Restricted Domination Parameters in Graphs

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Abstract

In a graph G, a vertex dominates itself and its neighbors. A subset $S \subseteq V(G)$ is an m-tuple dominating set if S dominates every vertex of G at least m times, and an m-dominating set if S dominates every vertex of $G - S$ at least m times. The minimum cardinality of a dominating set is γ, of an m-dominating set is γ_m, and of an m-tuple dominating set is $\gamma_{\times m}$. For a property π of subsets of $V(G)$, with associated parameter f_π, the k-restricted π-number $r_k(G, f_\pi)$ is the smallest integer r such that given any subset K of (at most) k vertices of G, there exists a π set containing K of (at most) cardinality r. We show that for $1 \le k \le n$ where n is the order of G: (a) if G has minimum degree m, then $r_k(G, \gamma_m) \le (mn+k)/(m+1)$; (b) if G has minimum degree 3, then $r_k(G, \gamma) \le (3n + 5k)/8$; and (c) if G is connected with minimum degree at least 2, then $r_k(G, \gamma_{\times 2}) \le 3n/4 + 2k/7$. These bounds are sharp.

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1 Introduction

In this paper we continue the study of restricted dominating sets started by Sanchis [16]: the restricted version of a parameter considers the case when certain vertices are specified to be in the set. We prove a general result which gives sharp bounds for several domination-like parameters, including domination and m-domination. We also establish a sharp bound for the case of double domination.

Suppose π is a property of sets of vertices (for example, being a dominating set). Suppose that f_π is the associated parameter: the minimum/maximum cardinality of a π-set; for definiteness, assume that f_π is the minimum cardinality. Then, for a graph G and a subset K of the vertex set, we define $r(G, K, f_\pi)$ as the minimum cardinality of a π-set containing K. The k-restricted π-number $r_k(G, f_\pi)$ is the maximum value of $r_k(G, f_\pi)$ taken over all subsets K of G of cardinality k. Note that the 0-restricted π-number is just f_π. (If f_π is a maximum, then swap minimum and maximum in the above definitions.)

In this paper we focus on parameters related to domination. For a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{ u \in V \mid uv \in E \}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a *dominating set* if each vertex in $V - S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

In [4] Fink and Jacobsen defined a subset S of V to be an *m-dominating set* (mDS) of G if for every vertex $v \in V - S$, $|N(v) \cap S| \geq m$. The *m-domination number* $\gamma_m(G)$ is the minimum cardinality of an mDS. Cockayne, Gamble and Shepherd provided a sharp upper bound:

Theorem 1 *([2]) If H is a graph of order n with minimum degree at least m, then $\gamma_m(H) \leq mn/(m + 1)$.*

In [7] Harary and Haynes defined a subset S of V to be a *double dominating set* (DDS) of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a DDS. More generally one can consider k-tuple domination; see for example [12, 13]. In [11] a sharp upper bound of the double domination of a connected graph with minimum degree at least 2 was established (and the graphs achieving the bound were characterized):

Theorem 2 *([11]) If $H \neq C_5$ is a connected graph of order n with minimum degree at least 2, then $\gamma_{\times 2}(H) \leq 3n/4$.*

The concept of restricted domination in graphs, where we restrict the dominating sets to contain any given subset of vertices, was introduced by Sanchis in [16] and studied further in [9, 17]. Restricted total domination in graphs was introduced and studied in [10]. Bounds on m-domination or double domination are given in [1, 5, 6, 11, 18] and elsewhere. For more on domination, see the book [8].

In this paper, we show that for $1 \le k \le n$ where n is the order of G:

- If minimum degree $\delta \ge 3$, then $r_k(G, \gamma) \le (3n + 5k)/8$, and this is sharp.
- If minimum degree $\delta \ge m$, then $r_k(G, \gamma_m) \le (mn + k)/(m + 1)$, and this is sharp.
- If minimum degree $\delta \ge 2$ and G is connected, then $r_k(G, \gamma_{\times 2}) \le 3n/4 + 2k/7$, and this is sharp.

The first two results are a consequence of a general result which derives a bound on the restricted domination number from a bound on the unrestricted version. For example, this gives a quick proof of the main result of [9]. It is to be noted that there are no useful bounds for smaller minimum degree: there are graphs G with $\delta = m - 1$ and $\gamma_m(G) = n$, and (hence) with $\delta = 1$ and $\gamma_{\times 2}(G) = n$.

