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RESEARCH NOTE

Response of an infinite elastic transversely isotropic medium to a point force. An analytical solution in Hankel space

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SUMMARY
The displacement field of an elastic transversely isotropic medium due to a harmonic force, located at one point of an infinite 3-D space, is given in this paper. The Kupradze method allows the number of unknown displacement functions to be reduced to only one potential scalar function and permits an analytical solution in Hankel space.

Key words: Displacement field, Green's functions, transversely isotropic medium

1 INTRODUCTION

To solve wave problems, boundary methods have increasingly been used in elastic isotropic media and more recently in anisotropic media. But this approach is dependent on progress being made in our knowledge of fundamental solutions which are trivial for the isotropic case, but partly known or non-existent in more complicated media. It is remarkable, for instance, that even in the simple case of a transversely isotropic medium no general fundamental solution exists.

Several authors have made partial approaches to this problem. Kraut (1963) gives Green displacement functions at a distance from the force source using an infinite number of plane waves. Other authors (Mikhaïlenko, Martinov & Mikhaïlenko 1984) give a numerical–analytical solution at any distance from the source for a vertical force, an explosion and a dipole, using separate variables and the finite-difference method. More recently, Ha (1986) gives elaborate algorithms for far-field wave computations in the case of a layered or transversely isotropic medium.

The aim of this article is to obtain general fundamental solutions or so-called Green's functions which give the response at any point of an infinite transversely isotropic elastic medium to a harmonic point force source.

In the next section fundamental dynamic movement equations are recalled in the frequency space and in a transformed geometrical space. In Section 3, the Kupradze method is developed, giving the displacement scalar potential in this transformed geometrical space. In Section 4 the analytical solution in Hankel space is given, using an inverse Fourier transform with respect to the anisotropy axis. This is followed by a brief overview of methods for returning to the original space.

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2 EQUATIONS OF MOTION

In principal axes i_1, i_2, i_3 of a transversely isotropic medium with axis i_3 giving the direction of anisotropy, the balance of momentum and elastic stress–strain relations are given, after a time Fourier transform, by:

$$\sigma_{ij,j} = -\rho\omega^2 u_i - F_i \tag{2.1}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} * \begin{bmatrix} e_{11}\\ e_{22}\\ e_{33}\\ e_{23}\\ e_{13}\\ e_{12}\end{bmatrix} \tag{2.2}$$

$$C_{11} = C_{12} + C_{66},$$

where σ_{ij} are the components of the stress tensor $\boldsymbol{\sigma}$, u_i the components of the displacement $\mathbf{u}$ in the frequency space, e_{ij} the components of the strain tensor $\mathbf{e}$, ω the pulsation, ρ the mass density, c_{11} to c_{66} the elastic constants of the transversely isotropic medium and F_i the components of the mass force $\mathbf{F}$.

The displacement field $\mathbf{u}(\mathbf{x}, \omega)$ at point $\mathbf{x}$ (x_1, x_2, x_3) is a solution of the differential equation

$$\mathscr{L}\mathbf{u} = -\mathbf{F}. \tag{2.3}$$

$\mathscr{L}$ is the differential operator of components $\mathscr{L}_{ij}$ given by Table 1 or under a symbolic form as follows, with

Table 1. Differential operator $\mathscr{L}$.

$c_{11}\frac{\partial^2}{\partial x_1^2}+\frac{1}{2}c_{66}\frac{\partial^2}{\partial x_2^2}+\frac{1}{2}c_{44}\frac{\partial^2}{\partial x_3^2}+\rho\omega^2$	$(c_{12}+\frac{1}{2}c_{66})\frac{\partial^2}{\partial x_1\,\partial x_2}$	$(c_{13}+\frac{1}{2}c_{44})\frac{\partial^2}{\partial x_1\,\partial x_3}$
symmetric	$c_{11}\frac{\partial^2}{\partial x_2^2}+\frac{1}{2}c_{66}\frac{\partial^2}{\partial x_1^2}+\frac{1}{2}c_{44}\frac{\partial^2}{\partial x_3^2}+\rho\omega^2$	$(c_{13}+\frac{1}{2}c_{44})\frac{\partial^2}{\partial x_2\,\partial x_3}$
symmetric	symmetric	$+\frac{1}{2}c_{44}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+c_{33}\frac{\partial^2}{\partial x_3^2}+\rho\omega^2$

