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Response and Correlation in Fokker-Planck Dynamics. I

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The perturbative expansion of the correlation and response functions in Fokker-Planck dynamics is carried out through a Wick's theorem for a general non-stationary situation. The stochastic force is introduced dynamically by means of noncommuting variables and the equivalence of Langevin dynamics and Fokker-Planck dynamics in the Heisenberg picture is explicitly proved. No restriction is imposed on the initial value distribution, so that initial non-Gaussian correlations are considered. It is concluded that the whole effect of the stochastic force is the appearance of a stochastic propagator that dresses the free correlation function. Wyld's equation for the turbulence problem is derived.

§ 1. Introduction

The increasing attention[1)~6)] that classical perturbation theory has received recently, is mainly related to phenomenological and model equations of motion for a set of gross variables.[7)] It is the aim of this paper to establish a consistent perturbation theory for these systems of wide physical interest on the basis of a classical Wick's theorem. The explicit unrenormalized perturbation scheme for the correlation and response functions achieved with such procedure enables one to discuss the hypothesis and approximations involved in particular calculations. Our work is related to the perturbation theory for canonical systems developed some years ago[8)] and to the proof of a classical Wick's theorem restricted to canonical systems.[1)] Specifically, we prove here the existence of a Wick's theorem for Fokker-Planck dynamics in a general non-stationary situation that allows us to examine carefully the role of a stochastic force.

We consider in what follows equations of motion of the type

$$\dot{\psi}_\mu(t)=\lambda_\mu\psi_\mu(t)+F_\mu(\psi_1(t),\cdots,\psi_N(t),t)+\xi_\mu(t),\ \ \mu=1,\cdots,N \tag{1·1}$$

where $F_\mu(\psi(t),t)$ is an arbitrary polynomial in $\psi(t)$, and $\xi_\mu(t)$ is a stationary stochastic force assumed to be independent of $\psi(t)$, Gaussian with zero mean and white noise spectrum:

$$E_\xi\{\xi_\mu(t)\xi_\nu(t')\}=\int\delta\xi\mathscr{P}(\xi)\xi_\mu(t)\xi_\nu(t')=2D_{\mu\nu}\delta(t-t'), \tag{1·2}$$

where $\mathscr{P}(\xi)$ is the Gaussian probability distribution functional of $\xi(t)$.

Equation (1·1) can describe, for a suitable selection of the parameters λ_μ

and the arbitrary nonlinear function F_μ, a variety of physical situations that includes critical dynamics,[9)~12)] hydromagnetic turbulence phenomena,[13)~15)] structural phase transitions,[6)] and the nonlinear dynamics of general gross variables studied in transport theory.[5),16)]

The non-equilibrium statistical mechanics treatment of Eq. $(1\cdot1)$ is accomplished in this paper by the evaluation of the correlation and response functions, which are here defined as averages, in the Heisenberg picture of motion, over the distribution of initial conditions. Our definition of response function is a generalization of the one given within the MSR formalism[2)] and it applies to situations in which the external field is coupled to the gross variables.*) Besides the average over initial values, essential to define propagators in the Heisenberg picture of motion, there remains the second average over realizations of the stochastic force to be considered. This second average is taken into account in the present work in a dynamical manner from the very beginning, by replacing $\xi_\mu(t)$ in $(1\cdot1)$ by a derivative operator. We are then led to non-commuting classical gross variables which include the stochastic average in themselves, precisely because they do not commute. The validity of such procedure is proved in the Appendix for any order of perturbation theory. The equation of motion for these non-commuting variables defines what it is here understood by Fokker-Planck dynamics.

We make no restriction at all on the probability distribution of initial conditions, and therefore initial non-Gaussian correlations come in. In order to arrive at a useful expression for Wick's theorem, initial correlations are manipulated through a cluster decomposition which is the classical counterpart of the procedure followed by Fujita[17)] for the quantum case.

We proceed in §§ 4 and 5 to develop Wick's theorem in two steps without any stationary assumption. The first step is operational. It does not involve initial conditions and gives rise to two propagators, the free response function R^0 and a stochastic propagator U^0. The second step is the statistical one, and it consists in performing the average over initial conditions. The average to be considered over realizations of the stochastic force, is already included in the Fokker-Planck equations of motion satisfied by the gross variables $\psi_\mu(t)$ and its whole effect is a dressing of the free correlation function by the stochastic propagator U^0. The role of this stochastic propagator has not been discussed, to our knowledge, in the existing literature.

The formalism presented is applied, as an example, to the turbulence problem in § 6. Wyld's equations[13)] are derived clarifying the assumptions needed and its range of validity for other physical situations.

The connections of this work with other related approaches, which will not be discussed further, deserve some comments. Indeed, the goal of Kawasaki's[5)]

*) As stressed in Ref. 12), the response function defined in Ref. 2) is not the physical one. See further comments in § 3.

and Enz's[6] papers is similar to ours. Kawasaki's scheme makes use of the classical second quantization representation of Zwanzig.[16] Therefore it is based on a Gaussian local distribution function acting as a ground state. For a general nonequilibrium situation, this implies the assumption of an adiabatic switch-on condition. Finally, the development of Ref. 6) makes use of a similar switch-on condition, and it is incomplete in the sense that the results of Ref. 1) are taken for granted once the free response functions are extracted. Indeed, the result of Ref. 1) needs much more elaboration in the present case since now we handle classical noncommuting variables that give rise to the stochastic propagator which is not considered in Ref. 6).

In summary, in this paper we define, for general dynamics, a response function to an external field arbitrarily coupled and we prove the existence of a Wick's theorem for Fokker-Planck dynamics that enables to expand perturbatively the correlation and response functions in a general nonstationary situation in which non-Gaussian correlations are taken into account. On these grounds we conclude that the introduction of the stochastic force only accounts for the appearance of a stochastic propagator that dresses the free correlation function. This is one of our main conclusions. Wyld's equations are derived from the general formalism. In a forthcoming paper, the specialization to the stationary situation is carried out, allowing us to discuss a number of fluctuation dissipation theorems.

