Resonant Hypergeometric Systems and Mirror Symmetry

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Abstract

In Part I the Γ-series of [11] are adapted so that they give solutions for certain resonant systems of Gel'fand-Kapranov-Zelevinsky hypergeometric differential equations. For this some complex parameters in the Γ-series are replaced by nilpotent elements from a ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}$. The adapted Γ-series is a function $\Psi_{\mathcal{T},\beta}$ with values in the finite dimensional vector space $\mathcal{R}_{\mathsf{A},\mathcal{T}}\otimes_{\mathbb{Z}}\mathbb{C}$. Part II describes applications of these results in the context of toric Mirror Symmetry. Building on Batyrev's work [2] we show that a certain relative cohomology module $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ is a GKZ hypergeometric $\mathcal{D}$-module which over an appropriate domain is isomorphic to the trivial $\mathcal{D}$-module $\mathcal{R}_{\mathsf{A},\mathcal{T}}\otimes\mathcal{O}_{\mathcal{T}}$, where $\mathcal{O}_{\mathcal{T}}$ is the sheaf of holomorphic functions on this domain. The isomorphism is explicitly given by adapted Γ-series. As a result one finds the periods of a holomorphic differential form of degree d on a d-dimensional Calabi-Yau manifold, which are needed for the B-model side input to Mirror Symmetry. Relating our work with that of Batyrev and Borisov [3] we interpret the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ as the cohomology ring of a toric variety and a certain principal ideal in it as a subring of the Chow ring of a Calabi-Yau complete intersection. This interpretation takes place on the A-model side of Mirror Symmetry.

Contents

PART I

Introduction I

A GKZ hypergeometric system [11] depends on four parameters: two positive integers N and n, a set $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ of vectors in $\mathbb{Z}^n$ and a vector β in $\mathbb{C}^n$. The standard assumptions [11] are

Condition 1 [1] *$\mathfrak{a}_1, \ldots, \mathfrak{a}_N$ generate a rank n sub-lattice $\mathbb{M}$ in $\mathbb{Z}^n$* (1)

$\exists\, \mathfrak{a}_0^\vee \in \mathbb{Z}^{n\vee}$ such that $\mathfrak{a}_0^\vee \cdot \mathfrak{a}_i = 1 \;\; (i = 1, \ldots, N)$ (2)

The GKZ system with these parameters is the following system of partial differential equations for functions Φ on a torus with coordinates $v_1, \ldots, v_N$:

$$\left(-\beta + \sum_{j=1}^{N} \mathfrak{a}_j \, v_j \frac{\partial}{\partial v_j} \right) \Phi \;=\; 0 \tag{3}$$

$$\left(\prod_{\ell_j > 0} \left[\frac{\partial}{\partial v_j} \right]^{\ell_j} - \prod_{\ell_j < 0} \left[\frac{\partial}{\partial v_j} \right]^{-\ell_j} \right) \Phi \;=\; 0 \quad \text{for } \ell \in \mathbb{L} \tag{4}$$

where (3) is in fact a system of n equations and

$$\mathbb{L} \;:=\; \{\, \ell = (\ell_1, \ldots, \ell_N)^t \in \mathbb{Z}^N \mid \ell_1 \mathfrak{a}_1 + \ldots + \ell_N \mathfrak{a}_N \;=\; 0 \,\}\,. \tag{5}$$

Some of the above data are displayed in the following short exact sequence in which $\mathcal{A}$ denotes the linear map $\mathcal{A} : \mathbb{Z}^N \to \mathbb{Z}^n$, $\mathcal{A}(\lambda) \;=\; \lambda_1 \mathfrak{a}_1 + \ldots + \lambda_N \mathfrak{a}_N$.

$$0 \to \mathbb{L} \longrightarrow \mathbb{Z}^N \xrightarrow{\mathcal{A}} \mathbb{M} \to 0 \tag{6}$$

We are going to construct solutions for GKZ systems with $\beta \in \mathbb{M}$. Of special interest for applications to mirror symmetry are the cases $\beta = 0$ and $\beta = -\mathfrak{a}_0$ with $\mathfrak{a}_0$ as in the definition of reflexive Gorenstein cone (definition 5).

The idea is as follows. Gel'fand-Kapranov-Zelevinskii [11] give solutions for (3)-(4) in the form of so-called Γ-series

$$\sum_{\ell \in \mathbb{L}} \prod_{j=1}^{N} \frac{v_j^{\gamma_j + \ell_j}}{\Gamma(\gamma_j + \ell_j + 1)} \tag{7}$$

[1] $\mathbb{Z}^n, \mathbb{R}^n, \mathbb{C}^n$ resp. $\mathbb{Z}^{n\vee}, \mathbb{R}^{n\vee}, \mathbb{C}^{n\vee}$ denote spaces of column vectors resp. row vectors.

Γ is the usual Γ-function, $\ell = (\ell_1, \ldots, \ell_N)^t \in \mathbb{L} \subset \mathbb{Z}^N$. The series depends on an additional parameter $\gamma = (\gamma_1, \ldots, \gamma_N)^t \in \mathbb{C}^N$ which must satisfy

$$\gamma_1 \mathfrak{a}_1 + \ldots + \gamma_N \mathfrak{a}_N \; = \; \beta \tag{8}$$

Allowing the obvious formal rules for differentiating such Γ-series one sees that the functional equations of the Γ-function guarantee that (7) satisfies the differential equations (4) and that condition (8) on γ takes care of (3). *The issue is to interpret the* Γ*-series (7) as a function on some domain.* In order that (7) can be realized as a function γ must satisfy more conditions. Gel'fand-Kapranov-Zelevinskii obtain convenient conditions from a triangulation $\mathcal{T}$ of the convex hull of $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$. However, if β is in $\mathbb{M}$ and the triangulation has more than one maximal simplex, the vectors γ which satisfy these extra conditions do not provide enough Γ-series solutions for the GKZ system. This phenomenon is called *resonance* [11]. An extreme case of resonance, in which all Γ-series coincide, occurs when β is in $\mathbb{M}$ and $\mathcal{T}$ is unimodular.

Definition 1 *(cf. [24]) A triangulation is called* unimodular *if all its maximal simplices have volume* 1; *the volume of a maximal simplex* $\mathrm{conv}\,\{\mathfrak{a}_{i_1}, \ldots, \mathfrak{a}_{i_n}\}$ *is defined as* $|\det(\mathfrak{a}_{i_1}, \ldots, \mathfrak{a}_{i_n})|$.

To get around the resonance problem for $\beta \in \mathbb{M}$ we proceed as follows. Fixing a solution $\gamma^\circ \in \mathbb{Z}^N$ for equation (8) we write the general solution of (8) as $\gamma = \gamma^\circ + \mathsf{c}$ with $\mathsf{c} = (c_1, \ldots, c_N)^t$ such that

$$c_1 \mathfrak{a}_1 + \ldots + c_N \mathfrak{a}_N = 0 \tag{9}$$

and note $\gamma + \mathbb{L} = \mathsf{c} + \mathcal{A}^{-1}(\beta)$. Thus (7) becomes $\sum_{\lambda \in \mathcal{A}^{-1}(\beta)} \; \prod_{j=1}^N \frac{v_j^{c_j+\lambda_j}}{\Gamma(c_j+\lambda_j+1)}$. Multiplying this by $\prod_{j=1}^N \Gamma(c_j+1)$ we obtain

$$\Phi_{\mathcal{T},\beta}(\mathsf{v}) \; := \sum_{\lambda \in \mathcal{A}^{-1}(\beta)} \; Q_\lambda(\mathsf{c}) \; \cdot \; \prod_{j=1}^N v_j^{\lambda_j} \; \cdot \; \prod_{j=1}^N v_j^{c_j} \tag{10}$$

where

$$Q_\lambda(\mathsf{c}) \; := \; \frac{\prod_{\lambda_j<0} \prod_{k=0}^{-\lambda_j-1} (c_j - k)}{\prod_{\lambda_j>0} \prod_{k=1}^{\lambda_j} (c_j + k)} \, . \tag{11}$$

The key observation is that (11) and (10) also make sense when $c_1, \ldots, c_N$ are taken from a $\mathbb{Q}$-algebra in which they are nilpotent. The expression $v_j^{c_j}$ can still be interpreted as $\exp(c_j \log v_j)$.

Definition 2 *Let* $\mathsf{A} = (a_{ij})$ *denote the* $n \times N$*-matrix with columns* $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$. *For a regular triangulation* $\mathcal{T}$ *(cf. § 1.1) of the polytope* $\Delta := \mathrm{conv}\,\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ *we define:*

$$\mathcal{R}_{\mathsf{A},\mathcal{T}} \; := \; \mathbb{Z}[D^{-1}][C_1, \ldots, C_N] \, / \, \mathcal{J} \tag{12}$$

where $\mathcal{J}$ is the ideal generated by the linear forms

$$a_{i1}C_1 + \ldots + a_{iN}C_N \quad \textit{for } i = 1, \ldots, n \tag{13}$$

and by the monomials

$$C_{i_1} \cdot \ldots \cdot C_{i_s} \quad \textit{with} \quad \mathrm{conv}\,\{\mathfrak{a}_{i_1}, \ldots, \mathfrak{a}_{i_s}\} \quad \textit{not a simplex in } \mathcal{T}; \tag{14}$$

D is the product of the volumes of the maximal simplices of $\mathcal{T}$.
We write c_i for the image of C_i in $\mathcal{R}_{\mathsf{A},\mathcal{T}}$.

In theorem 3 we show that $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is a free $\mathbb{Z}[D^{-1}]$-module with rank equal to the number of maximal simplices in the triangulation. This implies that $c_1, \ldots, c_N$ are nilpotent and hence

$$Q_\lambda(\mathsf{c}) \in \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$$

Theorem 1 *With this interpretation of $Q_\lambda(\mathsf{c})$ the function $\Phi_{\mathcal{T},\beta}(\mathsf{v})$ defined by (10) takes values in the* algebra $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{C}$. ⊠

The domain of definition of the function $\Phi_{\mathcal{T},\beta}(\mathsf{v})$ is discussed hereafter. Relation (13) ensures that this function $\Phi_{\mathcal{T},\beta}(\mathsf{v})$ satisfies the differential equations (3); it automatically satisfies (4). Relation (14) ensures that the series expansion for $\Phi_{\mathcal{T},\beta}(\mathsf{v})$ only contains λ's (i.e. $Q_\lambda(\mathsf{c}) \neq 0$) which satisfy

$$\mathcal{A}\,\lambda = \beta \quad \text{and} \quad \mathrm{conv}\,\{\mathfrak{a}_i \mid \lambda_i < 0\} \quad \text{is a simplex in the triangulation } \mathcal{T}. \tag{15}$$

This is important for determining a domain of definition for $\Phi_{\mathcal{T},\beta}(\mathsf{v})$.

As we tried to distinguish a kind of regular behavior for the λ's which satisfy (15), we were led to triangulations for which the intersection of the maximal simplices is not empty. We call

$$\mathrm{core}\,\mathcal{T} := \text{ intersection of the maximal simplices of } \mathcal{T} \tag{16}$$

the core of the triangulation $\mathcal{T}$. We use the short notation $i \in \mathrm{core}\,\mathcal{T}$ for $\mathfrak{a}_i \in \mathrm{core}\,\mathcal{T}$. The following result is corollary 3 in section 5.

Theorem 2 *Assume* $\mathrm{core}\,\mathcal{T} \neq \emptyset$ *and* $\beta = \sum_{i \in \mathrm{core}\,\mathcal{T}} m_i \mathfrak{a}_i$ *with all $m_i < 0$. Then the function $\Phi_{\mathcal{T},\beta}(\mathsf{v})$ takes values in the* principal ideal $c_{\mathrm{core}}\,\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{C}$ *where*

$$c_{\mathrm{core}} \;:=\; \prod_{i \in \mathrm{core}\,\mathcal{T}} c_i$$

Multiplication by c_{core} on $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ induces a linear isomorphism

$$\mathcal{R}_{\mathsf{A},\mathcal{T}} / \mathrm{Ann}\, c_{\mathrm{core}} \xrightarrow{\simeq} c_{\mathrm{core}}\, \mathcal{R}_{\mathsf{A},\mathcal{T}} \tag{17}$$

Thus one can also say that the function $\Phi_{\mathcal{T},\beta}(\mathsf{v})$ takes values in the algebra $\mathcal{R}_{\mathsf{A},\mathcal{T}} / \mathrm{Ann}\, c_{\mathrm{core}} \otimes \mathbb{C}$.

⊠

By composing $\Phi_{\mathcal{T},\beta}$ with a linear map $\mathcal{R}_{\mathsf{A},\mathcal{T}} \to \mathbb{C}$ one obtains a $\mathbb{C}$-multi-valued function which satisfies the system of differential equations (3)-(4). When $\beta = 0$ and $\mathcal{T}$ is unimodular all solutions of (3)-(4) can be obtained in this way; see theorem 5.

For $\beta \neq 0$ not all solutions of (3)-(4) can be obtained in this way. Yet what we need for mirror symmetry are the solutions which can be obtained in this way for appropriate β and $\mathcal{T}$; see theorem 10. Our proof of this theorem makes essential use of the relation:

$$\frac{\partial}{\partial v_i}\Phi_{\mathcal{T},\beta}(\mathsf{v}) \;=\; \Phi_{\mathcal{T},\beta-\mathfrak{a}_i}(\mathsf{v})\,. \tag{18}$$

which follows imediately from the formulas (10) and (11).

Remark 1 The ideal generated by the monomials in (14) is known as the *Stanley-Reisner ideal* and has been defined for finite simplicial complexes in general [22]. It is well-known [5, 10, 21] that the cohomology ring of a toric variety constructed from a complete simplicial fan has a presentation by generators and relations as in (13)-(14). Unimodular triangulations whose core is not empty and is not contained in the boundary of Δ, give rise to such toric varieties and in that case $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is indeed the cohomology ring of a toric variety; see theorem 9. However not all triangulations to which the present discussion applies are of this kind. For instance for the triangulation $\mathcal{T}_5$ in figure 1 we find $\mathcal{R}_{\mathsf{A},\mathcal{T}_5} = \mathbb{Z}[c_1, c_2, c_5]/(c_1^2,\, c_2^2,\, c_5^2,\, c_1c_2,\, c_2c_5)$. An element like c_2 which annihilates the whole degree 1 part of $\mathcal{R}_{\mathsf{A},\mathcal{T}_5}$ can not exist in the cohomology of a toric variety.

Remark 2 Our method for solving GKZ systems in the resonant case evolved directly from the Γ-series of Gel'fand-Kapranov-Zelevinskii. In hindsight it can also be viewed as a variation on the classical method of Frobenius [9]. The latter would view $\gamma_1, \ldots, \gamma_N$ in (7) or $c_1, \ldots, c_N$ in (10) as variables with a restriction given by (8) or (9); then differentiate (repeatedly if necessary) with respect to these variables and set $\gamma = (\gamma_1, \ldots, \gamma_N)$ in the derivatives equal to its special value γ°, c.q. set $c_1 = \ldots = c_N = 0$, to obtain solutions for (3)-(4). Frobenius [9] considered only functions in one variable. In the case with more variables one also needs a good bookkeeping device for the linear relations between the solutions of the differential equations. The rings $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ resp. $\mathcal{R}_{\mathsf{A},\mathcal{T}}/\operatorname{Ann} c_{\mathrm{core}}$ are such a bookkeeping devices. Hosono-Klemm-Theisen-Yau have applied Frobenius' method directly in the situation of the Picard-Fuchs equations of certain families of Calabi-Yau threefolds; see [17] formulas (4.9) and (4.10). In their work the cohomology ring of the mirror Calabi-Yau threefold plays a similar role of bookkeeper; in fact $\mathcal{R}_{\mathsf{A},\mathcal{T}}/\operatorname{Ann} c_{\mathrm{core}}$ is the cohomology ring of the mirror Calabi-Yau manifold . The way in which we arrive at our result looks quite different from that in [17] §4. Moreover the formulation in op. cit. is restricted to the situation of Calabi-Yau threefolds.

Remark 3 Some of our $\Phi_{\mathcal{T},\beta}$'s are similar to expressions presented by Givental in [14] theorems 3 and 4; more specifically, $\vec{g}_l$ in [14] thm. 4 is a special case of $\Phi_{\mathcal{T},\beta}$ in our theorem 2 with $\beta = -\sum_{i\in\mathrm{core}\,\mathcal{T}} \mathfrak{a}_i$, whereas in [14] thm. 3 there is a difference in that the input data are not subject to (2) in condition 1. The algebra H in [14] thm. 3 is the cohomology algebra of a toric variety while our $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ for appropriate $\mathcal{T}$ is also the cohomology algebra of a toric variety. The algebra H in [14] thm. 4 is the algebra $\mathcal{R}_{\mathsf{A},\mathcal{T}}/\operatorname{Ann} c_{\mathrm{core}}$ in our theorem 2.

For a proper treatment of the logarithms which appear in (10) we set

$$\begin{array}{lcl} v_j &:=& \exp(2\pi\mathsf{i}\, z_j) \quad (j = 1, \ldots, N) \\ \mathsf{z} &:=& (z_1, \ldots, z_N) \in \mathbb{C}^{N\vee} \\ \mathsf{c} &:=& (c_1, \ldots, c_N)^t \in \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Z}^N\,; \end{array} \tag{19}$$

by (9) c lies in fact in $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{L}$. Instead of (10) we now consider

$$\Psi_{\mathcal{T},\beta}(\mathsf{z}) := \sum_{\lambda\in\mathcal{A}^{-1}(\beta)} Q_\lambda(\mathsf{c}) \mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\lambda} \cdot \mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\mathsf{c}}\,. \tag{20}$$

Note that $\mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\mathsf{c}}$ is just a polynomial, but $\sum_{\lambda\in\mathcal{A}^{-1}(\beta)} Q_\lambda(\mathsf{c})\mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\lambda}$ is really a series. In section 3 we analyse the convergence of this series and give a domain $\mathcal{V}_\mathcal{T}$ in $\mathbb{C}^{N\vee}$ on which the function $\Psi_{\mathcal{T},\beta}$ is defined; see theorem 4.