$i, j, k, m = 1, 2, 3$:

$$\begin{aligned}\mathscr{L}_{ij} = [&(c_{11}-\tfrac{1}{2}c_{66})(\delta_{i1}+\delta_{i2})(\delta_{j1}+\delta_{j2})\delta_{ik}\delta_{jm}\\ &+\tfrac{1}{2}c_{66}\delta_{ij}(\delta_{i1}+\delta_{i2})(\delta_{k1}+\delta_{k2})\delta_{km}\\ &+\tfrac{1}{2}c_{44}\delta_{ij}(\delta_{i1}\delta_{k3}+\delta_{i2}\delta_{k3}+\delta_{i3}(\delta_{k1}+\delta_{k2}))\delta_{km}\\ &+(c_{13}+\tfrac{1}{2}c_{44})(\delta_{i3}(\delta_{j1}+\delta_{j2})+\delta_{j3}(\delta_{i1}+\delta_{i2}))\delta_{ik}\delta_{jm}\\ &+c_{33}\delta_{i3}\delta_{j3}\delta_{k3}\delta_{m3}]\ \partial^2/\partial x_k\,\partial x_m+\rho\omega^2\delta_{ij},\end{aligned}$$

with δ_{ij} the Kronecker symbol ($=1$ only for $i=j$).

Green's function u_{kj} represents the displacement component in direction k due to a Dirac force of pulsation ω, in direction j, which may be placed, in view of some half-space problems and without reducing generality, at point $P(0, 0, c)$. It is given by:

$$\mathscr{L}_{ik}u_{kj} = \delta_{ij}\Delta(0, 0, c)\cdot f(\omega), \qquad (2.4)$$

where $\mathscr{L}_{ik}$ are the components of operator $\mathscr{L}$ and Δ the Dirac function.

In order to solve analytically the differential equation (2.4) for a given force, in direction j, a 3-D Fourier transform is used, vector $\boldsymbol{x}$ becoming vector ξ and displacement components u_{kj} becoming u^*_{kj} according to the transform formula (2.5):

$$u^*_{kj} = \iiint_{-\infty}^{+\infty} u_{kj}\exp(-i\boldsymbol{x}\cdot\xi)\,\mathrm{d}\boldsymbol{x}. \qquad (2.5)$$

Then equation (2.4) becomes:

$$\mathscr{L}^*_{ik}u^*_{kj} = \delta_{ij}\exp(-ic\xi_3)f(\omega), \qquad (2.6)$$

where $\mathscr{L}^*_{ik}$ are the components of the algebraic operator given in Table 2.

Table 2. Algebraic operator $\mathscr{L}^*$.

$-c_{11}\xi_1^2-\frac{1}{2}c_{66}\xi_2^2-\frac{1}{2}c_{44}\xi_3^2+\rho\omega^2$	$-(c_{12}+\frac{1}{2}c_{66})\xi_1\xi_2$	$-(c_{13}+\frac{1}{2}c_{44})\xi_1\xi_3$
$-(c_{12}+\frac{1}{2}c_{66})\xi_1\xi_2$	$-c_{11}\xi_2^2-\frac{1}{2}c_{66}\xi_1^2-\frac{1}{2}c_{44}\xi_3^2+\rho\omega^2$	$-(c_{13}+\frac{1}{2}c_{44})\xi_2\xi_3$
$-(c_{13}+\frac{1}{2}c_{44})\xi_1\xi_3$	$-(c_{13}+\frac{1}{2}c_{44})\xi_2\xi_3$	$-\frac{1}{2}c_{44}(\xi_1^2+\xi_2^2)-c_{33}\xi_3^2+\rho\omega^2$

3 DISPLACEMENT FIELD IN THE 3-D FOURIER TRANSFORMED SPACE

Equation (2.4) shows that six functions u_{kj} are unknown, the operator $\mathscr{L}_{ik}$ being symmetrical. Kupradze gives a method (Kupradze 1979) for finding out these functions using one scalar potential function $\varphi(\mathbf{x}, \omega)$ defined by the following expression:

$$u_{kj} = \mathscr{L}^+_{kj}\varphi \qquad (3.1)$$

in which the operator $\mathscr{L}^+$ is the differential operator (3×3) given by the cofactors of $\mathscr{L}$. In the case of a tranversely isotropic medium, operator $\mathscr{L}^+$ is given in Table 3. Equation (2.4) becomes:

$$(\mathscr{L}_{ik}\mathscr{L}^+_{kj})\varphi = (\delta_{ij}\det\mathscr{L})\varphi = \delta_{ij}\Delta(0, 0, c)\cdot f(\omega). \qquad (3.2)$$