§ 2. Fokker-Planck dynamics

In this section, the basic equations of motion and commutation relations implied by a Fokker-Planck dynamics are studied. We take $\psi_\mu(t)$ to stand for a set of N gross variables[7] whose dynamics is closed in themselves, except for a possible stochastic force representing the effect of the irrelevant variables. The dynamical evolution is specified by a generalized Liouvillian[1] $L(\psi(t), \tilde{\psi}(t), t)$, in such a way that

$$\dot{\psi}_\mu(t) = [L(\psi(t), \tilde{\psi}(t), t), \psi_\mu(t)], \tag{2·1}$$

$$\dot{\tilde{\psi}}_\mu(t) = [L(\psi(t), \tilde{\psi}(t), t), \tilde{\psi}_\mu(t)], \quad \mu = 1, \cdots, N \tag{2·2}$$

where the square bracket denotes a commutator.

We proceed in the Heisenberg picture stated in (2·1) and (2·2) with $\psi_\mu \equiv \psi_\mu(0)$. $\tilde{\psi}_\mu(t)$ is given by Eq. (2·2) with $\tilde{\psi}_\mu \equiv \tilde{\psi}_\mu(0) \equiv \partial/\partial\psi_\mu(0)$. The operator $\tilde{\psi}_\mu(t)$ was already introduced in Ref. 8) where it is interpreted as a derivative operator with respect to the value of ψ at time t.

Equation (1·1) can be obtained from (2·1) when the following Liouvillian is used:

$$L(\psi(t), \tilde{\psi}(t), t) = \sum_\mu (\lambda_\mu \psi_\mu(t) + F_\mu(\psi(t), t) + \xi_\mu(t)) \tilde{\psi}_\mu(t). \tag{2·3}$$

We are not going to study the dynamics associated with (2·3) but rather the dynamics associated with the adjoint*) of the Fokker-Planck operator corresponding to the Langevin equation (1·1). Such adjoint operator is found to be equal to (2·3) when the replacement

$$\xi_\mu(t) \to \sum_\nu D_{\mu\nu}\hat{\psi}_\nu(t) \tag{2·4}$$

is made.

Therefore, we start considering the dynamics defined by the following generalized Liouvillian operator:

$$L(\psi(t), \hat{\psi}(t), t) = \sum_\mu (\lambda_\mu \psi_\mu(t) + F_\mu(\psi(t), t) + \sum_\nu D_{\mu\nu}\hat{\psi}_\nu(t))\hat{\psi}_\mu(t). \tag{2·5}$$

When we do so, account is taken in a dynamical manner of the average that should be evaluated over the realizations of the stochastic force. This statement is proved to be true in the Appendix. In fact, replacement (2·4) is implicitly used in Refs. 3), 6) and 9), and has been remarked in Refs. 2), 4) and 5). It is our purpose to explore fully its consequences in this paper.

The equations of motion (2·1) and (2·2) become, when (2·5) is used,

$$\dot{\psi}_\mu(t) = \sum_\nu 2D_{\mu\nu}\hat{\psi}_\nu(t) + \lambda_\mu \psi_\mu(t) + F_\mu(\psi(t), t), \tag{2·6}$$

$$\dot{\hat{\psi}}_\mu(t) = -\lambda_\mu \hat{\psi}_\mu(t) - \sum_\nu [\hat{\psi}_\mu(t), F_\nu(\psi(t), t)]\hat{\psi}_\nu(t). \tag{2·7}$$

Since we plan to study perturbation theory, we have to select the unperturbed generalized Liouvillian L_0. We take it to be

$$L_0(\psi(t), \hat{\psi}(t)) = \sum_\mu (\lambda_\mu \psi_\mu(t) + \sum_\nu D_{\mu\nu}\hat{\psi}_\nu(t))\hat{\psi}_\mu(t) \tag{2·8}$$

so that it contains the stochastic part of the problem. The solutions of the free evolution equations of motion define the interaction picture. We take it to be coincident at $t=0$ with the Heisenberg picture. Explicitly

$$\psi_\mu{}^0(t) = W_\mu(t)\psi_\mu + \sum_\nu V_{\mu\nu}(t)\hat{\psi}_\nu, \tag{2·9}$$

$$\hat{\psi}_\mu{}^0(t) = W_\mu(-t)\hat{\psi}_\mu. \tag{2·10}$$

We have defined

$$W_\mu(t) \equiv e^{\lambda_\mu t}, \tag{2·11}$$

$$V_{\mu\nu}(t) \equiv \frac{2D_{\mu\nu}}{\lambda_\mu + \lambda_\nu}(W_\mu(t) - W_\nu(-t)). \tag{2·12}$$

The solutions (2·9) and (2·10) are reached by writing

$$\psi_\mu{}^0(t) = \exp(L_0(\psi, \hat{\psi})t)\psi_\mu \exp(-L_0(\psi, \hat{\psi})t) \tag{2·13}$$

and similarly for $\hat{\psi}_\mu{}^0(t)$, and summing up the resulting commutator series.

*) The definition of adjoint operator is the well-known one given, for example, in H. Mori and H. Fujisaka, Prog. Theor. Phys. **49** (1973), 764.

From (2·9) and (2·10) follow the commutation relations

$$[\hat{\psi}_\mu^0(t), \hat{\psi}_\nu^0(t')] = 0, \tag{2·14}$$

$$[\hat{\psi}_\mu^0(t), \psi_\nu^0(t')] = \delta_{\mu\nu} W_\nu(t') W_\mu(-t), \tag{2·15}$$

$$[\psi_\mu^0(t), \psi_\nu^0(t')] = V_{\mu\nu}(t-t'). \tag{2·16}$$

Therefore, the $\psi_\mu^0(t)$ variables do not longer commute at different times when their evolution is of the Fokker-Planck type. This non-commutativity accounts for the introduction of the stochastic average in the dynamics by means of the quadratic term in $\hat{\psi}_\mu(t)$ of the Liouvillian (2·5).