For notation and graph theory terminology we in general follow [8]. The *subdivision* of a graph H, denoted by $S(H)$, is the graph obtained from H by subdividing every edge exactly once. A *vertex cover* S in G is a set of vertices such that every edge is incident with a vertex in S. The minimum cardinality of a vertex cover in G is denoted by $\alpha(G)$.

2 A General Restricted Domination Result

Given thresholds t_v at each vertex, S is a *threshold ordinary dominating* (TOD) set if $|N(v) \cap S| \ge t_v$ for all $v \in V - S$. Examples include: ordinary domination ($t_v \equiv 1$), m-domination ($t_v \equiv m$), and α-domination (t_v is α times the degree of v: see [3]).

We say that a family of graphs is *closed under pointwise identification*, if given G_1 and G_2 in the family, the graph formed by taking the disjoint union of G_1 and G_2 and identifying one vertex on each copy is in the set. For example, the set of trees is so closed. The notation $|G|$ represents the order of G.

We are now in a position to present a general restricted domination result.

Theorem 3 *Consider a threshold ordinary dominating parameter $\tilde{\gamma}$. Assume $\mathcal{F}$ is a set of graphs such that:*
(a) *$\mathcal{F}$ is closed under pointwise identifications;*
(b) *there is a constant c such that $\tilde{\gamma}(G) \leq c|G|$ for all $G \in \mathcal{F}$;*
(c) *there is a graph $F \in \mathcal{F}$ and a vertex $v \in V(F)$ such that $\tilde{\gamma}(F) = \tilde{\gamma}(F - v) = c|F|$ and there is a $\tilde{\gamma}$-set of F containing v.*
Then

$$r_k(G, \tilde{\gamma}) \leq c|G| + (1 - c)k$$

for all $G \in \mathcal{F}$ and for all k with $0 \leq k \leq |G|$. Furthermore this bound is sharp in that for every k there are infinitely many connected $G \in \mathcal{F}$ such that $r_k(G, \tilde{\gamma}) = c|G| + (1 - c)k$.

Proof. Let graph $G \in \mathcal{F}$ be given of order n. Say it has k specified vertices $K = \{w_1, \ldots, w_k\}$.

Construct a supergraph G' as follows. Take k disjoint copies of F; say these are $F_1, \ldots, F_k$ with the vertex corresponding to v called $v_1, \ldots, v_k$ in the corresponding copy. Then, identify v_i and w_i for $1 \leq i \leq k$. By the closure property of $\mathcal{F}$, the resultant graph G' is in $\mathcal{F}$.

Now, consider a minimum TOD-set S' of G'. There are at least $\tilde{\gamma}(F - v)$ vertices of S' in each copy of F. (We assume that the threshold t_v is nondecreasing as a function of the degree of v.) Since $\tilde{\gamma}(F - v) = \tilde{\gamma}(F)$ and there is a $\tilde{\gamma}$-set of F containing v, we may therefore assume that S' contains all of $w_1, \ldots, w_k$. It follows that $S = S' \cap V(G)$ is a TOD-set of G, containing K.

Hence,

$$r_k(G, \tilde{\gamma}) \leq |S| = |S'| - k(\tilde{\gamma}(F) - 1) \leq c(n + k(|F| - 1)) - k(c|F| - 1) = cn + (1 - c)k,$$

as required.

For the sharpness, note that by repeated pointwise identification with F it follows that $\mathcal{F}$ contains arbitrarily large connected graphs. Take such a graph and specify any k vertices. Add pointwise copies of F to the remaining vertices to form the graph G. It follows that $r_k(G, \tilde{\gamma}) = c|G| + (1 - c)k$. QED

2.1 Applications

As a consequence of Theorem 3, we can derive bounds on the restricted m-domination number and the restricted domination number from bounds on the unrestricted versions.