The domain $\mathcal{V}_\mathcal{T}$ is invariant under translations by elements of $\mathbb{Z}^{N\vee}$ and by elements of $\mathbb{M}^\vee_\mathbb{C} := \operatorname{Hom}(\mathbb{M}, \mathbb{C}) \subset \mathbb{C}^{N\vee}$. From (20) one immediately sees

$$\Psi_{\mathcal{T},\beta}(\mathsf{z}+\mu) = \mathsf{e}^{2\pi\mathsf{i}\,\mu\cdot\mathsf{c}} \cdot \Psi_{\mathcal{T},\beta}(\mathsf{z}) \quad \forall\, \mu \in \mathbb{Z}^{N\vee} \tag{21}$$

$$\Psi_{\mathcal{T},\beta}(\mathsf{z}+\mathsf{m}) = \mathsf{e}^{2\pi\mathsf{i}\,\mathsf{m}\cdot\beta} \cdot \Psi_{\mathcal{T},\beta}(\mathsf{z}) \quad \forall\, \mathsf{m} \in \mathbb{M}^\vee_\mathbb{C}\,. \tag{22}$$

The functional equation (21) gives the monodromy of $\Phi_{\mathcal{T},\beta}$, when viewed as a multivalued function on $\mathcal{V}_\mathcal{T}/\mathbb{Z}^{N\vee}$ with values in the vector space $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{C}$. Because of (13) elements of $\mathbb{M}^\vee_\mathbb{Z} := \operatorname{Hom}(\mathbb{M}, \mathbb{Z})$ give trivial monodromy and the actual monodromy comes from $\mathbb{L}^\vee_\mathbb{Z} := \operatorname{Hom}(\mathbb{L}, \mathbb{Z})$.

As $\mathbb{M}^\vee_\mathbb{Z}$ acts trivially, the translation action of $\mathbb{M}^\vee_\mathbb{C}$ descends to an action of the torus $\mathbb{M}^\vee_\mathbb{C}/\mathbb{M}^\vee_\mathbb{Z} = \operatorname{Hom}(\mathbb{M}, \mathbb{C}^*)$. The functional equation (22), whose infinitesimal analogues are the differential equations (3), means that $\Psi_{\mathcal{T},\beta}$ is an eigenfunction with character β.

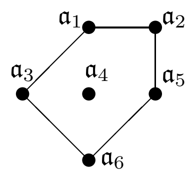
If one wants an invariant function for $\beta \neq 0$ one must replace the range of values of $\Psi_{\mathcal{T},\beta}$ by $(\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{C})/\mathbb{C}^*$, the orbit space for the natural $\mathbb{C}^*$-action on the vector space $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{C}$. On a possibly slightly smaller domain of definition the invariant function even takes values in the projective space $\mathbb{P}(\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{C})$. The $\mathbb{M}^\vee_\mathbb{C}$-invariant function $\Psi_{\mathcal{T},\beta}$ mod $\mathbb{C}^*$ is defined on the domain $\mathbb{L}^\vee_\mathbb{R} + \sqrt{-1}\mathcal{B}_\mathcal{T}$ in $\mathbb{L}^\vee_\mathbb{C}$; cf. formula (39). The (multivalued) function $\Phi_{\mathcal{T},\beta}$ mod $\mathbb{C}^*$ is defined on a domain in the torus $\operatorname{Hom}(\mathbb{L}, \mathbb{C}^*)$.

For a good overall picture it is appropriate to point out here that the pointed secondary fan (the construction of which is recalled in section 1.2) defines a toric variety which compactifies the torus $\mathrm{Hom}(\mathbb{L}, \mathbb{C}^*)$. To each regular triangulation of Δ corresponds a special point in the boundary of this compactification. The domain of definition of $\Phi_{\mathcal{T},\beta}$ mod $\mathbb{C}^*$ is the intersection of the torus $\mathrm{Hom}(\mathbb{L}, \mathbb{C}^*)$ and a neighborhood of the special point corresponding to $\mathcal{T}$; see the end of section 3.

Example 1 Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_6$ be the columns of the following matrix A:

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathsf{B} = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{pmatrix}$$

$\Delta =$ 

These satisfy conditions (1) and (2) with $\mathbb{M} = \mathbb{Z}^3$ and $\mathfrak{a}_0^\vee = (1, 0, 0)$.

Figure 1 shows all regular triangulations of the polytope Δ, with two triangulations joined by an edge iff the corresponding cones in the pointed secondary fan are adjacent.

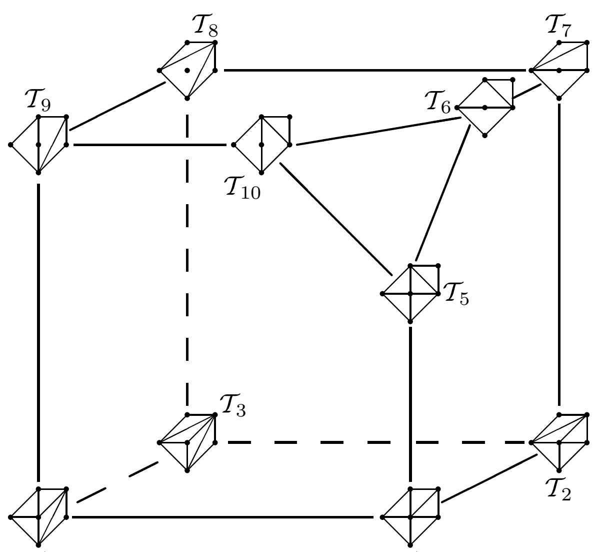


figure 1

The columns of the matrix B^t constitute a $\mathbb{Z}$-basis for $\mathbb{L}$ by means of which one can identify $\mathbb{L}$ with $\mathbb{Z}^3$ and $\mathbb{L}_{\mathbb{R}}^\vee$ with $\mathbb{R}^{3\vee}$. The rows $\mathfrak{b}_1, \ldots, \mathfrak{b}_6$ of the matrix B^t are then identified with the images of the standard basis vectors under the projection $\mathbb{R}^{6\vee} \to \mathbb{L}_{\mathbb{R}}^\vee$, dual to the inclusion $\mathbb{L} \subset \mathbb{Z}^6$. Thus one finds $\ell_j = \mathfrak{b}_j \cdot \ell$ for every $\ell \in \mathbb{L}_{\mathbb{R}} \simeq \mathbb{R}^3$ and (15) becomes a condition on the signs of $\mathfrak{b}_1 \cdot \ell + \gamma_1^\circ, \ldots, \mathfrak{b}_6 \cdot \ell + \gamma_6^\circ$. The signs give a vector in $\{-1, 0, +1\}^6$

These sign vectors correspond exactly to the various strata in the stratification of $\mathbb{R}^3$ induced by the six planes $\mathfrak{b}_j \cdot \mathsf{x} + \gamma_j^\circ = 0$ $(j = 1, \ldots, 6)$. Figure 2 shows the zonotope spanned by $\mathfrak{b}_1, \ldots, \mathfrak{b}_6$. The $3-j$-dimensional faces of this zonotope correspond bijectively with the j-dimensional strata in the stratification for $\gamma^\circ = 0$. The stratum with sign vector $(s_1, \ldots, s_6)$ corresponds with the face whose centre is $s_1\mathfrak{b}_1 + s_2\mathfrak{b}_2 + s_3\mathfrak{b}_3 + s_4\mathfrak{b}_4 + s_5\mathfrak{b}_5 + s_6\mathfrak{b}_6$. The vertices 1–14 (resp. 15–28) of the zonotope have sign vectors $(s_1, \ldots, s_6)$ (resp. $-(s_1, \ldots, s_6)$) as given in table 1.

The sign vectors of all faces of the zonotope give all possible signs for $\ell = (\ell_1, \ldots, \ell_6) \in \mathbb{L}$. Thus by comparing this with (15) one can see for every triangulation $\mathcal{T}$ what types of terms are involved in the series of $\Psi_{\mathcal{T},0}$. For example for triangulation $\mathcal{T}_1$ the series of $\Psi_{\mathcal{T}_1,0}$ involves precisely those $\ell \in \mathbb{L}$ whose sign vector corresponds to a face of the zonotope containing at least one of the vertices 1, 2, 3 or 4.

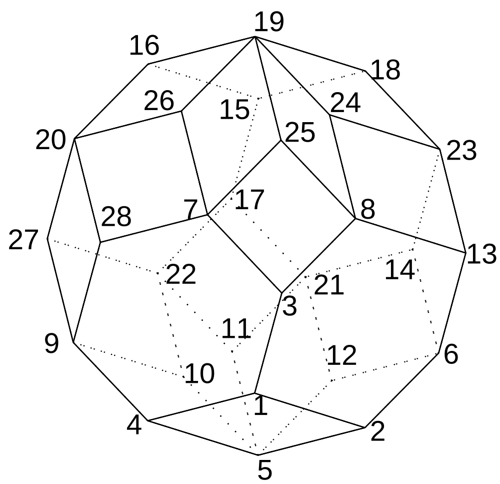


figure 2

1	+	+	+	−	+	+	15
2	−	+	+	−	+	+	16
3	+	−	+	−	+	+	17
4	+	+	+	−	−	+	18
5	−	+	+	−	−	+	19
6	−	+	+	−	+	−	20
7	+	−	−	−	+	+	21
8	+	−	+	−	+	−	22
9	+	+	−	−	−	+	23
10	−	+	−	−	−	+	24
11	−	+	+	+	−	+	25
12	−	+	+	−	−	−	26
13	−	−	+	−	+	−	27
14	−	+	+	+	+	−	28

table 1

The series $\Psi_{\mathcal{T}_1,-\mathfrak{a}_4}$ involves the same ℓ's with exception of $\ell = 0$ (which corresponds to the 3-dimensional the zonotope itself). Using the Pochhammer symbol notation $(x)_m := x\,(x+1)\cdot\ldots\cdot(x+m-1)$ we have

$$
\begin{aligned}
\Psi_{\mathcal{T}_1,-\mathfrak{a}_4} \quad = \quad & c_4\, \mathsf{e}^{-2\pi i z_4}\, T_1^{c_1} T_2^{c_2} T_5^{c_5} \times \\
\times \Bigg\{ & \sum_{p,q,r\geq 0} (-1)^q \frac{(2c_1+3c_2+2c_5+1)_{2p+3q+2r}}{(c_1)_p (c_2)_q (c_5)_r (c_1+c_2)_{p+q} (c_2+c_5)_{q+r}} T_1^p T_2^q T_5^r \\
& -c_1 \sum_{r\geq 0, -q\leq p<0} (-1)^{q+p} \frac{(2p+3q+2r)!(-p-1)!}{q!r!(p+q)!(q+r)!} T_1^p T_2^q T_5^r \\
& -c_5 \sum_{p\geq 0, -q\leq r<0} (-1)^{q+r} \frac{(2p+3q+2r)!(-r-1)!}{p!q!(p+q)!(q+r)!} T_1^p T_2^q T_5^r \\
& -c_2 \sum_{-p\leq q<0\,, -r\leq q<0} \frac{(2p+3q+2r)!(-q-1)!}{p!r!(p+q)!(q+r)!} T_1^p T_2^q T_5^r \quad \Bigg\}
\end{aligned}
$$

where

$$T_1 := \mathsf{e}^{2\pi i(z_1-2z_4+z_6)}\ ,\quad T_2 := \mathsf{e}^{2\pi i(z_2+z_3-3z_4+z_6)}\ ,\quad T_5 := \mathsf{e}^{2\pi i(z_3-2z_4+z_5)}$$

$$c_4 = -2c_1 - 3c_2 - 2c_5$$

and

$$\mathcal{R}_{\mathsf{A},\mathcal{T}_1} = \mathbb{Z}[c_1, c_2, c_5]\,/\,(c_1^2 - c_2^2,\ c_1^2 - c_5^2,\ c_1^2 + c_1c_2,\ c_1^2 + c_2c_5,\ c_1c_5)\,.$$

Note that $c_4c_1 = c_4c_2 = c_4c_5$. One may therefore simplify the expression for $\Psi_{\mathcal{T}_1,-\mathfrak{a}_4}$ and replace c_2 and c_5 by c_1.

1 Regular triangulations and the pointed secondary fan

In this section we review some results about regular triangulations and about the pointed secondary fan, essentially following [4]. One may take as a definition of *regular triangulations* that these are the triangulations produced by the construction in this section; see in particular proposition 1.

1.1 Regular triangulations

We start from a set of vectors $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ in $\mathbb{Z}^n$ satisfying condition 1. Let $\Delta = \operatorname{conv}\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ denote the convex hull of this set of points in $\mathbb{R}^n$. We are interested in triangulations of Δ such that all vertices are among the marked points $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$. The notation can be conveniently simplified by referring to a simplex $\operatorname{conv}\{\mathfrak{a}_{i_1}, \ldots, \mathfrak{a}_{i_m}\}$ by just the index set $\{i_1, \ldots, i_m\}$. We will allways take the indices in increasing order. If $\mathcal{T}$ is a triangulation, we write $\mathcal{T}^m$ for the set of simplices with m vertices. A triangulation is completely determined by its set of maximal simplices $\mathcal{T}^n$.

For the construction of a regular triangulation we take an N-tuple of positive real numbers $\mathsf{d} = (d_1, \ldots, d_N)$ and consider the polytope

$$\mathcal{P}_{\mathsf{d}} := \operatorname{conv}\{0, d_1^{-1}\mathfrak{a}_1, \ldots, d_N^{-1}\mathfrak{a}_N\} \ \subset\ \mathbb{R}^n\,. \tag{23}$$

Consider a subset $I = \{i_1, \ldots, i_n\}$ of $\{1, \ldots, N\}$ for which $\mathfrak{a}_{i_1}, \ldots, \mathfrak{a}_{i_n}$ are linearly independent. The affine hyperplane through $d_{i_1}^{-1}\mathfrak{a}_{i_1}, \ldots, d_{i_n}^{-1}\mathfrak{a}_{i_n}$ is given by the equation $D_{\mathsf{d},I}(\mathsf{x}) = 0$ with

$$D_{\mathsf{d},I}(\mathsf{x}) \ :=\ \det\begin{pmatrix} d_{i_1}^{-1}\mathfrak{a}_{i_1} & \ldots & d_{i_n}^{-1}\mathfrak{a}_{i_n} & \mathsf{x} \\ 1 & \ldots & 1 & 1 \end{pmatrix} \tag{24}$$

Write $I^* := \{1, \ldots, N\} \setminus I$. Then $\{d_{i_1}^{-1}\mathfrak{a}_{i_1}, \ldots, d_{i_n}^{-1}\mathfrak{a}_{i_n}\}$ lies in a codimension 1 face of $\mathcal{P}_{\mathsf{d}}$ if and only if for all $j \in I^*$:

$$D_{\mathsf{d},I}(d_j^{-1}\mathfrak{a}_j) \cdot D_{\mathsf{d},I}(0) \geq 0 \tag{25}$$

This face is a simplex with vertices $d_{i_1}^{-1}\mathfrak{a}_{i_1}, \ldots, d_{i_n}^{-1}\mathfrak{a}_{i_n}$ iff $D_{\mathsf{d},I}(d_j^{-1}\mathfrak{a}_j) \neq 0$ for every $j \in I^*$. Thus if d does not lie on any hyperplane in $\mathbb{R}^N$ given by the vanishing of $D_{\mathsf{d},I}(d_j^{-1}\mathfrak{a}_j)$ for some I and j with $j \notin I$, then all faces of $\mathcal{P}_{\mathsf{d}}$ opposite to the vertex 0 are simplicial.

In this case the projection with center 0 projects the boundary of $\mathcal{P}_{\mathsf{d}}$ onto *a triangulation $\mathcal{T}$ of Δ*. The maximal simplices of $\mathcal{T}$ are those $I = \{i_1, \ldots, i_n\}$ for which $D_{\mathsf{d},I}(d_j^{-1}\mathfrak{a}_j) \cdot D_{\mathsf{d},I}(0) > 0$ holds for every $j \in I^*$.

Let $\mathsf{A} = (a_{ij})$ denote the $n \times N$-matrix with columns $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$. The triangulation obviously depends only on d modulo the row space of A. Let us reformulate the above construction accordingly.

Take $\mathbb{L} = \ker \mathsf{A} \subset \mathbb{Z}^N$ as in (5). Assumption (2) implies $\ell_1 + \ldots + \ell_N = 0$ for every $\ell = (\ell_1, \ldots, \ell_N)^t \in \mathbb{L}$. Take an $(N-n) \times N$-matrix B with entries in $\mathbb{Z}$ such that columns of B^t constitute a basis for $\mathbb{L}$.

Let $w \in \mathbb{R}^{N-n}$. Then there exists a row vector of positive real numbers $\mathsf{d} = (d_1, \ldots, d_N)$ such that $w = \mathsf{B}\mathsf{d}^t$. Take the matrices

$$\widetilde{\mathsf{A}} := \left(\begin{array}{c|c} \mathsf{A} & 0 \\ \hline \mathsf{d} & 1 \end{array} \right) \qquad \text{and} \qquad \widetilde{\mathsf{B}} := \left(\begin{array}{c|c} \mathsf{B} & -w \end{array} \right).$$

Denote by $\widetilde{\mathsf{A}}_K$ (resp. $\widetilde{\mathsf{B}}_K$) the submatrix of $\widetilde{\mathsf{A}}$ (resp. $\widetilde{\mathsf{B}}$) composed of the entries with column index in a subset K of $\{1, \ldots, N+1\}$. Since $\operatorname{rank} \widetilde{\mathsf{A}} = n+1$, $\operatorname{rank} \widetilde{\mathsf{B}} = N-n$ and $\widetilde{\mathsf{A}} \cdot \widetilde{\mathsf{B}}^t = 0$ there is a non-zero $r \in \mathbb{Q}$ such that for every $J \subset \{1, \ldots, N+1\}$ of cardinality $n+1$ and $J' = \{1, \ldots, N+1\} \setminus J$

$$\det(\widetilde{\mathsf{A}}_J) = (-1)^{\sum_{j \in J} j}\, r \det(\widetilde{\mathsf{B}}_{J'})$$

One sees that (25) is equivalent to

$$(-1)^{\sharp\{h \in I^* \mid h > j\}} \det\left(\left(\begin{array}{c|c} \mathsf{B}_{I^* \setminus \{j\}} & w \end{array} \right) \right) \cdot \det(\mathsf{B}_{I^*}) \geq 0\,; \tag{26}$$

here B_{I^*} resp. $\mathsf{B}_{I^* \setminus \{j\}}$ is the submatrix of B consisting of the entries with column index in I^* resp. $I^* \setminus \{j\}$.

Thus the triangulation $\mathcal{T}$ can also be constructed from (26).

