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Response of multiple horizontal soil layers using higher order frequency response functions

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Abstract

For several years the Equivalent Linear Method was the most used method to solve dynamic problems of stratified soil deposits. This is valid for soils showing small or medium nonlinearities. It is an iterative procedure without a mathematical approach associated to it. The proposed mathematical method is based on the use of the First and Higher Order Frequency Response functions as required by the Volterra series. This method is based in the determination of the functions that relates the input motion to the response. In this study, up to the third order frequency response functions (kernel) are developed. The use of these expressions in some numerical examples and a comparison with the Equivalent Linear Method will be presented. Also, a sensitivity analysis is done to show how the series is affected with the quantity of terms in it. It will be demonstrated that at low or medium shear strain, three terms in the series are acceptable to obtain the result produced by the Equivalent Linear Method.

1 Introduction

In practice most soil deposits do not consist of only one layer. Usually there are a number of horizontal layers with different properties and thickneses and therefore, multiple degrees of freedom models (MDOFS) are needed. These models are used in this paper, in which a combination of the shear beam model and the hysteretic damping model are employed to determine the seismic response of a soil deposit.

For comparison purposes, the response is calculated using two methods: the equivalent linear method and the higher order frequency response function method based on the Volterra series. Finally, some numerical examples are presented to confirm that the use of the Volterra series-based method is an acceptable mathematical tool to calculate the response of soil deposits with multiple horizontal layers.

2 Equation of motion of a MDOFS with hysteretic damping

The equations of motion of a MDOF system with hysteretic damping are

$$[M]\,\underset{\sim}{\ddot{w}}(t)+\left[K^*\right]\underset{\sim}{w}(t)=\underset{\sim}{F}(t) \tag{1}$$

where

$[M]$ = global mass matrix

$\left[K^*\right]$ = global complex stiffness matrix

$\underset{\sim}{\ddot{w}}(t)$ = relative acceleration vector

$\underset{\sim}{w}(t)$ = relative displacement vector

$\underset{\sim}{F}(t)$ = vector of nodal forces

Each degree of freedom $w_i(t)$ represents one interface between two consecutive soil layers. If the system is subjected to a bedrock excitation, the load vector is calculated as

$$\underset{\sim}{F}(t)=-[M]\underset{\sim}{r}\,\ddot{x}_g(t) \tag{2}$$

where $\ddot{x}_g(t)$ is the acceleration time history of the bedrock and $\underset{\sim}{r}$ is the influence vector, a column vector of one's for this model. The equations of motion for a MDOF system with hysteretic damping and for an upward wave propagation from the bedrock motion are

$$[M]\,\underset{\sim}{\ddot{w}}(t)+\left[K^*\right]\underset{\sim}{w}(t)=-[M]\underset{\sim}{r}\,\ddot{x}_g(t) \tag{3}$$

3 First order frequency response matrix

If a unit harmonic load $e^{i\Omega t}$ acts on the "p" degree of freedom, the response of the "j" degree of freedom can be obtained as

$$u_j = H_1^{jp}(\Omega)e^{i\Omega t} \qquad 1 \le j \le n \tag{4}$$

where

j = degree of freedom where the response is calculated

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p = loaded degree of freedom

$H_1^{jp}(\Omega)$ = first order Frequency Response Function (FRF)

in which each column of matrix $\left[H_1(\Omega)\right]$ correspond to a loaded degree of freedom. Using the following expression for the complex shear modulus

$$G_j^* = G_j\left(1 + 2i\xi_j\right) \tag{5}$$

and (eqn 4) in (eqn 3), matrix $\left[H_1(\Omega)\right]$ has the form

$$\left[H_1(\Omega)\right] = \begin{bmatrix} A_1 & B_1 & 0 & 0 & 0 & 0 \\ B_1 & A_1 + A_2 & B_2 & 0 & 0 & 0 \\ 0 & B_2 & A_2 + A_3 & B_3 & 0 & 0 \\ 0 & 0 & B_3 & A_3 + \cdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots + A_{n-1} & B_{n-1} \\ 0 & 0 & 0 & 0 & B_{n-1} & A_{n-1} + A_n \end{bmatrix}^{-1} \tag{6}$$

where

$$A_j = -m_j\Omega^2 + \frac{G_{0j}}{h_j} + i\frac{2G_{0j}\xi_{0j}}{h_j}$$

$$B_j = -\frac{m_j}{2}\Omega^2 - \frac{G_{0j}}{h_j} - i\frac{2G_{0j}\xi_{0j}}{h_j}$$

$$m_j = \frac{\rho_j h_j}{3}$$

The size of matrix $\left[H_1(\Omega)\right]$ depend on the number of degrees of freedom of the model. In practice, the analysis will be carried out using a set of discrete frequencies Ω_k. In this case there will be one matrix for each frequency in the spectrum.

