solutions only when the point (a, q) lies on one of the full lines of Fig. 1. These periodic solutions divide regions of the (a, q) plane in which all solutions are stable from those in which exponentially growing solutions exist. The unstable solutions occur when the natural frequency $\sqrt{a}$ lies close to a multiple of half the driving frequency. The larger the range of frequencies $[(a+2q)^{1/2} - (a-2q)^{1/2}]$ through which the instantaneous frequency sweeps, the greater the likelihood that an unstable solution exists. The first two instability strips are defined by the conditions (Abramowitz & Stegun 1965)

$$-q - \tfrac{1}{8} q^2 < a - 1 < q - \tfrac{1}{8} q^2, \tag{12a}$$

$$-\tfrac{1}{12} q^2 < a - 4 < \tfrac{5}{12} q^2, \tag{12b}$$

where terms of order q^3 and smaller have been neglected.

3 Eccentric orbits in an axisymmetric potential

In the absence of a non-axisymmetric component of the potential, equation (9) becomes Mathieu's equation (11) where

$$a = 4\nu_0^2/\kappa^2 \tag{13a}$$

$$q = \frac{2A}{\kappa^2}\frac{\partial \nu_0^2}{\partial R} \tag{13b}$$

$$\tau = \tfrac{1}{2}\kappa t \tag{13c}$$

From equation (4) it follows that A is half the star's amplitude of oscillation in the radial direction. One expects $\nu_0 \gtrsim \kappa$ for a flattened galaxy, so that resonant coupling between the z-oscillations and the epicyclic motion will occur only when $a = 4$, 9 etc. Consider the case $a = 4$. For a galaxy that has a flat rotation curve, all frequencies will be roughly proportional to R^{-1}, so that $|q| \approx 4A/R_0(\nu_0/\kappa)^2 \approx 4A/R_0$ near the resonance. Thus equation (12b) suggests that z-motions may develop if

$$-\tfrac{1}{3}\left(\frac{A}{R_0}\right)^2 < 2\left(\frac{\nu_0}{\kappa} - 1\right) < \tfrac{5}{3}\left(\frac{A}{R_0}\right)^2. \tag{14}$$

When $\nu_0 = \kappa$ the unstable solution to equation (9) grows by a factor $(1+\delta)$ during each complete epicycle, where (Ambromowitz & Stegun 1965)

$$\delta \approx 0.12q + 0.083 q^2 + \ldots \approx 0.5\,(A/R_0). \tag{15}$$

Thus it would appear that this might be an interesting process. A word of caution is in order, however, because the condition (14) for the existence of an unstable solution depends on $(A/R_0)^2$, while equation (9) is valid only through order A/R_0. In particular, though there can be no secular growth in z for a star moving in a Keplerian force field, one has in this case that $\nu_0 = \kappa$ and therefore that the resonant condition $a = 4$ is exactly satisfied. Evidently for a Keplerian force field, coupling of the x- and z-modes is quenched by the deviation of the x-oscillations from simple harmonicity. I have investigated the possibility that such quenching occurs also for a non-Keplerian force field by numerically following the motion of particles in the potential

$$\Phi = \ln\,(R^2 + z^2/u^2), \tag{16}$$

with $u = 0.71$. Each orbit is characterized by the radius $R_0 = L/\sqrt{2}$ of the circular orbit with the same angular momentum L, and by the difference x_{max} between R_0 and the maximum radius reached by the particle. Fig. 2 shows the evolution of the z-energies $E_z \equiv \frac{1}{2}\dot{z}^2 + z^2/(uR)^2$ during 40 epicyclic periods for the cases $x_{max}/R_0 = 0.25$ and $x_{max}/R_0 = 0.75$. The range of values of u within which appreciable growth in E_z is detectable in these numerical experiments is rather narrower than Mathieu's equation would predict; when $x_{max}/R_0 = 0.5$, and therefore $q = 2$, convincing growth in E_z is found only for $u \in (0.7\text{–}0.725)$, compared with the range $u \in (0.65\text{–}0.73)$ that one derives from Mathieu's equation. The peak growth rate observed is also rather smaller than that predicted by Mathieu's equation. Evidently deviation of the x-motion from simple harmonicity does tend to dampen the $a = 4$ instability, but for a flat or outward-increasing rotation curve the x-motion is sufficiently near simple harmonicity to allow the instability to proceed.