1.2 The pointed secondary fan

For a more intrinsic formulation which does not refer to a choice of a basis for $\mathbb{L}$ we consider the $(N-n)$-dimensional real vector space $\mathbb{L}_{\mathbb{R}}^{\vee} := \operatorname{Hom}(\mathbb{L}, \mathbb{R})$. Let $\mathfrak{b}_1, \ldots, \mathfrak{b}_N \in \mathbb{L}_{\mathbb{R}}^{\vee}$ be the images of the standard basis vectors of $\mathbb{R}^{N\vee}$ under the surjection $\mathbb{R}^{N\vee} \twoheadrightarrow \mathbb{L}_{\mathbb{R}}^{\vee}$ dual to the inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^N$. Let $\mathfrak{B}$ (resp. $\mathfrak{D}$) be the collection of those subsets J of $\{1, \ldots, N\}$ of cardinality $N-n$ (resp. $N-n-1$) for which the vectors $\mathfrak{b}_j$ ($j \in J$) are linearly independent. For $K = \{k_1, \ldots, k_{N-n}\} \in \mathfrak{B}$ and $J = \{j_1, \ldots, j_{N-n-1}\} \in \mathfrak{D}$ we write

$$\begin{aligned} \mathcal{C}_K &:= \{t_1\mathfrak{b}_{k_1} + \ldots + t_{N-n}\mathfrak{b}_{k_{N-n}} \in \mathbb{L}_{\mathbb{R}}^{\vee} \mid t_1, \ldots, t_{N-n} \in \mathbb{R}_{\geq 0}\} \\ H_J &:= \{t_1\mathfrak{b}_{j_1} + \ldots + t_{N-n-1}\mathfrak{b}_{j_{N-n-1}} \in \mathbb{L}_{\mathbb{R}}^{\vee} \mid t_1, \ldots, t_{N-n-1} \in \mathbb{R}\}. \end{aligned}$$

Choosing a basis for $\mathbb{L}$ as before one can identify $\mathbb{L}_{\mathbb{R}}^{\vee}$ with $\mathbb{R}^{N-n\vee}$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_N$ with the rows of matrix B^t. The inequality (26) becomes equivalent to the statement $w \in \mathcal{C}_{I^*}$. The condition $D_{\mathsf{d},I}(d_j^{-1}\mathfrak{a}_j) \neq 0$ for the left hand factor in (25) becomes equivalent to $w \notin H_J$ for $J = \{1, \ldots, N+1\} \setminus (I \cup \{j\})$.

Thus the preceding discussion shows:

Proposition 1 (cf. [4] lemma 4.3.) *For $w \in \mathbb{L}_{\mathbb{R}}^{\vee} \setminus \bigcup_{J \in \mathfrak{D}} H_J$ the set*

$$\mathcal{T}^n := \{ I \mid I^* \in \mathfrak{B} \text{ and } w \in \mathcal{C}_{I^*} \}$$

is the set of maximal simplices of a regular triangulation $\mathcal{T}$ of Δ.
(Recall the notation $I^* := \{1, \ldots, N\} \setminus I$.) ⊠

If $\mathcal{T}$ is a regular triangulation of Δ write

$$\mathcal{C}_{\mathcal{T}} = \bigcap_{I \in \mathcal{T}^n} \mathcal{C}_{I^*}. \tag{27}$$

Then every $w \in \mathcal{C}_{\mathcal{T}} \setminus \bigcup_{J \in \mathfrak{D}} H_J$ leads by the above construction to the same triangulation $\mathcal{T}$.

The cones $\mathcal{C}_{\mathcal{T}}$ one obtains in this way from all regular triangulations of Δ constitute the collection of maximal cones of a complete fan in $\mathbb{L}_{\mathbb{R}}^{\vee}$. This fan is called *the pointed secondary fan.*

Remark 4 The dual (or polar) set of $\mathcal{P}_{\mathsf{d}}$ in (23) is (e.g. [1] def.4.1.1, [10] p.24)

$$\mathcal{P}_{\mathsf{d}}^{\vee} := \{ \mathsf{y} \in \mathbb{R}^{n\vee} \mid \mathsf{y} \cdot \mathsf{x} \geq -1 \text{ for all } \mathsf{x} \in \mathcal{P}_{\mathsf{d}} \} \tag{28}$$

It is the intersection of half-spaces given by the inequalities

$$\mathsf{y} \cdot \mathfrak{a}_i + d_i \geq 0 \qquad (i = 1, \ldots, N)$$

$\mathcal{P}_{\mathsf{d}}^{\vee}$ is an unbounded polyhedron. Its vertices correspond with the codimension 1 faces of $\mathcal{P}_{\mathsf{d}}$ which do not contain 0.

Adding to d an element of the row space of matrix A amounts to just a translation of the polyhedron $\mathcal{P}_{\mathsf{d}}^{\vee}$ in $\mathbb{R}^{n\vee}$. If d gives rise to a unimodular triangulation, then $\mathcal{P}_{\mathsf{d}}^{\vee}$ is an (unbounded) *Delzant polyhedron* in the sense of [16] p.8. Thus, by the constructions in [16] *a point in the real cone $\mathcal{C}_{\mathcal{T}}$ for a unimodular regular triangulation $\mathcal{T}$ can be interpreted as a parameter for the symplectic structure of a toric variety. In view of formula (39) this applies in particular to the imaginary part of the variable z in (20).*

2 The ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}$.

Theorem 3 *Consider the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ as in definition 2.*

(i). *$\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is a free $\mathbb{Z}[D^{-1}]$-module of rank $\sharp\,\mathcal{T}^n$.*

(ii). *$\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is a graded ring. Let $\mathcal{R}^{(k)}_{\mathsf{A},\mathcal{T}}$ denote its homogeneous component of degree k. Then the Poincaré series of $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is:*

$$\sum_{k\geqslant 0}\left(\operatorname{rank}\mathcal{R}^{(k)}_{\mathsf{A},\mathcal{T}}\right)\tau^k \;=\; \sum_{m=0}^{n}\sharp\,(\,\mathcal{T}^m\,)\,\tau^m(1-\tau)^{n-m}$$

where $\sharp\,(\,\mathcal{T}^m\,)\;=$ the number of simplices with m vertices; $\sharp\,(\,\mathcal{T}^0\,)=1$ by convention. In particular

$$\mathcal{R}^{(k)}_{\mathsf{A},\mathcal{T}}=0 \quad \textit{for} \quad k\geq n\,.$$

(iii). *$\{c_I \mid I\in\mathcal{T}^n\}$ is a $\mathbb{Z}[D^{-1}]$-basis for $\mathcal{R}_{\mathsf{A},\mathcal{T}}$. (cf. formula (30))*

The **proof of theorem 3** closely follows the proofs of Danilov ([5] § 10) and Fulton ([10] § 5.2) for the analogous presentation of the Chow ring of a complete simplicial toric variety. We include a proof here in order check that it needs no reference to algebraic cycles and also works when the simplicial complex is homeomorphic to a ball instead of a sphere as in [5, 10].

For the construction of a basis for $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ we choose a vector ξ in Δ which should be linearly independent from every $n-1$-tuple of vectors in $\{\mathfrak{a}_1,\ldots,\mathfrak{a}_N\}$. If I is a maximal simplex, then $\{\mathfrak{a}_i\}_{i\in I}$ is a basis of $\mathbb{R}^n$ and $\xi\;=\;\sum_{i\in I}x_i\mathfrak{a}_i$ with all $x_i\neq 0$. We define

$$I^- \quad := \quad \{\,i\in I\mid x_i<0\,\}\,, \tag{29}$$

$$c_I \quad := \quad \prod_{i\in I^-} c_i\,. \tag{30}$$

Let $\mathcal{T}$ be associated with $\mathsf{d}=(d_1,\ldots,d_N)$ as in section 1.1. For $I\in\mathcal{T}^n$ let p_I be the positive real number such that $p_I\xi$ lies in the affine hyperplane through the points $d_i^{-1}\mathfrak{a}_i$ with $i\in I$, i.e. $D_{\mathsf{d},I}(p_I\xi)=0$. We may assume that d is chosen such that $p_{I_1}\neq p_{I_2}$ whenever $I_1\neq I_2$; indeed, for $I_1\neq I_2$ the equality $p_{I_1}=p_{I_2}$ amounts to a non-trivial linear equation for $d_1,\ldots,d_N$. As in [10] we define a total ordering on $\mathcal{T}^n$:

$$I_1<I_2 \qquad \text{iff} \qquad p_{I_1}<p_{I_2}\,. \tag{31}$$

Lemma 1 (cf. [10] p.101(∗)) *If $I_1^-\subset I_2$ then $I_1\leq I_2$.*

proof: By definition of p_{I_1} there exist $s_j\in\mathbb{R}$ such that $p_{I_1}\xi\;=\sum_{j\in I_1}s_jd_j^{-1}\mathfrak{a}_j$ and $1\;=\sum_{j\in I_1}s_j$. If $I_1\neq I_2$ and $I_1^-\subset I_2$ then $s_j>0$ for every $j\in I_1\setminus I_2$. Using this and (25) for I_2 one checks: $D_{\mathsf{d},I_2}(p_{I_1}\xi)\cdot D_{\mathsf{d},I_2}(0)>0$. This shows that 0 and $p_{I_1}\xi$ lie on the same side of the affine hyperplane through the points $d_i^{-1}\mathfrak{a}_i$ with $i\in I_2$. Hence: $p_{I_1}<p_{I_2}$. ⊠

Lemma 2 (cf. [10] p.102) *Let J be a simplex in $\mathcal{T}$. Then: $I^- \subset J \subset I$ where $I := \min\{I' \in \mathcal{T}^n \mid J \subset I'\}$.*

proof: The conclusion is clear if $I = J$. So assume $I \neq J$ and take $i \in I \setminus J$. Then $I \setminus \{i\}$ is a codim 1 simplex in the triangulation, which either is contained in the boundary of Δ or is contained in another maximal simplex $I' \neq I$.

If $I \setminus \{i\}$ is a boundary simplex, then ξ and $\mathfrak{a}_i$ are on the same side of the linear hyperplane in $\mathbb{R}^n$ spanned be the vectors $\mathfrak{a}_j$ with $j \in I \setminus \{i\}$. This implies $x_i > 0$ in the expansion $\xi = \sum_{j \in I} x_j \mathfrak{a}_j$. So $i \notin I^-$.

If $I \setminus \{i\}$ is contained in a maximal simplex $I' \neq I$, then $J \subset I'$ and hence $I < I'$. Now look at the two expansions $\xi = x_i \mathfrak{a}_i + \sum_{j \in I \cap I'} x_j \mathfrak{a}_j$ and $\xi = y_k \mathfrak{a}_k + \sum_{j \in I \cap I'} y_j \mathfrak{a}_j$ where $\{k\} = I' \setminus (I \cap I')$. Then $y_k < 0$ because $I'^- \not\subset I$ by the preceding lemma. On the other hand, x_i and y_k have different signs because $\mathfrak{a}_i$ and $\mathfrak{a}_k$ lie on different sides of the linear hyperplane spanned by the vectors $\mathfrak{a}_j$ with $j \in I \cap I'$. We see $x_i > 0$ and $i \notin I^-$.

Conclusion: $I^- \subset J$. ⊠

Proposition 2 *The elements c_I ($I \in \mathcal{T}^n$) generate $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ as a $\mathbb{Z}[D^{-1}]$-module.*

proof: $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is linearly generated by monomials in $c_1, \ldots, c_N$. For one $I_0 \in \mathcal{T}^n$ we have $I_0^- = \emptyset$, hence $c_{I_0} = 1$. Therefore we only need to show that for every j and every I_1 the product $c_j \cdot c_{I_1}$ can be written as a linear combination of c_I's. If $j \in I_1$ one can use the linear relations (13) to express every c_i with $i \in I_1$ as a $\mathbb{Z}[D^{-1}]$-linear combination of c_k's with $k \notin I_1$. Since this works for c_j in particular, the problem can be reduced to showing that a monomial of the form $\prod_{i \in J} c_i$ with J a simplex of the triangulation, can be written as a linear combination of c_I's. Given such a J take $I_J \in \mathcal{T}^n$ such that $I_J^- \subset J \subset I_J$; see lemma 2. If $J = I_J^-$, then $\prod_{i \in J} c_i = c_{I_J}$ and we are done. If $J \neq I_J^-$ take $m \in J \setminus I_J^-$ and use the linear relations (13) to rewrite c_m as a $\mathbb{Z}[D^{-1}]$-linear combination of c_k's with $k \notin I_J$. This leads to an expression for $\prod_{i \in J} c_i$ as a $\mathbb{Z}[D^{-1}]$-linear combination of monomials of the form $\prod_{i \in K} c_i$ with K a simplex of the triangulation and $I_J^- \subsetneq K$. Given such a K take $I_K \in \mathcal{T}^n$ such that $I_K^- \subset K \subset I_K$. Then, according to lemma 1, $I_J < I_K$. We proceed by induction. ⊠

Next we follow Danilov's arguments in [5] remark 3.8 to prove

$$\sum_{k \geqslant 0} \left(\dim_{\mathbb{Q}} \mathcal{R}^{(k)}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q} \right) \tau^k = \sum_{m=0}^{n} \sharp \, (\mathcal{T}^m) \, \tau^m (1-\tau)^{n-m} \tag{32}$$

We have added a few references of which [20] is most relevant because it deals with a triangulation of a polytope, while [5] deals with a triangulation of a sphere. In [22, 20] the Stanley-Reisner ring $\mathbb{Q}[\mathcal{T}]$ of the simplicial complex $\mathcal{T}$ over the field $\mathbb{Q}$ is defined as the quotient of the polynomial ring $\mathbb{Q}[C_1, \ldots, C_N]$ modulo the ideal generated by the monomials (14). $\mathbb{Q}[\mathcal{T}]$ is a Cohen-Macaulay ring of Krull dimension n; see [20] thm.2.2 and [22] thm 1.3. By definition 2 there is a natural homomorphism $\mathbb{Q}[\mathcal{T}] \to \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$ with kernel generated by the n elements $\alpha_i := a_{i1}C_1 + a_{i2}C_2 + \ldots + a_{iN}C_N$. By proposition 2 the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space and hence has Krull dimension 0. It also follows from proposition 2 that $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$ and $\mathbb{Q}[\mathcal{T}]$ are local rings. We can now apply [19] thm.16.B and see that $\alpha_1, \ldots, \alpha_n$ is a regular sequence. As pointed out in [5] remark 3.8b this implies that the Poincaré series of $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$ is equal to $(1-\lambda)^n$ times the Poincaré series of $\mathbb{Q}[\mathcal{T}]$. The latter is $\sum_{m=0}^{n} \sharp(\mathcal{T}^m)\,\lambda^m(1-\lambda)^{-m}$ by [22] thm. 1.4 (where it is called Hilbert series). Formula (32) follows.

We see that $\dim_{\mathbb{Q}} \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q} = \sharp\,\mathcal{T}^n$ and hence that the elements c_I $(I \in \mathcal{T}^n)$ are linearly independent over $\mathbb{Q}$. *This completes the proof of theorem 3* ⊠

Corollary 1 *If $\mathcal{T}$ is unimodular, then*

(i). *$\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is a free $\mathbb{Z}$-module with rank equal to* $\mathrm{vol}\,\Delta$.

(ii). $\Delta \cap \mathbb{Z}^n = \mathcal{T}^1 = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$

(iii). *there is an isomorphism $\mathcal{R}^{(1)}_{\mathsf{A},\mathcal{T}} \xrightarrow{\sim} \mathbb{L}^\vee_{\mathbb{Z}}$ such that $c_j \leftrightarrow \mathfrak{b}_j$ $(j = 1, \ldots, N)$*

proof: **(i)** immediately follows from theorem 3.
(ii) Assume that there is a lattice point in Δ which is not a vertex of $\mathcal{T}$. This point lies in some maximal simplex and gives rise to a decomposition of this simplex into at least two integral simplices. This contradicts the assumption.
(iii) Because of (ii) all monomials in (14) have degree ≥ 2. Consequently, $\mathcal{R}^{(1)}_{\mathsf{A},\mathcal{T}}$ is just the quotient of $\mathbb{Z}\,C_1 \oplus \ldots \oplus \mathbb{Z}\,C_N$ modulo the span of the linear forms in (13). This quotient is $\mathbb{L}^\vee_{\mathbb{Z}}$. ⊠

3 A domain of definition for the function $\Psi_{\mathcal{T},\beta}$.

We first investigate for which λ's one possibly has $Q_\lambda(\mathsf{c}) \neq 0$ in $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$.

For $I \in \mathcal{T}^n$ let A_I denote the $n \times n$-submatrix of A with columns $\mathfrak{a}_i$ $(i \in I)$. By p_I we denote the $N \times N$-matrix whose entries with row index not in I are all 0 and whose $n \times N$-submatrix of entries with row index in I is $\mathsf{A}_I^{-1}\mathsf{A}$. This p_I is an idempotent linear operator on $\mathbb{R}^N$. Now define:

$$\mathfrak{P}_{\mathcal{T}} := \mathrm{conv}\,\{\mathsf{p}_I \mid I \in \mathcal{T}^n\} \quad \text{in} \quad \mathrm{Mat}_{N\times N}(\mathbb{R})\,. \tag{33}$$

The image of the idempotent operator $\mathsf{1} - \mathsf{p}_I$ is $\mathbb{L}_{\mathbb{R}}$. Therefore all elements of $\mathsf{1} - \mathfrak{P}_{\mathcal{T}} = \mathrm{conv}\,\{\mathsf{1} - \mathsf{p}_I \mid I \in \mathcal{T}^n\}$ are idempotent operators on $\mathbb{R}^N$ with image $\mathbb{L}_{\mathbb{R}}$. Hence all elements of $\mathfrak{P}_{\mathcal{T}}$ are also idempotent operators on $\mathbb{R}^N$.

For every $\lambda \in \mathbb{Z}^N$ one has the polytope $\mathfrak{P}_{\mathcal{T}}(\lambda)$ which is the convex hull of $\{\mathsf{p}_I(\lambda) \mid I \in \mathcal{T}^n\}$ in $\mathbb{R}^N$. This obviously depends only on $\lambda \bmod \mathbb{L}$.