Corollary 1 *For positive integer m, if graph G has minimum degree at least m, then $r_k(G, \gamma_m) \leq (mn + k)/(m + 1)$ for $0 \leq k \leq n$.*

Proof. Apply Theorem 3 to the bound of Cockayne et al. given in Theorem 1 with $F = K_{m+1}$. QED

A special case of this is the result for ordinary domination:

Corollary 2 *([9]) If G is a graph with no isolated vertex, then $r_k(G, \gamma) \leq (n + k)/2$ for $0 \leq k \leq n$.*

Corollary 3 *([9]) If G is a connected graph of order n with minimum degree $\delta \geq 2$, then $r_k(G, \gamma) \leq (2n + 3k)/5$ for $1 \leq k \leq n$.*

Proof. McCuaig and Shepherd [14] showed that $\gamma(G) \leq 2n/5$ for a connected graph G with minimum degree at least 2, apart for seven exceptional graphs (one of order four and six of order seven). So apply Theorem 3 with $\mathcal{F}$ the connected graphs with minimum degree 2 apart from the seven exceptions, and with $F = C_5$. Then check by hand that the exceptional graphs are not exceptions when $k \geq 1$. QED

Corollary 4 *If G is a connected graph of order n with minimum degree $\delta \geq 3$, then $r_k(G, \gamma) \leq (3n + 5k)/8$ for $0 \leq k \leq n$.*

Proof. Reed [15] showed that $\delta \geq 3$ implies $\gamma(G) \leq 3n/8$. So apply Theorem 3 with F being the cubic nonplanar graph $\mathcal{C}$ on 8 vertices shown in Figure 1 with v any of the vertices of the triangle. QED

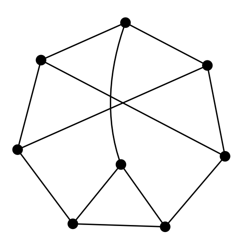

Figure 1: A cubic graph $\mathcal{C}$ with domination number 3

To illustrate the sharpness of Corollary 4, let H be a connected graph with K a set of $k \geq 0$ specified vertices each of which has degree at least 3 in H. For each vertex v of $V(H)-K$, add a (disjoint) copy of the graph $\mathcal{C}$ of Figure 1 and identify any one of its vertices that is in a triangle with v. Then, for G the resulting graph, we have $r_k(G, \gamma) = (3n+5k)/8$.

For minimum degree 4 we might expect a similar result, once the minimum degree bound is solved!

3 Restricted Double Domination

Our aim in this section is to establish the following result.

Theorem 4 *Let G be a connected graph of order n with minimum degree at least 2, and let k be an integer with $1 \le k \le n$. Then,*

$$r_k(G, \gamma_{\times 2}) \le \frac{3n}{4} + \frac{2k}{7}.$$

By Theorem 2, the upper bound of does not necessarily hold if G is a disconnected graph, unless we insist that no component is a 5-cycle or that the subset K of k vertices of G contains at least one vertex from each component.

The bound of Theorem 4 is sharp for graphs of arbitrarily large order. To see this, let G be a connected graph of order $n = 28r$ obtained from the disjoint union of $r \ge 1$ copies of $S(K_7)$, the subdivision of K_7, by adding any number of edges joining non-degree-2 vertices. Let K be the set of degree-2 vertices in G, and let $k = |K| = 21r$. Then every DDS of G that contains K must contain all but one vertex from each copy of $S(K_7)$ in G, and so $r(G, K, \gamma_{\times 2}) = n - r$. Thus, by Theorem 4, $r_k(G, \gamma_{\times 2}) = n - r = 3n/4 + 2k/7$.

3.1 Proof of Theorem 4

The value of $\gamma_{\times 2}(C_n)$ for a cycle C_n was established by Harary and Haynes [7] who showed that for $n \ge 3$, $\gamma_{\times 2}(C_n) = \lceil 2n/3 \rceil$. Using a similar proof (which we omit) we can determine the double domination number of a path P_n.

Proposition 5 *For $n \ge 2$, $\gamma_{\times 2}(P_n) = \lceil 2(n + 1)/3 \rceil$.*

The proof of Theorem 4 is in general by induction. We need to handle one case separately.

3.1.1 Subdivision Graphs

We will need the following lemma:

Lemma 6 *Let H be a graph with p vertices and q edges. Let $a = 3/4$ and $b = 1/28$. Then the vertex cover number $\alpha(H) \le a\,p + b\,q$.*

Proof. By induction. There are two (overlapping) possibilities.