After about 100 epicyclic periods (70 galactic years) the z-velocity dispersion σ_z along the more violently unstable of the orbits plotted in Fig. 2 becomes comparable with the radial dispersion σ_R ($\sigma_z \approx 0.7\,\sigma_R$). σ_z then begins to decline, returning to of order twice its original small value ($\sigma_{z0} \approx 10^{-3}\,\sigma_{R0}$) after a further 100 epicyclic periods. When one averages σ_z and σ_R over this long cycle, one obtains $\bar{\sigma}_z/\bar{\sigma}_R = 0.15$. The slower initial rate

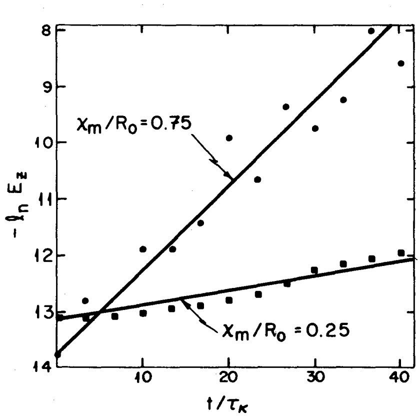

Figure 2. The evolution of $E_z = \frac{1}{2}\dot{z}^2 + z^2/(uR)^2$ along two orbits in the axisymmetric potential of equation (16) with $u = 0.71$. The straight lines correspond to $\delta = 0.0125$ [$(x_{max}/R_0) = 0.25$] and $\delta = 0.075$ [$(x_{max}/R_0) = 0.75$].

of growth of σ_z along the more stable of the orbits of Fig. 2 is maintained through over 200 epicyclic periods; after 200 cycles $\sigma_z/\sigma_R \approx 0.3$.

4 Closed orbits in rotating triaxial potentials

4.1 ANALYTICAL CONSIDERATIONS

The second case in which equation (9) reduces to Mathieu's equation is when the free–epicyclic amplitude A vanishes and the orbit becomes closed in the frame that rotates with the potential. Thus equation (9) then reduces to equation (11), where

$$a = \nu_0^2/(\Omega - \omega_p)^2, \tag{17a}$$

$$q = \tfrac{1}{2}\left[\nu_1^2 - \frac{2\Omega\Phi_1\,\partial\nu_0^2/\partial R}{R_0(\Omega-\omega_p)[\kappa^2 - 4(\Omega-\omega_p)^2]}\right] \bigg/ (\Omega-\omega_p)^2, \tag{17b}$$

$$\tau = (\Omega - \omega_p)t. \tag{17c}$$

Consider the import of equation (17a) for stars on prograde orbits. Between $R = 0$ and corotation a rises from $a = (\nu_0/\Omega)^2$ without limit. Outside corotation a falls steadily such that for a flat circular velocity curve $a(R) \sim R^{-2}$ at large R. Thus one has in principle to investigate all resonant conditions $a = n^2$. However, I shall concentrate on the two lowest and strongest resonances $n = 1$ and $n = 2$.

The situation as regards the $n = 2$ resonance is very similar to that of the free epicyclic motion discussed above. Indeed, equation (17b) indicates that unless the potential is very strongly bar-like, so that ν_1/Ω is appreciable, q will be large only when $\kappa \approx 2|\Omega - \omega_p|$, or $\Omega \approx \omega_p$, i.e. when the star is near one of the classical resonances. But as $\Omega \approx \omega_p \Rightarrow a \gg 4$, one has that resonant coupling will occur for $n = 2$ only when $\kappa \approx 2(\Omega - \omega_p) \approx \nu_0$. That is, the resonance that occurs at $a = 4$ for a star that moves in a slightly bar-like potential, differs from the resonance experienced by a star on an eccentric orbit in an axisymmetric potential only in that in the former case the amplitude of the star's epicyclic motion is determined by the strength of the bar and by the star's nearness to a Lindblad resonance. However, unless ν_0 is very different from κ, one expects stars on closed orbits that are sufficiently close to a Lindblad resonance to develop large z-motions.