The proof of Wick's theorem for classical-canonical systems[1] relies upon the fact that (2·14) holds and that the $\psi_\mu^0(t)$ variables commute. Therefore an extension of that result is not immediate, and this makes the results of Ref. 6) incomplete. However before we prove the corresponding Wick's theorem for Fokker-Planck dynamics, we turn in the next section to define the propagators and to evaluate them in the interaction picture. The latter is the fundamental tools used to introduce a perturbation theory for classical mechanics.

§ 3. Propagators

The statistical-mechanical description of a system of gross variables obeying Eqs. (2·6) and (2·7) is given by a correlation function $G_{\mu\nu}(t, t')$ and a response function $\bar{R}_{\mu\nu}(t, t')$. $G_{\mu\nu}(t, t')$ is defined for a general non-stationary state as the following average over initial conditions:

$$G_{\mu\nu}(t, t') \equiv \langle \bar{T}(\psi_\mu(t)\psi_\nu(t'))\rangle \equiv \int \prod_\sigma d\psi_\sigma \rho(\psi_1, \cdots, \psi_N; 0)\, \bar{T}(\psi_\mu(t)\psi_\nu(t')), \tag{3·1}$$

where $\rho(\psi_1, \cdots, \psi_N; 0)$ is any given probability distribution of the initial values ψ_μ at time $t=0$, and $\bar{T}$ is the time antiordering operator.[1] Any other average over the realizations of the stochastic force is already included in the equations satisfied by the gross variables $\psi_\mu(t)$, not making irrelevant the time antiordering in (3·1). Then, $G_{\mu\nu}(t, t')$ takes account of correlations arising from the initial conditions and those originated by the stochastic driving force. In this way we have introduced the Heisenberg picture of motion for the definition of propagators.

To study the response of the system to an external field $f_\nu(t)$ coupled to the $\psi_\mu(t)$ variables through a function $\Gamma_{\mu\nu}(\psi(t), t)$, let us add to the Liouvillian (2·5) the term $L' = \sum_{\mu\nu} \Gamma_{\mu\nu}(\psi(t), t) f_\nu(t) \hat{\psi}_\mu(t)$. The variables $\psi_\mu^f(t)$ whose evolution is given by $L+L'$ are related to $\psi_\mu(t)$ by[1),8)]

$$\psi_\mu^f(t) = S^f(0, t)\psi_\mu(t) S^f(t, 0), \tag{3·2}$$

$$S^f(0, t) = \bar{T} \exp\left\{\int_0^t dt' \sum_{\mu\nu} \Gamma_{\mu\nu}(\psi(t'), t') f_\nu(t') \hat{\psi}_\mu(t')\right\}. \tag{3·3}$$

In the linear approximation, we have

$$\langle \psi_\mu^f(t)\rangle = \langle \psi_\mu(t)\rangle$$
$$+\int_0^t dt'\theta(t-t')\sum_{\nu\delta}\langle[\Gamma_{\delta\nu}(\psi(t'),t')\hat{\psi}_\delta(t'),\psi_\mu(t)]\rangle f_\nu(t'). \qquad (3\cdot4)$$

Therefore, the linear response function is

$$\bar{R}_{\mu\nu}(t,t')\equiv\left\langle\frac{\delta\psi_\mu^f(t)}{\delta f_\nu(t')}\right\rangle_{f\to0}=\sum_\delta\langle\bar{T}(\psi_\mu(t)\Gamma_{\delta\nu}(\psi(t'),t')\hat{\psi}_\delta(t'))\rangle. \qquad (3\cdot5)$$

On the other hand, since

$$\frac{\delta S^f(0,t)}{\delta f_\nu(t')}=\theta(t-t')S^f(0,t')\sum_\delta\Gamma_{\delta\nu}(\psi(t'),t')\hat{\psi}_\delta(t')S^f(t',t), \qquad (3\cdot6)$$

we have from (3·2) that, for any value of the external field $f_\nu(t)$

$$\frac{\delta\psi_\mu^f(t)}{\delta f_\nu(t')}=\theta(t-t')\sum_\delta[\Gamma_{\delta\nu}(\psi^f(t'),t')\hat{\psi}_\delta^f(t'),\psi_\mu^f(t)]. \qquad (3\cdot7)$$

Thus, the nonlinear response function is given by (3·5) if $\psi_\mu(t)$ is substituted by $\psi_\mu^f(t)$. However, it is evident, that no similar expression to (3·4) can be written for the nonlinear response function.

We should like to point out that if the external field term is considered to be already included in the Liouvillian (2·5), we can think of the response function (3·5) either as the linear response function to small perturbations of the external field, or as the nonlinear response function to the external field.

In previous papers dealing with the MSR formalism[1)~4)] $\Gamma_{\mu\nu}(\psi(t),t)$ is taken as unity and then, $\bar{R}_{\mu\nu}(t,t')$ only gives the response to an external field not coupled to the variables, which is usually taken to be a stochastic force.*) We shall denote the response function by $R_{\mu\nu}$ when $\Gamma_{\mu\nu}=\delta_{\mu\nu}$.

Going over to the interaction picture in (3·1) and (3·5), we arrive at[1)]

$$G_{\mu\nu}(t,t')=\langle\bar{T}(S(0,T_M)\psi_\mu^0(t)\psi_\nu^0(t'))\rangle, \qquad (3\cdot8)$$

$$\bar{R}_{\mu\nu}(t,t')=\sum_\delta\langle\bar{T}(S(0,T_M)\psi_\mu^0(t)\Gamma_{\delta\nu}(\psi^0(t'),t')\hat{\psi}_\delta^0(t'))\rangle. \qquad (3\cdot9)$$

T_M is any time larger than t and t', and

$$S(0,T_M)=\bar{T}\exp\left\{\int_0^{T_M}dt\sum_\mu F_\mu(\psi^0(t),t)\hat{\psi}_\mu^0(t)\right\}. \qquad (3\cdot10)$$

It has been possible to obtain a time antiordering product when going over the interaction picture because the perturbing Liouvillian $L_1=\sum_\mu F_\mu\ (\psi(t),t)\ \hat{\psi}_\mu(t)$ is a differential operator, and hence, $S(t_1,t_2)$ is unity when it stands completely to

*) One of the shortcomings of Wyld's scheme[13)] is the lack of an analytical definition of the response function. An expression for it, which is not perturbatively expanded, is given by Ma and Mazenko.[11)] That expression is recovered from (3·5) under their special requirements on $F_\mu(\psi(t),t)$.

the right. This would not be true if the evolution were given by the Fokker-Planck operator, which is the adjoint of (2·5).