Case 1: H has maximum degree $\Delta \geq 7$. Let v be a vertex of maximum degree and let H' be the graph $H - v$. Then

$$\alpha(H) \leq \alpha(H') + 1 \leq a(p-1) + b(q - \Delta) + 1 = a\,p + b\,q + R(\Delta)$$

where $R(\Delta) = 1 - a - b\Delta$. Clearly $R(\Delta)$ is decreasing in Δ and $R(7) = 0$.

Case 2: H has minimum degree $\delta \leq 6$. Let v be a vertex of minimum degree and let H' be the graph formed from H by the deletion of v and its δ neighbors. Then

$$\alpha(H) \leq \alpha(H') + \delta \leq a(p - (\delta+1)) + b(q - \tbinom{\delta+1}{2}) + \delta = a\,p + b\,q + S(\delta)$$

where $S(\delta) = \delta - a(\delta+1) - b\binom{\delta+1}{2}$. It is easily checked that $S(\delta)$ is increasing as a function of δ for $\delta \leq 6$, and $S(6) = 0$. This gives the desired result. QED

Equality in Lemma 6 occurs for H being K_7 or K_8. (We comment that Lemma 6 also holds for $a = d(d+3)/(d^2+5d+2)$ and $b = 2/(d^2+5d+2)$ where d is any nonnegative integer.)

Lemma 7 *Let H be any loopless multigraph with minimum degree at least 2, and let $S(H)$ be its subdivision. Let K be the subdivision vertices, $|K| = k$. Then*

$$r(S(H), K, \gamma_{\times 2}) \leq 3n/4 + 2k/7,$$

where n is the order of $S(H)$.

Proof. Let C be a vertex cover of H. Then $C \cup K$ is a DDS of $S(H)$: every original vertex is adjacent to at least two members of K, and every vertex of K is adjacent to at least one member of C.

By the above lemma applied to the underlying simple graph of H, it follows that

$$r(S(H), K, \gamma_{\times 2}) \leq k + (3p/4 + k/28) = 3n/4 + 2k/7$$

where p is the order of H. QED

3.1.2 Main Lemma

We define a vertex as *small* if it has degree 2; otherwise it is *large*. We say a prescribed vertex is *troublesome* if it is small but both its neighbors are large. The main induction is provided by the following lemma.

Lemma 8 *Let G be a connected graph of order n with minimum degree at least 2, such that the set $\mathcal{L}$ of large vertices is an independent set. Let K be a subset of the vertices, $|K| = k$ such that if $G = C_5$ then K is nonempty. Let T denote those vertices in K that are troublesome, $|T| = t$. Then,*

$$r(G, K, \gamma_{\times 2}) \leq f(n, k, t) := \frac{3n + k}{4} + \frac{t}{28}.$$

We proceed by induction on n. If $n = 3$, then $G = C_3$ and $t = 0$, and it is straightforward to check that $r_k(C_3, K, \gamma_{\times 2}) \leq (3n + k)/4$ (with equality if and only if $k = 3$). This establishes the base cases. Further, if $k = 0$, then the result is given by Theorem 2.

So let G be a connected graph of order $n \geq 4$ with minimum degree at least 2 and K a nonempty subset of the vertices. Let $\mathcal{L}$ denote the set of large vertices of G. By assumption, $\mathcal{L}$ is an independent set.

The following observation is trivial.

Observation 1 *If graph G has a subgraph H that has 5 vertices and contains a 5-cycle, then any DDS S of $G - H$ can be extended to a DDS of G by adding at most three vertices provided some vertex of $N_G(H) - H$ is in S.*

We observe that we may apply the inductive hypothesis of Lemma 8 to a disconnected graph G provided every component of G that is a C_5 contains a prescribed vertex. (The values n, k, t are simply the sum of the values of the components.)

Observation 2 *We may assume that any small vertex in K that has a small neighbor is in a triangle.*

Proof. Assume that b is a small vertex in K with small neighbor c that is not in a triangle. Let a be b's other neighbor and let d be c's other neighbor. Since b is not in a triangle, $a \neq d$.