4 Second order frequency response matrix

Following the same procedure as in the first order case, the expression of the response is

$$u_j = H_1^{jp}(\Omega_1)e^{i\Omega_1 t} + H_1^{jp}(\Omega_2)e^{i\Omega_2 t} + 2H_2^{jp}(\Omega_1,\Omega_2)e^{i(\Omega_1+\Omega_2)t} \tag{7}$$

Taking the appropriate derivatives of (eqn 7) and making the substitution in the equation of motion, the second order FRM has the form

$$\left[H_2(\Omega_1,\Omega_2)\right] = \left[H_1(\Omega_1 + \Omega_2)\right]\left[\alpha\right] \tag{8}$$

Therefore, the second order FR matrix depends on the first order FRM evaluated at $(\Omega_1 + \Omega_2)$ and a matrix $[\alpha]$. Matrix $[\alpha]$ has the form

$$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ -\alpha_{11} + \alpha_{21} & -\alpha_{12} + \alpha_{22} & -\alpha_{13} + \alpha_{23} & \cdots \\ -\alpha_{21} + \alpha_{31} & -\alpha_{22} + \alpha_{32} & -\alpha_{23} + \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_{(n-1)1} + \alpha_{n1} & -\alpha_{(n-1)2} + \alpha_{n2} & -\alpha_{(n-1)3} + \alpha_{n3} & \cdots \end{bmatrix} \tag{9}$$

where

$$\alpha_{jm} = \hat{S}_j\{H_1^{jm}(\Omega_1)H_1^{jm}(\Omega_2) - H_1^{jm}(\Omega_1)H_1^{(j+1)m}(\Omega_2) - H_1^{(j+1)m}(\Omega_1)H_1^{jm}(\Omega_2) + H_1^{(j+1)m}(\Omega_1)H_1^{(j+1)m}(\Omega_2)\} \qquad 1 \le j \le (n-1)$$

$$\alpha_{jm} = \hat{S}_j\{H_1^{jm}(\Omega_1)H_1^{jm}(\Omega_2)\} \qquad j = n$$

$$\hat{S}_j = \frac{G_{0j}}{h_j^2\gamma_{rj}} - \frac{4G_{0j}}{3\pi h_j^2\gamma_{rj}} + \frac{i2G_{0j}\xi_{0j}}{h_j^2\gamma_{rj}}$$

The nonlinear behavior of each layer of the deposit is included through the term $\hat{S}_j$. Also one matrix $[H_2]$ exist for each combination of Ω_1 and Ω_2.

5 Third order frequency response matrix

The solution becomes more complex as the terms in the expression of the response increases, the third order case becomes

$$\begin{aligned} u_j = {} & H_1^{jp}(\Omega_1)e^{i\Omega_1 t} + H_1^{jp}(\Omega_2)e^{i\Omega_2 t} + H_1^{jp}(\Omega_3)e^{i\Omega_3 t} + 2H_2^{jp}(\Omega_1,\Omega_2)e^{i(\Omega_1+\Omega_2)t} \\ & + 2H_2^{jp}(\Omega_1,\Omega_3)e^{i(\Omega_1+\Omega_3)t} + 2H_2^{jp}(\Omega_2,\Omega_3)e^{i(\Omega_2+\Omega_3)t} \\ & + 6H_3^{jp}(\Omega_1,\Omega_2,\Omega_3)e^{i(\Omega_1+\Omega_2+\Omega_3)t} \end{aligned} \tag{10}$$

Substituting (eqn 10) and its time derivatives in the equation of motion and rearranging terms we obtain

$$[H_3(\Omega_1,\Omega_2,\Omega_3)] = [H_1(\Omega_1 + \Omega_2 + \Omega_3)][\Psi] \tag{11}$$

where

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$$[\Psi] = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \cdots \\ -\Psi_{11} + \Psi_{21} & -\Psi_{12} + \Psi_{22} & -\Psi_{13} + \Psi_{23} & \cdots \\ -\Psi_{21} + \Psi_{31} & -\Psi_{22} + \Psi_{32} & -\Psi_{23} + \Psi_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ -\Psi_{(n-1)1} + \Psi_{n1} & -\Psi_{(n-1)2} + \Psi_{n2} & -\Psi_{(n-1)3} + \Psi_{n3} & \cdots \end{bmatrix}$$