Beyond the outer Lindblad resonance a passes through the value $a = 1$ associated with the powerful fundamental resonance of Mathieu's equation. To lowest order in q, condition (12a) for the development of large z-motions becomes when $(\omega_p - \Omega) \approx \nu_0 \gg \kappa$

$$-q < \frac{\nu_0^2}{(\omega_p - \Omega)^2} - 1 < q, \tag{18a}$$

where

$$q \approx \tfrac{1}{2}\left[\left(\frac{\nu_1}{\nu_0}\right)^2 + 4\frac{\Phi_1}{R_0^2}\,\frac{\Omega}{\nu_0(4\nu_0^2 - \kappa^2)}\right]. \tag{18b}$$

Here I have assumed that $\nu_0 \sim R_0^{-1}$ at large radii. The second term in square brackets in equation (18b) will always be smaller than the first, since $4\nu_0^2 \gg \kappa^2$, $|\Phi_1/R_0^2| \lesssim \nu_1^2$ and $\Omega < \nu_0$. Thus the range of values of Ω, and therefore of R_0, within which the excitation of z-motion should occur, is proportional to $(\nu_1/\nu_0)^2 \approx \Phi_1/\Phi_0$.

For a star on a retrograde orbit $(\omega_p < 0)$ a falls from $a \approx (\nu_0/\Omega)^2$ at $R_0 \approx 0$, to zero at large radii. The $n = 1$ resonance is therefore the one of greatest interest, although in a very highly flattened potential resonance might occur when $n=2$ very near the centre. A similar line of argument to that followed above in connection with equation (18b) shows that the second term on the right-hand side of equation (17b) is negligible compared to the first. This resonance of retrograde stars was discussed in Paper I in connection with galactic warps.

4.2 ORBITS IN A MODEL POTENTIAL

It is interesting to verify these analytical results by numerically following orbits in a suitable force-field. Consider closed orbits in the potential

$$\Phi = \ln(X^2 + Y^2/u_1^2 + Z^2/u_2^2 + C^2). \tag{19}$$

Here (X, Y, Z) are a set of Cartesian coordinates that rotate with constant angular velocity ω_p. The circular-velocity curve rises linearly for $R \equiv (X^2 + Y^2/u_1^2)^{1/2} \ll C$ to a constant value $\sqrt{2}$ for $R \gg C$. The only scale radius other than C associated with the potential may be taken to be the corotation radius $R_c = \sqrt{2}/\omega_p$. When $R_c \gg C$ one may distinguish five main sequencies of simple closed orbits: (1) The retrograde orbits. (2) The prograde orbits at large radius. These become very non-circular and finally die out near the radius $R_{OL} = (1+1/\sqrt{2})R_c$ of the outer Lindblad resonance. (3) The prograde orbits of small radius. These die out at the radius $R_{IL} = (1-1/\sqrt{2})R_c$ of the inner Lindblad radius. (4) The X-axial orbits. These extend from small radii, where they are nearly linear, through oval-shaped figures in the region R_{IL}, to complex shapes just interior to R_c. (5) The Y-axial orbits. These exist only interior to R_{IL}.

The mass distribution associated with the potential of equation (19) is positive definite provided the axis ratios of the mass distribution

$$\frac{Y_0}{X_0} = u_1^2 \left[\frac{1+1/u_2^2 - 1/u_1^2}{1/u_1^2 + 1/u_2^2 - 1}\right]^{1/2} \tag{20a}$$

$$\frac{Z_0}{X_0} = u_2^2 \left[\frac{1+1/u_1^2 - 1/u_2^2}{1/u_1^2 + 1/u_2^2 - 1}\right]^{1/2}, \tag{20b}$$

are themselves positive. For example, if $u_1 = 0.95$ and $u_2 = 0.75$, the axial ratios of the associated matter distribution are $Y_0/X_0 = 0.89$ and $Z_0/X_0 = 0.24$; the ellipticities of the equipotential surfaces in the (X, Z) and (X, Y) planes are roughly 1/3 and 1/2 of the corresponding ellipticities of the isodensity surfaces.

Fig. 3 indicates the approximate range of axial ratios u_2 for which closed orbits of the last four families listed above develop large z-motions through the resonance associated with the condition $a \approx 4$. In all these calculations $u_1 = 0.95$, which corresponds to a very mildly triaxial matter distribution. With this value of u_1 the retrograde orbits are always so nearly circular that they are scarcely affected by the $a=4$ resonance. However, when u_2 is such that $\nu_0 \approx \kappa$, all closed orbits near either the ILR or the OLR are unstable to the development of z-motions for essentially the same reason that eccentric orbits in an axisymmetric potential are unstable; the only function of the non-axisymmetric component of the potential is to drive epicyclic motion at a frequency that is close to the natural epicyclic frequency.