Suppose $c \notin K$ or d is small. Then let $G' = (G - b) \cup \{ac\}$ (that is, with b contracted out). Then, G' satisfies the hypothesis of the lemma. By the assumption of the case, the contraction does not create a troublesome vertex. Applying the inductive hypothesis to G', there exists a DDS S' of G containing $K \setminus \{b\}$ with $|S'| \leq f(n-1, k-1, t) = f(n, k, t) - 1$. Let $S = S' \cup \{b\}$; then since at least one of a and c is in S', S is a DDS of G containing K. Thus $r(G, K, \gamma_{\times 2}) \leq f(n, k, t)$.

Suppose $c \in K$ and d is large. Then let $G' = G - \{b, c\}$. By the assumption on G, this removal does not create a troublesome vertex. If each component of G' satisfies the lemma hypothesis, then applying the inductive hypothesis to each component of G', there exists a DDS S' of G containing $K \setminus \{b, c\}$ with $|S'| \leq f(n-2, k-2, t) = f(n, k, t) - 2$. Let $S = S' \cup \{b, c\}$; this is a DDS of G containing K, and so $r(G, K, \gamma_{\times 2}) \leq f(n, k, t)$.

If some component C of G' is a K-free C_5, then let $G'' = G' - C$. At least one vertex of C is adjacent to $\{b, c\}$; thus one gets a DDS of G by adding b, c and only three vertices to a DDS of G''. The arithmetic is similar to above. QED

By Observation 2, we may assume that G is not a cycle. Let C be any component of $G - \mathcal{L}$; it is a path. If C has only one vertex, or has at least two vertices but the two ends of C are adjacent in G to different large vertices, then we say that C is a 2-*path*. Otherwise we say that C is a 2-*handle*.

Observation 3 *We may assume that any small vertex that has two large neighbors is in K.*

Proof. Assume that b is a small vertex with two large neighbors u and w that is not in K. Let $G' = G - b$ and $K' = K \cup \{u, w\}$. Then, $\delta(G') \geq 2$ and G' has at most t troublesome vertices.

If G' is disconnected, then both components contain a prescribed vertex. Applying the inductive hypothesis to (each component of) G', there exists a DDS S' of G containing K' with $|S'| \leq f(n-1, k+2, t) < f(n, k, t)$. Since $\{u, w\} \subseteq S'$, the set S' is also a DDS of G, and so $r(G, K, \gamma_{\times 2}) < f(n, k, t)$. QED

Observation 4 *We may assume that every 2-path has only one vertex.*

Proof. Assume that there is a 2-path $P\colon v_1, v_2, \ldots, v_r$ with $r \geq 2$. Let u be the large vertex adjacent to v_1 and v the large vertex adjacent to v_r. By Observation 2, none of the vertices on P is in K.

Let $G' = G - V(P)$ and $K' = K \cup \{u, v\}$. Then, $\delta(G') \geq 2$ and G' has at most t troublesome vertices. Every component of G' satisfies the hypothesis of the lemma; thus by the inductive hypothesis, there exists a DDS set S' of G' containing K' with $|S'| \leq f(n-r, k+2, t)$.

Suppose $r \geq 4$. Then adding to the set S' a minimum DDS of the path $P - \{v_1, v_r\}$ produces a DDS of G (as a DDS of the path necessarily contains the end-vertices). Thus, by Proposition 5, one obtains a DDS of G containing K of cardinality

$$|S'| + \gamma_{\times 2}(P_{r-2}) \leq f(n-r, k+2, t) + \lceil 2(r-1)/3 \rceil = f(n, k, t) + A(r)$$

where $A(r) = (2-3r)/4 + \lceil 2(r-1)/3 \rceil$. We observe that $A(r) \leq 0$ with equality if and only if $r = 6$. Thus, one obtains a DDS of G containing K of cardinality at most $f(n,k,t)$.

Suppose $r = 2$. Then $S' \cup \{v_1\}$ is a DDS of G containing K of cardinality $|S'| + 1 \leq f(n-2, k+2, t) + 1 = f(n,k,t)$.