According to the Wick's theorem to be proved in the next sections, there are two kinds of free propagators appearing in the perturbation series. They are readily evaluated by means of (2·9) and (2·10)

$$R^0_{\mu\nu}(t,t') \equiv \langle \overline{T}(\psi_\mu{}^0(t)\hat{\psi}_\nu{}^0(t'))\rangle = \theta(t-t')\delta_{\mu\nu}W_\mu(t-t'), \tag{3·11}$$

$$G^0_{\mu\nu}(t,t') \equiv \langle \overline{T}(\psi_\mu{}^0(t)\psi_\nu{}^0(t'))\rangle = g^0_{\mu\nu}(t,t') + U^0_{\mu\nu}(t,t'), \tag{3·12}$$

where

$$g^0_{\mu\nu}(t,t') = W_\mu(t)W_\nu(t')\langle\psi_\mu\psi_\nu\rangle, \tag{3·13}$$

$$U^0_{\mu\nu}(t,t') = \theta(t-t')V_{\nu\mu}(t')W_\mu(t) + \theta(t'-t)V_{\mu\nu}(t)W_\nu(t'). \tag{3·14}$$

Due to the commutativity (2·14) the free response function does not depend on the stochastic term. It also does not depend on initial conditions and it exhibits invariance under time translation.

The free correlation function has two well-differentiated parts.[20] $g^0_{\mu\nu}(t,t')$ takes account of correlations arising from initial conditions and it corresponds to the non-stochastic term of L_0. $U^0_{\mu\nu}(t,t')$ arises from correlations originated by the stochastic term. The appearance of $U^0_{\mu\nu}(t,t')$ in (3·12) is the whole effect of the stochastic term in the perturbation series to be developed. $U^0_{\mu\nu}(t,t')$ will be referred to as the stochastic propagator. In fact, it comes out explicitly due to the second term on the right-hand side of Eq. (2·9). The stochastic propagator does not depend on initial conditions and its dynamical nature, on the same footing as $R^0_{\mu\nu}(t,t')$, will be made apparent in the next section.

In the next section we prove a Wick's theorem for the perturbative expansion of (3·1) and (3·5) under quite general circumstances, without any stationary assumption.

§ 4. Operational Wick's theorem

The perturbation expansion for the correlation and response functions comes out when (3·10) is substituted in (3·8) and (3·9). Each term of this expansion has the form $\langle \overline{T}(\phi_1{}^0(t_1)\cdots\phi_n{}^0(t_n))\rangle$, where $\phi_\mu{}^0(t_\mu)$ stands either for $\psi_\mu{}^0(t_\mu)$ or $\hat{\psi}_\mu{}^0(t_\mu)$. The purpose of a Wick's theorem is the factorization of such terms into free correlation and free response functions. A part of this factorization is previous to the statistical average to be performed and therefore valid for any initial conditions. It follows from an operational Wick's theorem for $\overline{T}(\phi_1{}^0(t_1)\cdots\phi_n{}^0(t_n))$.

Any $\phi_\mu{}^0(t)$ is a linear time dependent combination of the initial values ψ_ν and $\hat{\psi}_\nu$ ($\nu=1,\cdots N$). Let us introduce a distributive normal ordering of the $\phi_\nu{}^0(t)$'s in which all the $\hat{\psi}_\nu$ are placed to the right of the ψ_ν. This normal ordering is denoted by the symbol N. A contraction between any two fields $\phi_\mu{}^0(t)$ is then defined as

$$\phi_\mu{}^0(t)\phi_\nu{}^0(t') \equiv \overline{T}(\phi_\mu{}^0(t)\phi_\nu{}^0(t')) - N(\phi_\mu{}^0(t)\phi_\nu{}^0(t')). \tag{4·1}$$

From (4·1), (2·9), (2·10), (2·14), (3·11) and (3·14) three different contractions are obtained

$$\hat{\phi}_\mu{}^0(t)\hat{\phi}_\nu{}^0(t') = 0\,, \tag{4·2}$$

$$\psi_\mu{}^0(t)\hat{\phi}_\nu{}^0(t') = R^0_{\mu\nu}(t,t')\,, \tag{4·3}$$

$$\psi_\mu{}^0(t)\psi_\nu{}^0(t') = U^0_{\mu\nu}(t,t')\,. \tag{4·4}$$

With the above definitions of normal ordering and contractions, the quantum operational zero-temperature Wick's theorem[21] applies, and so,

$$\overline{T}(\phi_1{}^0(t_1)\cdots\phi_n{}^0(t_n)) =$$

$$\delta_{K0}\left\{\sum_{P'}\frac{1}{2^N N!}\phi^0_{\gamma_1}(t_{\gamma_1})\phi^0_{\gamma_2}(t_{\gamma_2})\cdots\phi^0_{\gamma_{n-1}}(t_{\gamma_{n-1}})\phi^0_{\gamma_n}(t_{\gamma_n})\right\}$$

$$+\sum_{i=1}^{N}\sum_{P'}\frac{1}{(2i-K)!\,2^{N-i}(N-i)!}\phi^0_{\gamma_1}(t_{\gamma_1})\phi^0_{\gamma_2}(t_{\gamma_2})\cdots\phi^0_{\gamma_{2N-2i-1}}(t_{\gamma_{2N-2i-1}})\phi^0_{\gamma_{2N-2i}}(t_{\gamma_{2N-2i}})$$

$$\times N(\phi^0_{\gamma_{2N-2i+1}}(t_{\gamma_{2N-2i+1}})\cdots\phi^0_{\gamma_n}(t_{\gamma_n}))\,. \tag{4·5}$$

Here $\sum_{P'}$ stands for a sum over the permutations of the set $\gamma_\mu = \{1,\cdots,n\}$. $K=0$ if $n=2N$ and $K=1$ if $n=2N-1$ (N$=1,2,\cdots$).