Both the x- and y-axial orbits are unstable to the development of z-motions over a wide range of values of u_2.

The resonances associated with the condition $a=1$ are of a different nature and have no analogue in an axisymmetric potential. The orbits on which this resonance occurs are all

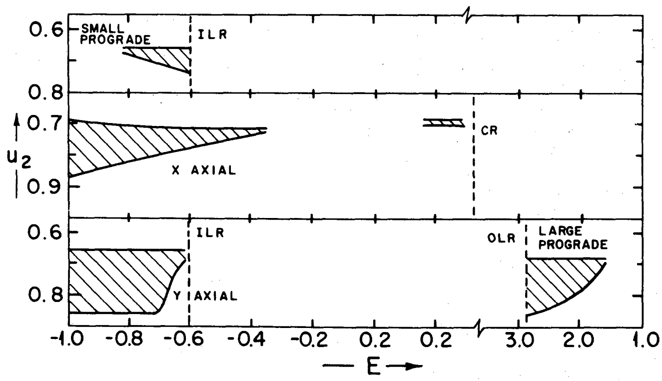

Figure 3. Susceptibility of closed orbits in the potential of equation (19) to the $a = 4$ resonance. When the energy of the orbit E and the axial ratio u_2 of the potential are such that (E, u_2) lies in one of the shaded regions, the orbit is unstable to the development of z-motion. The vertical dashed lines indicate the approximate energies at which resonances cause sequences of orbits to terminate. $u_1 = 0.95$ for all orbits.

rather nearly round, because stars on these orbits perceive the non-axisymmetric component of the potential at a frequency $2(\omega_p - \Omega) \approx 2\nu \gg \kappa$ that is much larger than their natural epicyclic frequency. In the potential of equation (19) this resonance occurs for prograde orbits that have radii near

$$R_r = \frac{\sqrt{2}}{\omega_p}(1 + 1/u_2). \tag{21}$$

The width of the resonant region is determined by u_1 through the inequality (18a). Comparing equations (8a) and (19) shows that the latter potential has

$$\nu_1^2 \approx \tfrac{1}{2}(1-u_1^2)\nu_0^2 \approx (1-u_1)\nu_0^2, \tag{22}$$

and hence

$$q \approx \tfrac{1}{2}(1-u_1). \tag{23}$$

The instability should occur when $|\Delta| \lesssim \tfrac{1}{2}(1-u_1)$, where

$$\Delta \equiv \left(\frac{\nu_0}{\omega_p - \Omega}\right)^2 - 1 \tag{24}$$

quantifies the fractional deviation of ν_0 from its resonant value. Approximating Δ by a linear function of radius around $R = R_r$ (where $\Delta = 0$) one finds that the width δR of the annulus within which the prograde closed orbits should be unstable to the development of z-motion,

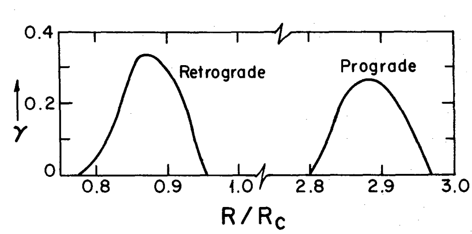

Figure 4. Susceptibility of closed orbits in the potential of equation (19) to the $a = 1$ resonance. The ration R/R_c of the X-axis intercept of closed orbits to the corotation radius, is plotted against $\gamma = d \ln E_z/d\tau$. These orbits are calculated with $u_1 = 0.9$ and $u_2 = 0.5$.

satisfies

$$\frac{\delta R}{R_r} \approx \tfrac{1}{2}\left(\frac{1-u_1}{1+u_2}\right). \tag{25}$$

Thus the annulus of unstable orbits is narrow unless the potential is strongly bar-like.

Fig. 4 shows the growth rate $\gamma \equiv d \ln E_z/d\tau$ of the z-energies of stars whose orbits are in the unstable annuli associated with prograde and regrograde orbits in the potential of equation (19) with $u_1 = 0.9$ and $u_2 = 0.5$. Near the centres of the annuli, the growth rate can be quite large.

5 Application of equation (9) to nearly spherical systems

The foregoing sections have been concerned with strongly flattened potentials such as those of disc galaxies. Here I want to use equation (9) to prove that there are no stable loop orbits about the middle axis of a typical slowly rotating triaxial galaxy. This result has already been demonstrated numerically by Heiligman & Schwarzschild (1979), but it is interesting to see how directly it follows from equation (9).