Lemma 3 *If $\lambda \in \mathbb{Z}^N$ is such that $Q_\lambda(\mathsf{c}) \neq 0$ in $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathbb{Q}$, then λ lies in the set $\mathfrak{P}_{\mathcal{T}}(\lambda) + \mathcal{C}_{\mathcal{T}}^\vee$. Here $\mathcal{C}_{\mathcal{T}}^\vee$ is the dual of the cone $\mathcal{C}_{\mathcal{T}}$ defined in (27):*

$$\mathcal{C}_{\mathcal{T}}^\vee := \{\ell \in \mathbb{L}_{\mathbb{R}} \mid \omega \cdot \ell \geq 0 \;\; \textit{for all} \;\; \omega \in \mathcal{C}_{\mathcal{T}}\}\,. \tag{34}$$

proof: If $Q_\lambda(\mathsf{c}) \neq 0$ then $\{i \,|\, \lambda_i < 0\}$ is contained in some maximal simplex I of $\mathcal{T}$. Let $\ell = (1 - \mathsf{p}_I)(\lambda)$. Then $\ell = (\ell_1, \ldots, \ell_N) \in \mathbb{L}_\mathbb{R}$ and $\mathfrak{b}_j \cdot \ell = \ell_j = \lambda_j \geq 0$ for all $j \in I^*$. This shows $(1 - \mathsf{p}_I)(\lambda) \in \mathcal{C}_{I^*}^\vee \subset \mathcal{C}_\mathcal{T}^\vee$. ⊠

Lemma 4 *The coefficients in the power series expansion*

$$\frac{\prod_{\lambda_j<0}\prod_{k=0}^{-\lambda_j-1}(k+x_j)}{\prod_{\lambda_j>0}\prod_{k=1}^{\lambda_j}(k-x_j)} = \sum_{m_1,\ldots,m_N\geq 0} K_{m_1,\ldots,m_N} x_1^{m_1}\cdot\ldots\cdot x_N^{m_N} \tag{35}$$

satisfy

$$0 \leq K_{m_1,\ldots,m_N} \;\leq\; N^{\|\lambda\|}\cdot 2^{\|m\|+N}\cdot N!\cdot(\max(1, N-\deg\lambda))! \tag{36}$$

with $\| \, m \, \| := \sum_{i=1}^N m_i$ *and* $\| \, \lambda \, \| := \sum_{i=1}^N |\lambda_i|$ *and* $\deg\lambda := \sum_{i=1}^N \lambda_i = \mathfrak{a}_0^\vee \cdot \beta$.

proof: Clearly $K_{m_1,\ldots,m_N} \geq 0$. Clearly also $2^{-\|m\|}K_{m_1,\ldots,m_N}$ is less than the value of the left hand side at $x_1 = \ldots = x_N = \frac{1}{2}$. Therefore

$$K_{m_1,\ldots,m_N} \;<\; 2^{\|m\|+S}\cdot\frac{\prod_{\lambda_j<0}(-\lambda_j)!}{\prod_{\lambda_j>0}(\lambda_j-1)!} \;\leq\; 2^{\|m\|+S}\cdot\frac{P!}{(R-S)!}\cdot S^{R-S}$$

where $P = -\sum_{\lambda_i<0}\lambda_i$ and $R = \sum_{\lambda_i>0}\lambda_i$ and $S = \sharp\{i \,|\, \lambda_i > 0\}$.

If $P \leq R-S$ then $\frac{P!}{(R-S)!} \leq 1$. If $P > R-S$ then $\frac{P!}{(R-S)!} \leq 2^P\cdot(P-R+S)!$. Combining these estimates one arrives at (36). ⊠

The sum of the series $\sum_{\lambda\in\mathcal{A}^{-1}(\beta)} Q_\lambda(\mathsf{c})\mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\lambda}$ in formula (20) should be computed as a limit for $L \to \infty$ of partial sums Σ_L taking only terms with $\| \, \lambda \, \| \leq L$. These sums only involve λ's with $Q_\lambda(\mathsf{c}) \neq 0$. According to lemma 3 such a λ is of the form $\lambda = \tilde{\lambda} + \ell$ with $\ell \in \mathcal{C}_\mathcal{T}^\vee$ and with $\tilde{\lambda}$ contained in a compact polytope which only depends on β. Therefore $\| \, \lambda \, \| \leq \| \, \ell \, \| +$ some constant which only depends on β. Since

$$Q_\lambda(\mathsf{c}) = (-1)^{\sharp\{i\,|\,\lambda_i<0\}}\sum_{m_1,\ldots,m_N\geq 0,\,\|m\|\leq n}(-1)^{\|m\|}K_{m_1,\ldots,m_N}c_1^{m_1}\cdot\ldots\cdot c_N^{m_N}$$

lemma 4 shows that the coordinates of $Q_\lambda(\mathsf{c})$ with respect to a basis of the vector space $\mathcal{R}_{\mathsf{A},\mathcal{T}}\otimes\mathbb{Q}$ are less than $N^{\|\ell\|}$ times some constant which only depends on β. Thus one sees that the limit of the partial sums exists if the imaginary part $\Im\,\mathsf{z}$ of z satisfies

$$\Im\,\mathsf{z}\cdot\ell \;>\; \frac{\log N}{2\pi}\,\|\,\ell\,\| \qquad \text{for all } \ell\in\mathcal{C}_\mathcal{T}^\vee \tag{37}$$

Let $p : \mathbb{R}^{N\vee} \to \mathbb{L}_\mathbb{R}^\vee$ denote the canonical projection. If $b \in \mathbb{L}_\mathbb{R}^\vee$ is any vector which satisfies

$$b\cdot\ell \;>\; \frac{\log N}{2\pi}\,\|\,\ell\,\| \qquad \text{for all } \ell\in\mathcal{C}_\mathcal{T}^\vee, \tag{38}$$

then $b \in \mathcal{C}_{\mathcal{T}}$ and every z with the property $p\,(\Im\,\mathsf{z}) \in b + \mathcal{C}_{\mathcal{T}}$ satisfies (37). Let us therefore define

$$\mathcal{B}_{\mathcal{T}} := \bigcup_{b \ \text{s.t. } (38)} (\, b \,+\, \mathcal{C}_{\mathcal{T}}\,) \tag{39}$$

The above discussion proves:

Theorem 4 *Formula (20):*

$$\Psi_{\mathcal{T},\beta}(\mathsf{z}) \;:=\; \sum_{\lambda\in\mathcal{A}^{-1}(\beta)} Q_{\lambda}(\mathsf{c})\mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\lambda}\cdot\mathsf{e}^{2\pi\mathsf{i}\,\mathsf{z}\cdot\mathsf{c}}$$

defines a function with values in $\mathcal{R}_{\mathsf{A},\mathcal{T}}\otimes\mathbb{C}$ *on the domain*

$$\mathcal{V}_{\mathcal{T}} := \{\mathsf{z}\in\mathbb{C}^{N\vee} \mid p\,(\Im\,\mathsf{z})\in\mathcal{B}_{\mathcal{T}}\}\,. \tag{40}$$

$\boxtimes$

In order to have a more global geometric picture of where the domain of definition of the function $\Psi_{\mathcal{T},\beta}$ is situated we give a brief description of *the toric variety associated with the pointed secondary fan.*

The pointed secondary fan is a complete fan of strongly convex polyhedral cones which are generated by vectors from the lattice $\mathbb{L}^{\vee}_{\mathbb{Z}}$. By the general theory of toric varieties [10, 21] this lattice-fan pair gives rise to a toric variety. In the case of $\mathbb{L}^{\vee}_{\mathbb{Z}}$ and the pointed secondary fan the general construction reads as follows.

For each regular triangulation $\mathcal{T}$ one has the cone $\mathcal{C}_{\mathcal{T}}$ in the secondary fan and one considers the monoid ring $\mathbb{Z}[\mathbb{L}_{\mathcal{T}}]$ of the sub-monoid $\mathbb{L}_{\mathcal{T}}$ of $\mathbb{L}$:

$$\mathbb{L}_{\mathcal{T}} := \mathbb{L}\cap\mathcal{C}_{\mathcal{T}}^{\vee} \;=\; \{\,\ell\in\mathbb{L}\mid\omega\cdot\ell\geq 0 \quad \text{for all } \omega\in\mathcal{C}_{\mathcal{T}}\}\,. \tag{41}$$

The affine schemes $\mathcal{U}_{\mathcal{T}} :=$ spec $\mathbb{Z}[\mathbb{L}_{\mathcal{T}}]$ for the various triangulations naturally glue together to form the toric variety for the pointed secondary fan.

A complex point of $\mathcal{U}_{\mathcal{T}}$ is just a homomorphism from the additive monoid $\mathbb{L}_{\mathcal{T}}$ to the multiplicative monoid $\mathbb{C}$. There is a special point in $\mathcal{U}_{\mathcal{T}}$, namely the homomorphism sending $0\in\mathbb{L}_{\mathcal{T}}$ to 1 and all other elements of $\mathbb{L}_{\mathcal{T}}$ to 0. A disc of radius r, $0<r<1$, about this special point consists of homomorphisms $\mathbb{L}_{\mathcal{T}}\to\mathbb{C}$ with image contained in the disc of radius r in $\mathbb{C}$.

A vector $\mathsf{z}\in\mathbb{C}^{N\vee}$ defines the homomorphism

$$\mathbb{L}\to\mathbb{C}^{*}\,,\quad \ell\mapsto\mathsf{e}^{2\pi i\mathsf{z}\cdot\ell}$$

and hence a point of the toric variety. The point lies in the disc of radius $r<1$ about the special point corresponding to a regular triangulation $\mathcal{T}$ iff $\Im\mathsf{z}\cdot\ell > -\frac{\log r}{2\pi}$ holds for every $\ell\in\mathbb{L}_{\mathcal{T}}$. It suffices of course to require this only for a set of generators of $\mathbb{L}_{\mathcal{T}}$.

If b is in $\mathcal{C}_{\mathcal{T}}$ and K is such that $K > b \cdot \ell$ for all ℓ from a set of generators of $\mathbb{L}_{\mathcal{T}}$, then the set $\{\, \mathsf{z} \in \mathbb{C}^{N\vee} \mid p\,(\Im \mathsf{z}) \in b + \mathcal{C}_{\mathcal{T}} \}$ contains the intersection of the disc of radius $\exp(-2\pi K)$ with the torus $\mathrm{Hom}(\mathbb{L},\, \mathbb{C}^*)$.

This shows that the domain of definition of the function $\Psi_{\mathcal{T},\beta}$ is situated about the special point associated with $\mathcal{T}$ in the toric variety of the pointed secondary fan.

4 The special case $\beta = 0$

The function $\Psi_{\mathcal{T},0}$ is invariant under the action of $\mathbb{M}^\vee_{\mathbb{C}}$; see (22). So it is in fact a function on the domain $\mathbb{L}^\vee_{\mathbb{R}} + \sqrt{-1}\mathcal{B}_{\mathcal{T}}$ in $\mathbb{L}^\vee_{\mathbb{C}}$. For $F \in \mathrm{Hom}(\mathcal{R}_{\mathsf{A},\mathcal{T}},\, \mathbb{C})$ we have the $\mathbb{C}$-valued function $F\Psi_{\mathcal{T},0}$ on $\mathbb{L}^\vee_{\mathbb{R}} + \sqrt{-1}\mathcal{B}_{\mathcal{T}}$.

Lemma 5 *If $F\Psi_{\mathcal{T},0}$ is the 0-function on $\mathbb{L}^\vee_{\mathbb{R}} + \sqrt{-1}\mathcal{B}_{\mathcal{T}}$ then $F = 0$.*

proof: By lemma 3 the series $\Psi_{\mathcal{T},0}$ involves only λ's in $\mathbb{L} \cap \mathcal{C}_{\mathcal{T}}{}^\vee$ and $\lambda = 0$ is really present with $Q_0(\mathsf{c}) = 1$. Moreover $\mathcal{B}_{\mathcal{T}}$ is contained in the interior of $\mathcal{C}_{\mathcal{T}}$. Therefore, if $F\Psi_{\mathcal{T},0} = 0$, then the polynomial function

$$\sum_{m_1,\ldots,m_N \geq 0} \frac{(2\pi\mathsf{i}\,)^{m_1+\ldots+m_N}}{m_1! \cdot \ldots \cdot m_N!} \cdot F(c_1^{m_1} \cdot \ldots \cdot c_N^{m_N}) \cdot z_1^{m_1} \cdot \ldots \cdot z_N^{m_N}$$

is bounded on an unbounded open domain in $\mathbb{C}^N$. So, this is the zero polynomial. Therefore $F(c_1^{m_1} \cdot \ldots \cdot c_N^{m_N}) = 0$ for all $m_1, \ldots, m_N \geq 0$. ⊠

Theorem 5 *If $\beta = 0$ and $\mathcal{T}$ is unimodular, then there is an isomorphism:*

$$\mathrm{Hom}(\mathcal{R}_{\mathsf{A},\mathcal{T}},\, \mathbb{C}) \xrightarrow{\sim} \textit{\textbf{solution space of (3)-(4)}}\;, \qquad F \mapsto F\Phi_{\mathcal{T},0}\;.$$

proof: Lemma 5 shows that the map is injective. From corollary 1 we know $\dim \mathrm{Hom}(\mathcal{R}_{\mathsf{A},\mathcal{T}},\, \mathbb{C}) = \mathrm{vol}\,\Delta$. Since the triangulation $\mathcal{T}$ is unimodular, the proof of [24] prop.13.15 shows that the normality condition for the correction in [12] to [11] thm. 5 is satisfied. Therefore the number of linearly independent solutions of the GKZ system (3)-(4) at a generic point equals $\mathrm{vol}\,\Delta$. ⊠

5 Triangulations with non-empty core

The intersection of all maximal simplices in a regular triangulation $\mathcal{T}$ of Δ is a remarkable structure. We call it the core of $\mathcal{T}$. It is a simplex in the triangulation $\mathcal{T}$. Since we identify simplices with their index sets, we view $\mathrm{core}\,\mathcal{T}$ also as a subset of $\{1, \ldots, N\}$.

Definition 3 $\qquad \mathrm{core}\,\mathcal{T} := \bigcap_{I \in \mathcal{T}^n} I$

Lemma 6 *A simplex which does not contain* $\operatorname{core}\mathcal{T}$ *lies in the boundary of* Δ *.*

proof: It suffices to prove this for simplices of the form $I\setminus\{j\}$ with $I\in\mathcal{T}^n$ and $j\in\operatorname{core}\mathcal{T}$. Since every maximal simplex contains j, I is the only maximal simplex which contains $I\setminus\{j\}$. Therefore $I\setminus\{j\}$ lies in the boundary of Δ . ⊠

Lemma 7 $\quad \operatorname{core}\mathcal{T}\;=\;\{j\,|\,\ell_j\leq 0\ \ \textit{for all}\ \ \ell\in\mathcal{C}^{\vee}_{\mathcal{T}}\,\}$

proof: $\supset$: assume $j\notin\operatorname{core}\mathcal{T}$, say $j\notin I$ for some $I\in\mathcal{T}^n$. Then there is a relation $\mathfrak{a}_j-\sum_{i\in I}x_i\mathfrak{a}_i\;=\;0$; whence an $\ell\in\mathbb{L}$ with $\ell_j>0$ and $\{i\,|\,\ell_i<0\}\subset I$. As in the proof of lemma 3 this implies $\ell\in\mathcal{C}^{\vee}_{\mathcal{T}}$.

$\subset$: assume $j\in\operatorname{core}\mathcal{T}$. First consider an $\ell\in\mathbb{L}_{\mathbb{R}}$ such that $\{i\,|\,\ell_i<0\}$ is a simplex. Let $L=\sum_{\ell_i>0}\ell_i=\sum_{\ell_i<0}-\ell_i$. The relation in (5) can be rewritten as

$$\sum_{\ell_i>0}\frac{\ell_i}{L}\mathfrak{a}_i\;=\;\sum_{\ell_i<0}\frac{-\ell_i}{L}\mathfrak{a}_i \tag{42}$$

Suppose $\ell_j>0$. Then the simplex $\{i\,|\,\ell_i<0\}$ lies in a boundary face of Δ . Take a linear functional F whose restriction to Δ attains its maximum exactly on this face. Evaluate F on both sides of (42). The value on the right hand side is $\max F$, but on the left hand side it is $<\max F$, because $F(\mathfrak{a}_j)<\max F$ and $\ell_j>0$. Contradiction! Therefore we conclude: $\ell_j\leq 0$ if ℓ is such that $\{i\,|\,\ell_i<0\}$ is a simplex. From the constructions in section 1.2 one sees that $\{i\,|\,\ell_i<0\}$ is a simplex if and only if $\ell\in\mathcal{C}^{\vee}_{I^*}$ for some $I\in\mathcal{T}^n$; note: $\ell_j=\mathfrak{b}_j\cdot\ell$. Since $\mathcal{C}^{\vee}_{\mathcal{T}}$ is the Minkowski sum of the cones $\mathcal{C}^{\vee}_{I^*}$ with $I\in\mathcal{T}^n$ we finally get: $\ell_j\leq 0$ for every $\ell\in\mathcal{C}^{\vee}_{\mathcal{T}}$. ⊠

Definition 4

$$c_{\mathrm{core}}\;:=\;\prod_{i\in\operatorname{core}\mathcal{T}}c_i \tag{43}$$

Corollary 2 *If* λ *is such that* $\mathcal{A}\lambda\;=\;\sum_{i\in\operatorname{core}\mathcal{T}}m_i\mathfrak{a}_i$ *with all* $m_i<0$ *then* $\lambda_i<0$ *for every* $i\in\operatorname{core}\mathcal{T}$ *and hence*

$$Q_\lambda(\mathsf{c})\;\in\;c_{\mathrm{core}}\,\mathcal{R}_{\mathsf{A},\mathcal{T}}$$

proof: Let $\mu=(\mu_1,\ldots,\mu_N)$ be defined by $\mu_i=m_i$ for $i\in\operatorname{core}\mathcal{T}$ and $\mu_i=0$ for $i\notin\operatorname{core}\mathcal{T}$. Then $\mathfrak{P}_{\mathcal{T}}(\lambda)=\mathfrak{P}_{\mathcal{T}}(\mu)$ in lemma 3. From the definitions one sees immediately that $\mathfrak{P}_{\mathcal{T}}(\mu)=\{\mu\}$. The result now follows from lemmas 3 and 7. ⊠

Corollary 3 *If* $\operatorname{core}\mathcal{T}$ *is not empty and* $\beta\;=\;\sum_{i\in\operatorname{core}\mathcal{T}}m_i\mathfrak{a}_i$ *with all* $m_i<0$ *then the function* $\Psi_{\mathcal{T},\beta}$ *takes values in the ideal* $c_{\mathrm{core}}\,\mathcal{R}_{\mathsf{A},\mathcal{T}}\otimes\mathbb{C}$ *.* ⊠

Theorem 6 *If* $\operatorname{core}\mathcal{T}$ *is not empty and* $\beta = \sum_{i\in\operatorname{core}\mathcal{T}} m_i\mathfrak{a}_i$ *with all* $m_i < 0$ *then the linear map*

$$\mathrm{Hom}(c_{\mathrm{core}}\mathcal{R}_{\mathsf{A},\mathcal{T}},\,\mathbb{C}) \longrightarrow \textit{\textbf{solution space of (3)-(4)}}\,, \qquad F \mapsto F\Phi_{\mathcal{T},\beta}$$

is injective.

proof: From lemma 3 and the proof of corollary 2 one sees that the series $\Psi_{\mathcal{T},\beta}$ involves only λ's in $\mu + (\mathbb{L}\cap\mathcal{C}_{\mathcal{T}}{}^{\vee})$ and that $\lambda = \mu$ is really present:

$$Q_\mu(\mathsf{c}) = c_{\mathrm{core}}\cdot U \quad \text{with} \quad U := \prod_{i\in\operatorname{core}\mathcal{T}}\prod_{k=1}^{-m_i-1}(c_i - k)\,.$$

The rest of the proof is analogous to the proof of lemma 5. In particular, if $F\Psi_{\mathcal{T},\beta}$ is the 0-function on $\mathcal{V}_{\mathcal{T}}$, then $F(c_{\mathrm{core}}\cdot U\cdot c_1^{n_1}\cdot\ldots\cdot c_N^{n_N}) = 0$ for all $n_1,\ldots,n_N \geq 0$. The desired result now follows because U is invertible in the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}\otimes\mathbb{Q}$. $\boxtimes$

PART II

Introduction II

One aspect of the mirror symmetry phenomenon (cf. [25, 15]) is that (generalized) Calabi-Yau manifolds seem to come in pairs (X, Y) with the geometries of X and Y related in a beautifully intricate way. On one side of the mirror - usually called *the B-side* - it is the geometry of complex structure, of periods of a holomorphic differential form, of variations of Hodge structure. On the other side - *the A-side* - it is the geometry of symplectic structure, of algebraic cycles and of enumerative questions about curves on the manifold.