The remarkable fact of (4·5) is that the whole problem of initial conditions is relegated to the normal ordered product, while the propagators R^0 and U^0 have been extracted through the contractions. In this manner, the factorization of the stochastic term in the Wick's theorem is already finished operationally, and it explicitly exhibits the inclusion of the stochastic average in the dynamics.

§ 5. Statistical Wick's theorem

The statistical Wick's theorem comes out by averaging (4·5) over the initial conditions specified by $\rho(\psi_1,\cdots,\psi_N;0)$. Only the normal ordered product depends on ψ_μ. Whatever the initial distribution, we have

$$\langle N(\phi_1{}^0(t_1)\cdots\phi_m{}^0(t_m))\rangle$$

$$=\begin{cases}0 \text{ if } \phi_\mu{}^0(t_\mu)=\hat{\phi}_\mu{}^0(t_\mu) \text{ for some } \mu=1,\cdots,m\,,\\ W_1(t_1)\cdots W_m(t_m)\langle\psi_1\cdots\psi_m\rangle \text{ if } \phi_\mu{}^0(t_\mu)=\psi_\mu{}^0(t_\mu) \text{ for all } \mu=1,\cdots,m\,.\end{cases} \tag{5·1}$$

Let l be the number of $\hat{\phi}_\mu{}^0(t_\mu)$ in (4·5). Due to (4·2) and (5·1) the average of (4·5) vanishes if $l>N$, or $l=N$ and n is odd. If $l=N$ and n is even, it reduces to the first term on the right-hand side of (4·5) that becomes a product of free response functions. For $l\leq N$, the average of (4·5) is written as

$$\langle \bar{T}(\hat{\psi}_1{}^0(t_1)\cdots\hat{\psi}_l{}^0(t_l)\psi^0_{l+1}(t_{l+1})\cdots\psi_n{}^0(t_n))\rangle$$
$$=\delta_{K0}\Big\{\sum_P \frac{1}{2^{N-l}(N-l)!}R^0_{\gamma_{l+1},1}(t_{\gamma_{l+1}},t_1)\cdots R^0_{\gamma_{2l},l}(t_{\gamma_{2l}},t_l)$$
$$\times U^0_{\gamma_{2l+1},\gamma_{2l+2}}(t_{\gamma_{2l+1}},t_{\gamma_{2l+2}})\cdots U^0_{\gamma_{n-1},\gamma_n}(t_{\gamma_{n-1}},t_{\gamma_n})\Big\}$$
$$+\sum_{i=1}^{N-l}\sum_P \frac{1}{(2i-K)!\,2^{N-i-l}(N-i-l)!}R^0_{\gamma_{l+1},1}(t_{\gamma_{l+1}},t_1)\cdots R^0_{\gamma_{2l},l}(t_{\gamma_{2l}},t_l)$$
$$\times U^0_{\gamma_{2l+1},\gamma_{2l+2}}(t_{\gamma_{2l+1}},t_{\gamma_{2l+2}})\cdots U^0_{\gamma_{2N-2i-1},\gamma_{2N-2i}}(t_{\gamma_{2N-2i-1}},t_{\gamma_{2N-2i}})$$
$$\times W_{\gamma_{2N-2i+1}}(t_{\gamma_{2N-2i+1}})\cdots W_{\gamma_n}(t_{\gamma_n})\langle\psi_{\gamma_{2N-2i+1}}\cdots\psi_{\gamma_n}\rangle\,, \tag{5·2}$$

where $\sum_P$ stands now for a sum over the permutations of the set $\gamma_j=\{l+1,\cdots,n\}$.

To specify the initial distribution is equivalent to giving the complete set of its moments (i.e., $\langle\psi_1\cdots\psi_j\rangle$ for any $i\cdots j$) that are explicitly present in (5·2). Therefore, without further assumptions on $\rho(\psi_1,\cdots,\psi_N;0)$, (5·2) is the final expression for Wick's theorem.

However, it is physically appealing to decompose the moments of the initial distribution in terms of the reducible initial correlations defined as the following combination of the cumulants[22] or truncated functions $\langle\psi_1\cdots\psi_m\rangle^T$ of the distribution $\rho(\psi_1,\cdots,\psi_N;0)$:

$$\chi_m(\psi_1\cdots\psi_m)=\langle\psi_1\cdots\psi_m\rangle^T$$
$$+\sum_{(l_i)}\sum_{P'}\frac{1}{l_1!\cdots l_q!\,q!}\langle\psi_{\gamma_1}\cdots\psi_{\gamma_{l_1}}\rangle^T\cdots\langle\psi_{\gamma_{m-l_q}}\cdots\psi_{\gamma_m}\rangle^T. \tag{5·3}$$