Hence,

$$(\det\mathscr{L})\varphi = \Delta(0, 0, c)\cdot f(\omega). \qquad (3.3)$$

The 3-D Fourier transform of equation (3.3) leads to:

$$(\det\mathscr{L}^*)\varphi^* = \exp(-ic\xi_3)f(\omega). \qquad (3.4)$$

This determinant $\det\mathscr{L}^*$ is an algebraic function of ξ_1, ξ_2, ξ_3 and may be developed according to ξ_3 as follows:

$$\det\mathscr{L}^* = -\left\{\frac{c_{66}}{2}\xi^2+\frac{c_{44}}{2}\xi_3^2-\rho\omega^2\right\}\{A\xi_3^4+B\xi_3^2+C\} \qquad (3.5)$$

with the following coefficients:

$$A = \frac{c_{33}c_{44}}{2}$$

$$B = \left\{c_{33}c_{11}+\frac{c_{44}^2}{4}-\left(c_{13}+\frac{c_{44}}{2}\right)^2\right\}\xi^2-\left\{c_{33}+\frac{c_{44}}{2}\right\}\rho\omega^2$$

$$C = \frac{c_{11}c_{44}}{2}\xi^4-\left\{c_{11}+\frac{c_{44}}{2}\right\}\xi^2\rho\omega^2+\rho^2\omega^4,$$

where $\xi^2 = \xi_1^2+\xi_2^2$.

Such a determinant is analogous to the left member of the classical Christoffel equation, the wave number k in the direction of propagation parallel to the vector (ξ_1, ξ_2, ξ_3) being given by $k^2 = \xi_1^2+\xi_2^2+\xi_3^2$.

The algebraic solution of equation (3.4) is:

$$\varphi^*(\xi_1, \xi_2, \xi_3) = -\frac{\exp(-ic\xi_3)f(\omega)}{\left\{\frac{c_{66}}{2}\xi^2+\frac{c_{44}}{2}\xi_3^2-\rho\omega^2\right\}\{A\xi_3^4+B\xi_3^2+C\}}. \qquad (3.6)$$

Using a 3-D Fourier transform on operator $\mathscr{L}^+$ given in Table 3, it would be possible to obtain the six fundamental displacement functions u^*_{kj} in the transformed geometrical space. But to return to the original space, applying inverse transformation to φ^* first, is more convenient, as shown below.

Table 3. Differential operator $\mathscr{L}^+$

$\left\{c_{11}\dfrac{\partial^2}{\partial x_2^2}+\tfrac{1}{2}c_{66}\dfrac{\partial^2}{\partial x_1^2}+\tfrac{1}{2}c_{44}\dfrac{\partial^2}{\partial x_3^2}\rho\omega^2\right\}$ $*\left\{\tfrac{1}{2}c_{44}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)+c_{33}\dfrac{\partial^2}{\partial x_3^2}+\rho\omega^2\right\}$ $-(c_{13}+\tfrac{1}{2}c_{44})^2\dfrac{\partial^4}{\partial x_2^2\,\partial x_3^2}$	$(c_{13}+\tfrac{1}{2}c_{44})^2\dfrac{\partial^4}{\partial x_1\,\partial x_2\,\partial x_3^2}$ $-\left\{\tfrac{1}{2}c_{44}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)+c_{33}\dfrac{\partial^2}{\partial x_3^2}+\rho\omega^2\right\}$ $*(c_{12}+\tfrac{1}{2}c_{66})\dfrac{\partial^2}{\partial x_1\,\partial x_2}$	$(c_{12}+\tfrac{1}{2}c_{66})(c_{13}+\tfrac{1}{2}c_{44})\dfrac{\partial^4}{\partial x_1\,\partial x_2^2\,\partial x_3}$ $-\left\{c_{11}\dfrac{\partial^2}{\partial x_2^2}+\tfrac{1}{2}c_{66}\dfrac{\partial^2}{\partial x_1^2}+\tfrac{1}{2}c_{44}\dfrac{\partial^2}{\partial x_3^2}\rho\omega^2\right\}$ $*(c_{13}+\tfrac{1}{2}c_{44})\dfrac{\partial^2}{\partial x_1\,\partial x_3}$
symmetric	$\left\{c_{11}\dfrac{\partial^2}{\partial x_1^2}+\tfrac{1}{2}c_{66}\dfrac{\partial^2}{\partial x_2^2}+\tfrac{1}{2}c_{44}\dfrac{\partial^2}{\partial x_3^2}\rho\omega^2\right\}$ $*\left\{\tfrac{1}{2}c_{44}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)+c_{33}\dfrac{\partial^2}{\partial x_3^2}+\rho\omega^2\right\}$ $-(c_{13}+\tfrac{1}{2}c_{44})^2\dfrac{\partial^4}{\partial x_1^2\,\partial x_3^2}$	$(c_{12}+\tfrac{1}{2}c_{66})(c_{13}+\tfrac{1}{2}c_{44})\dfrac{\partial^4}{\partial x_1^2\,\partial x_2\,\partial x_3}$ $-\left\{c_{11}\dfrac{\partial^2}{\partial x_1^2}+\tfrac{1}{2}c_{66}\dfrac{\partial^2}{\partial x_2^2}+\tfrac{1}{2}c_{44}\dfrac{\partial^2}{\partial x_3^2}\rho\omega^2\right\}$ $*(c_{13}+\tfrac{1}{2}c_{44})\dfrac{\partial^2}{\partial x_2\,\partial x_3}$
symmetric	symmetric	$\left\{c_{11}\dfrac{\partial^2}{\partial x_1^2}+\tfrac{1}{2}c_{66}\dfrac{\partial^2}{\partial x_2^2}+\tfrac{1}{2}c_{44}\dfrac{\partial^2}{\partial x_3^2}\rho\omega^2\right\}$ $*\left\{c_{11}\dfrac{\partial^2}{\partial x_2^2}+\tfrac{1}{2}c_{66}\dfrac{\partial^2}{\partial x_1^2}+\tfrac{1}{2}c_{44}\dfrac{\partial^2}{\partial x_3^2}\rho\omega^2\right\}$ $-(c_{12}+\tfrac{1}{2}c_{66})^2\dfrac{\partial^4}{\partial x_1^2\,\partial x_2^2}$