$$\Psi_{jm} = \left[L_j - iP_j\right]A_{jm} - \left[M_j + iQ_j\right]B_{jm} \qquad 1 \le j \le (n-1)$$

$$\Psi_{jm} = \left[L_j - iP_j\right]D_{jm} - \left[M_j + iQ_j\right]E_{jm} \qquad j = n$$

$$L_j = \frac{2G_{0j}}{3h_j^2\gamma_{rj}} \qquad M_j = \frac{G_{0j}}{h_j^3\gamma_{rj}^2}$$

$$Q_j = -\frac{2G_{0j}}{\pi h_j^3\gamma_{rj}^2} \qquad P_j = \frac{4}{3h_j}\left[\frac{2G_{0j}}{3\pi h_j\gamma_{rj}} - \frac{G_{0j}\xi_{0j}}{h_j\gamma_{rj}}\right]$$

$$\begin{aligned} A_{jm} = {} & H_1^{jm}(\Omega_1)H_2^{jm}(\Omega_2,\Omega_3) + H_1^{jm}(\Omega_2)H_2^{jm}(\Omega_1,\Omega_3) + H_1^{jm}(\Omega_3)H_2^{jm}(\Omega_1,\Omega_2) \\ & - H_1^{jm}(\Omega_1)H_2^{(j+1)m}(\Omega_2,\Omega_3) - H_1^{jm}(\Omega_2)H_2^{(j+1)m}(\Omega_1,\Omega_3) \\ & - H_1^{jm}(\Omega_3)H_2^{(j+1)m}(\Omega_1,\Omega_2) - H_1^{(j+1)m}(\Omega_1)H_2^{jm}(\Omega_2,\Omega_3) \\ & - H_1^{(j+1)m}(\Omega_2)H_2^{jm}(\Omega_1,\Omega_3) - H_1^{(j+1)m}(\Omega_3)H_2^{jm}(\Omega_1,\Omega_2) \\ & + H_1^{(j+1)m}(\Omega_1)H_2^{(j+1)m}(\Omega_2,\Omega_3) + H_1^{(j+1)m}(\Omega_2)H_2^{(j+1)m}(\Omega_1,\Omega_3) \\ & H_1^{(j+1)m}(\Omega_3)H_2^{(j+1)m}(\Omega_1,\Omega_2) \end{aligned}$$

$$\begin{aligned} B_{jm} = {} & H_1^{jm}(\Omega_1)H_1^{jm}(\Omega_2)H_1^{jm}(\Omega_3) - H_1^{jm}(\Omega_1)H_1^{jm}(\Omega_2)H_1^{(j+1)m}(\Omega_3) \\ & - H_1^{jm}(\Omega_1)H_1^{jm}(\Omega_3)H_1^{(j+1)m}(\Omega_2) - H_1^{jm}(\Omega_2)H_1^{jm}(\Omega_3)H_1^{(j+1)m}(\Omega_1) \\ & + H_1^{(j+1)m}(\Omega_1)H_1^{(j+1)m}(\Omega_2)H_1^{jm}(\Omega_3) + H_1^{(j+1)m}(\Omega_1)H_1^{(j+1)m}(\Omega_3)H_1^{jm}(\Omega_2) \\ & + H_1^{(j+1)m}(\Omega_2)H_1^{(j+1)m}(\Omega_3)H_1^{jm}(\Omega_1) - H_1^{(j+1)m}(\Omega_1)H_1^{(j+1)m}(\Omega_2)H_1^{(j+1)m}(\Omega_3) \end{aligned}$$

$$E_{jm} = H_1^{jm}(\Omega_1)H_1^{jm}(\Omega_2)H_1^{jm}(\Omega_3)$$

$$D_{jm} = H_1^{jm}(\Omega_1)H_2^{jm}(\Omega_2,\Omega_3) + H_1^{jm}(\Omega_2)H_2^{jm}(\Omega_1,\Omega_3) + H_1^{jm}(\Omega_3)H_2^{jm}(\Omega_1,\Omega_2)$$

It becomes apparent one of the main obstacles facing the implementation of a third or greater order solution, namely the huge amount of data that is necessary to manipulate. Note that, for each combination of Ω_1, Ω_2 and Ω_3 a matrix has to be assembled. In the solution process it is necessary to multiply the Frequency Response matrix by the Fourier transform of the excitation. Then, calculating the inverse transform of the first, second and third order solution, taking the diagonal of the second and third order case, the solution in the time domain is obtained.