Here $\sum_{(l_i)}$ indicates a sum over all the possible partitions of m such that $\sum_{i=1}^q l_i=m$ and $l_i\neq1$, $l_i\neq2$. $\sum_{P'}$ stands for a summation over the permutations of the set $\gamma_\mu=\{1,\cdots,m\}$. For simplicity, it is assumed that the $\psi_\mu(t)$ variables are defined in such a way that $\langle\psi_\mu\rangle=0$, $\mu=1,\cdots,N$. For $m=2$

$$\langle\psi_\mu\psi_\nu\rangle=\chi_2(\psi_\mu\psi_\nu)=\langle\psi_\mu\psi_\nu\rangle^T \tag{5·4}$$

and

$$g^0_{\mu\nu}(t,t')=W_\mu(t)\,W_\nu(t')\,\chi_2(\psi_\mu\psi_\nu). \tag{5·5}$$

When the resummation indicated in (5·3) is carried out in the expansion of a moment of order m in its corresponding smaller order cumulants, an expansion in terms of the irreducible initial correlation functions is obtained. Namely,

$$\langle\psi_1\cdots\psi_m\rangle=\delta_{K0}\Big\{\sum_P\frac{1}{2^N N!}\langle\psi_{\gamma_1}\psi_{\gamma_2}\rangle\cdots\langle\psi_{\gamma_{m-1}}\psi_{\gamma_m}\rangle\Big\}$$
$$+\sum_{i=2}^{N}\sum_{P'}\frac{1}{(2i-K)!\,2^{N-i}(N-i)!}\langle\psi_{\gamma_1}\psi_{\gamma_2}\rangle\cdots\langle\psi_{\gamma_{2N-2i-1}}\psi_{\gamma_{2N-2i}}\rangle$$
$$\times\chi_{2i-K}(\psi_{\gamma_{2N-2i+1}}\cdots\psi_{\gamma_m})\,. \tag{5·6}$$

N and K are related to m in the same way that they are related to n in (4·5).

The functions χ_m are called reducible in the sense that they contain initial correlations in any order smaller than m, except those contributions involving a first order correlation (i.e., $m=2$). In contraposition the cumulant of order m is irreducible because it does not contain any correlations of order inferior to m.

The free time dependence can be restored in (5·6) by multiplying both members by $W_1(t_1)\cdots W_m(t_m)$. Nevertheless, to substitute ψ_μ by $\psi_\mu{}^0(t_\mu)$ in (5·6) the normal ordering should be taken into account inside the brackets and the χ functions.

Replacing the expansion (5·6) in (5·2) two kind of terms come out. In one of them, no reducible initial correlations appear. This is the "normal" term and it is the only one that contributes for a Gaussian $\rho(\psi;0)$. The other term, or "spurious" term, contains the χ functions, and it is present whenever the initial distribution is not Gaussian. The spurious term is often disregarded, by using a switch-on condition[1),5),6)] or by assuming a unique inverse for the full propagator when we deal with functional methods[2)] (see Refs. 18) and 19).)

The normal term vanishes if n is odd. If $n=2N$ it reads

$$\begin{aligned}
&\langle \overline{T}\,(\hat{\psi}_1{}^0(t_1)\cdots\hat{\psi}_l{}^0(t_l)\,\psi^0_{l+1}(t_{l+1})\cdots\psi_n{}^0(t_n))\rangle_n \\
&\quad=\sum_P R^0_{\gamma_{l+1},1}(t_{\gamma_{l+1}},t_1)\cdots R^0_{\gamma_{2l},l}(t_{\gamma_{2l}},t_l) \\
&\qquad\times\Big\{\frac{1}{2^{N-l}(N-l)!}U^0_{\gamma_{2l+1},\gamma_{2l+2}}(t_{\gamma_{2l+1}},t_{\gamma_{2l+2}})\cdots U^0_{\gamma_{n-1},\gamma_n}(t_{\gamma_{n-1}},t_{\gamma_n}) \\
&\qquad+\sum_{i=1}^{N-l}\frac{1}{(2i)!\,2^{N-l}(N-i-l)!\,i!}U^0_{\gamma_{2l+1},\gamma_{2l+2}}(t_{\gamma_{2l+1}},t_{\gamma_{2l+2}})\cdots U^0_{\gamma_{2N-2i-1},\gamma_{2N-2i}}(t_{\gamma_{2N-2i-1}},t_{\gamma_{2N-2i}}) \\
&\qquad\times\Big[\sum_\sigma g^0_{\delta_{2N-2i+1},\delta_{2N-2i+2}}(t_{\delta_{2N-2i+1}},t_{\delta_{2N-2i+2}})\cdots g^0_{\delta_{n-1},\delta_n}(t_{\delta_{n-1}},t_{\delta_n})\Big]\Big\}\,. \qquad (5\cdot 7)
\end{aligned}$$

Here $\sum_\sigma$ stands for a sum over the permutation of the set $\delta_i=\{\gamma_{2N-2i+1},\cdots,\gamma_n\}$. By considering the sum over all permutations that do not change $\{\gamma_{l+1},\cdots,\gamma_{2l}\}$, the response functions appear as a multiplicative factor of a sum of products of the U^0's and the g^0's. This sum is just the product of $N-l$ binomials, each of which is the sum of some U^0 with its corresponding g^0 with the same indices. No index is repeated in different binomials. Therefore in view of (3·12) we arrive at

$$\begin{aligned}
\langle \overline{T}\,(\hat{\psi}_1{}^0(t_1)\cdots\hat{\psi}_l{}^0(t_l)\,\psi^0_{l+1}(t_{l+1})\cdots\psi_n{}^0(t_n))\rangle_n&=\sum_P\frac{1}{2^{N-l}(N-l)!} \\
\times R^0_{\gamma_{l+1},1}(t_{\gamma_{l+1}},t_1)\cdots R^0_{\gamma_{2l},l}(t_{\gamma_{2l}},t_l)\,G^0_{\gamma_{2l+1},\gamma_{2l+2}}&(t_{\gamma_{2l+1}},t_{\gamma_{2l+2}})\cdots G^0_{\gamma_{n-1},\gamma_n}(t_{\gamma_{n-1}},t_{\gamma_n})\,.
\end{aligned} \qquad (5\cdot 8)$$

For the spurious term a similar resummation of U^0 and g^0 can be performed. We finally arrive at

$$\langle \overline{T}\,(\hat{\psi}_1^0(t_1)\cdots\hat{\psi}_l^0(t_l)\,\psi_{l+1}^0(t_{l+1})\cdots\psi_n^0(t_n))\rangle_s$$
$$=\sum_P\sum_{j=2}^{N-l}\frac{1}{2^{N-l-j}(N-l-j)!(2j-K)!}R^0_{\gamma_{l+1},1}(t_{\gamma_{l+1}},t_1)\cdots R^0_{\gamma_{2l},l}(t_{\gamma_{2l}},t_l)$$
$$\times G^0_{\gamma_{2l+1},\gamma_{2l+2}}(t_{\gamma_{2l+1}},t_{\gamma_{2l+2}})\cdots G^0_{\gamma_{2N-2j-1},\gamma_{2N-2j}}(t_{\gamma_{2N-2j-1}},t_{\gamma_{2N-2j}})$$
$$\times W_{\gamma_{2N-2j+1}}(t_{\gamma_{2N-2j+1}})\cdots W_{\gamma_n}(t_{\gamma_n})\,\chi_{2j-K}(\psi_{\gamma_{2N-2j+1}}\cdots\psi_{\gamma_n})\,. \tag{5·9}$$

Therefore the statistical Wick's theorem is expressed for $l\leq N$ as the sum of (5·8) and (5·9), bearing in mind that the normal part vanishes whenever n is odd.