Batyrev [1] showed that behind many examples of the mirror symmetry phenomenon one can see a simple combinatorial duality. Batyrev and Borisov gave a generalization of this combinatorial duality and formulated a *mirror symmetry conjecture for generalized Calabi-Yau manifolds* in arbitrary dimension ([3] 2.17). The fundamental combinatorial structure is a *reflexive Gorenstein cone.*

Definition 5 ([3] definitions 2.1-2.8.) *A cone* Λ *in* $\mathbb{R}^n$ *is called a* Gorenstein cone *if it is generated, i.e.*

$$\Lambda = \mathbb{R}_{\geq 0}\mathfrak{a}_1 + \ldots + \mathbb{R}_{\geq 0}\mathfrak{a}_N\,, \tag{44}$$

by a finite set $\{\mathfrak{a}_1,\ldots,\mathfrak{a}_N\}\subset\mathbb{Z}^n$ *which satisfies condition 1. It is called a* reflexive Gorenstein cone *if both* Λ *and its dual* $\Lambda^{\vee}$ *are Gorenstein cones,*

$$\Lambda^{\vee} := \{\,\mathsf{y}\in\mathbb{R}^{n\vee}\,|\,\forall\mathsf{x}\in\Lambda:\,\mathsf{y}\cdot\mathsf{x}\geq 0\,\}\,, \tag{45}$$

i.e. there should also exist a vector $\mathfrak{a}_0 \in \mathbb{Z}^n$ *and a set* $\{\mathfrak{a}_1^\vee, \ldots, \mathfrak{a}_{N'}^\vee\} \subset \mathbb{Z}^{n\vee}$ *of generators for* $\Lambda^\vee$ *such that* $\mathfrak{a}_i^\vee \cdot \mathfrak{a}_0 = 1$ *for* $i = 1, \ldots, N'$*. The vectors* $\mathfrak{a}_0^\vee$ *and* $\mathfrak{a}_0$ *are uniquely determined by* Λ*. The integer* $\mathfrak{a}_0^\vee \cdot \mathfrak{a}_0$ *is called* the index *of* Λ.

For a reflexive Gorenstein cone one has one new datum in addition to the data for GKZ systems; namely $\mathfrak{a}_0$. It has the very important property

$$\operatorname{interior}(\Lambda) \cap \mathbb{Z}^n \;=\; \mathfrak{a}_0 + \Lambda\,. \tag{46}$$

Our aim is to show that in the case of a mirror pair (X, Y) *associated with a reflexive Gorenstein cone* Λ *and a unimodular regular triangulation* $\mathcal{T}$ *whose core is not empty and is not contained in the boundary of* Δ*, the periods of a holomorphic differential form on* X *are given by the function* $\Phi_{\mathcal{T},-\mathfrak{a}_0}$ *which takes values in the ring* $\mathcal{R}_{\mathsf{A},\mathcal{T}}/\operatorname{Ann} c_{\mathrm{core}} \otimes \mathbb{C}$ *and that the ring* $\mathcal{R}_{\mathsf{A},\mathcal{T}}/\operatorname{Ann} c_{\mathrm{core}}$ *is isomorphic with a subring of the Chow ring of* Y.

This project naturally has a B-side and an A-side which we develop separately in Part II B and Part II A. Our method puts some natural restrictions on the generality. For Part II B we must eventually assume that there is a unimodular triangulation $\mathcal{T}$ of the polytope

$$\Delta \;:=\; \operatorname{conv}\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\} = \{\mathsf{x} \in \Lambda \mid \mathfrak{a}_0^\vee \cdot \mathsf{x} \,=\, 1\,\}\,. \tag{47}$$

This restriction which comes from the use of theorem 5, also implies

$$\mathfrak{a}_0 \in \mathbb{Z}_{\geq 0}\mathfrak{a}_1 + \ldots + \mathbb{Z}_{\geq 0}\mathfrak{a}_N\,. \tag{48}$$

So, $\Phi_{\mathcal{T},-\mathfrak{a}_0}$ is defined in Part I. For Part II A we must additionally assume that the core of $\mathcal{T}$ is not empty and is not contained in the boundary of Δ.

PART II B

Introduction II B

For a Gorenstein cone Λ we denote the monoid algebra $\mathbb{C}[\Lambda \cap \mathbb{Z}^n]$ by $\mathcal{S}_\Lambda$ and view it as a subalgebra of the algebra $\mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$ by identifying $\mathsf{m} = (m_1, \ldots, m_n)^t \in \Lambda \cap \mathbb{Z}^n$ with the Laurent monomial $\mathsf{u}^{\mathsf{m}} := u_1^{m_1} \cdot \ldots \cdot u_n^{m_n}$. For $\mathsf{m} \in \mathbb{Z}^n$ we put $\deg \mathsf{u}^{\mathsf{m}} := \deg \mathsf{m} := \mathfrak{a}_0^\vee \cdot \mathsf{m}$. Thus $\mathcal{S}_\Lambda$ becomes a graded ring. The scheme $\mathbb{P}_\Lambda := \operatorname{Proj} \mathcal{S}_\Lambda$ is a projective toric variety. If Λ is a reflexive Gorenstein cone, the zero set in $\mathbb{P}_\Lambda$ of a global section of $\mathcal{O}_{\mathbb{P}_\Lambda}(1)$ is called a *generalized Calabi-Yau manifold* of dimension $n-2$ ([3] 2.15).

The toric variety $\mathbb{P}_\Lambda$ is a compactification of the $n-1$-dimensional torus

$$\mathbb{T} \;:=\; \widetilde{\mathbb{T}}/(\mathbb{Z}\mathfrak{a}_0^\vee \otimes \mathbb{C}^*) \tag{49}$$

where

$$\widetilde{\mathbb{T}} := \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*) = \mathbb{Z}^{n\vee} \otimes \mathbb{C}^* \tag{50}$$

is the n-dimensional torus of $\mathbb{C}$-points of Spec $\mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$. A global section of $\mathcal{O}_{\mathbb{P}_\Lambda}(1)$ is given by a Laurent polynomial

$$\mathsf{s} = \sum_{\mathsf{m} \in \Delta \cap \mathbb{Z}^n} v_{\mathsf{m}} \mathsf{u}^{\mathsf{m}} \,. \tag{51}$$

with Δ as in (47). As in [2] we assume from now on

Condition 2 $\qquad \{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\} = \Delta \cap \mathbb{Z}^n$

The Laurent polynomial s gives a function on $\widetilde{\mathbb{T}}$ which is homogeneous of degree 1 for the action of $\mathbb{Z}\mathfrak{a}_0^\vee \otimes \mathbb{C}^*$. Let

$$\mathsf{Z}_\mathsf{s} := \{ \text{ zero locus of } \mathsf{s} \} \subset \mathbb{T} \tag{52}$$

Over the complementary set $\mathbb{T} \setminus \mathsf{Z}_\mathsf{s}$ there is a section of $\widetilde{\mathbb{T}} \to \mathbb{T}$ which identifies $\mathbb{T} \setminus \mathsf{Z}_\mathsf{s}$ with the zero set $\widetilde{\mathsf{Z}}_{\mathsf{s}-1}$ of $\mathsf{s} - 1$ in $\widetilde{\mathbb{T}}$:

$$\mathbb{T} \setminus \mathsf{Z}_\mathsf{s} \simeq \widetilde{\mathsf{Z}}_{\mathsf{s}-1} \subset \widetilde{\mathbb{T}} \tag{53}$$

One may say that according to Batyrev [2] *the geometry on the B-side of mirror symmetry is encoded in the weight n part $\mathcal{W}_n H^{n-1}(\mathbb{T} \setminus \mathsf{Z}_\mathsf{s})$ of the Variation of Mixed Hodge Structure of* $H^{n-1}(\mathbb{T} \setminus \mathsf{Z}_\mathsf{s})$; the variation comes from varying the coefficients v_m in (51).

Remark 5 One usually formulates Mirror Symmetry with on the B-side the Variation of Hodge Structure on the d-th cohomology of a d-dimensional Calabi-Yau manifold. For a CY hypersurface in a toric variety the Poincaré residue mapping gives an isomorphism with the $d+1$-st cohomology of the hypersurface complement, at least on the primitive parts (see [2] prop.5.3). For a CY complete intersection of codimension > 1 in a toric variety one needs besides the Poincaré residue mapping also corollary 3.4 and remark 3.5 in [3] to relate the CYCI's cohomology to the cohomology of the complement of a generalized Calabi-Yau hypersurface in a toric variety, i.e. to the situation we are studying in this paper. Our investigations do however also allow on this B-side of the mirror generalized Calabi-Yau hypersurfaces which are not related to CY complete intersections, although on the other A-side we do eventually want a Calabi-Yau complete intersection (see [3] §5 for an example of mirror symmetry with such an asymmetry between the two sides).

In [2] Batyrev described the weight and Hodge filtrations of this Variation of Mixed Hodge Structure (VMHS) in terms of the combinatorics of Λ . In particular, $\mathcal{W}_n H^{n-1}(\mathbb{T}\setminus \mathsf{Z_s})$ corresponds with the ideal $\mathbb{C}[\mathrm{interior}(\Lambda)\cap\mathbb{Z}^n]$ in $\mathcal{S}_\Lambda$. If Λ is a reflexive Gorenstein cone of index κ, this ideal is the principal ideal generated by $\mathsf{u}^{\mathfrak{a}_0}$ (cf. (46)) and the part of weight n and Hodge type $(n-\kappa,\kappa)$ has dimension 1.

Batyrev [2] also showed that the periods of the rational $(n-1)$-form

$$\omega_\mu \;:=\; \frac{\mathsf{u}^\mu}{\mathsf{s}^{\deg\mu}}\,\frac{dt_2}{t_2}\wedge\ldots\wedge\frac{dt_n}{t_n} \tag{54}$$

($\mu\in\Lambda\cap\mathbb{Z}^n$, $t_2,\ldots,t_n$ coordinates on $\mathbb{T}$) as functions of the coefficients v_{m} satisfy a GKZ system of differential equations (3)-(4) with parameters $\{\mathfrak{a}_1,\ldots,\mathfrak{a}_N\}$ and $\beta=-\mu$. However, *not all solutions of this system are $\mathbb{C}$-linear combinations of the periods of ω_μ. Theorem 10 shows precisely which solutions of this system are $\mathbb{C}$-linear combinations of the periods of ω_μ in case $\mu\in\mathbb{Z}_{\geq 0}\mathfrak{a}_1+\ldots+\mathbb{Z}_{\geq 0}\mathfrak{a}_N$.*

The key point of our method is to study the VMHS on $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$. This has the advantage that *if $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ generate $\mathbb{Z}^n$, then $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ is a hypergeometric $\mathcal{D}$-module* as in [11] with parameters $\{\mathfrak{a}_1,\ldots,\mathfrak{a}_N\}$ and $\beta=0$; see theorem 8.

If s is Λ-regular (cf. definition 6) there is an exact sequence of mixed Hodge structures

$$0\to H^{n-1}(\widetilde{\mathbb{T}})\to H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1})\to H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})\to H^n(\widetilde{\mathbb{T}})\to 0 \tag{55}$$

The left hand 0 results from a theorem of Bernstein-Danilov-Khovanskii [8, 2]. On the right we used $H^n(\widetilde{\mathsf{Z}}_{\mathsf{s}-1})=0$ because $\widetilde{\mathsf{Z}}_{\mathsf{s}-1}$ is an affine variety of dimension $n-1$. Writing as usual $\mathbb{Q}(m)$ for the 1-dimensional $\mathbb{Q}$-Hodge structure which is purely of weight $-2m$ and Hodge type $(-m,-m)$ one has

$$H^{n-1}(\widetilde{\mathbb{T}})\simeq\mathbb{Q}^n\otimes\mathbb{Q}(1-n)\,,\qquad H^n(\widetilde{\mathbb{T}})\simeq\mathbb{Q}(-n)\,. \tag{56}$$

Morphisms of mixed Hodge structures are strictly compatible with the weight filtrations ([6] thm. 2.3.5). Thus the sequence (55) in combination with (53) gives the isomorphisms

$$\mathcal{W}_i H^{n-1}(\mathbb{T}\setminus\mathsf{Z_s})\xrightarrow{\simeq}\mathcal{W}_i H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1})\xrightarrow{\simeq}\mathcal{W}_i H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \tag{57}$$

for $i\leq 2n-3$. In particular if $n\geq 3$, the weight n part relevant for the geometry on the B-side of mirror symmetry will get a complete and simple description by our analysis of the GKZ hypergeometric $\mathcal{D}$-module $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$.

Remark 6 Though it plays no role in this paper I want to point out that there is an interesting relation with recent work of Deninger [7]. The group G of diagonal $n\times n$-matrices with entries ±1 acts naturally on $\widetilde{\mathbb{T}}=\mathrm{Hom}(\mathbb{Z}^n,\mathbb{C}^*)$. From the inclusion $\imath:\widetilde{\mathsf{Z}}_{\mathsf{s}-1}\hookrightarrow\widetilde{\mathbb{T}}$ one gets the G-equivariant map $G\times\widetilde{\mathsf{Z}}_{\mathsf{s}-1}\to\widetilde{\mathbb{T}}$,

$(g,z) \mapsto g \cdot \imath(z)$. Corresponding to this map there is an exact sequence of mixed Hodge structures with G-action analogous to (55). Taking isotypical parts for the character $\det : G \to \{\pm 1\}$ and using $H^{n-1}(\widetilde{\mathbb{T}})(\det) = 0$, $H^n(\widetilde{\mathbb{T}})(\det) \stackrel{\simeq}{\to} H^n(\widetilde{\mathbb{T}})$ and $H^{n-1}(G \times \widetilde{\mathsf{Z}}_{\mathsf{s}-1})(\det) \stackrel{\simeq}{\to} H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ one finds the short exact sequence

$$0 \to H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \to H^n(\widetilde{\mathbb{T}} \,\mathrm{rel}\, (G \times \widetilde{\mathsf{Z}}_{\mathsf{s}-1}))(\det) \to H^n(\widetilde{\mathbb{T}}) \to 0 \tag{58}$$

see [7] (12). In [7] remark 2.4 Deninger sketches how the extension (58) comes from a Steinberg symbol in the group $K_n(\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ in the algebraic K-theory of $\widetilde{\mathsf{Z}}_{\mathsf{s}-1}$; in our coordinates (see remark 8) this Steinberg symbol reads

$$\{u_1, u_2, \ldots, u_n\} \in K_n\left(\mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}] / (\mathsf{s} - 1)\right) \tag{59}$$

The exact sequence (55) decomposes into two short exact sequences

$$0 \to H^{n-1}(\widetilde{\mathbb{T}}) \to H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \to PH^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \to 0 \tag{60}$$
$$0 \to PH^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \to H^n(\widetilde{\mathbb{T}} \,\mathrm{rel}\, \widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \to H^n(\widetilde{\mathbb{T}}) \to 0 \tag{61}$$

which define the *primitive part* of cohomology ([2] def. 3.13). The relation between the various cohomology groups is best displayed in the following commutative diagram with injective horizontal and surjective vertical arrows:

$$\begin{array}{ccccc} H^{n-1}(\widetilde{\mathbb{T}}) & \to & H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) & \to & H^n(\widetilde{\mathbb{T}} \,\mathrm{rel}\, (G \times \widetilde{\mathsf{Z}}_{\mathsf{s}-1}))(\det) \\ & & \downarrow & & \downarrow \\ & & PH^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) & \to & H^n(\widetilde{\mathbb{T}} \,\mathrm{rel}\, \widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \\ & & & & \downarrow \\ & & & & H^n(\widetilde{\mathbb{T}}) \end{array} \tag{62}$$

With varying coefficients v_{m} the story plays in the category of Variations of Mixed Hodge Structures. With coefficients v_{m} fixed in some number field the story plays in a category of Mixed Motives. A challenge for further research is to combine these stories and our results on hypergeometric systems.

6 VMHS associated with a Gorenstein cone

In this section we prove theorem 8. This result is essentially implicitly contained in [2]. Our proof is mainly a review of constructions and results in [2].

Shifting emphasis from the polytope Δ to the cone Λ we write $\mathcal{S}_\Lambda$ (instead of S_Δ as in [2]) for the monoid algebra $\mathbb{C}[\Lambda \cap \mathbb{Z}^n]$ viewed as a subalgebra of $\mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$. The grading is given by $\deg \mathsf{u}^{\mathsf{m}} = \mathfrak{a}_0^\vee \cdot \mathsf{m}$ for $\mathsf{m} \in \mathbb{Z}^n$. A homogeneous element s of degree 1 in $\mathcal{S}_\Lambda$ is a Laurent polynomial as in (51):

$$\mathsf{s} = \sum_{i=1}^{N} v_i \, \mathsf{u}^{\mathfrak{a}_i} \tag{63}$$

with coefficients $v_i \in \mathbb{C}$. Let $\widetilde{\mathbb{T}}$, $\mathbb{T}$, Z_{s} and $\widetilde{\mathsf{Z}}_{\mathsf{s}-1}$ be as in (49)-(53).