4 INVERSE TRANSFORMATIONS

Coefficients of expression (3.5) show a symmetry around ξ_3 axis for $\varphi^*(\xi_1, \xi_2, \xi_3)$. Such a symmetry is not true for functions u_{ij} but is restored for scalar Kupradze potential φ, a fundamental property for what follows. When dealing with variables x_1 and x_2 a direct Fourier transform F may be replaced by a Hankel transform $\mathbf{H}_0$, using the following properties (Bracewell 1965, p. 247, footnote 7):

$$\mathrm{F}[f(\sqrt{x_1^2+x_2^2})] = \mathbf{H}_0[f(r)] \times 2\pi \quad \text{with} \quad r^2 = x_1^2 + x_2^2,$$

and with the following expression of the Hankel transform:

$$\mathbf{H}_0[f(r)] = \int_0^{+\infty} r f(r) J_0(qr)\, dr.$$

In the case of potential φ this identity becomes, for a given value of ξ_3:

$$\varphi^*(\xi_1, \xi_2, \xi_3) = 2\pi\varphi'^*(\xi, \xi_3) \quad \text{with} \quad \xi^2 = \xi_1^2 + \xi_2^2, \tag{4.1}$$

where φ'^* is the Hankel transform of $\varphi(x_1, x_2, \xi_3)$.

To return to the geometrical space, an inverse Fourier transform acting on ξ_3 is used as a first step to a return to the Hankel space; then, in a second step, an inverse Hankel transform acting on the variable ξ is needed. Finally, we have to come back to the time space.

4.1 Fourier inverse transformation on variable ξ_3-solution in Hankel space

Equation (3.5) may be written as follows:

$$\det \mathscr{L}^* = -\frac{Ac_{44}}{2}\{\xi_3^2 - \xi_{31}^2\}\{\xi_3^2 - \xi_{32}^2\}\{\xi_3^2 - \xi_{33}^2\}, \tag{4.2}$$

where the first root is:

$$\xi_{31}^2 = -\frac{c_{66}}{c_{44}}\xi^2 + \frac{2\rho\omega^2}{c_{44}} \tag{4.3}$$

and the two others are the roots of a second-degree equation.