It should be stressed that the free correlation function G^0 appears in the final results (5·8) and (5·9) only after the U^0 coming from the operational Wick's theorem is added to g^0. The term g^0 is a nondynamical propagator arising from the statistical average. The possible spuriosity of the initial distribution function is evidenced through the reducible initial correlations χ.

Comparison of the present Wick's theorem with that valid for classical canonical systems,[1] clarifies that the only effect of the stochastic force is the replacement of g^0 by G^0. Therefore, we are led to the conclusion that the whole effect of the stochastic force is accomplished by a shift of g^0 in the quantity U^0 which is the stochastic propagator. G^0 is the free correlation function dressed by the stochastic process. Thus, the analysis of diagrams and self-energies of Ref. 1) applies, up to spuriosities, to the Fokker-Planck dynamics for the G and R functions. Then to pass from canonical diagrams to Fokker-Planck diagrams, the free correlation function g^0 of the canonical case has to be substituted by $G^0=g^0+U^0$, while the free response function remains unchanged. Nevertheless, it should be remembered that the general response function $\overline{R}$ can involve an arbitrary number of variables $\psi(t')$ and the analysis of diagrams must then be made for each particular case.

§ 6. Turbulence problem

The homogeneous hydromagnetic turbulence problem can be represented by Eq. (1·1). In this case $\psi_\mu(t)$ stands for the real independent parts of the Fourier modes of the velocity and magnetic induction fields.[23] The λ_μ's are damping factors; $F_\mu(\psi(t))$ is a quadratic energy conserving term, and $\xi_\mu(t)$ represents an external stirring force exciting or maintaining the turbulence in the most random possible way.[13]

Even in a non-stationary situation, if initial correlations are somehow disregarded, Dyson's equations[1] are valid for the present problem with the free propagators replaced by those discussed in § 3. With some rearrangements of the self-energies we are able to obtain Wyld's equations,[13),2] including a non-stationary free correlation function.

Dyson's equations are formally written as[1]

$$R = R^0 + R^0 \Pi^T R\,, \tag{6·1}$$

$$R^T = R^{0T} + R^{0T} \Pi R^T\,, \tag{6·2}$$

$$G = G^0 + G^0 \Pi R^T + R^0 \Pi^T G + R^0 \Sigma R^T\,. \tag{6·3}$$

It should be remembered that now $G^0 = g^0 + U^0$, where U^0 is the stochastic propagator.

Equation (6·1) implies that

$$1 - R^0 \Pi^T = R^0 R^{-1} \tag{6·4}$$

and similarly from (6·2)

$$1 + \Pi R^T = (R^{0T})^{-1} R^T\,. \tag{6·5}$$

By substituting (6·5) and R^0 from (6·1) in Eq. (6·3), it reduces to

$$(1 - R^0 \Pi^T)\,(G - R \Sigma R^T) = G^0 (R^{0T})^{-1} R^T\,. \tag{6·6}$$

Replacing (6·5), we are finally led to an equation for G:

$$\begin{aligned} G_{\mu\nu}(t,t') = \sum_{\alpha\beta} \int dt_1 dt_2 \{ & R_{\mu\alpha}(t,t_1)\, \Sigma_{\alpha\beta}(t_1,t_2)\, R^T_{\beta\nu}(t_2,t') \\ & + R_{\mu\alpha}(t,t_1)(R^0_{\alpha\alpha}(t_1))^{-1} G^0_{\alpha\beta}(t_1,t_2)(R^{0T}_{\beta\beta}(t_2))^{-1} R^T_{\beta\nu}(t_2,t')\}\,. \end{aligned} \tag{6·7}$$

Since in any diagrams contributing to the response function, there is a trunk of R^0 lines and Σ is an irreducible self-energy, the second term on the right-side of (6·7) includes all those diagrams which can be split into two pieces by severing a single G^0 line (class A of diagrams in Ref. 13).)

U^0 given by (3·14), can be written by means of (2·11)~(2·16) as

$$U^0_{\mu\nu}(t,t') = \frac{2D_{\mu\nu}}{\lambda_\mu + \lambda_\nu} W_\mu(t)\, W_\nu(t') - \frac{2D_{\mu\nu}}{\lambda_\mu + \lambda_\nu} [R^0_{\mu\mu}(t-t') + R^0_{\nu\nu}(t'-t)]\,. \tag{6·8}$$

For $\mathrm{Re}\lambda_\mu < 0$, (3·12), (3·13) and (6·8) lead to

$$\begin{aligned} G^0_{\alpha\beta}(t_1,t_2) = {} & W_\alpha(t_1)\, W_\beta(t_2) \left\{ \langle \psi_\alpha \psi_\beta \rangle + \frac{2D_{\alpha\beta}}{\lambda_\alpha + \lambda_\beta} \right\} \\ & + 2D_{\alpha\beta} \int d\tau\, R^0_{\alpha\alpha}(t_1 - t_2 - \tau)\, R^0_{\beta\beta}(-\tau)\,. \end{aligned} \tag{6·9}$$