Suppose $r = 3$. Using the inductive hypothesis, we can show that there exists a DDS set S' of G' that contains $K \cup \{u\}$ or $K \cup \{v\}$ with $|S'| \leq f(n-3, k+1, t)$. (If G' is connected or if G' is disconnected with both components containing a vertex of K, this follows immediately by induction. Otherwise, if G' is disconnected and the component containing u has no vertex of K, then set $K' = K \cup \{u\}$, else set $K' = K \cup \{v\}$; then there exists a DDS set S' of G' that contains K' of the desired cardinality.) So without loss of generality we may assume that $S' \supseteq K \cup \{u\}$. Then $S' \cup \{v_2, v_3\}$ is a DDS of G containing K of cardinality $|S'| + 2 \leq f(n,k,t)$, as desired. QED

Observation 5 *We may assume that no vertex in a* 2*-handle is prescribed (i.e., in* K*).*

Proof. Assume that there is a small vertex $b \in K$ that lies in a 2-handle C. Assume the ends of C are adjacent to $v \in \mathcal{L}$. By Observation 2, we may assume that C is a triangle; say $V(C) = \{a, b, v\}$.

Suppose $\deg v \geq 4$. Let $G' = G - \{a, b\}$. Then, G' is connected with $\delta(G') \geq 2$. If $a \notin K$, then by the inductive hypothesis, there exists a DDS set S' of G' containing $(K \setminus \{b\}) \cup \{v\}$ with $|S'| \leq f(n-2, k, t)$. The set $S' \cup \{b\}$ is a DDS of G containing K of cardinality less than $f(n,k,t)$. If $a \in K$, then by the inductive hypothesis, there exists a DDS set S'' of G' that contains $K \setminus \{a, b\}$ with $|S''| \leq f(n-2, k-2, t)$. The set $S \cup \{a, b\}$ is a DDS of G containing K with $|S| \leq f(n,k,t)$.

Suppose $\deg v = 3$. Let c be the small neighbor of v not on C, and let u be the other neighbor of c. Since c is in a 2-path, by Observation 4, $u \in \mathcal{L}$. By Observation 3, $c \in K$. Let $G' = G - \{a, b, c, v\}$. Then, G' is a connected graph with $\delta(G') \geq 2$, and the number of troublesome vertices in G' is at most $t - 1$.

Define the set K' as follows. If $a \in K$, then $K' = (K \setminus \{a, b, c\}) \cup \{u\}$; otherwise $K' = K \setminus \{a, b, c\}$. Note that $|K'| \leq k-2$. Unless $K' = \emptyset$ and $G' = C_5$, by the inductive hypothesis, there exists a DDS set S' containing K' with $|S'| \leq f(n-4, k-2, t-1) < f(n,k,t) - 3$. Then, $S = S' \cup \{b, c, v\}$ is a DDS of G containing K with $|S| < f(n,k,t)$. The case that $K' = \emptyset$ and $G' = C_5$ is easily verified. QED

Observation 6 *We may assume that there is no* 2*-handle.*

Proof. Suppose there is a 2-handle C. Say its ends have common neighbor $v \in \mathcal{L}$. By the above observation, no vertex of C is in K.

Assume $\mathcal{L} = \{v\}$. Then, G can be constructed from $r \geq 2$ disjoint cycles by identifying a set of r vertices, one from each cycle, into one vertex v. By Observation 5, $K = \{v\}$. By Theorem 2, there exists a DDS S of G with $|S| \leq 3n/4$. By rotating the vertices of S around C, if necessary, we may assume that $v \in S$, and so the desired result follows.

Assume that $|\mathcal{L}| \geq 2$. Then by connectivity there exists a 2-path with an end adjacent to v. By Observation 4, this 2-path has only one vertex, say x. By Observation 3, $x \in K$. Let $G' = G - x$. Then, G' is a connected graph with $\delta(G') \geq 2$. Provided no component of G' is a K-free C_5, by the inductive hypothesis, there exists a DDS S' of G' that contains the set $K \setminus \{x\}$ with $|S'| \leq f(n-1, k-1, t-1) < f(n, k, t) - 1$. By rotating the vertices of S' around C, if necessary, we may assume that $v \in S'$. Hence, $S' \cup \{x\}$ is a DDS of G containing K with $|S'| + 1 < f(n, k, t)$. The case that one or more components of G' is a K-free C_5 is easily handled similarly. QED