6 Numerical examples

To show the implementation the proposed method of the Higher Order Frequency Response Matrix (HOFRM) and to compare it against the Equivalent Linear method (ELM) the soil deposit shown in Figure 1 will be analyzed. It consists of several horizontal layers over a bedrock. The base excitation is a half sine function with a period of 1.0 seconds and with an amplitude A_g= 0.01*g*. Then, a sensitivity analysis will be presented.

6.1 Multiple layer soil deposit

A soil system is now used to compared the proposed method and the LE method. It consists of the soil deposit with four layer shown in Figure 1. The properties of the layers are summarized in Table 1.

Table 1 Properties of the four layer soil deposit

Layer	G_0 (ksf)	Unit Weight (pcf)	γ_r	h (ft)
1	1000	100	0.0015	30
2	2000	100	0.0015	30
3	3000	120	0.0015	30
4	4000	120	0.0015	30

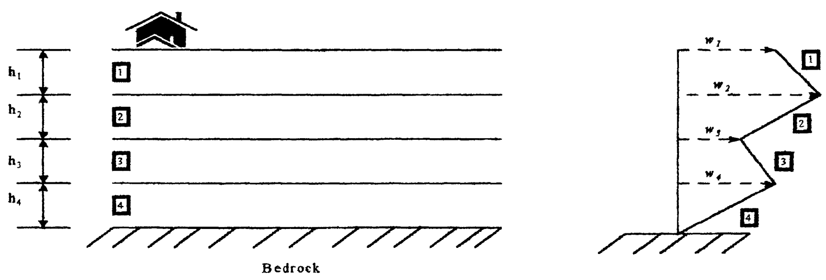

Figure 1 Four layer soils deposit

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The amplitude of the bedrock excitation A_g is equal to $0.01g$ and the damping ratio ξ is 0.08. Figure 2 shows the results when the input acceleration time history is applied at the bedrock and the Volterra series is used. When the same data was used with the LE method, a similar response was obtained. The results of the LE method are displayed in Figure 3. The maximum shear strains obtained during the time history analyses are presented in Table 2.

Table 2 Shear strain of four layers soils deposit

Method	HOFRM	LE
$(\gamma_1)_{max}$	0.0002	0.00016
$(\gamma_2)_{max}$	0.00028	0.00023
$(\gamma_3)_{max}$	0.0003	0.00025
$(\gamma_4)_{max}$	0.00028	0.00025

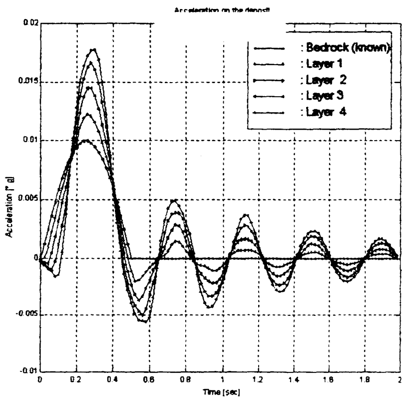

Figure 2 Response of a soil deposit with the HOFRF method

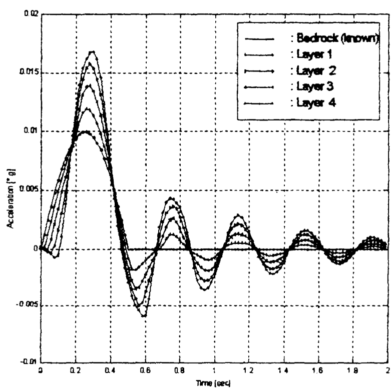

Figure 3 Response of a soil deposit with the LE method

6.2 Sensitivity of the Series

An analysis of the accuracy of the HOFRM method was made to know how many terms in the series are necessary to reach the true solution. It is reasonable to expect that the number of terms is directly proportional to the amplitude A_g of the input.

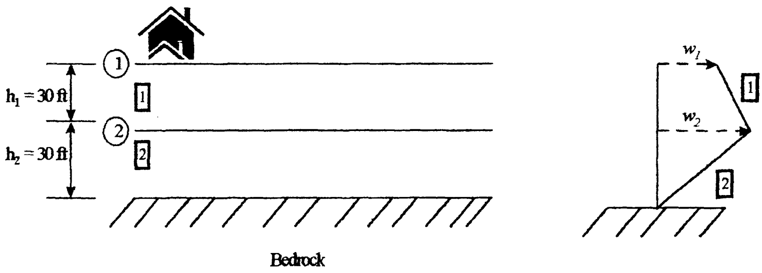

Figure 4 Two layer soil deposit

Table 3 present the properties of each layer in the soil deposit shown in Figure 4. The results of the analysis using two excitation amplitudes, $0.01g$ and $0.1g$ are presented in Figures 5 and 6. The absolute acceleration response of the bottom layer or the interface of both layers is presented in these figures.