Remark 7 When comparing with [2] one should keep in mind that in op.cit. n is the dimension of the polytope Δ whereas here n is the dimension of the cone Λ and the polytope Δ has dimension $n-1$. Also one has to make the following change of coordinates on $\mathbb{Z}^n$ and $\mathbb{Z}^{n\vee}$. The idempotent $n \times n$-matrix $\mathfrak{a}_1 \cdot \mathfrak{a}_0^\vee$ gives rise to a direct sum decomposition $\mathbb{Z}^{n\vee} = \mathbb{Z}\mathfrak{a}_0^\vee \oplus \Xi$ and thus to a basis $\{\mathfrak{a}_0^\vee, \alpha_2, \ldots, \alpha_n\}$ for $\mathbb{Z}^{n\vee}$. The coordinate change on $\mathbb{Z}^n$ amounts to multiplying vectors in $\mathbb{Z}^n$ by the matrix $M = (m_{ij})$ with rows $\mathfrak{a}_0^\vee, \alpha_2, \ldots, \alpha_n$. In particular, in the new coordinates $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$ all have first coordinate 1.

The above coordinate change also induces a change of coordinates on $\widetilde{\mathbb{T}}$: $u_j = \prod_{i=1}^n t_i^{m_{ij}}$. The map $\widetilde{\mathbb{T}} \to \mathbb{T}$ is then just omitting the coordinate t_1. In t-coordinates s takes the form $t_1 \cdot f$ where f is a Laurent polynomial in the variables $t_2, \ldots, t_n$. Thus s corresponds with F_0 and $\mathsf{s}-1$ with F in [2] def. 4.1.

Remark 8 When comparing with [7] one sees again a shift of dimensions from n in op. cit. to $n-1$ here; T^n with coordinates $t_1, \ldots, t_n$ in op. cit. is our $\mathbb{T}$ with coordinates $t_2, \ldots, t_n$. The polynomial P of op. cit. and our s are related by $\mathsf{s} = t_1 \cdot P$. The identification of $\mathbb{T} \setminus \mathsf{Z}_{\mathsf{s}}$ with $\widetilde{\mathsf{Z}}_{\mathsf{s}-1}$ now gives for the Steinberg symbols $\{P, t_2, \ldots, t_n\} = -\{t_1, t_2, \ldots, t_n\} = \{u_1, u_2, \ldots, u_n\}$ if the coordinates are ordered such that $\det M = -1$.

Before we can state Batyrev's results we need some definitions/notations. [2] def. 2.8 defines an ascending sequence of homogeneous ideals in $\mathcal{S}_\Lambda$:

$$I_\Delta^{(0)} \subset I_\Delta^{(1)} \subset \ldots \subset I_\Delta^{(n)} \subset I_\Delta^{(n+1)} \tag{64}$$

where $I_\Delta^{(k)}$ is generated by the elements u^{m} with m in $\Lambda \cap \mathbb{Z}^n$ but not in any codimension k face of Λ; in particular

$$I_\Delta^{(0)} = 0\,, \quad I_\Delta^{(1)} = \mathbb{C}[\text{interior}(\Lambda) \cap \mathbb{Z}^n]\,, \quad I_\Delta^{(n)} = \mathcal{S}_\Lambda^+\,, \quad I_\Delta^{(n+1)} = \mathcal{S}_\Lambda \tag{65}$$

$\mathcal{S}_\Lambda^+$ is the ideal in $\mathcal{S}_\Lambda$ generated by the monomials of degree > 0.
[2] p.379 defines a descending sequence of $\mathbb{C}$-vector spaces in $\mathcal{S}_\Lambda$:

$$\ldots \supset \mathcal{E}^{-k} \supset \mathcal{E}^{-k+1} \supset \ldots \supset \mathcal{E}^{-1} \supset \mathcal{E}^0 \supset \mathcal{E}^1 = 0 \tag{66}$$

where $\mathcal{E}^{-k}$ is spanned by the monomials u^{m} with $\deg \mathsf{u}^{\mathsf{m}} \leq k$.
[2] def. 7.2 defines the differential operators

$$D_i \;:=\; u_i \frac{\partial}{\partial u_i} \,+\, u_i \frac{\partial \mathsf{s}}{\partial u_i}\,, \qquad (i = 1, \ldots, n) \tag{67}$$

These operate on $\mathbb{C}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$, preserving $\mathcal{S}_\Lambda$ and $\mathcal{S}_\Lambda^+$.
[2] thm. 4.8 can be used as a definition:

Definition 6 s *is said to be* Λ-regular *if* $u_1\dfrac{\partial \mathsf{s}}{\partial u_1},\ u_2\dfrac{\partial \mathsf{s}}{\partial u_2}\ldots,\ u_n\dfrac{\partial \mathsf{s}}{\partial u_n}$ *is a* regular sequence *in* $\mathcal{S}_\Lambda$.

Theorem 7 (summary of results in [2])
If s *is* Λ*-regular, then there is a commutative diagram*

$$\begin{array}{ccccc} \mathcal{S}_\Lambda^+/\sum_{i=1}^n D_i\mathcal{S}_\Lambda^+ & \stackrel{\simeq}{\to} & H^{n-1}(\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) & \stackrel{\simeq}{\to} & H^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s}) \\ \downarrow & & \downarrow & & \\ \mathcal{S}_\Lambda/\sum_{i=1}^n D_i\mathcal{S}_\Lambda & \stackrel{\simeq}{\to} & H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) & & \end{array} \tag{68}$$

in which the horizontal arrows are isomorphisms. These isomorphisms restrict to the following isomorphisms relating (65) and (66) with the weight and Hodge filtrations on $H^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s})$ *and* $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$.
For $k=-1,0,1,\ldots,n,n+1$:

$$\begin{array}{lcl} \textit{image } I_\Delta^{(k)} \textit{ in } \mathcal{S}_\Lambda^+/\sum_{i=1}^n D_i\mathcal{S}_\Lambda^+ & \stackrel{\simeq}{\to} & \mathcal{W}_{k+n-1}H^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s}) \\ \textit{image } \mathcal{E}^{-k}\cap\mathcal{S}_\Lambda^+ \textit{ in } \mathcal{S}_\Lambda^+/\sum_{i=1}^n D_i\mathcal{S}_\Lambda^+ & \stackrel{\simeq}{\to} & \mathcal{F}^{n-k}H^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s}) \\ \textit{image } I_\Delta^{(k)} \textit{ in } \mathcal{S}_\Lambda/\sum_{i=1}^n D_i\mathcal{S}_\Lambda & \stackrel{\simeq}{\to} & \mathcal{W}_{k+n-1}H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \\ \textit{image } \mathcal{E}^{-k} \textit{ in } \mathcal{S}_\Lambda/\sum_{i=1}^n D_i\mathcal{S}_\Lambda & \stackrel{\simeq}{\to} & \mathcal{F}^{n-k}H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \end{array}$$

proof: The statements for $H^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s})$ are theorems 7.13, 8.1 and 8.2 in [2]. The statements about $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ can also be derived with the methods of op. cit., as follows. Recall that $H^*(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ is the cohomology of the cone of the natural map of DeRham complexes $\Omega^\bullet_{\widetilde{\mathbb{T}}}\to\Omega^\bullet_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}}$ and that this cone complex is in degrees i and $i+1$

$$\begin{array}{ccccc} \ldots\to & \Omega^i_{\widetilde{\mathbb{T}}}\oplus\Omega^{i-1}_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}} & \longrightarrow & \Omega^{i+1}_{\widetilde{\mathbb{T}}}\oplus\Omega^{i}_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}} & \to\ldots \\ & (\omega_1,\omega_2) & \mapsto & (-d\omega_1, d\omega_2+\omega_1|_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}}) & \end{array} \tag{69}$$

A basis for the $\mathbb{C}[u_1^{\pm1},\ldots,u_n^{\pm1}]$-module $\Omega^\bullet_{\widetilde{\mathbb{T}}}$ is given by the forms $\frac{du_{i_1}}{u_{i_1}}\wedge\ldots\wedge\frac{du_{i_r}}{u_{i_r}}$. Let $\Omega^\bullet_{\widetilde{\mathbb{T}},0}$ denote the subgroup of $\Omega^\bullet_{\widetilde{\mathbb{T}}}$ consisting of the linear combinations of the basic forms with coefficients in $\mathbb{C}$. The standard differential d on $\Omega^\bullet_{\widetilde{\mathbb{T}}}$ is 0 on $\Omega^\bullet_{\widetilde{\mathbb{T}},0}$. The inclusion of complexes $\Omega^\bullet_{\widetilde{\mathbb{T}},0}\hookrightarrow\Omega^\bullet_{\widetilde{\mathbb{T}}}$ is a quasi-isomorphism. So in (69) we may replace $\Omega^\bullet_{\widetilde{\mathbb{T}}}$ by $\Omega^\bullet_{\widetilde{\mathbb{T}},0}$.

For the proof of [2] thm.7.13 Batyrev uses the $\mathbb{C}$-linear map $\mathcal{R}:\ \mathcal{S}_\Lambda^+\to\Omega^{n-1}_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}}$, $\mathcal{R}(\mathsf{u}^\mathsf{m}) := (-1)^{\deg\mathsf{m}-1}(\deg\mathsf{m}-1)!\,\mathsf{u}^\mathsf{m}\frac{dt_2}{t_2}\wedge\ldots\wedge\frac{dt_n}{t_n}$ (cf. remark 7 for the t-coordinates). Let us extend this to a $\mathbb{C}$-linear map $\mathcal{R}:\ \mathcal{S}_\Lambda\to\Omega^n_{\widetilde{\mathbb{T}},0}\oplus\Omega^{n-1}_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}}$ by setting $\mathcal{R}(1)=\left(\frac{dt_1}{t_1}\wedge\ldots\wedge\frac{dt_n}{t_n}\,,\,0\right)$. This induces a surjective linear map

$\mathcal{S}_\Lambda \longrightarrow H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ with $\sum_{i=1}^n D_i\mathcal{S}_\Lambda^+$ in its kernel. Note $D_i(1) = u_i\dfrac{\partial\mathsf{s}}{\partial u_i}$. A direct calculation shows for $i = 1, \ldots, n$:

$$(-1)^{i-1}\mathcal{R}(t_i\frac{\partial\mathsf{s}}{\partial t_i}) \;=\; d\left(\frac{dt_1}{t_1}\wedge\ldots\wedge\widehat{\frac{dt_i}{t_i}}\wedge\ldots\wedge\frac{dt_n}{t_n}\,,\,0\right)$$

in $\Omega^n_{\widetilde{\mathbb{T}},0}\oplus\Omega^{n-1}_{\widetilde{\mathsf{Z}}_{\mathsf{s}-1}}$. Therefore $\mathcal{R}$ induces a surjective linear map

$$\mathcal{S}_\Lambda\,/\sum_{i=1}^n D_i\mathcal{S}_\Lambda \to H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})\,.$$

A simple dimension count now shows that this is in fact an isomorphism.

The statements about the Hodge filtration and the weight filtration on $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ follow from the corresponding statements for $H^{n-1}(\mathbb{T}\setminus\mathsf{Z}_\mathsf{s})$ and from (56). ⊠

The *principal* A-*determinant* of Gel'fand-Kapranov-Zelevinskii [13] is a polynomial $E_\mathsf{A}(v_1,\ldots,v_N)\in\mathbb{Z}[v_1,\ldots,v_N]$ such that (see [2] prop. 4.16):

$$\mathsf{s}\text{ is }\Lambda\text{-regular}\qquad\Longleftrightarrow\qquad E_\mathsf{A}(v_1,\ldots,v_N)\neq 0 \tag{70}$$

Now we want to vary the coefficients v_i in (63) and work over the ring

$$\mathbb{C}[\mathsf{v}] \;:=\; \mathbb{C}[v_1,\ldots,v_N,E_\mathsf{A}^{-1}]\,. \tag{71}$$

Let $\Omega^\bullet$ resp. $\widetilde{\Omega}^\bullet$ denote the DeRham complex of $\mathbb{C}[u_1^{\pm1},\ldots,u_n^{\pm1}]\otimes\mathbb{C}[\mathsf{v}]$ relative to $\mathbb{C}[\mathsf{v}]$ resp. relative to $\mathbb{C}$. Define on these complexes a new differential

$$\begin{aligned}&\delta:\Omega^i\to\Omega^{i+1}\text{ resp. }\widetilde{\Omega}^i\to\widetilde{\Omega}^{i+1}\\ &\delta\omega \;:=\; d\omega+d\mathsf{s}\wedge\omega\end{aligned} \tag{72}$$

where d is the ordinary differential on DeRham complexes.

As a basis for the $\mathbb{C}[u_1^{\pm1},\ldots,u_n^{\pm1}]\otimes\mathbb{C}[\mathsf{v}]$-module Ω^1 (resp. $\widetilde{\Omega}^1$) we take $\frac{du_1}{u_1},\ldots,\frac{du_n}{u_n}$ (resp. $\frac{du_1}{u_1},\ldots,\frac{du_n}{u_n},dv_1,\ldots,dv_N$) and extend it by taking wedge products to a basis for $\Omega^\bullet$ (resp. $\widetilde{\Omega}^\bullet$). Let $\Omega^\bullet_\Lambda$ (resp. $\Omega^\bullet_{\Lambda+}$) denote the subgroups of $\Omega^\bullet$ consisting of the linear combinations of the given basic forms with coefficients in $\mathcal{S}_\Lambda\otimes\mathbb{C}[\mathsf{v}]$ (resp. $\mathcal{S}_\Lambda^+\otimes\mathbb{C}[\mathsf{v}]$). Define $\widetilde{\Omega}^\bullet_\Lambda$ (resp. $\widetilde{\Omega}^\bullet_{\Lambda+}$) in the same way as subgroups of $\widetilde{\Omega}^\bullet$. The differential δ (72) preserves these subgroups. Thus we get the two complexes

$$\begin{aligned}(\Omega^\bullet_\Lambda,\delta)\quad &:\quad \Omega^0_\Lambda\xrightarrow{\delta}\Omega^1_\Lambda\xrightarrow{\delta}\ldots\xrightarrow{\delta}\Omega^{n-1}_\Lambda\xrightarrow{\delta}\Omega^n_\Lambda\\ (\widetilde{\Omega}^\bullet_\Lambda,\delta)\quad &:\quad \widetilde{\Omega}^0_\Lambda\xrightarrow{\delta}\widetilde{\Omega}^1_\Lambda\xrightarrow{\delta}\ldots\xrightarrow{\delta}\widetilde{\Omega}^{n-1}_\Lambda\xrightarrow{\delta}\widetilde{\Omega}^n_\Lambda\xrightarrow{\delta}\widetilde{\Omega}^{n+1}_\Lambda\xrightarrow{\delta}\ldots\xrightarrow{\delta}\widetilde{\Omega}^{N+n}_\Lambda\end{aligned}$$

Then

$$H^n(\Omega^\bullet_\Lambda,\delta) \;=\; \big(\mathcal{S}_\Lambda\,/\sum_{i=1}^n D_i\mathcal{S}_\Lambda\big)\otimes\mathbb{C}[\mathsf{v}] \tag{73}$$

The Gauss-Manin connection

$$\nabla \;:\; H^n(\Omega^\bullet_\Lambda, \delta) \to H^n(\Omega^\bullet_\Lambda, \delta) \otimes \Omega^1_{\mathbb{C}[\mathsf{v}]/\mathbb{C}} \tag{74}$$

on this module is described by the Katz-Oda construction (cf. [18] §1.4) as follows. Lift the given $\xi \in H^n(\Omega^\bullet_\Lambda, \delta)$ to an element $\tilde{\xi}$ in $\widetilde{\Omega}^n_\Lambda$. Then $\nabla\xi$ is the cohomology class of $\delta\tilde{\xi} \in \widetilde{\Omega}^{n+1}_\Lambda$ in $H^n(\Omega^\bullet_\Lambda, \delta) \otimes \Omega^1_{\mathbb{C}[\mathsf{v}]/\mathbb{C}}$. Having $\nabla\xi$ one defines $\frac{\partial}{\partial v_j}\xi \in H^n(\Omega^\bullet_\Lambda, \delta)$ by

$$\nabla\xi \;=\; \sum_{j=1}^{N} \left(\frac{\partial}{\partial v_j}\xi\right) \,\otimes\, dv_j \tag{75}$$

In particular for $\mu \in \Lambda \cap \mathbb{Z}^n$ and

$$\xi_\mu \;:=\; \text{cohomology class of } \mathsf{u}^\mu \cdot \frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_n}{u_n} \;\in H^n(\Omega^\bullet_\Lambda, \delta) \tag{76}$$

we find

$$\begin{aligned} \frac{\partial}{\partial v_j}\xi_\mu &= \text{cohomology class of } \mathsf{u}^{\mathfrak{a}_j+\mu} \cdot \frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_n}{u_n} && (77) \\ &= \xi_{\mu+\mathfrak{a}_j} && (78) \end{aligned}$$

The form ξ_μ for $\mu \neq 0$ corresponds via (73) and [2] thm.7.13 with the form ω_μ in (54); more precisely ξ_μ is the cohomology class of ω_μ modulo $H^{n-1}(\widetilde{\mathbb{T}})$.

Corollary 4

$$\begin{aligned} \left(\mu + \sum_{j=1}^{N} \mathfrak{a}_j\, v_j \frac{\partial}{\partial v_j}\right) \xi_\mu &= 0 && (79) \\ \left(\prod_{\ell_j>0} \left[\frac{\partial}{\partial v_j}\right]^{\ell_j} - \prod_{\ell_j<0} \left[\frac{\partial}{\partial v_j}\right]^{-\ell_j}\right) \xi_\mu &= 0 \quad \textit{for } \ell \in \mathbb{L} && (80) \end{aligned}$$

proof: On the level of differential forms in the complex $(\Omega^\bullet_\Lambda, \delta)$ the i-th equation of (79) reads

$$\begin{aligned} &\left(\mu_i + \sum_{j=1}^{N} a_{ij}\, v_j \frac{\partial}{\partial v_j}\right) \mathsf{u}^\mu \cdot \frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_n}{u_n} \;= \\ &= \;\delta\left((-1)^{i-1}\mathsf{u}^\mu \frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_{i-1}}{u_{i-1}} \wedge \frac{du_{i+1}}{u_{i+1}} \wedge \ldots \wedge \frac{du_n}{u_n}\right) \end{aligned}$$

(80) follows immediately from (77). ⊠

Remark 9 We have essentially repeated the proof of [2] thm. 14.2. There is however a small difference: Batyrev uses coefficients in $\mathcal{S}_\Lambda^+$ where we are using coefficients in $\mathcal{S}_\Lambda$. His differential equations hold for $H^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s}) = H^n(\Omega^\bullet_{\Lambda^+}, \delta)$ whereas ours only hold in the primitive part $PH^{n-1}(\mathbb{T}\setminus \mathsf{Z}_\mathsf{s})$. On the other hand we can also treat ξ_0. The following theorem shows that this gives an important advantage.