Observation 7 *We may assume that no large vertex is in K.*

Proof. Suppose there is a large vertex v in K. Let a be any neighbor of v. By our earlier observations, we may assume that a belongs to a 2-path of length 1 and is in K. Let $G' = G - a$. Then, G' is a graph with $\delta(G') \geq 2$. Provided no component of G' is a K-free C_5, by the inductive hypothesis, there exists a DDS S' of G' that contains the set $K \setminus \{a\}$ with $|S'| \leq f(n-1, k-1, t-1)$. Since $v \in S$, the set $S \cup \{a\}$ is a DDS of G containing K with $|S| < f(n, k, t)$. The case that one or more components of G' is a K-free C_5 is easily handled similarly QED

By the above observations, we may assume that G has the following form. It is $S(H)$ where H is a multigraph with $|\mathcal{L}|$ vertices and k edges with minimum degree at least 3. Further, K is the set of small (spliced) vertices of G, and $t = |K| = k$. By Lemma 7, $r(G, K, \gamma_{\times 2}) \leq 3n/4 + 2k/7$, which is what is required, since $2/7 = 1/4 + 1/28$.

3.1.3 Proof of Theorem 4

We proceed by induction. Let G be a connected graph with minimum degree at least 2, and K a nonempty subset of the vertices. If there is no edge joining two large vertices, then the theorem follows from Lemma 8. Hence we may assume that there is an edge e joining two large vertices. Since the restricted double domination number of a graph cannot decrease if edges are removed, we may remove the edge unless its removal creates a K-free C_5 component C. If both ends of e are in C, then G is the 5-vertex house graph and the result is easily checked. If one end of e, say v, is outside C, then let $G' = G - V(C)$ and $K' = K \cup \{v\}$. By the inductive hypothesis, there exists a DDS S' of G that contains K' with $|S'| \leq 3(n-5)/4 + 2(k+1)/7 < 3n/4 + 2k/7 - 3$. Since $v \in S$, the set S' can be extended to a DDS S of G that contains K with $|S| < 3n/4 + 2k/7$, as required.

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References

[1] G. Chartrand, L. Eroh, F. Harary, and P. Zhang, How large can the domination numbers of a graph be? *Australas. J. Combin.* **21** (2000), 23–35.

[2] E.J. Cockayne, B. Gamble, and B. Shepherd, An upper bound for the k-domination number of a graph. *J. Graph Theory* **9** (1985), 533–534.

[3] J.E. Dunbar, D.G. Hoffman, R.C. Laskar, and L.R. Markus, α-domination. *Discrete Math.* **211** (2000), 11–26.

[4] J.F. Fink, and M.S. Jacobson, n-domination in graphs. *Graph theory with Applications to Algorithms and Computer Science* (Kalamazoo 1984), 283–300, Wiley, New York, 1985.

[5] J. Harant and M.A. Henning, On double domination in graphs. *Discussiones Math. Graph Theory* **25** (2005), 29–34.

[6] J. Harant, A. Pruchnewski, and M. Voigt, On dominating sets and independent sets of graphs. *Combin. Prob. Comput.* **8** (1998), 547–553.

[7] F. Harary and T.W. Haynes, Double domination in graphs. *Ars Combin.* **55** (2000), 201–213.

[8] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[9] M.A. Henning, Restricted domination in graphs. *Discrete Math.* **254** (2002), 175–189.

[10] M.A. Henning, Restricted total domination in graphs. *Discrete Math.* **289** (2004), 25–44.

[11] M.A. Henning, Graphs with large double domination numbers. *Discussiones Math. Graph Theory* **25** (2005), 13–28.

[12] C.-S. Liao and G.J. Chang, Algorithmic aspect of k-tuple domination in graphs. *Taiwanese J. Math.* **6** (2002), 415–420.

[13] C.-S. Liao and G.J. Chang, k-tuple domination in graphs. *Inform. Process. Lett.* **87** (2003), 45–50.

[14] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two. *J. Graph Theory* **13** (1989), 749–762.

[15] B. Reed, Paths, stars and the number three. *Combin. Probab. Comput.* **5** (1996), 277–295.

[16] L.A. Sanchis, Bounds related to domination in graphs with minimum degree two. *J. Graph Theory* **25** (1997), 139–152.

[17] L.A. Sanchis, Relating the size of a connected graph to its total and restricted domination numbers. *Discrete Math.* **283** (2004), 205–216.

[18] C. Stracke and L. Volkmann, A new domination conception. *J. Graph Theory* **17** (1993), 315–323.