Formula (3.6) thus becomes, using identity (4.1) and expression (4.2),

$$\varphi'^*(\xi, \xi_3) = -\frac{f(\omega)\exp(-ic\xi_3)}{Ac_{44}(\xi_3^2 - \xi_{31}^2)(\xi_3^2 - \xi_{32}^2)(\xi_3^2 - \xi_{33}^2)} \times \frac{1}{\pi}. \tag{4.4}$$

Inverse Fourier transform on variable ξ_3, according to direct transform formula (2.5), is given by:

$$\varphi_1'^*(\xi, x_3) = -\frac{1}{2\pi^2}\frac{f(\omega)}{Ac_{44}} \times \int_{-\infty}^{+\infty} \frac{\exp(+ix_3\xi_3)\exp(-ic\xi_3)}{(\xi_3^2 - \xi_{31}^2)(\xi_3^2 - \xi_{32}^2)(\xi_3^2 - \xi_{33}^2)}\, d\xi_3. \tag{4.5}$$

Considering ξ_3 as a complex number and integrating along a circuit made by the real axis and a semi-infinite circle, we obtain:

$$\varphi_1'^*(\xi, x_3) = -\frac{i\,f(\omega)}{\pi Ac_{44}}\sum \text{residues},$$

which gives

$$\varphi_1'^*(\xi, x_3) = -\frac{i\,f(\omega)}{\pi Ac_{44}}\sum_{j=1}^{j=3}\frac{\exp(i\xi_{3j}(|x_3 - c|))}{2\xi_{3j}(\xi_{3j}^2 - \xi_{3j+1}^2)(\xi_{3j}^2 - \xi_{3j+2}^2)}, \tag{4.6}$$

with $\xi_{34}^2 = \xi_{31}^2$ and $\xi_{35}^2 = \xi_{32}^2$.

This expression takes into account the Sommerfeld condition and is written with values of poles in the first quadrant.

4.2 Hankel inverse transformation on variable ξ-solution in original space

The Potential solution given by an inverse Hankel transformation applied to $\varphi_1'^*(\xi, x_3)$, according to variable ξ, at pulsation ω, is:

$$\varphi(r, x_3) = -\frac{i f(\omega)}{\pi A c_{44}} \times \sum_{j=1}^{j=3} \int_0^{+\infty} \xi \frac{\exp[i\xi_{3j}(|x_3 - c|)]J_0(r\xi)}{2\xi_{3j}(\xi_{3j}^2 - \xi_{3j+1}^2)(\xi_{3j}^2 - \xi_{3j+2}^2)} d\xi \quad (4.7)$$

The fundamental solutions or Green's functions are finally given by:

$$u_{ij}(x_1, x_2, x_3, \omega) = \mathscr{L}_{ij}^+ \varphi((x_1^2 + x_2^2)^{1/2}, x_3, \omega), \quad (4.8)$$

where operators $\mathscr{L}_{ij}^+$ are given in Table 3. This approach involves numerical derivations of the preceding expression of the potential.

To avoid such additional numerical derivations, an alternative method consists in inversing the process of integration and derivation, allowing analytical derivations to be made before numerical integrations. Of course, the number of integrations is higher, and the integrand is composed of elementary terms as:

$$\int_0^{\infty} \xi^p \xi_{3j}^q \frac{\exp[i\xi_{3j}(|x_3 - c|)]J_0(\xi r)}{2\xi_{3j}(\xi_{3j}^2 - \xi_{3j+1}^2)(\xi_{3j}^2 - \xi_{3j+2}^2)} d\xi$$

$$\int_0^{\infty} \xi^p \xi_{3j}^q \frac{\exp[i\xi_{3j}(|x_3 - c|)]J_1(\xi r)}{2\xi_{3j}(\xi_{3j}^2 - \xi_{3j+1}^2)(\xi_{3j}^2 - \xi_{3j+2}^2)} d\xi.$$

Different numerical methods have been used to integrate similar expressions in the simpler case of an isotropic infinite medium. A first method consists in a discretization of the integrals according to the discrete wave-number method, described, for instance, in Bouchon (1980, 1981) and Bouchon & Aki (1977). A second method uses algorithms of Fast Fourier Transform (Chapel & Tsakalidis 1985).

4.3 Time Fourier inverse transform

In all cases discretizations of the integrals, as described above, introduce, when doing a time Fourier inverse transform, systematic errors which may be interpreted as the effect of periodically located virtual sources added to the original source. But if the distance of these sources to the original source is large enough, compared with the time interval, the attained fields $u_{ij}(x_1, x_2, x_3, t)$ are the desired solutions.

CONCLUSION

The solution given in this paper describes the method for computing the displacement field in an infinite elastic transversely isotropic medium, created by a point force source, without any restriction on the distance to the source. For each component of the displacement fields, the only required numerical computations are one Hankel transform and one time Fourier transform. The applications of this method are numerous in the field of fundamental geophysics but also in seismic engineering.

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