Theorem 8 *If $\Lambda \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}\mathfrak{a}_1 + \ldots + \mathbb{Z}_{\geq 0}\mathfrak{a}_N$, then ξ_0 generates $H^n(\Omega^\bullet_\Lambda, \delta)$ as a module over the ring $\mathcal{D} := \mathbb{C}[v_1, \ldots, v_N, E_\mathsf{A}^{-1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_N}]$. The annihilator of ξ_0 in $\mathcal{D}$ is the left ideal generated by the differential operators*

$$\sum_{j=1}^N a_{ij}\, v_j \frac{\partial}{\partial v_j} \quad \text{and} \quad \prod_{\ell_j > 0} \left[\frac{\partial}{\partial v_j}\right]^{\ell_j} - \prod_{\ell_j < 0} \left[\frac{\partial}{\partial v_j}\right]^{-\ell_j}$$

with $1 \leq i \leq n$ and $\ell \in \mathbb{L}$.

In other words, $H^n(\widetilde{\mathbb{T}} \operatorname{rel} \widetilde{\mathsf{Z}}_{\mathsf{s}-1}) = H^n(\Omega^\bullet_\Lambda, \delta)$ is the hypergeometric $\mathcal{D}$-module in the sense of [11] §*2.1 with parameters $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ and $\beta = 0$.*

proof: Let $\mathcal{M}_0$ denote the hypergeometric $\mathcal{D}$-module with parameters $\beta = 0$ and $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ as in [11] section 2.1. By corollary 4 and formula (78) we have a surjective homomorphism of $\mathcal{D}$-modules $\mathcal{M}_0 \to H^n(\Omega^\bullet_\Lambda, \delta)$. The filtration of $\mathcal{D}$ by the order of differential operators induces an ascending filtration on $\mathcal{M}_0$ and $H^n(\Omega^\bullet_\Lambda, \delta)$. It suffices to prove that the above surjection induces an isomorphism for the associated graded modules. According to [11] prop.3 $gr\,\mathcal{M}_0$ is isomorphic to the quotient of the ring $\mathbb{C}[x_1, \ldots, x_N] \otimes \mathbb{C}[\mathsf{v}]$ by the ideal generated by the linear forms $\sum_{j=1}^N a_{ij}x_j$ for $i = 1, \ldots, n$ and by the polynomials $\prod_{\ell_j>0} x_j^{\ell_j} - \prod_{\ell_j<0} x_j^{-\ell_j}$ with $\ell \in \mathbb{L}$. Via the substitution homorphism $x_j \mapsto \mathsf{u}^{\mathfrak{a}_j}$ this quotient ring is isomorphic to the quotient of the ring $\mathcal{S}_\Lambda \otimes \mathbb{C}[\mathsf{v}]$ by the ideal generated by $u_1\dfrac{\partial \mathsf{s}}{\partial u_1}$, $u_2\dfrac{\partial \mathsf{s}}{\partial u_2}, \ldots, u_n\dfrac{\partial \mathsf{s}}{\partial u_n}$. Using (77), (73) and (67) one checks that the latter quotient ring is isomorphic to $gr\,H^n(\Omega^\bullet_\Lambda, \delta)$. ⊠

PART II A

Introduction II A

In this Part II A we give our results the flavor of Mirror Symmetry by showing that for a regular triangulation $\mathcal{T}$ which satisfies conditions (81), (82), (83), the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is the cohomology ring of a toric variety constructed somehow from the dual Gorenstein cone $\Lambda^\vee$ and that the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}/\operatorname{Ann} c_{\mathrm{core}}$ is a subring of the Chow ring of a Calabi-Yau complete intersection in that toric variety; more precisely the subring is the image of the Chow ring of the ambient toric variety.

We construct several toric varieties which are also used in [3]. As we want to promote the use of triangulations we give a construction of these toric varieties as a quotient of an open part of $\mathbb{C}^d$ (d an appropriate dimension) by a torus. The torus is related to $\mathbb{L}$ and the open part is given by the triangulation $\mathcal{T}$. Such a construction of toric varieties is well known (see for instance [16]).

7 Triangulations with non-empty core and completely split reflexive Gorenstein cones.

Proposition 3 *Assume that $\mathcal{T}$ satisfies the following three conditions*

$$\operatorname{core}\mathcal{T} \textit{ is not empty and } \operatorname{core}\mathcal{T} = \{1,\ldots,\kappa\} \tag{81}$$

$$\operatorname{core}\mathcal{T} \textit{ is not contained in the boundary of } \Delta \tag{82}$$

$$\mathcal{T} \textit{ is unimodular} \tag{83}$$

Then $\Lambda := \mathbb{R}_{\geq 0}\mathfrak{a}_1 + \ldots + \mathbb{R}_{\geq 0}\mathfrak{a}_N$ is a reflexive Gorenstein cone of index κ and the dual cone $\Lambda^\vee$ is completely split *in the sense of [3] definition 3.9.*

proof: By lemma 6 and hypotheses (81) and (82) the $(n-2)$-dimensional simplices in the boundary of Δ are precisely the simplices $I \setminus \{i\}$ with $I \in \mathcal{T}^n$ and $i = 1,\ldots,\kappa$. It follows that the dual cone $\Lambda^\vee$ is generated by the set of row vectors $\{\mathfrak{a}_{I,i}^\vee \mid I \in \mathcal{T}^n,\ i = 1,\ldots,\kappa\}$ where

$$\mathfrak{a}_{I,i}^\vee \ := \ \text{the } i\text{-th row of the matrix } \mathsf{A}_I^{-1}$$

Hypothesis (83) implies $\mathfrak{a}_{I,i}^\vee \in \mathbb{Z}^{n\vee}$ for all I, i. By construction

$$\mathfrak{a}_{I,i}^\vee \cdot \mathfrak{a}_j \ = \ \begin{cases} \geq 0 & \text{for } j = 1,\ldots,N \\ 1 & \text{if } j = i \\ 0 & \text{if } 1 \leq j \leq \kappa\,,\ j \neq i \end{cases} \tag{84}$$

So if we take

$$\mathfrak{a}_0 \ := \ \mathfrak{a}_1 + \ldots + \mathfrak{a}_\kappa \ \in \mathbb{R}^n \tag{85}$$

then

$$\mathfrak{a}_{I,i}^\vee \cdot \mathfrak{a}_0 \ = \ 1 \qquad \text{for} \quad I \in \mathcal{T}^n,\ i = 1,\ldots,\kappa\,.$$

This shows that $\Lambda^\vee$ is a Gorenstein cone. Hence Λ is a reflexive Gorenstein cone with index $\mathfrak{a}_0^\vee \cdot \mathfrak{a}_0 = \kappa$.

Every element of $\Lambda^\vee$ can be written as $\sum_{I,i} s_{I,i}\mathfrak{a}_{I,i}^\vee$ with all $s_{I,i} \in \mathbb{R}_{\geq 0}$. Such a sum can be rearranged as $\sum_{i=1}^{\kappa} t_i\alpha_i$ with $t_i = \sum_I s_{I,i}$ and $\alpha_i \in \Box_i$ where

$$\Box_i \ := \ \operatorname{conv}\{\mathfrak{a}_{I,i}^\vee \mid I \in \mathcal{T}^n\}\,. \tag{86}$$

$\Box_i$ is a lattice polytope in the $(n-\kappa)$-dimensional affine subspace of $\mathbb{R}^{n\vee}$ given by the equations $\xi \cdot \mathfrak{a}_i = 1$ and $\xi \cdot \mathfrak{a}_j = 0$ if $1 \leq j \leq \kappa,\ j \neq i$ (cf. (84)). This shows that $\Lambda^\vee$ is a *completely split* reflexive Gorenstein cone of index κ in the sense of [3] definition 3.9.

Note that the dimension of $\Box_i$ equals $n-2$ minus the dimension of the minimal face of Δ which contains $\{\mathfrak{a}_j \mid j \in \operatorname{core}\mathcal{T} \setminus \{i\}\}$. $\boxtimes$

8 Triangulations and toric varieties

We assume from now on that $\mathcal{T}$ satisfies the conditions (81), (82), (83).

Take some $I_0 \in \mathcal{T}^n$ and consider the matrix $(u_{ij}) := \mathsf{A}_{I_0}^{-1}\mathsf{A}$. Then in definition 2 the linear forms

$$u_{i1}C_1 + u_{i2}C_2 + \ldots + u_{iN}C_N \qquad (i = 1, \ldots, n) \tag{87}$$

together with the monomials in (14) give another system of generators for the ideal $\mathcal{J}$. The corresponding relations in $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ for $i = 1, \ldots, \kappa$ express $c_1, \ldots, c_\kappa$ as linear combinations of $c_{\kappa+1}, \ldots, c_N$. The relations for $i = \kappa+1, \ldots, n$ do not involve $c_1, \ldots, c_\kappa$. Also the monomials in (14) do not involve $C_1, \ldots, C_\kappa$.

Let $\mathfrak{u}_{\kappa+1}, \ldots, \mathfrak{u}_N \in \mathbb{R}^{n-\kappa}$ be the columns of the matrix $(u_{ij})_{\kappa<i\leq n, \kappa<j\leq N}$. There is a simplicial fan $\mathcal{F}'$ in $\mathbb{R}^{n-\kappa}$ given by the cones

$$\mathbb{R}_{\geq 0}\mathfrak{u}_{i_1} + \ldots + \mathbb{R}_{\geq 0}\mathfrak{u}_{i_s} \qquad \text{with } i_1, \ldots, i_s > \kappa \text{ and } \{i_1, \ldots, i_s\} \in \mathcal{T} \tag{88}$$

i.e. the index set is a simplex in the triangulation $\mathcal{T}$.

The fan $\mathcal{F}'$ is complete iff $0 \in \mathbb{R}^{n-\kappa}$ is a linear combination with positive coefficients of the vectors $\mathfrak{u}_{\kappa+1}, \ldots, \mathfrak{u}_N$. This is equivalent to condition (82). Condition (83) implies that $\mathcal{F}'$ is a fan of regular simplicial cones, i.e. its maximal cones are spanned by a basis of $\mathbb{Z}^{n-\kappa}$.

Combining these considerations with [5] thm.10.8 or [10] prop.p.106 we find:

Theorem 9 *If the triangulation $\mathcal{T}$ satisfies conditions (81), (82), (83), then $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is isomorphic to the cohomology ring $H^*(\mathbb{P}_\mathcal{T}, \mathbb{Z})$ of the $(n-\kappa)$-dimensional smooth projective toric variety $\mathbb{P}_\mathcal{T}$ associated with the fan $\mathcal{F}'$ (see definition 8); more precisely:*

$$\mathcal{R}^{(m)}_{\mathsf{A},\mathcal{T}} \simeq H^{2m}(\mathbb{P}_\mathcal{T}, \mathbb{Z}) \,, \qquad m = 0, 1, \ldots, n-\kappa \,.$$

and $\mathcal{R}^{(m)}_{\mathsf{A},\mathcal{T}} = 0$ for $m > n - \kappa$. ⊠

There is much more geometry in those three conditions than was used for theorem 9. Consider in $\mathbb{R}^n$ the fan $\mathcal{F}$ consisting of the cones

$$\mathbb{R}_{\geq 0}\mathfrak{a}_{i_1} + \ldots + \mathbb{R}_{\geq 0}\mathfrak{a}_{i_s} \;, \qquad \{i_1, \ldots, i_s\} \in \mathcal{T}. \tag{89}$$

The standard constructions produce a toric variety $\mathbb{E}_\mathcal{T}$ from this fan. We recall the construction of the toric variety $\mathbb{E}_\mathcal{T}$ as a quotient of an open part of $\mathbb{C}^N$ by the torus $\mathbb{L} \otimes \mathbb{C}^*$. This torus appears here because $\mathbb{L}$ is the lattice of linear relations between the vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$; by condition (83) and corollary 1 these are exactly the generators of the 1-dim cones of the fan $\mathcal{F}$.

Take $\mathbb{C}^N$ with coordinates $x_1, \ldots, x_N$ and define

$$\begin{aligned} \mathbb{C}^N_I &:= \{ (x_1, \ldots, x_N) \in \mathbb{C}^N \mid x_j \neq 0 \text{ if } j \notin I \} \quad \text{for} \quad I \in \mathcal{T}^n \\ \mathbb{C}^N_\mathcal{T} &:= \bigcup_{I \in \mathcal{T}^n} \mathbb{C}^N_I \end{aligned} \tag{90}$$

The torus $\mathbb{C}^{*N}$ acts on $\mathbb{C}^N$ via coordinatewise multiplication. The inclusion $\mathbb{L} \subset \mathbb{Z}^N$ induces an inclusion of tori $\mathbb{L} \otimes \mathbb{C}^* \subset \mathbb{C}^{*N}$. Thus $\mathbb{L} \otimes \mathbb{C}^*$ acts on $\mathbb{C}^N$. For $\ell = (\ell_1, \ldots, \ell_N) \in \mathbb{L}$, $t \in \mathbb{C}^*$ the element $\ell \otimes t$ acts as

$$(\ell \otimes t) \cdot (x_1, \ldots, x_N) \;:=\; (t^{\ell_1} x_1, \ldots, t^{\ell_N} x_N) \tag{91}$$

Definition 7 $\quad \mathbb{E}_{\mathcal{T}} := \mathbb{C}^N_{\mathcal{T}} / \mathbb{L} \otimes \mathbb{C}^*$.

Take an $(N-n) \times N$-matrix B with entries in $\mathbb{Z}$ such that the columns of B^t constitute a basis for $\mathbb{L}$. For $I \subset \{1, \ldots, N\}$ we denote by A_I (resp. B_{I^*}) the submatrix of A (resp. B) composed of the entries with column index in I (resp. in $I^* := \{1, \ldots, N\} \setminus I$). Consider $I = \{i_1, \ldots, i_n\} \in \mathcal{T}^n$. Then $\det(\mathsf{B}_{I^*}) = \pm\det(\mathsf{A}_I) = \pm 1$ by condition (83). So B_{I^*} is invertible over $\mathbb{Z}$. From this one easily sees that there is an isomorphism

$$\begin{array}{rcl} \mathbb{C}^n & \xrightarrow{\simeq} & \mathbb{C}^N_I / \mathbb{L} \otimes \mathbb{C}^* \\ (y_1, \ldots, y_n) & \mapsto & (x_1, \ldots, x_N) \text{ with } x_j = \begin{cases} y_t & \text{if } j = i_t \in I \\ 1 & \text{if } j \notin I \end{cases} \end{array} \tag{92}$$

Hence $\mathbb{E}_{\mathcal{T}}$ is a smooth toric variety. The torus $\mathbb{C}^{*N} / \mathbb{L} \otimes \mathbb{C}^* = \mathbb{M} \otimes \mathbb{C}^*$ acts on $\mathbb{E}_{\mathcal{T}}$ and the variety $\mathbb{E}_{\mathcal{T}}$ contains $\mathbb{M} \otimes \mathbb{C}^*$ as a dense open subset.

One constructs in the same way the toric variety $\mathbb{P}_{\mathcal{T}}$ from the fan $\mathcal{F}'$ (see (88)). Now the lattice of linear relations between the generators $\mathfrak{u}_{\kappa+1}, \ldots, \mathfrak{u}_N$ of the 1-dimensional cones of the fan $\mathcal{F}'$ is the image of the composite map $\mathbb{L} \hookrightarrow \mathbb{Z}^N \twoheadrightarrow \mathbb{Z}^{N-\kappa}$. This map $\mathbb{L} \to \mathbb{Z}^{N-\kappa}$ is also injective. Take $\mathbb{C}^{N-\kappa}$ with coordinates $x_{\kappa+1}, \ldots, x_N$ and define

$$\begin{array}{rcl} \mathbb{C}^{N-\kappa}_I & := & \{ (x_{\kappa+1}, \ldots, x_N) \in \mathbb{C}^N \mid x_j \neq 0 \text{ if } j \notin I \} \quad \text{for} \quad I \in \mathcal{T}^n \\ \mathbb{C}^{N-\kappa}_{\mathcal{T}} & := & \displaystyle\bigcup_{I \in \mathcal{T}^n} \mathbb{C}^{N-\kappa}_I \end{array} \tag{93}$$

$\mathbb{L} \otimes \mathbb{C}^*$ is a subtorus of $\mathbb{C}^{*N-\kappa}$ and acts accordingly; i.e. as in (91) using only the coordinates with index $> \kappa$.

Definition 8 $\quad \mathbb{P}_{\mathcal{T}} := \mathbb{C}^{N-\kappa}_{\mathcal{T}} / \mathbb{L} \otimes \mathbb{C}^*$.

$\mathbb{P}_{\mathcal{T}}$ is a smooth projective toric variety: smooth for the same reason as $\mathbb{E}_{\mathcal{T}}$ and projective because the fan $\mathcal{F}'$ is complete. Projection onto the last $N - \kappa$ coordinates induces a surjective morphism

$$\pi \,:\, \mathbb{E}_{\mathcal{T}} \to \mathbb{P}_{\mathcal{T}} \tag{94}$$

As (90) puts no restriction on the coordinates $x_1, \ldots, x_\kappa$, the fibers of π are complex vector spaces of dimension κ; more precisely, (92) gives a trivialization

$$\mathbb{C}^N_I / \mathbb{L} \otimes \mathbb{C}^* \;\simeq\; \mathbb{C}^n \;\simeq\; \mathbb{C}^\kappa \times \mathbb{C}^{n-\kappa} \;\simeq\; \mathbb{C}^\kappa \times \left(\mathbb{C}^{N-\kappa}_I / \mathbb{L} \otimes \mathbb{C}^* \right)$$

Thus:

Proposition 4 *$\mathbb{E}_{\mathcal{T}}$ has the structure of a vector bundle of rank κ over $\mathbb{P}_{\mathcal{T}}$.* $\boxtimes$

The dual vector bundle $\mathbb{E}_{\mathcal{T}}^{\vee} \to \mathbb{P}_{\mathcal{T}}$ can be constructed as

$$\mathbb{E}_{\mathcal{T}}^{\vee} := \mathbb{C}_{\mathcal{T}}^{N} / (\mathbb{L} \otimes \mathbb{C}^{*})' \tag{95}$$

with $\mathbb{C}_{\mathcal{T}}^{N}$ as in definition 7, but with the action of $\mathbb{L} \otimes \mathbb{C}^{*}$ slightly modified from (91): the element $\ell \otimes t$ now acts as

$$(\ell \otimes t) \cdot' (x_1, \ldots, x_N) := (t^{-\ell_1} x_1, \ldots, t^{-\ell_\kappa} x_\kappa, t^{\ell_{\kappa+1}} x_{\kappa+1}, \ldots, t^{\ell_N} x_N) \tag{96}$$

For the sake of completeness we also describe the construction of the bundle of projective spaces $\mathbb{P}\mathbb{E}_{\mathcal{T}} \to \mathbb{P}_{\mathcal{T}}$ associated with the vector bundle $\mathbb{E}_{\mathcal{T}} \to \mathbb{P}_{\mathcal{T}}$. Take as before $\mathbb{C}^N$ with coordinates $x_1, \ldots, x_N$. Define for $i \in \operatorname{core}\mathcal{T}$ and $I \in \mathcal{T}^n$

$$\begin{aligned} \mathbb{C}_{i,I}^{N} \quad &:= \quad \{ (x_1, \ldots, x_N) \in \mathbb{C}^N \mid x_i \neq 0 \text{ and } x_j \neq 0 \text{ if } j \notin I \} \\ \mathbb{C}_{\mathcal{T}\circ}^{N} \quad &:= \quad \bigcup_{i \in \operatorname{core}\mathcal{T},\, I \in \mathcal{T}^n} \mathbb{C}_{i,I}^{N} \end{aligned} \tag{97}$$

Write $\mathsf{k} := (k_1, \ldots, k_N)^t$ with $k_j = 1$ if $j \in \operatorname{core}\mathcal{T}$ resp. $k_j = 0$ if $j \notin \operatorname{core}\mathcal{T}$, i.e. $\mathsf{k} = (1, \ldots, 1, 0, \ldots, 0)^t$. Clearly $\mathsf{k} \notin \mathbb{L}$. Hence $\mathbb{Z}\cdot\mathsf{k}\oplus\mathbb{L} \subset \mathbb{Z}^N$ and $(\mathbb{Z}\cdot\mathsf{k}\oplus\mathbb{L})\otimes\mathbb{C}^{*} \subset \mathbb{C}^{*N}$. Then

$$\mathbb{P}\mathbb{E}_{\mathcal{T}} := \mathbb{C}_{\mathcal{T}\circ}^{N} / (\mathbb{Z} \cdot \mathsf{k} \oplus \mathbb{L}) \otimes \mathbb{C}^{*} . \tag{98}$$

with the morphism $\mathbb{P}\mathbb{E}_{\mathcal{T}} \to \mathbb{P}_{\mathcal{T}}$ induced from projection onto the last $N - \kappa$ coordinates.