Thus,

$$(R^0_{\alpha\alpha}(t_1))^{-1} G^0_{\alpha\beta}(t_1,t_2) = 2D_{\alpha\beta} R^{0T}_{\beta\beta}(t_1 - t_2)\,. \tag{6·10}$$

Substituting (6·10) in (6·7) and recalling (6·2), we have

$$\begin{aligned} G_{\mu\nu}(t,t') = {} & \sum_{\alpha\beta} \int dt_1 dt_2\, R_{\mu\alpha}(t,t_1)\, \Sigma_{\alpha\beta}(t_1,t_2)\, R^T_{\beta\nu}(t_2,t') \\ & + \sum_{\alpha\beta} \int dt_1\, R_{\mu\alpha}(t,t_1)\, 2D_{\alpha\beta} R^T_{\beta\nu}(t_1,t')\,. \end{aligned} \tag{6·11}$$

This is Wyld's equation[13] whose second order approximation is the direct interaction approximation of Kraichnan. Equation (6·11) as derived above, applies to many other physical situations, namely, to those described by Eq. (1·1) in a general non-stationary condition whenever the real part of λ_μ represents a damping and the spurious term (5·9) can be disregarded. This derivation is an alternative to the one given by Martin et al.,[2] derivation that clarifies the exact content and range of applicability of (6·11).

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Appendix

Replacement (2·4) has led us to classical non-commuting variables that already contain in their dynamics an average over the realizations of the stochastic force. To stress the different dynamics generated by (2·3) and (2·5), we denote in this appendix by $\psi_\mu{}^L(t)$ the variables satisfying (1·1) and whose evolution is then governed by (2·3). Substitution (2·4) follows from[4]

$$E_\xi\{\xi_\mu(t')A(\psi^L(t))\}=\theta(t-t')E_\xi\{\sum_\nu 2D_{\mu\nu}[\hat{\psi}_\nu(t'), A(\psi^L(t))]\} \tag{A·1}$$

and then

$$\langle E_\xi\{\xi_\mu(t')A(\psi^L(t))\}\rangle=\sum_\nu 2D_{\mu\nu}\langle \overline{T}(E_\xi\{\hat{\psi}_\nu(t')A(\psi^L(t))\})\rangle, \tag{A·2}$$

where $A(\psi^L(t))$ is any functional of $\psi^L(t)$.

To obtain Eq. (A·1) use is made of Novikov's theorem[24] in the form

$$E_\xi\{\xi_\mu(t')A(\psi^L(t))\}=E_\xi\left\{\sum_\nu 2D_{\mu\nu}\frac{\delta A(\psi^L(t))}{\delta\xi_\nu(t')}\right\}. \tag{A·3}$$

The functional derivative of $A(\psi^L(t))$ with respect to the random force is then evaluated similarly to (3·7) by taking the random force as an uncoupled external field.

We feel that a complete understanding of (2·4) is given by proving its validity in each order of perturbation theory as we now do. The explicit manner in which the stochastic propagator is constructed from the Langevin equation gives to (2·4) its exact meaning.

Starting with the Langevin Liouvillian (2·3) we take now the free part to be

$$L_0(\psi,\hat{\psi},t)=\sum_\mu(\lambda_\mu\psi_\mu+\xi_\mu(t))\hat{\psi}_\mu. \tag{A·4}$$

The equation of motion for $\hat{\psi}_\mu(t)$ is the same as that calculated with the Liouvillian $(2\cdot5)$, and therefore we do not distinguish between $\hat{\psi}_\mu(t)$ and $\hat{\psi}_\mu{}^L(t)$. The $\psi_\mu{}^L(t)$ variables do now commute, and the free evolution is

$$\psi_\mu{}^{0L}(t)=W_\mu(t)\psi_\mu+h_\mu(t), \tag{A$\cdot$5}$$

where

$$h_\mu(t)=\int_0^t W_\mu(t-s)\xi_\mu(s)\,ds\,. \tag{A$\cdot$6}$$

The definitions of the propagators involve now one further average over the realizations of $\xi_\mu(t)$. They are

$$G_{\mu\nu}(t,t')=\langle E_\xi\{\psi_\mu{}^L(t)\psi_\mu{}^L(t')\}\rangle, \tag{A$\cdot$7}$$

$$R_{\mu\nu}(t,t')=\sum_\delta\langle\bar{T}(E_\xi\{\psi_\mu{}^L(t)\Gamma_{\delta\nu}(\psi^L(t'),t')\hat{\psi}_\delta(t')\})\rangle. \tag{A$\cdot$8}$$

The free response function is again given by $(3\cdot11)$ and

$$G^0_{\mu\nu}(t,t')=W_\mu(t)W_\nu(t')\langle\psi_\mu\psi_\nu\rangle+E_\xi\{h_\mu(t)h_\nu(t')\}. \tag{A$\cdot$9}$$

According to $(1\cdot2)$,

$$E_\xi\{h_\mu(t)\ h_\nu(t')\}=U^0_{\mu\nu}(t,t') \tag{A$\cdot$10}$$

and therefore $(3\cdot12)$ is still valid.

Equation $(\mathrm{A}\cdot10)$ establishes in an explicit manner the desired connection between the statistical treatment of the Langevin and adjoint Fokker-Planck Liouvillians.

The operational Wick's theorem of §4 holds for the Langevin Liouvillian but only the contraction giving rise to $R^0_{\mu\nu}(t,t')$ does not vanish. In the corresponding statistical Wick's theorem for the Langevin equation $(1\cdot1)$ there are two averages to be evaluated. We perform first the average over $\xi_\mu(t)$. A tedious combinatorial calculation leads to an expression that only differs from $(4\cdot5)$ in terms that will vanish when the second average over initial conditions is carried out, whatever the initial distribution may be. (This is due to the Gaussian character of $\xi_\mu(t)$ and to Eq. $(\mathrm{A}\cdot10)$.) Those are terms in which the $\hat{\psi}_\mu(t)$ variables at time $t=0$ are placed at the extreme right. Thus, for any $\rho(\psi;0)$ we obtain again Eq. $(5\cdot2)$.

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