It is convenient now to expand the potential in spherical polar coordinates as

$$\Phi(r,\theta,\phi) = \Phi_{00}(r) + \Phi_{20}(r)P_2(\cos\theta) + \Phi_{22}(r)P_{22}(\cos 2\theta)\cos 2\phi, \tag{26}$$

where the directions $\theta = 0$ and ($\theta = \pi/2$, $\phi = 0$) are aligned with some two principal axes of the galaxy. The quantities in this expansion are related to those of the expansion (3) in terms of cylindrical coordinates by

$$\begin{aligned} \Phi_0(R,z) &= \Phi_{00}(r) + \Phi_{20}(r)P_2(\cos 2\theta), \\ \Phi_1(R,z) &= \Phi_{22}(r)P_{22}(\cos\theta). \end{aligned} \tag{27}$$

Differentiating these relations with respect to z and setting $\cos\theta = 0$ one obtains

$$\nu_0^2 = \frac{1}{r}\left(\frac{\partial\Phi_{00}}{\partial r} - \tfrac{1}{2}\frac{\Phi_{20}}{\partial r}\right) + \frac{3\Phi_{20}}{r^2}, \tag{28a}$$

$$\nu_1^2 = \frac{3}{r}\frac{\partial\Phi_{22}}{\partial r} - 6\frac{\Phi_{22}}{r^2}. \tag{28b}$$

Thus if $\omega_p = 0$ the quantities a and q defined by equations (17) are in this case given by

$$\Omega^2 a = \Omega^2 + 3\Phi_{20}/r^2, \tag{29a}$$

$$\Omega^2 q = -\frac{3}{r^2}\Phi_{22} + \frac{3}{2r}\frac{\partial\Phi_{22}}{\partial r} - \frac{\Omega(3\Phi_{22}/r^2)r(\partial\nu_0^2/\partial r)}{\Omega[\kappa^2 - 4\Omega^2]}. \tag{29b}$$

In the special case of a galaxy whose isodensity surfaces are similar ellipses and are such that the mass density $\rho(r) \sim r^{-2}$, one has both that $\kappa^2 = 2\Omega^2$ and that Φ_{20} and Φ_{22} are independent of r. Also $\Phi_{00} \approx \ln r$ in this case. Therefore in this simple but rather realistic case, condition (12a) for the occurrence of the $n = 1$ instability of Mathieu's equation becomes to lowest order in q

$$\Phi_{22}\left(1 + \frac{\nu_0^2}{\Omega^2}\right) < \Phi_{20} < -\Phi_{22}\left(1 + \frac{\nu_0^2}{\Omega^2}\right). \tag{30}$$

One may now ask after the implication of condition (30) for the orientation of the galaxy's principal axis with respect to the assumed rotation axis of the star in question. Evaluating the potential of equations (26) distance r from the centre along the three principal axes, one obtains the values

$$\Phi_x = \Phi_{00}(r) - \tfrac{1}{2}\Phi_{20}(r) + 3\Phi_{22}(r),$$
$$\Phi_y = \Phi_{00} - \tfrac{1}{2}\Phi_{20} - 3\Phi_{22}, \quad (31)$$
$$\Phi_z = \Phi_{00} + \Phi_{20}.$$

Thus the z-axis, that is the axis about which the star is presumed to be rotating, is the middle axis of the potential if

$$|\Phi_{20}| < |2\Phi_{22}| \quad (32)$$

But this condition is to lowest order in Φ_{20}/Φ_{00} the same as the instability condition (30). Therefore stars that initially move nearly in the fundamental plane that is perpendicular to the middle axis of a non-rotating galactic potential, will not remain near that plane.

When the potential rotates slowly, the instability condition (30) becomes to first order in (ω_p/Ω)

$$S < \Phi_{20} + \tfrac{2}{3} v_c^2 \frac{\omega_p}{\Omega} < -S;$$
$$S \equiv \Phi_{22}\left[1 + \frac{\nu_0^2}{\Omega^2}\left(1 + 5\frac{\omega_p}{\Omega}\right)\right]. \quad (33)$$

Here $v_c \equiv r\Omega$ is the local circular velocity. Much the most important change introduced by a finite rotation rate between conditions (30) and (33) is the presence of the term $\tfrac{2}{3} v_c^2(\omega_p/\Omega)$ in (33). If the system's rotation axis is shorter than the average of the two axes in the equatorial plane, one has $\Phi_{20} > 0$, and thus that a small pattern speed ω_p tends to destabilize retrograde and stabilize prograde orbits, and conversely in the case that the rotation axis is longer than the average of the two axes in the equatorial plane. Note, however, that a sufficiently large pattern speed in either sense will stabilize orbits about the middle axis.