There are two kinds of codim 1 simplices in the triangulation $\mathcal{T}$: those which do contain $\operatorname{core}\mathcal{T}$ and those which do not. Those which do not contain $\operatorname{core}\mathcal{T}$ are precisely the ones of the form $I \setminus \{i\}$ with $I \in \mathcal{T}^n$ and $i \in \operatorname{core}\mathcal{T}$. Notice the relation with (97). The codim 1 simplices which do not contain $\operatorname{core}\mathcal{T}$ constitute a triangulation of the boundary of Δ . Let as in (85)

$$\mathfrak{a}_0 := \mathfrak{a}_1 + \ldots + \mathfrak{a}_\kappa .$$

Then $\mathbb{Z}\cdot\mathsf{k}\oplus\mathbb{L} \subset \mathbb{Z}^N$ is precisely the lattice of linear relations between the vectors $\mathfrak{a}_1 - \frac{1}{\kappa}\mathfrak{a}_0,\ \mathfrak{a}_2 - \frac{1}{\kappa}\mathfrak{a}_0, \ldots, \mathfrak{a}_N - \frac{1}{\kappa}\mathfrak{a}_0$. Thus we see:

Proposition 5 *$\mathbb{P}\mathbb{E}_{\mathcal{T}}$ is the $(n-1)$-dimensional smooth projective toric variety associated with the lattice $\mathbb{Z}(\mathfrak{a}_1 - \frac{1}{\kappa}\mathfrak{a}_0) + \ldots + \mathbb{Z}(\mathfrak{a}_N - \frac{1}{\kappa}\mathfrak{a}_0)$ and the fan consisting the cones with apex 0 over the simplices of the triangulation of the boundary of $-\frac{1}{\kappa}\mathfrak{a}_0 + \Delta$ induced by $\mathcal{T}$.* $\boxtimes$

9 Calabi-Yau complete intersections in toric varieties

According to proposition 3 conditions (81), (82), (83) imply that $\Lambda^{\vee}$ is a completely split reflexive Gorenstein cone. In [3] Batyrev and Borisov relate this splitting property to complete intersections in toric varieties. Formulated in our present context this relation is as follows.

A (global) section of $\mathbb{E}_{\mathcal{T}}^{\vee} \to \mathbb{P}_{\mathcal{T}}$ is given by polynomials $P_i(x_{\kappa+1}, \ldots, x_N)$ $(i = 1, \ldots, \kappa)$ which satisfy the homogeneity condition

$$P_i(t^{\ell_{\kappa+1}} \cdot x_{\kappa+1}, \ldots, t^{\ell_N} \cdot x_N) \;=\; t^{-\ell_i} \cdot P_i(x_{\kappa+1}, \ldots, x_N) \tag{99}$$

for every $t \in \mathbb{C}^*$ and $\ell = (\ell_1, \ldots, \ell_N)^t \in \mathbb{L}$. The vector bundle is a direct sum of line bundles and the polynomial P_i gives a section of the i-th line bundle.

The polynomial P_i is a linear combination of monomials $x_{\kappa+1}^{m_{\kappa+1}} \cdot \ldots \cdot x_N^{m_N}$ such that

$$\ell_{\kappa+1} m_{\kappa+1} + \ldots + \ell_N m_N \;=\; -\ell_i \qquad \text{for all } \ell = (\ell_1, \ldots, \ell_N) \in \mathbb{L}\,.$$

These monomials correspond bijectively to the elements $(m_1, \ldots, m_N)$ in the row space of matrix A which satisfy $m_i = 1$, $m_j = 0$ if $1 \le j \le \kappa, j \ne i$ and $m_j \ge 0$ if $j > \kappa$. Equivalently, these monomials correspond bijectively to the elements $\mathsf{w} \in \mathbb{Z}^{n\vee}$ which satisfy

$$\mathsf{w} \cdot \mathfrak{a}_j \;=\; \begin{cases} \ge 0 & \text{for } j = 1, \ldots, N \\ 1 & \text{if } j = i \\ 0 & \text{if } 1 \le j \le \kappa\,,\ j \ne i \end{cases} \tag{100}$$

So the monomials in the polynomial P_i correspond bijectively to the integral lattice points in the polytope $\square_i$; see (86).

The zero locus of the section of $\mathbb{E}_{\mathcal{T}}^{\vee} \to \mathbb{P}_{\mathcal{T}}$ corresponding to the polynomials $P_i(x_{\kappa+1}, \ldots, x_N)$ $(i = 1, \ldots, \kappa)$ is clearly the complete intersection in $\mathbb{P}_{\mathcal{T}}$ with (homogeneous) equations

$$P_i(x_{\kappa+1}, \ldots, x_N) = 0 \qquad (i = 1, \ldots, \kappa) \tag{101}$$

If the coefficients of these polynomials satisfy a $\Lambda^{\vee}$-regularity condition, then this complete intersection is a Calabi-Yau variety Y of dimension $n - 2\kappa$.

The ring $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is isomorphic to the cohomology ring of the toric variety $\mathbb{P}_{\mathcal{T}}$. The elements $-c_1, \ldots, -c_\kappa$ are the Chern classes of the hypersurfaces associated with the polynomials $P_1, \ldots, P_\kappa$. With as before $c_{\mathrm{core}} = c_1 \cdot \ldots \cdot c_\kappa$, the ring $\mathcal{R}_{\mathsf{A},\mathcal{T}} / \operatorname{Ann} c_{\mathrm{core}}$ is isomorphic to the image of $H^*(\mathbb{P}_{\mathcal{T}}, \mathbb{Z})$ in $H^*(Y, \mathbb{Z})$.

Conclusions

Consider the map $\mathsf{v} : \mathbb{C}^{N\vee} \to \mathbb{C}^{N\vee}$, $\mathsf{v}(z_1, \ldots, z_N) := (\mathsf{e}^{2\pi i\, z_1}, \ldots, \mathsf{e}^{2\pi i\, z_N})$. According to [13] p.304 cor.1.7 there is a vector $b \in \mathcal{C}_{\mathcal{T}}$ such that

$$E_{\mathsf{A}}(\mathsf{v}(\mathsf{z})) \ne 0 \quad \text{for all} \quad \mathsf{z} \in \mathbb{C}^{N\vee} \quad \text{such that} \quad p\,(\Im\, \mathsf{z}) \in b + \mathcal{C}_{\mathcal{T}}; \tag{102}$$

here $p : \mathbb{R}^{N\vee} \to \mathbb{L}_{\mathbb{R}}^{\vee}$ denotes the surjection dual to the inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^n$. This shows how one can replace the domain of definition $\mathcal{V}_{\mathcal{T}}$ of the functions $\Psi_{\mathcal{T},\beta}$

(cf. (40)) by a slightly smaller domain $\mathcal{V}'_{\mathcal{T}}$ such that on $\mathsf{v}(\mathcal{V}'_{\mathcal{T}})$ the function E_{A} is nowhere zero. The $\mathcal{D}$-module $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})$ is therefore defined on $\mathsf{v}(\mathcal{V}'_{\mathcal{T}})$; cf. theorem 8. Its pullback to $\mathcal{V}'_{\mathcal{T}}$ is the $\mathcal{D}_{\mathcal{T}}$-module $H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \otimes \mathcal{O}_{\mathcal{T}}$, where $\mathcal{O}_{\mathcal{T}}$ denotes the ring of holomorphic functions on $\mathcal{V}'_{\mathcal{T}}$ and $\mathcal{D}_{\mathcal{T}}$ denotes the corresponding ring of differential operators.

The functions $\Psi_{\mathcal{T},\beta}$ are also defined on the domain $\mathcal{V}'_{\mathcal{T}}$ and

$$\Psi_{\mathcal{T},\beta} \;\in\; \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathcal{O}_{\mathcal{T}} .$$

$\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathcal{O}_{\mathcal{T}}$ is a $\mathcal{D}_{\mathcal{T}}$-module with $\mathcal{R}_{\mathsf{A},\mathcal{T}}$ as its group of horizontal sections.

The following theorem summarizes the results of this paper:

Theorem 10 *Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ be a finite subset of $\mathbb{Z}^n$ which satisfies condition 1. Let $\Lambda := \mathbb{R}_{\geq 0}\mathfrak{a}_1 + \ldots + \mathbb{R}_{\geq 0}\mathfrak{a}_N$ be the associated Gorenstein cone and $\Delta :=$* $\mathrm{conv}\,\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$.

(i). *If there exists a unimodular regular triangulation of Δ, then condition 2 is satisfied and $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$ generate $\mathbb{Z}^n$, i.e.*

$$\Delta \cap \mathbb{Z}^n = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\} \quad \text{and} \quad \mathbb{M} = \mathbb{Z}^n . \tag{103}$$

(ii). *For every unimodular regular triangulation $\mathcal{T}$ there is an isomorphism of $\mathcal{D}_{\mathcal{T}}$-modules on $\mathcal{V}'_{\mathcal{T}}$:*

$$H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \otimes \mathcal{O}_{\mathcal{T}} \;\simeq\; \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathcal{O}_{\mathcal{T}} \tag{104}$$

through which ξ_0 corresponds with $\Psi_{\mathcal{T},0}$. More generally ξ_μ corresponds with $\Psi_{\mathcal{T},-\mu}$ if $\mu \in \Lambda \cap \mathbb{Z}^n$.

(iii). *In particular if Λ is a reflexive Gorenstein cone of index κ and $\mathcal{T}$ is a unimodular regular triangulation, then $\mathcal{W}_n H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \otimes \mathcal{O}_{\mathcal{T}}$ is generated as a $\mathcal{D}_{\mathcal{T}}$-module by $\xi_{\mathfrak{a}_0}$ and corresponds via (104) with the sub-$\mathcal{D}_{\mathcal{T}}$-module of $\mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathcal{O}_{\mathcal{T}}$ generated by $\Psi_{\mathcal{T},-\mathfrak{a}_0}$.*

Moreover $\xi_{\mathfrak{a}_0}$ has weight n and Hodge type $(n-\kappa, \kappa)$.

(iv). *If Λ is a reflexive Gorenstein cone and $\mathcal{T}$ is a unimodular regular triangulation with non-empty core, then (104) induces an isomorphism*

$$\begin{aligned} \mathcal{W}_n H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1}) \otimes \mathcal{O}_{\mathcal{T}} &\;\simeq\; c_{\mathrm{core}}\, \mathcal{R}_{\mathsf{A},\mathcal{T}} \otimes \mathcal{O}_{\mathcal{T}} \\ &\;\simeq\; \mathcal{R}_{\mathsf{A},\mathcal{T}} / \mathrm{Ann}\, c_{\mathrm{core}} \;\otimes \mathcal{O}_{\mathcal{T}} \end{aligned} \tag{105}$$

(v). *Now assume $\mathcal{T}$ satisfies conditions (81), (82), (83), i.e. $\mathcal{T}$ is a unimodular regular triangulation whose core is not empty and is not contained in the boundary of Δ . Then*

(a) Λ is a reflexive Gorenstein cone.

(b) *$\mathcal{R}_{\mathsf{A},\mathcal{T}}$ is isomorphic to the cohomology ring $H^*(\mathbb{P}_{\mathcal{T}},\mathbb{Z})$ of the $(n-\kappa)$-dimensional smooth projective toric variety $\mathbb{P}_{\mathcal{T}}$:*

$$\mathcal{R}^{(m)}_{\mathsf{A},\mathcal{T}} \simeq H^{2m}(\mathbb{P}_{\mathcal{T}},\mathbb{Z})\ , \qquad m=0,1,\ldots,n-\kappa \tag{106}$$

and in particular for $m=1$: $\mathbb{L}^{\vee}_{\mathbb{Z}} \simeq Pic(\mathbb{P}_{\mathcal{T}})$.

(c) *$c_{\mathrm{core}} = c_{\kappa}(\mathbb{E}_{\mathcal{T}})$, the top Chern class of the vectorbundle $\mathbb{E}_{\mathcal{T}}$.*

(d) *The zero locus of a general section of the dual vector bundle $\mathbb{E}^{\vee}_{\mathcal{T}}$ is an $n-2\kappa$-dimensional Calabi-Yau complete intersection in $\mathbb{P}_{\mathcal{T}}$.*

(e)

$$\begin{aligned} H^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})\otimes\mathcal{O}_{\mathcal{T}} &\simeq H^*(\mathbb{P}_{\mathcal{T}},\mathbb{Z})\otimes\mathcal{O}_{\mathcal{T}} \qquad (107)\\ \mathcal{W}_nH^n(\widetilde{\mathbb{T}}\,\mathrm{rel}\,\widetilde{\mathsf{Z}}_{\mathsf{s}-1})\otimes\mathcal{O}_{\mathcal{T}} &\simeq c_{\kappa}(\mathbb{E}_{\mathcal{T}})\,H^*(\mathbb{P}_{\mathcal{T}},\mathbb{Z})\otimes\mathcal{O}_{\mathcal{T}} \qquad (108)\end{aligned}$$

(f) *The monodromy representation is isomorphic to the representation of $Pic(\mathbb{P}_{\mathcal{T}})$ on $H^*(\mathbb{P}_{\mathcal{T}},\mathbb{Z})$ (resp. on $c_{\kappa}(\mathbb{E}_{\mathcal{T}})\,H^*(\mathbb{P}_{\mathcal{T}},\mathbb{Z})$) in which the Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ acts as multiplication by* $\exp(c_1(\mathcal{L}))$.

Proof: **(i)**: corollary 1. **(ii)**: theorems 5 and 8, formulas (18) and (78). **(iii)**: formulas (46), (48), (65) and theorem 7. **(iv)**: corollary 3 and theorem 6. **(v*a*)**: proposition 3. **(v*b*)**: theorem 9 and corollary 1. **(v*c*)**: section 9. **(v*d*)**: section 9. **(v*e*)**: (104) and (105). **(v*f*)**: formula (21). ⊠

All cases which have on the A-side of mirror symmetry a smooth complete intersection Calabi-Yau variety in a smooth projective toric variety, are covered by this theorem. Indeed, a smooth projective toric variety $\mathbb{P}$ of dimension d can be constructed from a complete simplicial fan in which every maximal cone is generated by a basis of the lattice $\mathbb{Z}^d$. Let $\mathsf{u}_1,\ldots,\mathsf{u}_p\in\mathbb{Z}^d$ be the generators of the 1-dimensional cones in the fan and let

$$\overline{\mathbb{L}} := \{(m_1,\ldots,m_p)\in\mathbb{Z}^p \mid m_1\mathsf{u}_1+\ldots m_p\mathsf{u}_p=0\,\}$$

The toric variety $\mathbb{P}$ can also be obtained as the quotient of a certain open part of $\mathbb{C}^p$ by the action of the subtorus $\overline{\mathbb{L}}\otimes\mathbb{C}^*$ of $(\mathbb{C}^*)^p$. The Calabi-Yau complete intersection Y of codimension κ in $\mathbb{P}$ is the common zero locus of polynomials $P_1,\ldots,P_\kappa$ which are homogeneous for the action of $\overline{\mathbb{L}}\otimes\mathbb{C}^*$. The homogeneity of P_i is given by a character of this torus, i.e. by a linear map $\chi_i:\overline{\mathbb{L}}\to\mathbb{Z}$. Now set $N=p+\kappa$ and $n=d+\kappa$. Let

$$\mathbb{L} := \{(-\chi_1(\mathsf{m}),\ldots,-\chi_\kappa(\mathsf{m}),m_1,\ldots,m_p)\in\mathbb{Z}^N \mid \mathsf{m}=(m_1,\ldots,m_p)\in\overline{\mathbb{L}}\,\}\,.$$

Then $\mathbb{L}$ has rank $N-n$. The Calabi-Yau condition for Y implies $\ell_1+\ldots+\ell_N=0$ for every $\ell=(\ell_1,\ldots,\ell_N)\in\mathbb{L}$.

Let B be an $(N-n)\times N$-matrix with entries in $\mathbb{Z}$ such that the columns of B^t constitute a basis for $\mathbb{L}$. Let A be an $n\times N$-matrix of rank n with entries in $\mathbb{Z}$ such that $\mathsf{A} \cdot \mathsf{B}^t = 0$. Then the columns $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$ of A satisfy condition 1. One obtains a regular triangulation of $\Delta := \mathrm{conv}\,\{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$ which satisfies the three conditions (81), (82), (83), by taking as its maximal simplices all $\mathrm{conv}\,\{\mathfrak{a}_1, \ldots, \mathfrak{a}_\kappa, \mathfrak{a}_{\kappa+i_1}, \ldots, \mathfrak{a}_{\kappa+i_d}\}$ for which $\mathsf{u}_{i_1}, \ldots, \mathsf{u}_{i_d}$ span a maximal cone in the fan defining $\mathbb{P}$.

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