Table 3 Properties of two layer soil deposit

Layer	G_0 (ksf)	Unit Weight (pcf)	γ_r	h (ft)
1	4000	100	0.0015	30
2	4000	100	0.0015	30

The graphs show that increasing the amplitude A_g from $0.01g$ to $0.1g$ and the maximum shear strain is greater than a medium range, increases the need to use more terms. When $A_g < 0.01g$, produces small shear strains and only one term in the series is necessary, that is, a linear solution suffices.

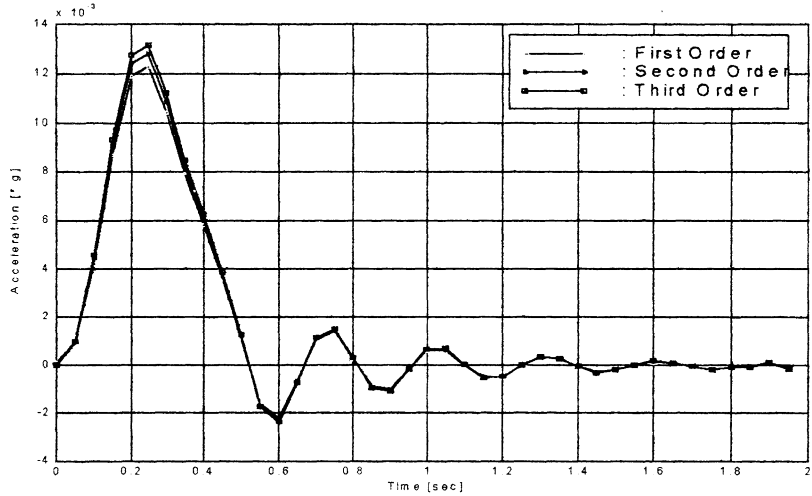

Figure 5 Sensitivity analysis, $A_g = 0.01g$

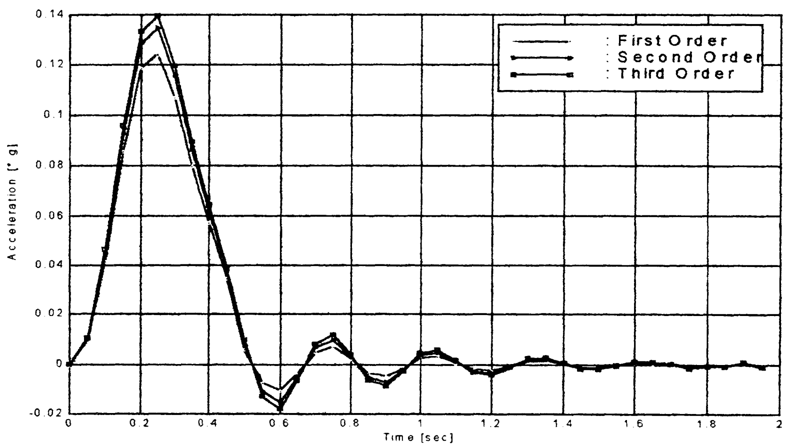

Figure 6 Sensitivity Analysis, $A_g = 0.1g$

7 Conclusions

In this paper a method to solve the problem of the propagation of a seismic wave through a soil deposit with multiple horizontal layers in the frequency domain is presented. The method is named the Higher Order Frequency Response Matrix (HOFRM) method and it is based upon the Volterra series for nonlinear systems. The proposed method can serve as a more rational and accurate alternative to the iterative Equivalent Linear (EL) method. Several examples were prepared to establish the accuracy of the proposed method. It was found that for low or medium shear strains, the use of three terms in the series was acceptable. The results obtained compared very well with those calculated via the EL method. The amplification of the input signal when it travels to the free surface or the signal reduction when the wave travels to the bedrock can be obtained using the HOFRM method.

The examples showed that the HOFRM method can be employed to calculate the response of a multiple layer soil deposit subjected to a prescribed acceleration of seismic origin strong enough to induce a nonlinear response in the medium.

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