6 Significance for real galaxies

I have shown that certain orbits in principal planes of a range of galactic potentials are unstable to the development of large oscillations in the direction perpendicular to those planes (the z-direction). The mechanism involved is the resonant coupling of oscillations in the z-direction with either epicyclic motion or with orbital motion with respect to a non-axisymmetric potential.

A simple analytical treatment suffices to show that stable planar orbits do not exist about the middle axis of a mildly triaxial non-rotating potential. The obvious application of this result is to the dynamics of the disc of dark material sometimes seen near the centres of radio-active elliptical galaxies (Kotanyi & Ekers 1978). Indeed, sufficienctly near the centres of these galaxies the angular velocity of the figure of the potential should be negligible in comparison with the angular velocity of orbiting particles. Therefore one might infer that the observed discs are orbiting either the longest, or the shortest of the principal axes of the system. In favourable cases this might lead to a unique interpretation of the photometry of the galaxy's central brightness profile in terms of the three-dimensional shape of the system. Furthermore, an elliptical galaxy may sometimes capture gas that initially settles on to nearly closed retrograde orbits about the galaxy's shortest axis. The gas then gradually spirals inwards until the associated closed orbits become unstable. According to equation (17) this occurs in the potential of equation (19) at the radius R_r that is a fraction $(1/u_2 - 1)$ of the corotation radius R_c. It seems probable that at this radius the gas starts to circulate about the longest axis of the galaxy on the family of anomalous orbits discussed by van Albada, Kotanyi & Schwarzschild (1981).

I have discussed two classes of resonance that may occur on orbits in strongly flattened potentials. The first class of resonance, the $a = 4$ resonance, is of importance only for stars on highly eccentric orbits. The eccentricity of these orbits may represent free radial oscillation in either an axisymmetric or a triaxial potential, or it may represent radial oscillation that is forced by a triaxial potential. This class of resonance occurs when the epicyclic frequency κ differs from the frequency ν_0 of oscillation in the z-direction by an amount that becomes large only for highly eccentric orbits.

The second class of resonance, the $a = 1$ resonance, occurs for stars whose nearly circular orbits lie in one of two annuli; the inner annulus is associated with stars on retrograde orbits, and the outer annulus with prograde stars. In a strongly flattened potential these annuli are narrow, but the rate at which stars lying within them acquire z-motion can be large.

Two considerations indicate that one should hesitate to apply the simplistic treatment of this paper to real galaxies: (a) The potential, which I have assumed to be given, is in a real galaxy partly determined by orbiting disc stars, and therefore will change if appreciable numbers of disc stars develop large z-oscillations. (b) The potentials I have investigated have z-frequencies ν_0 which vary slowly, if at all, with amplitude z_m of z-oscillation, whereas most stars in the disc of a galaxy like our own have z-frequencies that depend strongly on z_m.

As an illustration of the likely importance of this last point consider the situation in the solar neighbourhood. Stars that make very small amplitude z-oscillations are characterized by z-frequency $\nu_0(0) \approx 3\Omega$ that is considerably larger than the local circular frequency Ω. On the other hand stars whose z-amplitudes z_m are greater than the scale height of h of the mass-bearing population of the disc have lower z-frequencies. Fig. 5 shows that in the solar neighbourhood $\nu_0(z_m) \approx \kappa$ for $z_m \approx 4h$. Thus at any radius within the galactic disc there are stars having a wide range of z-frequencies. However, resonant coupling to a star's z-oscillations is likely to be of lasting importance only if ν_0 does not move rapidly out of the resonant region as soon as little energy is pumped into the star's z-motion. This requires that $d\nu_0/dz_m \approx 0$, which in turn requires either that $z_m \ll h$, or that z_m is so much greater than h that the local disc material makes a negligible contribution to the star's z-frequency.

It is clear that in a galaxy like our own that has a thin, self-gravitating disc, $\nu_0(0)$ will be too high to couple to κ through the $a = 4$ resonance. The only circumstance in which this resonance is likely to be of any importance is in the central region of an SB0 galaxy. Observations by Kormendy (1980) suggest that most of the stars that make up the lens components of SB0 galaxies are on highly eccentric orbits that will be susceptible to the

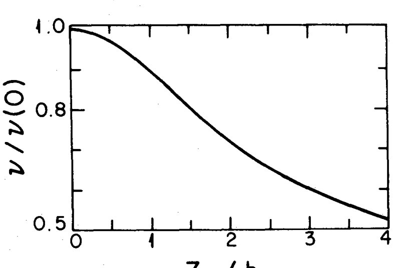

Figure 5. Frequency of z-oscillation as a function of maximum distance z_m from the plane. The (infinite) plane is assumed to have volume density $\rho(z) = \rho_0 \exp(-z^2/2h^2)$.

$a = 4$ instability if the potentials of these systems are sufficiently bulge-dominated. Furthermore, Burstein's (1979) work suggest that it is just possible that this last condition might be fulfilled in some SB0 galaxies.

However, it is likely that the $a = 1$ resonance that is experienced by stars on large prograde orbits is much more significant for galactic dynamics than any $a = 4$ resonance. Indeed a resonance of the $a = 1$ type will occur in every disc galaxy that contains a rotating central bar. If the pattern speed of the bar is such that the bar ends near corotation (e.g. Sellwood 1980), the resonant annulus should occur at a radius equal to two or three times the half-length of the bar (*cf.* Fig. 4). This typically places the resonant annulus where large warps develop in the gaseous discs of even isolated disc galaxies (Sancisi 1976), and possibly places the resonance at the edge of the stellar disc; van de Kruit & Searle (1980) have recently emphasized that stellar discs often terminate rather abruptly at what must be a characteristic radius for the system.

Unfortunately an adequate discussion of the excitation of the $a = 1$ resonance by a central bar requires that one take into account cooperative effects. In a spiral galaxy $\nu_0(0)$ is determined almost entirely by the self-gravity of the stellar disc; that is $\nu_0(0)$ is the frequency at which a single star will execute oscillations through the equatorial plane of the system when all the other stars of the disc stay put. If the other stars in some annulus of width δR are caused to move in sympathy with the first star, the frequency at which this collection of stars will oscillate will be lower than by an amount that depends on δR; for δR greater than the thickness of the disc $\nu_0 \sim (\delta R)^{-1/2}$. Furthermore, the concerted movement of an entire annulus of stars will result in a corrugation wave propagating away from that annulus. This wave may cause stars at other radii, which are not in resonance with the bar, to acquire additional energy of z-motion. Further investigation of these processes requires modification of the analysis of self-consistent corrugation waves of Hunter & Toomre (1969) and will not be attempted here. But it is worth remarking that resonant excitation of the type discussed in this paper may overcome the difficulty encountered by Hunter & Toomre in their discussion of the longevity of tidally induced warps: Hunter & Toomre concluded that the eigenfrequency spectrum of realistic stellar disc is sufficiently dense that many different modes will be excited by a tidal impulse. These modes will soon get out of phase with one another, thus eliminating all large-scale structure. Could the very narrow resonant annulus of the inactive disc of this paper translate to a narrow band of resonantly excited modes when one goes over to a self-gravitating disc? If this hypothesis were correct, one would have a natural explanation of long-lived warps and corrugations (Quiroga 1974; Solomon & Sanders 1979) even in isolated galaxies.

However, even if the band of modes that are resonantly excited by a central bar proves not to be narrow, resonantly excited corrugation waves may play a role in the acceleration of disc stars. It is widely believed (*cf.* Wielen 1977) that the correlations observed between the kinematic and spectral properties of stars in the solar neighbourhood should be understood in terms of the progressive stochastic acceleration of disc stars. It is less clear by what means the stars are actually accelerated, although scattering off clouds of interstellar matter (Spitzer & Schwarzschild 1953) or off transient spiral features (Barbanis & Woltjer 1967) are possibilities. However, it is unlikely that these mechanisms are able to accelerate disc stars to velocities $v > 30\,\mathrm{km\,s^{-1}}$ in the lifetime of the Galaxy as is required by the observations. Resonant coupling of the oscillations of stars about closed orbits to a large-scale, periodic component of the potential might offer a way out of this difficulty. In this picture resonance involving motion perpendicular to the galactic plane would be as important for the interpretation of the observations as the classical Lindblad resonances by which a non-axisymmetric potential couples to motion within the galaxian plane.

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